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Asymptotic stability of viscous shock profiles to Burgers equation with singular super-fast diffusion

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Abstract

This paper is concerned with the large time behaviors of solutions to the Burgers equation of porous-media type in the form of $u_t + f(u)_x = (u^{m-1}u_x)_x$, where the diffusion $(u^{m-1}u_x)_x = (\frac{u_x}{u^{1+|m|}})_x$ with $m < 0$ possesses the strong singularity of fast-diffusion at $u = 0$. The main issue of the paper is to show the asymptotic stability of viscous shock profiles with the constant states $u_- > u_+ = 0$, where the strong singularity exhibits for the equation when the viscous shock wave reaches the singular point $u_+ = 0$. To overcome such a strong singularity for wave stability, we first need to analyze the rate of the viscous shock wave to $u_+ = 0$, then we artfully choose some weight functions which are closely dependent on the decay rate of the viscous shock wave to the singular point $u_+ = 0$, and further show the wave stability by the weighted-energy-method.

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1. Introduction

Subsequently to the previous studies [21,31], we investigate the Cauchy problem for the one-dimensional Burgers equation with singular super-fast diffusion

$$\begin{cases} u_t + f(u)_x = (u^{m-1}u_x)_x, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x) \to u_\pm, & x \in \mathbb{R},\ \text{as } x \to \pm\infty, \end{cases} \tag{1.1}$$

where f is a smooth flux function, m is the index for porous media diffusion with $m < 0$, the so-called super-fast diffusion, and $u_\pm$ are the state constants satisfying $0 = u_+ < u_-$. Thus, the equation (1.1) possesses the singularity at $u = 0$. The main issue of the paper is to show the asymptotic stability of the viscous shock waves to the Burgers equation (1.1) with the singular super-fast diffusion $m < 0$.

When $m = 1$, the equation (1.1) is reduced to the regular Burgers equation:

$$u_t + f(u)_x = u_{xx}.$$

It was initially proposed by Bateman [1] to model the behavior of viscous fluids and was later adopted by Burgers [3,4] to study shock profiles in turbulence, eventually becoming known as the Burgers equation. As a significant nonlinear partial differential equation in fluid dynamics, a wide range of physical phenomena, including traffic flow [29], jet flow [20] and molecular interface growth [17], frequently incorporate its application.

When $m < 1$, the equation (1.1) is represented for the porous media flow [27] in the form of

$$u_t + f(u)_x = (\frac{u_x}{u^{1-m}})_x,$$

which is the case of fast diffusion. While, when $0 < m < 1$, the equation is called the regular-fast diffusion equation; and when $m = 0$, it is said to be the critical-fast diffusion:

$$u_t + f(u)_x = (\ln u)_{xx};$$

and when $m < 0$, it is then said to be the super-fast diffusion:

$$u_t + f(u)_x = (\frac{u_x}{u^{1+|m|}})_x.$$

All of these fast diffusion equations possess the singularity at $u = 0$, and the singularity is strongest for the case of super-fast diffusion with $m < 0$.

On the other hand, when $m > 1$, that is the case of slow diffusion for the porous media flow, the equation (1.1) becomes degenerate in nonlinear diffusion, and the solutions usually loss their regularity at $u = 0$, namely, these solutions are not differentiable.

Since the pioneering work of Burgers [3,4], the study of shock waves $U(x - st)$, where s is the speed of shocks, has been one of the most important research topics in fluid dynamics, driven by the observation of numerous wave phenomena in nature, such as the propagation of seismic waves and the formation of ocean waves. In mathematics, most of the achievements are concerned with the case of linear diffusion (i.e. $m = 1$) by various methods.

When $m = 1$, Il'in-Oleinik [11] first employed the maximum principle to study the stability of viscous shock profiles, provided the flux function $f(u)$ is convex. Independently, Sattinger utilizing the spectral analysis showed a similar result in [28]. Furthermore, Kawashima-Matsumura [18] and Nishihara [26] introduced a new different approach based on an energy method to establish the stability as well as convergence rates. In some circumstances the decay rates are optimal. In all afore-mentioned papers, the proof relied on the convexity of $f(u)$. However, Jones-Gardner-Kapitula [12] removed this convexity condition to prove the stability of shock profiles by applying the spectral analysis. Almost at the same time, Kawashima-Matsumura [19] further investigated the stability of shock waves by L^2-weighted energy method when $f(u)$ is nonconvex, without convergence rates, which were then obtained by Matsumura-Nishihara [23] and Mei [24], respectively, while, who also addressed the case of degenerate shock condition $f'(u_+) = s < f'(u_-)$ for the first time. The L^1 stability of these shock waves was further established by Freistühler-Serre [6]. Moreover, Howard [7–9], from the construction of Green's function, gave the details of stability for these shock waves. More recently, the relative entropy method, developed by Kang-Vasseur [14] and Kang-Vasseur-Wang [15,16], was formulated to prove the L^2 contraction of the solution towards the shock waves. For the absence of genuine nonlinearity, the stability was obtained by Weinberger [30]. Very recently, in the non-convex flux case, Huang-Wang-Zhang [10] significantly proved the new stability of the compositions of viscous shock waves and rarefaction waves.

When the Burgers equation (1.1) with porous media diffusion (i.e. $m \neq 1$), the understanding of its dynamics is quite limited. For the slow diffusion case (i.e. $m > 1$), since the viscous shock waves become sharp caused by the degeneracy, and lose their regularity, there is only one work related to the stability of sharp shocks of Burgers equation, see the L^1-stability by Osher-Ralston [27] based on the semigroup method.

Regarding the fast diffusion case (i.e. $m < 1$), recently we [21] first considered the Burgers equation with critical-fast diffusion (i.e. $m = 0$), and proved the stability of shock waves of (1.1) by the weighted energy method. Then, the third author of this paper joint with Xu et al. [31] treated the regular-fast diffusion case with $0 < m < 1$, where the chosen weights for the case of nondegenerate shock condition are stronger than that of the degenerate shock condition. Furthermore, a convexity assumption was proposed in [31] for the stability of nondegenerate shock in the regular-fast diffusion case ($0 < m < 1$).

For the super-fast diffusion case (i.e. $m < 0$), the stability of viscous shock waves $U(x - st)$, s is the wave speed, has been open as we know. To investigate such a problem is the main issue of this paper. Here, we are going to achieve two objectives: one is to show the existence and uniqueness (up to a shift) of monotone shock profiles connecting the zero-end-state, in particular, the decay rates of the viscous shocks $U(x - st)$ to $u_+ = 0$ as $\xi := x - st \to \infty$; the second to prove the time-asymptotic stability of these shock profiles. Here, the main difficulty is the strong singularity of the equation at $u = 0$. In order to overcome such an obstacle, we adopt the technical weighted-energy method and the cut-off technique, where, to choose different weights in different stages plays a key role in the proof. By observing the structure of the equation, we recognize that these weights should be carefully selected, which are closely dependent on the decay properties to $u_+ = 0$ of the viscous shock waves as $x \to \infty$. On the other hand, the cut-off technique is applied to establish the *a priori* estimates, in order to avoid the singularity at $u = 0$ when $x \to +\infty$. This work develops the previous studies on the stability of shock profiles to

Burgers equation with fast diffusion in the cases $m = 0$ [21] and $0 < m < 1$ [31]. Remarkably, considered in this paper, the singularity of super-fast diffusion $\left(u^{m-1}u_x\right)_x$ with $m < 0$ is much stronger than that of regular-fast diffusion with $0 < m < 1$ and of critical-fast diffusion with $m = 0$, and the convexity condition proposed in [31] can be removed.

Regarding the reaction-diffusion equations with fast/slow diffusions, they have been intensively studied recently. When $m > 1$ (the slow diffusion case), Biró [2] and Kamin-Rosenau [13] first showed the stability of sharp traveling waves for Fisher-KPP equations, see also recent significant developments by Du-Quirós-Zhou [5] for spreading waves, and by Lou-Zhou [22] for a complete classification of the asymptotic behaviors of solutions for porous medium equation with monostable, bistable and combustion types of reactions, and by Xu-Ji-Mei-Yin [32–35] for the stability of sharp traveling waves with monostable/bistable/combustion types of reactions. The reaction terms in these reaction-diffusion equations are crucial for proving the stability of sharp traveling waves, since specific reactions ensure the validity of comparison principle. In contrast, the Burgers equation (1.1) lacks of comparison principle, and renders the monotonic techniques inapplicable. To the best of our knowledge, the L^∞ stability of sharp viscous shock waves for Burgers equation still remains open.

The rest of this paper is arranged as follows. In Section 2, we state our main results on the existence and nonlinear stability of viscous shock wave solutions to the Burgers equation (1.1). Section 3 is devoted to the proof of the existence of shock profiles. In Section 4, we prove the nonlinear stability results.

Notations. Let C be a generic positive constant that may vary between lines. We abbreviate the integrals $\int_{\mathbb{R}} f(x)\mathrm{d}x$ and $\int_0^t\int_{\mathbb{R}} f(x,\tau)\mathrm{d}x\mathrm{d}\tau$ as $\int f(x)$ and $\iint_0^t f(x,\tau)$, respectively. $H^k(\mathbb{R})$ is the k-th order Sobolev space on $\mathbb{R}$ with the norm $\|f\|_{H^k(\mathbb{R})} := \left(\sum_{j=0}^k \|\partial_x^j f\|^2_{L^2(\mathbb{R})}\right)^{1/2}$. $H^k_w(\mathbb{R})$ represents the weighted Sobolev space consisting of measurable functions f such that $\sqrt{w}\partial_x^j f \in L^2$ for $0 \le j \le k$ with the norm $\|f\|_{H^k_w(\mathbb{R})} := \left(\sum_{j=0}^k \int w(x)|\partial_x^j f|^2 dx\right)^{1/2}$. Denote $\langle x\rangle := \sqrt{1+x^2}$,

$$\langle x\rangle_+ = \begin{cases} \sqrt{1+x^2}, & x > 0, \\ 1, & x \le 0, \end{cases} \text{ and } \langle x\rangle_- = \begin{cases} 1, & x > 0, \\ \sqrt{1+x^2}, & x \le 0. \end{cases}$$

The weighted space $L^2_w(\mathbb{R})$ for the weight function $w = \langle x\rangle_+^\alpha$ is denoted by $L^2_{\langle x\rangle_+^\alpha}(\mathbb{R})$, where $\alpha > 0$ is a constant, and the corresponding norm is $\|\cdot\|_{L^2_{\langle x\rangle_+^\alpha}(\mathbb{R})}$. The weighted Sobolev space $H^k_{\langle x\rangle_+^\alpha}(\mathbb{R})$ are defined similarly. For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_w := \|\cdot\|_{L^2_w(\mathbb{R})}$, $\|\cdot\|_{\langle x\rangle_+^\alpha} := \|\cdot\|_{L^2_{\langle x\rangle_+^\alpha}(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$, $\|\cdot\|_{k,w} := \|\cdot\|_{H^k_w(\mathbb{R})}$ and $\|\cdot\|_{k,\langle x\rangle_+^\alpha} := \|\cdot\|_{H^k_{\langle x\rangle_+^\alpha}(\mathbb{R})}$. When two positive functions $f(x)$ and $g(x)$ satisfy $cg(x) \le f(x) \le Cg(x)$ with some positive constants c and C for x in an interval I or $x \to a$, then we simply write $f(x) \sim g(x)$ for $x \in I$ or $x \to a$.

2. Preliminaries and main results

Let $u_\pm$ be the state constants such that $u_- > u_+ = 0$. The shock profile of (1.1) is a non-constant self-similar solution in the form of

$$u(x,t) = U(z), \quad z = x - st, \tag{2.1}$$

where s is the speed of shock profile. Substituting this ansatz into (1.1), we get the equation of shock profiles

$$\begin{cases} -sU_z - \left(U^{m-1}U_z\right)_z + f(U)_z = 0, & z \in \mathbb{R}, \\ U(+\infty) = 0, \; U(-\infty) = u_-. \end{cases} \tag{2.2}$$

It is expected that $\lim\limits_{z\to\pm\infty} U^{m-1}U_z(z) = 0$. Then, integrating (2.2) in z over $\mathbb{R}$ yields the Rankine-Hugoniot condition

$$s = \frac{f(0) - f(u_-)}{0 - u_-}. \tag{2.3}$$

On the other hand, integrating (2.2) in z and combining the fact that $U(\pm\infty) = u_\pm$, we get

$$-sU - U^{m-1}U_z + f(U) = -su_\pm + f(u_\pm), \tag{2.4}$$

which gives an ordinary differential equation (ODE) satisfied by U

$$U_z = U^{1-m}\left[f(U) - f(u_\pm) - s(U - u_\pm)\right]. \tag{2.5}$$

Theorem 2.1 *(Existence of shock profiles). Let $m < 0$ and $u_- > u_+ := 0$. Denote*

$$g(u) \triangleq f(u) - f(u_\pm) - s(u - u_\pm). \tag{2.6}$$

Suppose that $f \in C^{\max\{(k_\pm+1),\, 3\}}(\mathbb{R})$ with integers $k_\pm \geq 0$, and that $g'(u_\pm) = \cdots = g^{(k_\pm)}(u_\pm) = 0$ with $g^{(k_\pm+1)}(u_\pm) \neq 0$, while u_- and s satisfy the Rankine-Hugoniot condition (2.3) and the generalized shock condition

$$g(u) < 0, \quad \text{for } u \in (0, u_-). \tag{2.7}$$

Then the equation (1.1) has a unique (up to a shift) smooth monotone shock profile $U(z)$ satisfying

$$U_z(z) < 0, \quad \forall z \in \mathbb{R}. \tag{2.8}$$

Moreover, it holds that

$$\begin{cases} U(z) \sim |z|^{-\frac{1}{1-m}}, & \text{as } z \to +\infty, \\ u_- - U(z) \sim \mathrm{e}^{-\lambda_-|z|}, & \text{as } z \to -\infty, \end{cases} \quad \text{if } f'(0) < s < f'(u_-), \tag{2.9}$$

$$\begin{cases} U(z) \sim |z|^{-\frac{1}{k_+ + 1 - m}}, & \text{as } z \to +\infty, \\ u_- - U(z) \sim \mathrm{e}^{-\lambda_-|z|}, & \text{as } z \to -\infty, \end{cases} \quad \text{if } f'(0) = s < f'(u_-), \tag{2.10}$$

and

$$\begin{cases} U(z) \sim |z|^{-\frac{1}{1-m}}, & \text{as } z \to +\infty, \\ u_- - U(z) \sim |z|^{-\frac{1}{k_-}}, & \text{as } z \to -\infty, \end{cases} \quad \text{if } f'(0) < s = f'(u_-), \tag{2.11}$$

where $\lambda_- = u_-^{1-m}(f'(u_-) - s)$.

Remark 2.1. Due to the Rankine-Hugoniot condition $s = \frac{f(u_+)-f(u_-)}{u_+-u_-}$ with $u_+ = 0$, we have

$$su_+ - f(u_+) = su_- - f(u_-).$$

Thus

$$g(u) = f(u) - f(u_+) - s(u - u_+) = f(u) - f(u_-) - s(u - u_-).$$

Remark 2.2. As $z \to +\infty$, the wave U exhibits an algebraic decay rate, differing from the classical Burgers equation, where the waves decay at an exponential rate (see Ref. [23]). This behavior verifies the principle of "fast diffusion"commonly associated with the range $m < 1$, and indicates that the solution diffuses faster for smaller m. Thus, in this sense the diffusion in the case $m < 0$ is referred to as "super-fast diffusion".

Remark 2.3. One can see that the generalized shock condition (2.7) leads to Lax's degenerate entropy condition

$$f'(0) \le s \le f'(u_-), \tag{2.12}$$

which includes the nondegenerate shock condition $f'(0) < s < f'(u_-)$, and the degenerate shock conditions $f'(0) = s < f'(u_-)$ and $f'(0) < s = f'(u_-)$. For more details of the proof, we refer to Remark 2.1 of [21].

Since nondegenerate and degenerate shock conditions correspond to different decay rates of shock profiles, we investigate the stability of $U(x - st)$ into three different cases. The singularity of the equation (1.1), being the most pronounced among the fast diffusion cases, presents a significant challenge for the problem. In this paper, we shall develop some novel ideas to resolve this strong singularity.

Theorem 2.2 *(Stability of shock profiles). Let* $m < 0$ *and* $U(z)$ $(z = x - st)$ *be a shock profile obtained in Theorem 2.1. Assume that* $u_0(x) - U(x) \in L^1(\mathbb{R})$. *Set*

$$\phi_0(x) := \int_{-\infty}^{x} (u_0(y) - U(y + x_0))\,\mathrm{d}y, \tag{2.13}$$

where $x_0 := -\frac{1}{u_-}\int_{-\infty}^{+\infty}(u_0(x) - U(x))\,\mathrm{d}x$. *Then the following stability results hold.*

(i) *If* $f'(0) < s < f'(u_-)$, *there exists a constant* $\epsilon_1 > 0$ *such that if* $\|\phi_0\| + \|\phi_{0x}\|_{\langle x\rangle_+^{\alpha_1}} + \|\phi_{0xx}\|_{\langle x\rangle_+^{\alpha_2}} \le \epsilon_1$, *then the Cauchy problem* (1.1) *has a unique global solution* $u(x,t)$ *satisfying*

$$u - U \in C\left([0,\infty); L^2_{\langle z\rangle_+^{\alpha_1}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle_+}\right),$$

$$(u-U)_x \in C\left([0,\infty); L^2_{\langle z\rangle_+^{\alpha_2}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\alpha_3}}\right), (u-U)_{xx} \in L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\alpha_4}}\right),$$

where $U = U(x - st + x_0)$ is the shifted shock profile, $\alpha_1 = \frac{2}{1-m}$, $\alpha_2 = 2$, $\alpha_3 = \frac{3-m}{1-m}$ and $\alpha_4 = \frac{3-3m}{1-m}$. Furthermore, the solution has the following asymptotic convergence

$$\sup_{x\in\mathbb{R}} |u(x,t) - U(x - st + x_0)| \to 0 \quad \text{as } t \to +\infty. \tag{2.14}$$

(ii) *If $f'(0) = s < f'(u_-)$, there exists a constant $\epsilon_2 > 0$ such that if $\|\phi_0\|_{\langle x\rangle_+^{\beta_1}} + \|\phi_{0x}\|_{\langle x\rangle_+^{\beta_2}} + \|\phi_{0xx}\|_{\langle x\rangle_+^{\beta_3}} \le \epsilon_2$, then the Cauchy problem (1.1) has a unique global solution $u(x,t)$ satisfying*

$$u - U \in C\left([0,\infty); L^2_{\langle z\rangle_+^{\beta_2}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle_+}\right),$$

$$(u-U)_x \in C\left([0,\infty); L^2_{\langle z\rangle_+^{\beta_3}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\beta_4}}\right), (u-U)_{xx} \in L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\beta_5}}\right),$$

where $U = U(x - st + x_0)$ is the shifted shock profile, $\beta_1 = \frac{k_+}{k_++1-m}$, $\beta_2 = \frac{2}{k_++1-m}$, $\beta_3 = \frac{2(1-m)}{k_++1-m}$, $\beta_4 = \frac{3-m}{k_++1-m}$ and $\beta_5 = \frac{3-3m}{k_++1-m}$. Furthermore, the solution has the following asymptotic convergence

$$\sup_{x\in\mathbb{R}} |u(x,t) - U(x - st + x_0)| \to 0 \quad \text{as } t \to +\infty. \tag{2.15}$$

(iii) *If $f'(0) < s = f'(u_-)$, there exists a constant $\epsilon_3 > 0$ such that if $\|\phi_0\|_{\langle x\rangle_-} + \|\phi_{0x}\|_{\langle x\rangle_+^{\alpha_1}} + \|\phi_{0xx}\|_{\langle x\rangle_+^{\alpha_2}} \le \epsilon_3$, then the Cauchy problem (1.1) has a global solution $u(x,t)$ satisfying*

$$u - U \in C\left([0,\infty); H^1_{\langle z\rangle_+^{\alpha_1}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle}\right),$$

$$(u-U)_x \in C\left([0,\infty); L^2_{\langle z\rangle_+^{\alpha_2}}\right) \cap L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\alpha_3}}\right), (u-U)_{xx} \in L^2\left((0,\infty); L^2_{\langle z\rangle_+^{\alpha_4}}\right),$$

where $U = U(x - st + x_0)$ is the shifted shock profile. Furthermore, the solution has the following asymptotic convergence

$$\sup_{x\in\mathbb{R}} |u(x,t) - U(x - st + x_0)| \to 0 \quad \text{as } t \to +\infty. \tag{2.16}$$

3. Existence of the shock profile

In this section, we prove Theorem 2.1, namely the existence of a shock profile $U(x-st)$ connecting 0 and u_-. We shall show that the equation (1.1) admits a shock profile if u_- and s satisfy the Rankine-Hugoniot condition (2.3) and the generalized shock condition (2.7).

Proof of Theorem 2.1. Define

$$h(U) \triangleq U^{1-m}\left[f(U)-f(u_\pm)-s(U-u_\pm)\right]. \tag{3.1}$$

Thus, $h(U)=U^{1-m}g(U)$, where g is given by (2.6). By virtue of (2.5) and (3.1), one can see that $U_z=h(U)$. We only need to solve the following ordinary differential equation globally

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}U}=\dfrac{1}{h(U)}, \\ z(\frac{u_-}{2})=0. \end{cases} \tag{3.2}$$

Here, we assume that $U(0)=\frac{u_-}{2}$. Integrating (3.2) gives

$$z=\int_{\frac{u_-}{2}}^{U}\frac{1}{h(\tau)}d\tau \triangleq H(U).$$

By the generalized shock condition (2.7), it is evident that the function $z=H(U)$ is monotonically decreasing for any given $U\in(0,u_-)$. Noting $g(0)=0$, $g'(0)=\cdots=g^{(k_+)}(0)=0$ and $g^{(k_++1)}(0)\neq 0$ with an integer $k_+\geq 0$, by Taylor's formula, we have

$$\begin{aligned} g(U)&=g(0)+g'(0)U+\cdots+\frac{g^{(k_+)}(0)}{k_+!}U^{k_+}+\frac{g^{(k_++1)}(\xi)}{(k_++1)!}U^{k_++1} \\ &=\frac{g^{(k_++1)}(\xi)}{(k_++1)!}U^{k_++1} \text{ for some } \xi\in(0,U). \end{aligned}$$

Since $g\in C^{\max\{(k_\pm+1),\ 3\}}(\mathbb{R})$, one can see that

$$\lim_{U\to 0^+} g^{(k_++1)}(\xi)=g^{(k_++1)}(0)\neq 0.$$

Thus,

$$|g(U)|\sim U^{1+k_+} \text{ as } U\to 0^+. \tag{3.3}$$

Similarly, we have

$$|g(U)|\sim |U-u_-|^{1+k_-} \text{ as } U\to u^-. \tag{3.4}$$

By (3.3) and the fact that $m<0$, we get

$$\lim_{U\to 0^+}\int_{\frac{u_-}{2}}^{U}\frac{1}{h(\tau)}\mathrm{d}\tau=\lim_{U\to 0^+}\int_{U}^{\frac{u_-}{2}}\frac{1}{\tau^{1-m}|g(\tau)|}\mathrm{d}\tau=+\infty.$$

By (3.4), there exists a small constant $\delta_0>0$ such that for any $U\in(u_--\delta_0,u_-)$, it holds that

$$|g(U)|\le C|U-u_-|^{1+k_-}\text{ for some constant }C>0. \tag{3.5}$$

Since $h(U)<0$, it follows that for $U\in(\frac{u_-}{2},u_-)$

$$\begin{aligned}\int_{\frac{u_-}{2}}^{U}\frac{1}{h(\tau)}\mathrm{d}\tau=-\int_{\frac{u_-}{2}}^{U}\frac{1}{\tau^{1-m}|g(\tau)|}\mathrm{d}\tau\le-\frac{1}{u_-^{1-m}}\int_{\frac{u_-}{2}}^{u_--\delta_0}\frac{1}{|g(\tau)|}\mathrm{d}\tau-\frac{1}{u_-^{1-m}}\int_{u_--\delta_0}^{U}\frac{1}{|g(\tau)|}\mathrm{d}\tau\\ \triangleq I_1+I_2.\end{aligned}\tag{3.6}$$

Obviously, I_1 is a finite constant. By (3.5), I_2 satisfies

$$I_2\le-\frac{1}{Cu_-^{1-m}}\int_{u_--\delta_0}^{U}\frac{1}{(u_--\tau)^{1+k_-}}\mathrm{d}\tau=\begin{cases}\frac{\ln(u_--U)-\ln\delta_0}{Cu_-^{1-m}}, & \text{if } k_-=0,\\ \frac{\delta_0^{-k_-}-(u_--U)^{-k_-}}{Ck_-u_-^{1-m}}, & \text{if } k_->0,\end{cases}$$

which along with (3.6) leads to

$$\lim_{U\to u_-^-}\int_{\frac{u_-}{2}}^{U}\frac{1}{h(\tau)}\mathrm{d}\tau=\lim_{U\to u_-^-}(I_1+I_2)=-\infty.$$

Therefore, according to the inverse function theorem, there exists a unique continuous function H^{-1} such that

$$U=H^{-1}(z),\quad \forall z\in(-\infty,+\infty).$$

Thus, the existence of the shock profile $U(z)$ is proved. Furthermore, by virtue of (2.7) and (2.5), we get (2.8).

We now analyze the convergence rates of $U(z)$ as $z\to\pm\infty$. For the case $f'(0)<s<f'(u_-)$, we write (2.5) as

$$\frac{dU}{dz}=U^{2-m}(f'(0)-s)+U^{3-m}F_1(U)\text{ on } z>0, \tag{3.7}$$

where $F_1(U)\triangleq\frac{f(U)-f(0)-f'(0)U}{U^2}$. Recalling $f\in C^{\max\{(k_\pm+1),\,3\}}(\mathbb{R})$, by Taylor's formula, we get

$$|F_1(U(z))|\le C_1,\quad \forall z\in\mathbb{R}.$$

Since $\lim_{z\to+\infty} U(z)=0$, by continuity, given $0<\epsilon<\frac{s-f'(0)}{1+2C_1}$, there exists $z_1>0$ such that

$$0<U(z)<\epsilon, \quad \forall z\in[z_1,+\infty).$$

It then follows from (3.7) that

$$\frac{dU}{dz}\le U^{2-m}\left(f'(0)-s\right)+C_1\epsilon U^{2-m}=(f'(0)-s+C_1\epsilon)U^{2-m}, \quad \forall z\in[z_1,+\infty).$$

Solving this differential inequality yields

$$U(z)\le\left[U^{m-1}(z_1)+(1-m)\left(s-f'(0)-C_1\epsilon\right)z\right]^{-\frac{1}{1-m}} \text{ for } z\in[z_1,+\infty).$$

Applying a similar argument, we have

$$U(z)\ge\left[U^{m-1}(z_1)+(1-m)\left(s-f'(0)+C_1\epsilon\right)z\right]^{-\frac{1}{1-m}} \text{ for } z\in[z_1,+\infty).$$

Thus,

$$U(z)\sim z^{-\frac{1}{1-m}}, \ \text{ as } z\to+\infty.$$

When $z\to-\infty$, we write (2.5) as

$$\frac{d(u_--U)}{dz}=\lambda_-(u_--U)+F_2(U)(u_--U)^2 \text{ on } z<0, \tag{3.8}$$

where $\lambda_-=u_-^{1-m}\left(f'(u_-)-s\right)$ and $F_2(U)\triangleq\frac{U^{1-m}[f(U)-f(u_-)+s(U-u_-)]-u_-^{1-m}\left(f'(u_-)-s\right)(U-u_-)}{-(u_--U)^2}$.
By Taylor's formula, we get

$$|F_2(U(z))|\le C_2, \quad \forall z\in(-\infty,0).$$

Since $\lim_{z\to-\infty} U(z)=u_-$, given $0<\epsilon<\frac{\lambda_-}{1+2C_2}$, there exists $z_2<0$ such that

$$0<u_--U(z)<\epsilon, \quad \forall z\in(-\infty,z_2].$$

It then follows from (3.8) that

$$\frac{d(u_--U)}{dz}\ge(\lambda_--C_2\epsilon)(u_--U).$$

Solving this differential inequality yields

$$u_--U(z)\le(u_--U(z_2))e^{-(\lambda_--C_2\epsilon)z_2}e^{(\lambda_--C_2\epsilon)z}\triangleq Ae^{Bz} \text{ for } z\in(-\infty,z_2],$$

where $A:=(u_--U(z_2))e^{-(\lambda_--C_2\epsilon)z_2}$ and $B:=\lambda_--C_2\epsilon$. Substituting this estimate into (3.8) again, we have

$$\frac{d(u_- - U)}{dz} \geq (\lambda_- - Ae^{Bz})(u_- - U) \text{ for } z \leq z_2 < 0. \tag{3.9}$$

Solving this differential inequality gives

$$\begin{aligned} u_- - U(z) &\leq e^{-\lambda_- z_2}(u_- - U(z_2))e^{\frac{A}{B}(e^{Bz_2} - e^{Bz})}e^{\lambda_- z} \\ &\leq e^{-\lambda_- z_2}(u_- - U(z_2))e^{\frac{A}{B}e^{Bz_2}}e^{\lambda_- z} \text{ for } z \leq z_2 < 0. \end{aligned}$$

Applying a similar argument, we have

$$u_- - U(z) \geq Ce^{\lambda_- z} \text{ for } z \leq z_2 < 0.$$

Therefore,

$$u_- - U(z) \sim \mathrm{e}^{\lambda_- z}, \quad \text{as } z \to -\infty.$$

In the case $s = f'(0)$ or $s = f'(u_-)$, we have $|g(U)| \sim |U - u_\pm|^{1+k_\pm}$ for $k_\pm \geq 1$ as $z \to \pm\infty$ and this indicates the convergence rates $|z|^{-\frac{1}{k_+ + 1 - m}}$ as $z \to +\infty$ and $|z|^{-\frac{1}{k_-}}$ as $z \to -\infty$. The proof is complete. □

Corollary 3.1. *Let $m < 0$ and $U(z)$ $(z = x - st)$ be a shock profile obtained in Theorem 2.1.*

(i) *If $f'(0) < s < f'(u_-)$, it is easy to see that*

$$|U_z(z)| \leq CU^{2-m}(z), \quad |U_{zz}(z)| \leq CU^{3-2m}(z), \quad \forall z \in \mathbb{R}. \tag{3.10}$$

(ii) *If $f'(0) = s < f'(u_-)$, for any $k_+ \geq 1$, we have*

$$|U_z(z)| \leq CU^{k_+ + 2 - m}(z) \leq CU^{2-m}(z), \quad \forall z \in [0, +\infty), \tag{3.11}$$

$$|U_{zz}(z)| \leq CU^{k_+ + 3 - 2m}(z) \leq CU^{3-2m}(z), \quad \forall z \in [0, +\infty). \tag{3.12}$$

(iii) *If $f'(0) < s = f'(u_-)$, for any $k_- \geq 1$, we have*

$$|U_z(z)| \leq Cu_-^{1-m} |U(z) - u_-|^{1+k_-}, \quad \forall z \in (-\infty, 0]. \tag{3.13}$$

Proof. Since $u_- > 0$, by (2.5), it is easy to see that the first conclusion of (3.10) holds. Furthermore, a direct calculation by (2.5) gives

$$U_{zz} = (1 - m)\frac{U_z^2}{U} + (f'(U) - s)U^{1-m}U_z. \tag{3.14}$$

Then, we deduce that

$$|U_{zz}(z)| \leq C\left(\frac{|U_z(z)|}{U(z)} + U^{1-m}(z)\right)|U_z(z)| \leq CU^{3-2m}(z), \quad \forall z \in \mathbb{R},$$

which implies the second estimate of (3.10). Recalling that $U_z = U^{1-m}g(U)$, for the case $s = f'(0)$ or $s = f'(u_-)$, we have $|g(U)| \sim |U - u_\pm|^{1+k_\pm}$ for $k_\pm \geq 1$ as $z \to \pm\infty$. This together with (3.14) yields (3.11)-(3.13). □

4. Asymptotic stability

In this section, we prove Theorem 2.2, namely the asymptotic stability of the shock profile of (1.1). We first utilize the technique of anti-derivatives to rewrite the problem and establish the asymptotic stability of the transformed equation.

4.1. Reformulation of the problem

Let $U(x - st)$ be the shock profile obtained in Theorem 2.1, and $U(x - st + x_0)$ represents the asymptotic profile of the solution $u(x,t)$ for Cauchy problem (1.1) as $t \to +\infty$, where the shift $x_0 := -\frac{1}{u_-}\int_{-\infty}^{+\infty}(u_0(x) - U(x))\,dx$. It follows from the equations (1.1) and (2.2) that

$$(u - U)_t + (f(u) - f(U))_x - \left(u^{m-1}u_x - U^{m-1}U_x\right)_x = 0. \tag{4.1}$$

Thanks to Theorem 2.1, the shock profile U satisfies

$$\lim_{x\to+\infty} U(x - st + x_0) = 0 \text{ and } \lim_{x\to-\infty} U(x - st + x_0) = u_-.$$

Although (1.1) is a Cauchy problem, we expect that its global solution $u(x,t)$ satisfies

$$\lim_{x\to+\infty} u(x,t) = 0 \text{ and } \lim_{x\to-\infty} u(x,t) = u_-, \ \forall t > 0. \tag{4.2}$$

We will demonstrate in Remark 4.1 that our constructed solution satisfies (4.2). Thus,

$$\lim_{x\to\pm\infty} (f(u(x,t)) - f(U(x - st + x_0))) = 0, \ \forall t > 0.$$

Since U satisfies $\lim_{x\to-\infty} U_x(x - st + x_0) = 0$, we get

$$\lim_{x\to-\infty} U^{m-1}U_x(x - st + x_0) = 0.$$

It follows from Corollary 3.1 that $|U^{m-1}U_x(x - st + x_0)| \leq CU(x - st + x_0)$. Thus, we also have

$$\lim_{x\to+\infty} U^{m-1}U_x(x - st + x_0) = 0.$$

It is also expected that the solution $u(x,t)$ of (1.1) satisfies

$$\lim_{x\to\pm\infty} u^{m-1}u_x(x,t) = 0, \ a.e.\ t > 0. \tag{4.3}$$

See Remark 4.1 below for details. Then integrating the equation (4.1) over $(-\infty, +\infty) \times [0, t]$, we have

$$\int_{-\infty}^{+\infty} (u(x,t) - U(x - st + x_0))\, dx = \int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))\, \mathrm{d}x, \quad a.e.\ t > 0. \tag{4.4}$$

A direct calculation gives

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))\, \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\, \mathrm{d}x - \int_{-\infty}^{+\infty} (U(x + x_0) - U(x))\, \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\, \mathrm{d}x - \int_{-\infty}^{+\infty} \int_{0}^{x_0} U'(x + \eta)\mathrm{d}\eta \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\, \mathrm{d}x + x_0 u_- \\
&= 0,
\end{aligned}$$

due to $x_0 = -\frac{1}{u_-} \int_{-\infty}^{+\infty} (u_0(x) - U(x))\, \mathrm{d}x$. Now by (4.4), we have

$$\int_{-\infty}^{+\infty} (u(x,t) - U(x - st + x_0))\, dx = \int_{-\infty}^{+\infty} (u_0(x) - U(x + x_0))\, dx = 0, \quad a.e.\ t > 0. \tag{4.5}$$

Set

$$u(x,t) = U(x - st + x_0) + \phi_z(z,t), \quad z = x - st. \tag{4.6}$$

That is

$$\phi(z,t) = \int_{-\infty}^{z} (u(y,t) - U(y - st + x_0))\, \mathrm{d}y, \quad \forall z \in \mathbb{R},\ t > 0.$$

It then follows from (4.5) that

$$\phi(\pm\infty, t) = 0. \tag{4.7}$$

Let $z = x - st$, the equation (1.1) can be rewritten as

$$u_t - su_z + f(u)_z = (u^{m-1}u_z)_z. \tag{4.8}$$

Now substituting (4.6) into (4.8) and noting U satisfies (2.2), we have

$$\phi_{zt} - s\phi_{zz} + [f(U+\phi_z) - f(U)]_z = \frac{1}{m}[(U+\phi_z)^m - U^m]_{zz}.$$

Integrating the above equation over $(-\infty, z)$ and noting that $\phi(-\infty,t) = \phi_z(-\infty,t) = 0$, $U(z) \to u_-$ and $U^{m-1}U_z(z) \to 0$ as $z \to -\infty$, we derive the equation of ϕ:

$$\phi_t - s\phi_z + f(U+\phi_z) - f(U) = \frac{1}{m}[(U+\phi_z)^m - U^m]_z. \tag{4.9}$$

Observing that if $|\phi_z| \ll 1$, then the leading order term of $f(U+\phi_z) - f(U)$ is $f'(U)\phi_z$. Hence, we write the third term of (4.9) as

$$f(U+\phi_z) - f(U) = f'(U)\phi_z + \left(f(U+\phi_z) - f(U) - f'(U)\phi_z\right).$$

Similarly, if $|\frac{\phi_z}{U}| \ll 1$, then $[(U+\phi_z)^m - U^m]_z$ is dominated by $m(U^{m-1}\phi_z)_z$. Thus, we write the right hand side of (4.9) as

$$\frac{1}{m}[(U+\phi_z)^m - U^m]_z = \left(\frac{\phi_z}{U^{1-m}}\right)_z + \frac{1}{m}\left[(U+\phi_z)^m - U^m - m\frac{\phi_z}{U^{1-m}}\right]_z.$$

Substituting the last two identities into (4.9), we obtain

$$\phi_t + g'(U)\phi_z - \left(\frac{\phi_z}{U^{1-m}}\right)_z = F + \frac{1}{m}G_z, \tag{4.10}$$

where

$$\begin{aligned} g'(U) &= f'(U) - s, \\ F &:= -\left(f(U+\phi_z) - f(U) - f'(U)\phi_z\right), \end{aligned} \tag{4.11}$$

and

$$G := (U+\phi_z)^m - U^m - m\frac{\phi_z}{U^{1-m}}. \tag{4.12}$$

The initial condition for ϕ is given by

$$\phi_0(z) := \phi(z,0) = \int_{-\infty}^{z} (u_0(y) - U(y+x_0))\,dy \text{ with } \phi(\pm\infty, 0) = 0. \tag{4.13}$$

One may observe that the perturbation equation (4.10) still has a singular diffusivity near $z = +\infty$ since $U(+\infty) = 0$ and $m < 0$. In the sequel we shall carefully select weighted functional space to resolve the singularity.

4.2. Energy estimates

We search for solutions of (4.10)-(4.13) in the following space

$$X(0,T) := \Big\{\phi(z,t) \mid \phi \in C\left([0,T]; L^2_{w_1}\right),\ \phi_z \in C\left([0,T]; L^2_{w_2}\right) \cap L^2\left((0,T); L^2_{w_3}\right), \\ \phi_{zz} \in C\left([0,T]; L^2_{w_4}\right) \cap L^2\left((0,T); L^2_{w_5}\right),\ \phi_{zzz} \in L^2\left((0,T); L^2_{w_6}\right)\Big\}, \tag{4.14}$$

where the weight functions w_i, $i = 1, \dots, 6$, are defined by

$$w_1(U) = \frac{U\,(U-u_-)}{g(U)},\ w_2(U) = U^{-2},\ w_3(U) = \frac{U^m\,(U-u_-)}{g(U)}, \\ w_4(U) = U^{2m-2},\ w_5(U) = U^{m-3},\ w_6(U) = U^{3m-3} \tag{4.15}$$

with $U = U(z + x_0)$.

Denote

$$N(t) := \sup_{\tau\in[0,t]} \left(\|\phi(\cdot,\tau)\|_{w_1} + \|\phi_z(\cdot,\tau)\|_{w_2} + \|\phi_{zz}(\cdot,\tau)\|_{w_4}\right). \tag{4.16}$$

We remark that (1) if $f'(0) < s < f'(u_-)$, $|g(U)| \sim |U - u_\pm|$ as $U \to u_\pm$, then we have $w_1(U) \sim C$; (2) if $f'(0) = s < f'(u_-)$, $|g(U)| \sim |U|^{1+k_+}$ for $k_+ \geq 1$ as $U \to 0$, then we have $w_1(U) = \frac{U(U-u_-)}{g(U)} \sim U^{-k_+}$ as $z \to +\infty$; (3) if $f'(0) < s = f'(u_-)$, $|g(U)| \sim |U - u_-|^{1+k_-}$ for $k_- \geq 1$ as $U \to u_-$, then we have $w_1(U) = \frac{U(U-u_-)}{g(U)} \sim |U - u_-|^{-k_-}$ as $z \to -\infty$. Thus, in any case, we have $w_1(U) \geq 1$. Clearly, if $\phi_z \in L^2_{w_2}$, then $\phi_z \in L^2$ due to $w_2(U) \geq 1$. By virtue of the Sobolev inequality $H^1(R) \hookrightarrow C^0(R)$, it holds that

$$\sup_{\tau\in[0,t]} \|\phi(\cdot,\tau)\|_{L^\infty} \leq CN(t) \tag{4.17}$$

and

$$\|\sqrt{w_2(U)}\phi_z(\cdot,t)\|^2_{L^\infty} \leq C\|\sqrt{w_2(U)}\phi_z(\cdot,t)\|^2_1. \tag{4.18}$$

By the definition of H^1 norm, we get

$$\|\sqrt{w_2(U)}\phi_z\|_1^2 = \left\|\frac{\phi_z}{U}\right\|_1^2 = \int \frac{\phi_z^2}{U^2} + \int \left|\left(\frac{\phi_z}{U}\right)_z\right|^2 = \int \frac{\phi_z^2}{U^2} + \int \left|\frac{\phi_{zz}}{U} - \frac{U_z\phi_z}{U^2}\right|^2 \\ \leq \int \frac{\phi_z^2}{U^2} + 2\int \frac{\phi_{zz}^2}{U^2} + 2\int \frac{U_z^2}{U^2}\frac{\phi_z^2}{U^2}, \tag{4.19}$$

where we have used the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ in the last inequality. We next estimate the integral $\int \frac{U_z^2}{U^2}\frac{\phi_z^2}{U^2}$. Thanks to (3.10), (3.11) and $m < 0$, one has

$$|U_z^2/U^2(z)| \leq CU^{2-2m}(z) \leq C \text{ for all } z.$$

Then

$$\int \frac{U_z^2}{U^2}\frac{\phi_z^2}{U^2} \le C\int \frac{\phi_z^2}{U^2}.$$

Therefore, by (4.19), we have

$$\|\sqrt{w_2(U)}\phi_z\|_1^2 = \left\|\frac{\phi_z}{U}\right\|_1^2 \le C\left(\int \frac{\phi_z^2}{U^2} + \int \frac{\phi_{zz}^2}{U^2}\right) \le C\|\phi_z\|_{w_2}^2 + \|\phi_{zz}\|_{w_4}^2. \tag{4.20}$$

This along with (4.16) and (4.18) gives rise to

$$\sup_{\tau\in[0,t]} \left\|\sqrt{w_2}\phi_z(\cdot,\tau)\right\|_{L^\infty} = \sup_{\tau\in[0,t]} \left\|\frac{\phi_z}{U}(\cdot,\tau)\right\|_{L^\infty} \le CN(t). \tag{4.21}$$

Then we have the following global well-posedness result for the Cauchy problem (4.10)-(4.13).

Theorem 4.1. *Assume that $m<0$, $\phi_0 \in L^2_{w_1}(\mathbb{R})$, $\phi_{0z} \in L^2_{w_2}(\mathbb{R})$ and that $\phi_{0zz} \in L^2_{w_4}(\mathbb{R})$. There exists a positive constant δ_1 such that if $N(0) \le \delta_1$, then the Cauchy problem (4.10)-(4.13) has a unique global solution $\phi \in X(0,\infty)$ satisfying*

$$\begin{aligned} &\|\phi(\cdot,t)\|_{w_1}^2 + \|\phi_z(\cdot,t)\|_{w_2}^2 + \|\phi_{zz}(\cdot,t)\|_{w_4}^2 + \int_0^t \left(\|\phi_z(\cdot,\tau)\|_{w_3}^2 + \|\phi_{zz}(\cdot,\tau)\|_{w_5}^2 + \|\phi_{zzz}(\cdot,\tau)\|_{w_6}^2\right) \\ &\qquad \le C\left(\|\phi_0\|_{w_1}^2 + \|\phi_{0z}\|_{w_2}^2 + \|\phi_{0zz}\|_{w_4}^2\right) \le CN^2(0), \end{aligned} \tag{4.22}$$

for any $t \in [0,\infty)$, where w_i, $i=1,\cdots,6$, are defined by (4.15). Moreover, it holds that

$$\sup_{x\in\mathbb{R}} |\phi_z(z,t)| \to 0 \quad \text{as } t \to +\infty. \tag{4.23}$$

Remark 4.1. Recalling that $u(x,t) = U(x-st+x_0) + \phi_z(z,t)$, one can easily see that $u(x,t)$ satisfies (4.2). We next show that $u(x,t)$ satisfies (4.3). We write $u^{m-1}u_x$ as

$$u^{m-1}u_x = (U+\phi_z)^{m-1}U_z + (U+\phi_z)^{m-1}\phi_{zz}.$$

Thanks to (4.21) and the estimate (4.22), when $N(0)$ is small, there is a constant $c_1 > 0$ such that

$$c_1 U < U + \phi_z < \frac{1}{c_1}U, \ \forall (x,t).$$

Thus

$$|u^{m-1}u_x| \le CU^{m-1}|U_z| + CU^{m-1}|\phi_{zz}|. \tag{4.24}$$

Noting $\lim_{x\to-\infty} U(x-st+x_0) = u_- > 0$, we hence have

$$\lim_{x\to-\infty} u^{m-1}u_x(x,t) = \lim_{x\to-\infty}(U+\phi_z)^{m-1}U_z + \lim_{x\to-\infty}(U+\phi_z)^{m-1}\phi_{zz} = 0.$$

To analyze the limit of $u^{m-1}u_x(x,t)$ as $x\to+\infty$, one can see from (4.22) that ϕ_{zz} satisfies

$$\int U^{2(m-1)}\phi_{zz}^2 + \iint_0^t U^{3(m-1)}\phi_{zzz}^2 < +\infty,\ \forall t>0.$$

It then follows that

$$\int U^{2(m-1)}\phi_{zz}^2 + \int U^{3(m-1)}\phi_{zzz}^2 < +\infty,\ a.e.\ t>0. \tag{4.25}$$

Thus, by Corollary 3.1, we obtain

$$\begin{aligned}\int |(U^{m-1}\phi_{zz})_z|^2 &= \int |(m-1)U^{m-2}U_z\phi_{zz} + U^{m-1}\phi_{zzz}|^2\\ &\le 2(m-1)^2\int U^{2(m-2)}|U_z|^2|\phi_{zz}|^2 + 2\int U^{2(m-1)}|\phi_{zzz}|^2\\ &\le C\int|\phi_{zz}|^2 + C\int U^{3(m-1)}|\phi_{zzz}|^2 < +\infty,\end{aligned}$$

which along with (4.25) implies that $U^{m-1}\phi_{zz}\in H^1(\mathbb{R})$. Thus,

$$\lim_{x\to+\infty} U^{m-1}\phi_{zz} = 0.$$

It then follows from (4.24) that

$$\lim_{x\to+\infty} u^{m-1}u_x(x,t) = 0.$$

Owing to the decomposition (4.6), Theorem 2.2 is a consequence of Theorem 4.1. The local well-posedness of the system (4.10)-(4.13) is standard (see Ref. [25] for instance). To prove Theorem 4.1, it suffices to establish the following *a priori* estimate.

Proposition 4.1 *(*A priori *estimate). Assume that $m<0$ and $\phi\in X(0,T)$ is a solution of* (4.10)-(4.13) *for a positive constant T. Then there exists a positive constant δ_2 ($\le\delta_0$), independent of T, such that if*

$$N(t)\le\delta_2 \textit{ for all } 0\le t\le T, \tag{4.26}$$

then the estimate (4.22) *holds for all $t\in[0,T]$.*

Before establishing the *a priori* estimate in Proposition 4.1, we first present some preliminary calculations.

Lemma 4.1. *Under the same assumptions of Proposition 4.1, if $N(T)$ is sufficiently small, then there exists a constant $C > 0$ independent of T such that*

$$|F| \le C\phi_z^2, \quad |G| \le C\frac{\phi_z^2}{U^{2-m}}, \tag{4.27}$$

$$|F_z| \le C\left(|U_z|\phi_z^2 + |\phi_z||\phi_{zz}|\right), \tag{4.28}$$

$$|G_z| \le C\left(\frac{|U_z|}{U^{3-m}}\phi_z^2 + \frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right), \tag{4.29}$$

$$|G_{zz}| \le C\left[\left(\frac{U_z^2}{U^{4-m}} + \frac{|U_{zz}|}{U^{3-m}}\right)\phi_z^2 + \frac{\phi_{zz}^2}{U^{2-m}} + \frac{|U_z|}{U^{3-m}}|\phi_z||\phi_{zz}| + \frac{|\phi_z||\phi_{zzz}|}{U^{2-m}}\right]. \tag{4.30}$$

Proof. Owing to (4.21) and $U(z) \in (0, u_-)$ for $z \in \mathbb{R}$, it holds that

$$\|\phi_z(\cdot,t)\|_{L^\infty} \le u_- \left\|\frac{\phi_z}{U}(\cdot,t)\right\|_{L^\infty} \le CN(t).$$

Recalling that $F = -\left(f(U+\phi_z) - f(U) - f'(U)\phi_z\right)$, then applying Taylor's expansion along with $f \in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$, we get the first inequality of (4.27). From (4.12), it is easy to see that $G = (U+\phi_z)^m - U^m - m\frac{1}{U^{-m}}\frac{\phi_z}{U}$. Combining (4.21) and the Taylor's expansion yields the second inequality of (4.27).

A direct calculation gives

$$\begin{aligned} F_z &= -f'(U+\phi_z)(U_z+\phi_{zz}) + f'(U)U_z + f''(U)U_z\phi_z + f'(U)\phi_{zz} \\ &= -\left(f'(U+\phi_z) - f'(U) - f''(U)\phi_z\right)U_z - \left(f'(U+\phi_z) - f'(U)\right)\phi_{zz}, \end{aligned}$$

and

$$\begin{aligned} G_z &= m(U+\phi_z)^{m-1}(U_z+\phi_{zz}) - mU^{m-1}U_z - mU^{m-1}\phi_{zz} - m(m-1)\frac{U_z}{U^{2-m}}\phi_z \\ &= m\left[\frac{1}{(U+\phi_z)^{1-m}} - \frac{1}{U^{1-m}} - (m-1)\frac{1}{U^{1-m}}\frac{\phi_z}{U}\right]U_z + m\left(\frac{1}{(U+\phi_z)^{1-m}} - \frac{1}{U^{1-m}}\right)\phi_{zz}. \end{aligned}$$

Therefore, by (4.21) and $f \in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$, (4.28)-(4.29) immediately follow from Taylor's expansion. Similarly, by (4.12), one can derive

$$\begin{aligned} G_{zz} &= m\left[(U+\phi_z)^{m-1}(U_z+\phi_{zz}) - U^{m-1}U_z - U^{m-1}\phi_{zz} - (m-1)\frac{U_z}{U^{2-m}}\phi_z\right]_z \\ &= m(m-1)\left[\frac{1}{(U+\phi_z)^{2-m}} - \frac{1}{U^{2-m}} - (m-2)\frac{\phi_z}{U^{3-m}}\right]U_z^2 + m(m-1)\frac{\phi_{zz}^2}{(U+\phi_z)^{2-m}} \\ &\quad + m\left(\frac{1}{(U+\phi_z)^{1-m}} - \frac{1}{U^{1-m}}\right)\phi_{zzz} + 2m(m-1)\left(\frac{1}{(U+\phi_z)^{2-m}} - \frac{1}{U^{2-m}}\right)U_z\phi_{zz} \\ &\quad + m\left[\frac{1}{(U+\phi_z)^{1-m}} - \frac{1}{U^{1-m}} - (m-1)\frac{\phi_z}{U^{2-m}}\right]U_{zz}. \end{aligned}$$

Then a simple calculation gives (4.30). □

We now derive the basic L^2 estimate for ϕ.

Lemma 4.2. *Let the assumptions of Proposition 4.1 hold. If $N(T)$ is sufficiently small, then there exists a constant $C > 0$ independent of T such that*

$$\|\phi(\cdot,t)\|_{w_1}^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{w_3}^2 \le C\|\phi_0\|_{w_1}^2, \tag{4.31}$$

for any $t \in [0,T]$.

Proof. Multiplying (4.10) by $w_1(U)\phi(z,t)$ with $w_1(U) = \frac{U(U-u_-)}{g(U)}$ yields

$$\begin{aligned}&\left(\frac{1}{2}w_1(U)\phi^2\right)_t + \left(\frac{1}{2}(w_1g)'(U)\phi^2 - \frac{w_1(U)}{U^{1-m}}\phi\phi_z - \frac{1}{m}w_1(U)G\phi\right)_z + \frac{w_1(U)}{U^{1-m}}\phi_z^2\\ &\quad - \frac{1}{2}(w_1g)''(U)U_z\phi^2 = w_1(U)F\phi - \frac{1}{m}w_1'(U)U_zG\phi - \frac{1}{m}w_1(U)G\phi_z.\end{aligned} \tag{4.32}$$

We observe that when one integrates (4.32) with respect to z over $\mathbb{R}$, the second term in (4.32) introduces a non-zero boundary term at $z = +\infty$ due to singularity. To address this, we choose a smooth cut-off function $\eta(z)$ satisfying

$$\eta(z) = 1 \text{ for } |z| < L,\ \eta(z) = 0 \text{ for } |z| > 2L,\ 0 \le \eta(z) \le 1 \text{ and } |\eta_z(z)| \le \frac{C}{L} \text{ for } z \in \mathbb{R}. \tag{4.33}$$

Multiplying (4.32) by $\eta^2(z)$, integrating the result with respect to z and noting that $(w_1g)''(U) = 2$, we obtain

$$\begin{aligned}&\frac{1}{2}\int_{-2L}^{2L} w_1(U)\phi^2\eta^2 + \int_0^t\int_{-2L}^{2L} |U_z|\phi^2\eta^2 + \int_0^t\int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}}\phi_z^2\eta^2\\ &= \frac{1}{2}\int_{-2L}^{2L} w_1(U)\phi_0^2\eta^2 + \int_0^t\int_{-2L}^{2L} 2\eta\eta_z(z)\left(\frac{1}{2}(w_1g)'(U)\phi^2 - \frac{w_1(U)}{U^{1-m}}\phi\phi_z - \frac{1}{m}w_1(U)G\phi\right)\\ &\quad + \int_0^t\int_{-2L}^{2L}\left(w_1(U)F\phi\eta^2 - \frac{1}{m}w_1'(U)U_zG\phi\eta^2 - \frac{1}{m}w_1(U)G\phi_z\eta^2\right).\end{aligned} \tag{4.34}$$

Since $\phi \in X(0,T)$ defined in (4.14), we see that for $t \in [0,T]$,

$$\int_{-2L}^{2L} w_1(U)\phi^2 \le \int w_1(U)\phi^2 \le N^2(t). \tag{4.35}$$

Since $(w_1 g)'(U) = 2U - u_-$, together with $w_1(U) \geq 1$, it holds that

$$\left| \int_0^t \int_{-2L}^{2L} (w_1 g)'(U)\phi^2 \eta\eta_z(z) \right| \leq C \int_0^t \int_{-2L}^{2L} \phi^2 |\eta_z(z)| \leq \frac{C}{L} \int_0^t \int_{-2L}^{2L} w_1(U)\phi^2 \leq \frac{CTN^2(T)}{L}. \quad (4.36)$$

Furthermore, by Cauchy-Schwarz inequality, one has

$$\left| 2\int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi\phi_z \eta\eta_z(z) \right| \leq \int_{-2L}^{2L} w_1(U)\phi^2 \frac{|\eta_z(z)|^{\frac{3}{2}}}{U^{1-m}} + \int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi_z^2 \eta^2 |\eta_z(z)|^{\frac{1}{2}}. \quad (4.37)$$

By Theorem 2.1, it is clear that (1) if $f'(0) < s \leq f'(u_-)$, then $U^{1-m}(z+x_0) \sim |z+x_0|^{-1} \sim |z|^{-1}$ as $z \to +\infty$ and thus $\frac{|\eta_z(z)|^{\frac{3}{2}}}{U^{1-m}} \leq \frac{C}{\sqrt{L}}$; (2) if $f'(0) = s < f'(u_-)$, then $U^{1-m}(z+x_0) \sim |z+x_0|^{-\frac{1-m}{k_+ + 1 - m}} \sim |z|^{-\frac{1-m}{k_+ + 1 - m}}$ as $z \to +\infty$, which implies $\frac{|\eta_z(z)|^{\frac{3}{2}}}{U^{1-m}} \leq \frac{C}{L^{\frac{3}{2} - \frac{1-m}{k_+ + 1 - m}}} \leq \frac{C}{\sqrt{L}}$. In any case, we have

$$\frac{|\eta_z(z)|^{\frac{3}{2}}}{U^{1-m}} \leq \frac{C}{\sqrt{L}}. \quad (4.38)$$

Thus,

$$\int_{-2L}^{2L} w_1(U)\phi^2 \frac{|\eta_z(z)|^{\frac{3}{2}}}{U^{1-m}} \leq \frac{C}{\sqrt{L}} \int_{-2L}^{2L} w_1(U)\phi^2 \leq \frac{CN^2(t)}{\sqrt{L}}, \quad (4.39)$$

and

$$\int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi_z^2 \eta^2 |\eta_z(z)|^{\frac{1}{2}} \leq \frac{C}{\sqrt{L}} \int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi_z^2 \eta^2. \quad (4.40)$$

Substituting (4.39)-(4.40) into (4.37) gives

$$\left| 2\int_0^t \int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi\phi_z \eta\eta_z(z) \right| \leq \frac{CTN^2(T)}{\sqrt{L}} + \frac{C}{\sqrt{L}} \int_0^t \int_{-2L}^{2L} \frac{w_1(U)}{U^{1-m}} \phi_z^2 \eta^2. \quad (4.41)$$

Moreover, in view of (4.27), (4.21) and (4.41), it has

$$
\begin{aligned}
\left|\frac{2}{m}\int_0^t\int_{-2L}^{2L}\eta\eta_z(z)w_1(U)G\phi\right| &\le C\int_0^t\int_{-2L}^{2L}\frac{w_1(U)}{U^{1-m}}|\phi\phi_z\eta\eta_z(z)|\cdot\frac{|\phi_z|}{U}\\
&\le CN(t)\int_0^t\int_{-2L}^{2L}\frac{w_1(U)}{U^{1-m}}|\phi\phi_z\eta\eta_z(z)| \\
&\le \frac{CTN^3(T)}{\sqrt{L}}+\frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{w_1(U)}{U^{1-m}}\phi_z^2\eta^2.
\end{aligned}
\tag{4.42}
$$

Adding (4.36)-(4.42) with (4.34), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{-2L}^{2L}w_1(U)\phi^2\eta^2+\int_0^t\int_{-2L}^{2L}|U_z|\phi^2\eta^2+\left(1-\frac{C+CN(T)}{\sqrt{L}}\right)\int_0^t\int_{-2L}^{2L}\frac{w_1(U)}{U^{1-m}}\phi_z^2\eta^2\\
&\le\frac{1}{2}\int_{-2L}^{2L}w_1(U)\phi_0^2\eta^2+\int_0^t\int_{-2L}^{2L}\left(w_1(U)F\phi\eta^2-\frac{1}{m}w_1'(U)U_zG\phi\eta^2-\frac{1}{m}w_1(U)G\phi_z\eta^2\right)\\
&\quad+\frac{CTN^2(T)}{\sqrt{L}}.
\end{aligned}
\tag{4.43}
$$

Letting $L\to+\infty$ yields

$$
\begin{aligned}
&\int w_1(U)\phi^2+\iint_0^t|U_z|\phi^2+\iint_0^t\frac{w_1(U)}{U^{1-m}}\phi_z^2\\
&\le C\left(\int w_1(U)\phi_0^2+\iint_0^t|w_1(U)F\phi|+\iint_0^t\left|w_1'(U)U_zG\phi\right|+\iint_0^t|w_1(U)G\phi_z|\right).
\end{aligned}
\tag{4.44}
$$

By the first inequality of (4.27) and the fact that $\|\phi(\cdot,t)\|_{L^\infty}\le CN(t)$, one can see that

$$
\iint_0^t|w_1(U)F\phi|\le CN(t)\iint_0^t w_1(U)\phi_z^2\le CN(t)\iint_0^t\frac{w_1(U)}{U^{1-m}}\phi_z^2. \tag{4.45}
$$

We next estimate the third term on the right hand side of (4.44). Utilizing the second inequality of (4.27) and $\|\phi(\cdot,t)\|_{L^\infty}\le CN(t)$, we get

$$
\begin{aligned}
\iint_0^t \left|w_1'(U)U_z G\phi\right| &\le CN(t) \iint_0^t \frac{|w_1'(U)||U_z|}{U^{2-m}}\phi_z^2 \\
&= CN(t) \iint_0^t \frac{w_1(U)}{U^{1-m}}\phi_z^2 \cdot \left|\frac{w_1'(U)}{w_1(U)}\right|\left|\frac{U_z}{U}\right|.
\end{aligned}
\tag{4.46}
$$

We claim that

$$
\left|\frac{w_1'(U)}{w_1(U)}\right|\left|\frac{U_z}{U}\right| \le C, \quad \forall z \in (-\infty, \infty). \tag{4.47}
$$

In fact, if $f'(0) \le s < f'(u_-)$, since $|g(U)| \sim |U|^{1+k_+}$ with $k_+ \ge 0$ as $z \to +\infty$, it holds that

$$
\begin{aligned}
\left|\frac{w_1'(U)}{w_1(U)}\right| &= \left|\frac{(2U-u_-)\,g(U) - U\,(U-u_-)\,g'(U)}{U\,(U-u_-)\,g(U)}\right| \\
&\le C\frac{|U|^{1+k_+}\,(|2U-u_-| + |U-u_-|)}{|U|^{2+k_+}\,|U-u_-|} \\
&\le \frac{C}{U}.
\end{aligned}
\tag{4.48}
$$

When $z \to -\infty$, $|g(U)| \sim |U-u_-|$, we get

$$
\begin{aligned}
\left|\frac{w_1'(U)}{w_1(U)}\right| &= \left|\frac{(2U-u_-)\,g(U) - U\,(U-u_-)\,g'(U)}{U\,(U-u_-)\,g(U)}\right| \\
&\le C\frac{|U-u_-|\,(|2U-u_-| + |U|)}{|U-u_-|^2\,|U|} \\
&\le \frac{C}{|U-u_-|}.
\end{aligned}
\tag{4.49}
$$

In view of (2.5) and (2.6), it follows for $k_+ \ge 0$ that

$$
\left|\frac{U_z}{U}\right| \le C\,|U|^{k_++1-m} \text{ as } z \to +\infty \text{ and } \left|\frac{U_z}{U}\right| \le C\,|U-u_-| \text{ as } z \to -\infty.
$$

This along with (4.48), (4.49) and the fact that $m < 0$ gives

$$
\left|\frac{w_1'(U)}{w_1(U)}\right|\left|\frac{U_z}{U}\right| \le C\,|U-0|^{k_+-m} \le C, \quad \text{as } z \to +\infty \tag{4.50}
$$

and

$$
\left|\frac{w_1'(U)}{w_1(U)}\right|\left|\frac{U_z}{U}\right| \le C, \quad \text{as } z \to -\infty. \tag{4.51}
$$

If $f'(0) < s = f'(u_-)$, when $z \to +\infty$, (4.50) also holds, and when $z \to -\infty$ we have

$$\left|\frac{w_1'(U)}{w_1(U)}\right| \le C\frac{|U-u_-|^{1+k_-}\left(|2U-u_-|+U\right)}{|U-u_-|^{2+k_-}\,U} \le \frac{C}{|U-u_-|},$$

which in combination with (3.13) implies for $k_- \ge 1$ that

$$\left|\frac{w_1'(U)}{w_1(U)}\right|\left|\frac{U_z}{U}\right| \le C\,|U-u_-|^{k_-} \le C, \quad \text{as } z\to-\infty. \tag{4.52}$$

Hence, thanks to (4.50), (4.51) and (4.52), we get (4.47). By virtue of (4.46) and (4.47), we then arrive at

$$\iint_0^t \left|w_1'(U)U_zG\phi\right| \le CN(t)\iint_0^t \frac{w_1(U)}{U^{1-m}}\phi_z^2.$$

For the last term on the right hand side of (4.44), it follows from (4.27) and (4.21) that

$$\iint_0^t |w_1(U)G\phi_z| \le C\iint_0^t \frac{w_1(U)}{U^{2-m}}|\phi_z^3| \le CN(t)\iint_0^t \frac{w_1(U)}{U^{1-m}}\phi_z^2. \tag{4.53}$$

Substituting (4.45)-(4.53) into (4.44), we have

$$\int w_1(U)\phi^2 + \iint_0^t |U_z|\phi^2 + (1-CN(t))\iint_0^t \frac{w_1(U)}{U^{1-m}}\phi_z^2 \le C\int w_1(U)\phi_0^2.$$

Taking $N(t)$ small enough such that $CN(t)\le 1/2$, we obtain (4.31). □

The next lemma gives the H^1 estimate for ϕ.

Lemma 4.3. *Let the assumptions of Proposition 4.1 hold. If $N(T)$ is sufficiently small, then there exists a constant $C>0$ independent of T such that*

$$\|\phi_z(\cdot,t)\|_{w_2}^2 + \int_0^t \|\phi_{zz}(\cdot,\tau)\|_{w_5}^2 \le C\left(\|\phi_0\|_{w_1}^2 + \|\phi_{0z}\|_{w_2}^2\right). \tag{4.54}$$

Proof. Multiplying (4.10) by $\left(-\frac{\phi_z}{U^2}\right)_z\eta^2$, where $\eta^2(z)$ is the cut-off function given by (4.33), integrating the result, we get

$$\begin{aligned}
&\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_z^2}{U^2}\eta^2+\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{3-m}}\eta^2+2(1-m)\int_0^t\int_{-2L}^{2L}\frac{U_z^2}{U^{5-m}}\phi_z^2\eta^2\\
&=\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{0z}^2}{U^2}\eta^2+(3-m)\int_0^t\int_{-2L}^{2L}\frac{U_z}{U^{4-m}}\phi_z\phi_{zz}\eta^2+\int_0^t\int_{-2L}^{2L}g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2\\
&\quad-\int_0^t\int_{-2L}^{2L}F\left(\frac{\phi_z}{U^2}\right)_z\eta^2-\frac{1}{m}\int_0^t\int_{-2L}^{2L}G_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2-2\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_t\phi_z}{U^2}.
\end{aligned}\tag{4.55}$$

We next estimate the last term of (4.55). By (4.10),

$$\begin{aligned}
|\phi_t|&=\left|\left(-g'(U)-(1-m)\frac{U_z}{U^{2-m}}\right)\phi_z+\frac{\phi_{zz}}{U^{1-m}}+F+\frac{1}{m}G_z\right|\\
&\le\left|g'(U)+(1-m)\frac{U_z}{U^{2-m}}\right||\phi_z|+\frac{|\phi_{zz}|}{U^{1-m}}+|F|+|\frac{1}{m}G_z|,
\end{aligned}\tag{4.56}$$

where, it follows from Corollary 3.1 and (4.29) that

$$\left|g'(U)+(1-m)\frac{U_z}{U^{2-m}}\right||\phi_z|\le C|\phi_z|,\tag{4.57}$$

and

$$|\frac{1}{m}G_z|\le C\left(\frac{\phi_z^2}{U}+\frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right).\tag{4.58}$$

Hence, utilizing (4.27), one has

$$\begin{aligned}
|\phi_t|&\le C\left(|\phi_z|+\frac{|\phi_{zz}|}{U^{1-m}}+\phi_z^2+\frac{\phi_z^2}{U}+\frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right)\\
&\le C\left(|\phi_z|+\frac{|\phi_{zz}|}{U^{1-m}}+\frac{\phi_z^2}{U}+\frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right).
\end{aligned}\tag{4.59}$$

Therefore,

$$\begin{aligned}
2\left|\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_t\phi_z}{U^2}\right|&\le C\int_0^t\int_{-2L}^{2L}\left(|\phi_z|+\frac{|\phi_{zz}|}{U^{1-m}}+\frac{\phi_z^2}{U}+\frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right)\frac{|\phi_z|}{U^2}\eta|\eta_z|\\
&\le C\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_z^2}{U^2}+\frac{|\phi_z||\phi_{zz}|}{U^{3-m}}\right)\eta|\eta_z|,
\end{aligned}\tag{4.60}$$

where we have used the fact $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t) \le C$ in the last inequality. Since $\phi \in X(0,T)$, we know that

$$\int_{-2L}^{2L} \frac{\phi_z^2}{U^2} \le \int \frac{\phi_z^2}{U^2} \le N^2(t) \text{ for any } t \in [0,T]. \tag{4.61}$$

Thus,

$$\int_0^t \int_{-2L}^{2L} \frac{\phi_z^2}{U^2}\eta|\eta_z| \le \frac{C}{L}\int_0^t \int_{-2L}^{2L} \frac{\phi_z^2}{U^2} \le \frac{CTN^2(T)}{L}. \tag{4.62}$$

By virtue of (4.38) and Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} \int_0^t \int_{-2L}^{2L} \frac{|\phi_z||\phi_{zz}|}{U^{3-m}}\eta|\eta_z| &\le \int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^{3-m}}\eta^2|\eta_z|^{\frac{1}{2}} + \int_0^t \int_{-2L}^{2L} \frac{\phi_z^2}{U^2}\frac{|\eta_z|^{\frac{3}{2}}}{U^{1-m}} \\ &\le \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^{3-m}}\eta^2 + \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_z^2}{U^2} \\ &\le \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^{3-m}}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}. \end{aligned} \tag{4.63}$$

Combining (4.62) and (4.63) with (4.60), we estimate the last term of (4.55) as

$$2\left|\int_0^t \int_{-2L}^{2L} \eta\eta_z \frac{\phi_t\phi_z}{U^2}\right| \le \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^{3-m}}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}. \tag{4.64}$$

Substituting (4.64) into (4.55) and passing the limit $L \to +\infty$, one obtains

$$\begin{aligned} \frac{1}{2}\int \frac{\phi_z^2}{U^2} + \iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} \le& \frac{1}{2}\int \frac{\phi_{0z}^2}{U^2} + (3-m)\iint_0^t \frac{|U_z|}{U^{4-m}}|\phi_z\phi_{zz}| + \left|\iint_0^t g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\right| \\ &+ \left|\iint_0^t F\left(\frac{\phi_z}{U^2}\right)_z\right| + \left|\frac{1}{m}\iint_0^t G_z\left(\frac{\phi_z}{U^2}\right)_z\right|. \end{aligned} \tag{4.65}$$

We now estimate the terms on the right hand side of (4.65). From Corollary 3.1,

$$
\begin{aligned}
(3-m)\iint_0^t \frac{|U_z|}{U^{4-m}}|\phi_z\phi_{zz}| &\le \frac{1}{4}\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} + C\iint_0^t \frac{U_z^2}{U^{5-m}}\phi_z^2 \\
&\le \frac{1}{4}\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} + C\iint_0^t \frac{\phi_z^2}{U^{1+m}}.
\end{aligned}
\tag{4.66}
$$

Similarly, thanks to $|g'(U)| \le C$ due to (2.6) and $f \in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$, we have

$$
\begin{aligned}
\left|\iint_0^t g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\right| &= \left|\iint_0^t \left(g'(U)\frac{\phi_z\phi_{zz}}{U^2} - g'(U)\frac{2U_z}{U^3}\phi_z^2\right)\right| \\
&\le \frac{1}{4}\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} + C\iint_0^t \frac{\phi_z^2}{U^{1+m}}.
\end{aligned}
\tag{4.67}
$$

Moreover, by virtue of $|U_z(z)| \le CU^{2-m}(z)$ and the fact that $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t)$, it follows from the Cauchy-Schwarz inequality and (4.27) that

$$
\begin{aligned}
\left|\iint_0^t F\left(\frac{\phi_z}{U^2}\right)_z\right| &= \left|\iint_0^t F\left(\frac{\phi_{zz}}{U^2} - \frac{2U_z}{U^3}\phi_z\right)\right| \\
&\le C\left(\iint_0^t \frac{\phi_z^2|\phi_{zz}|}{U^2} + \iint_0^t \frac{|U_z|}{U^3}|\phi_z|^3\right) \\
&\le CN(t)\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} + C\iint_0^t \left(\frac{\phi_z^2}{U^{m-1}} + \frac{\phi_z^2}{U^m}\right),
\end{aligned}
$$

and by (4.29), the last term on the right hand side of (4.65) can be estimated as

$$
\begin{aligned}
\left|\frac{1}{m}\iint_0^t G_z\left(\frac{\phi_z}{U^2}\right)_z\right| &= \left|\frac{1}{m}\iint_0^t G_z\left(\frac{\phi_{zz}}{U^2} - \frac{2U_z}{U^3}\phi_z\right)\right| \\
&\le C\left(\iint_0^t \frac{|U_z|}{U^{5-m}}\phi_z^2|\phi_{zz}| + \iint_0^t \frac{U_z^2}{U^{6-m}}|\phi_z|^3 + \iint_0^t \frac{|\phi_z|\phi_{zz}^2}{U^{4-m}}\right) \\
&\le CN(t)\left(\iint_0^t \frac{|\phi_z||\phi_{zz}|}{U^2} + \iint_0^t \frac{\phi_z^2}{U^{1+m}} + \iint_0^t \frac{\phi_{zz}^2}{U^{3-m}}\right)
\end{aligned}
$$

$$\leq CN(t)\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} + C\iint_0^t \frac{\phi_z^2}{U^{1+m}}. \tag{4.68}$$

We claim that

$$\frac{1}{U^{m-1}} \leq \frac{C}{U^m} \leq \frac{C}{U^{1+m}} \leq C\frac{w_1(U)}{U^{1-m}}, \quad \forall z \in (-\infty, +\infty). \tag{4.69}$$

In fact, (1) if $f'(0) \leq s < f'(u_-)$, then we have $w_1(U) \sim U^{-k_+}$ with $k_+ \geq 0$ as $z \to +\infty$ and $w_1(U) \sim C$ as $z \to -\infty$, it is easy to verify that (4.69) holds due to $m < 0$; (2) if $f'(0) < s = f'(u_-)$, by virtue of $w_1(U) \sim C$ as $z \to +\infty$ and $w_1(U) \sim |U - u_-|^{-k_-}$ as $z \to -\infty$, (4.69) holds. Substituting (4.66)-(4.68) into (4.65), noting that $\frac{w_1(U)}{U^{1-m}} = w_3(U)$, by (4.31) and (4.69), we have

$$\begin{aligned}\frac{1}{2}\int \frac{\phi_z^2}{U^2} + \left(\frac{1}{2} - CN(t)\right)\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}} &\leq \frac{1}{2}\int \frac{\phi_{0z}^2}{U^2} + C\iint_0^t \frac{w_1(U)}{U^{1-m}}\phi_z^2 \\ &\leq \frac{1}{2}\int \frac{\phi_{0z}^2}{U^2} + C\int w_1(U)\phi_0^2.\end{aligned}$$

Thus (4.54) is proved if $CN(t) \leq 1/4$ and we finish the proof of Lemma 4.3. □

To close the *a priori* estimate, we next give the estimate for ϕ_{zz}.

Lemma 4.4. *Let the assumptions of Proposition 4.1 hold. If $N(T)$ is sufficiently small, then there exists a constant $C > 0$ independent of T such that*

$$\|\phi_{zz}(\cdot,t)\|_{w_4}^2 + \int_0^t \|\phi_{zzz}(\cdot,\tau)\|_{w_6}^2 \leq C\left(\|\phi_0\|_{w_1}^2 + \|\phi_{0z}\|_{w_2}^2 + \|\phi_{0zz}\|_{w_4}^2\right). \tag{4.70}$$

Proof. We differentiate (4.10) with respect to z to get

$$\phi_{zt} + g''(U)U_z\phi_z + g'(U)\phi_{zz} - \left(\frac{\phi_z}{U^{1-m}}\right)_{zz} = F_z + \frac{1}{m}G_{zz}. \tag{4.71}$$

Multiplying (4.71) by $\left(-\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z \eta^2$, where $\eta^2(z)$ is the cut-off function, and integrating the result, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}}\eta^2+\int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^{3-3m}}\eta^2+4(1-m)^2\int_0^t\int_{-2L}^{2L}\frac{U_z^2}{U^{5-3m}}\phi_{zz}^2\eta^2\\
&=\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{0zz}^2}{U^{2(1-m)}}\eta^2-2\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_{zt}\phi_{zz}}{U^{2(1-m)}}+4(1-m)\int_0^t\int_{-2L}^{2L}\frac{U_z}{U^{4-3m}}\phi_{zz}\phi_{zzz}\\
&\quad-(1-m)\int_0^t\int_{-2L}^{2L}\left(\frac{(2-m)U_z^2}{U^{5-3m}}-\frac{U_{zz}}{U^{4-3m}}\right)\phi_z\phi_{zzz}\eta^2\\
&\quad+2(1-m)^2\int_0^t\int_{-2L}^{2L}\left(\frac{(2-m)U_z^3}{U^{6-3m}}-\frac{U_zU_{zz}}{U^{5-3m}}\right)\phi_z\phi_{zz}\eta^2+\int_0^t\int_{-2L}^{2L}g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\eta^2\\
&\quad+\int_0^t\int_{-2L}^{2L}g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\eta^2-\int_0^t\int_{-2L}^{2L}F_z\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\eta^2\\
&\quad-\frac{1}{m}\int_0^t\int_{-2L}^{2L}G_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\eta^2.
\end{aligned}
\tag{4.72}
$$

We next estimate the second term of (4.72). By (4.71),

$$
\begin{aligned}
|\phi_{zt}|&=\Bigg|\left[-g''(U)U_z+(1-m)\left((2-m)\frac{U_z^2}{U^{3-m}}-\frac{U_{zz}}{U^{2-m}}\right)\right]\phi_z\\
&\qquad-\left(g'(U)+2(1-m)\frac{U_z}{U^{2-m}}\right)\phi_{zz}+\frac{\phi_{zzz}}{U^{1-m}}+F_z+\frac{1}{m}G_{zz}\Bigg|\\
&\le\left|-g''(U)U_z+(1-m)\left((2-m)\frac{U_z^2}{U^{3-m}}-\frac{U_{zz}}{U^{2-m}}\right)\right||\phi_z|\\
&\qquad+\left|\left(g'(U)+2(1-m)\frac{U_z}{U^{2-m}}\right)\right||\phi_{zz}|+\frac{|\phi_{zzz}|}{U^{1-m}}+|F_z|+|\frac{1}{m}G_{zz}|,
\end{aligned}
\tag{4.73}
$$

where, utilizing Corollary 3.1, it follows from (4.28) and (4.30) that

$$
|F_z|\le C\left(U^{2-m}\phi_z^2+|\phi_z||\phi_{zz}|\right),
\tag{4.74}
$$

and

$$
|G_{zz}|\le C\left(\frac{\phi_z^2}{U^m}+\frac{\phi_{zz}^2}{U^{2-m}}+\frac{|\phi_z||\phi_{zz}|}{U}+\frac{|\phi_z||\phi_{zzz}|}{U^{2-m}}\right).
\tag{4.75}
$$

Thus, by (4.73), we have

$$|\phi_{zt}| \le C\left(\frac{|\phi_z|}{U^m} + |\phi_{zz}| + \frac{|\phi_{zzz}|}{U^{1-m}} + \frac{\phi_z^2}{U^m} + \frac{\phi_{zz}^2}{U^{2-m}} + \frac{|\phi_z||\phi_{zz}|}{U} + \frac{|\phi_z||\phi_{zzz}|}{U^{2-m}}\right). \tag{4.76}$$

And hence

$$\begin{aligned}
&2\left|\int_0^t\int_{-2L}^{2L} \eta\eta_z \frac{\phi_{zt}\phi_{zz}}{U^{2(1-m)}}\right| \\
&\le C\int_0^t\int_{-2L}^{2L}\left(\frac{|\phi_z||\phi_{zz}|}{U^{2-m}} + \frac{\phi_{zz}^2}{U^{2(1-m)}} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^{3-3m}}\right)\eta|\eta_z| + C\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^3}{U^{4-3m}}\eta|\eta_z| \\
&\quad + C\int_0^t\int_{-2L}^{2L}\left(\frac{|\phi_z||\phi_{zz}|}{U^{1-m}} + \frac{\phi_{zz}^2}{U^{2(1-m)}} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^{3-3m}}\right)\eta|\eta_z|\cdot\frac{|\phi_z|}{U} \\
&\le C\int_0^t\int_{-2L}^{2L}\left(\frac{|\phi_z||\phi_{zz}|}{U^{2-m}} + \frac{\phi_{zz}^2}{U^{2(1-m)}} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^{3-3m}}\right)\eta|\eta_z| + C\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^3}{U^{4-3m}}\eta|\eta_z|,
\end{aligned} \tag{4.77}$$

where we have used $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t) \le C$ and $\frac{|\phi_z||\phi_{zz}|}{U^{1-m}} \le C\frac{|\phi_z||\phi_{zz}|}{U^{2-m}}$ in the last inequality. Using $|\eta_z(z)| \le \frac{C}{L}$, we derive from the Cauchy-Schwarz inequality that

$$\int_0^t\int_{-2L}^{2L}\frac{|\phi_z\phi_{zz}|}{U^{2-m}}\eta|\eta_z| \le \frac{C}{L}\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_{zz}^2}{U^{2(1-m)}} + \frac{\phi_z^2}{U^2}\right) \le \frac{CTN^2(T)}{L}, \tag{4.78}$$

$$\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}}\eta|\eta_z| \le \frac{C}{L}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}} \le \frac{CTN^2(T)}{L}. \tag{4.79}$$

And thanks to (4.38), we get

$$\begin{aligned}
\int_0^t\int_{-2L}^{2L}\frac{|\phi_{zz}||\phi_{zzz}|}{U^{3-3m}}\eta|\eta_z| &\le \int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^{3-3m}}\eta^2|\eta_z|^{\frac{1}{2}} + \int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}}\frac{|\eta_z|^{\frac{3}{2}}}{U^{1-m}} \\
&\le \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^{3-3m}}\eta^2 + \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}} \\
&\le \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^{3-3m}}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}.
\end{aligned} \tag{4.80}$$

Moreover, by Hölder's inequality, $|\eta_z(z)| \le \frac{C}{L}$ and $\|\phi_{zz}(\cdot,t)/U^{1-m}\|_{L^2} \le CN(t)$, the last term of (4.77) can be estimated as

$$\begin{aligned}\int_0^t\int_{-2L}^{2L}\frac{|\phi_{zz}|^3}{U^{4-3m}}\eta|\eta_z| &\le C\int_0^t L^{-\frac14}\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^\infty}\|U^{-1+m}\phi_{zz}\|_{L^2}\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|^{\frac34}\|_{L^2}\\ &\le CN(t)\int_0^t L^{-\frac14}\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^\infty}\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|^{\frac34}\|_{L^2}.\end{aligned}\tag{4.81}$$

Since

$$\begin{aligned}\frac{\phi_{zz}^2}{U^{3-m}}\eta^2=\int_{-\infty}^z\left(\frac{\phi_{zz}^2}{U^{3-m}}\eta^2\right)_z &=\int_{-2L}^z\left(\frac{2\phi_{zz}\phi_{zzz}}{U^{3-m}}\eta^2-\frac{(3-m)U_z}{U^{4-m}}\phi_{zz}^2\eta^2+2\frac{\phi_{zz}^2}{U^{3-m}}\eta\eta_z\right)\\ &\le C\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^{3-m}}+\frac{\phi_{zz}^2}{U^{3-m}}\right)\eta^2+C\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{3-m}}\eta_z^2,\end{aligned}\tag{4.82}$$

which along with $m<0$ implies that

$$\begin{aligned}&\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^\infty}\\ &\le C\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^2}+\|U^{-\frac{3-3m}{2}}\phi_{zzz}\eta\|_{L^2}+C\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|\|_{L^2}\\ &\le C\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^2}+\|U^{-\frac{3-3m}{2}}\phi_{zzz}\eta\|_{L^2}+CL^{-\frac14}\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|^{\frac34}\|_{L^2}.\end{aligned}\tag{4.83}$$

It then follows from (4.81), (4.38) and the Cauchy-Schwarz inequality that

$$\begin{aligned}&\int_0^t\int_{-2L}^{2L}\frac{|\phi_{zz}|^3}{U^{4-3m}}\eta|\eta_z|\\ &\le CN(t)\int_0^t L^{-\frac14}\left(\|U^{-\frac{3-m}{2}}\phi_{zz}\eta\|_{L^2}+\|U^{-\frac{3-3m}{2}}\phi_{zzz}\eta\|_{L^2}\right)\cdot\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|^{\frac34}\|_{L^2}\\ &\quad+CN(t)\int_0^t L^{-\frac12}\|U^{-\frac{3-3m}{2}}\phi_{zz}|\eta_z|^{\frac34}\|_{L^2}^2\\ &\le CN(t)\int_0^t L^{-\frac12}\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^{3-3m}}+\frac{\phi_{zz}^2}{U^{3-m}}\right)\eta^2+CN(t)\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}}\frac{|\eta_z|^{\frac32}}{U^{1-m}}\\ &\le\frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^{3-3m}}+\frac{\phi_{zz}^2}{U^{3-m}}\right)\eta^2+\frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^{2(1-m)}}\end{aligned}$$

$$\leq \frac{CN(T)}{\sqrt{L}} \int_0^t \int_{-2L}^{2L} \left(\frac{\phi_{zzz}^2}{U^{3-3m}} + \frac{\phi_{zz}^2}{U^{3-m}} \right) \eta^2 + \frac{CTN^3(T)}{\sqrt{L}}. \tag{4.84}$$

Substituting (4.78)-(4.80), (4.84) into (4.77), the second term of (4.72) can be estimated as

$$2\left| \int_0^t \int_{-2L}^{2L} \eta\eta_z \frac{\phi_{zt}\phi_{zz}}{U^{2(1-m)}} \right| \leq \frac{C}{\sqrt{L}} \int_0^t \int_{-2L}^{2L} \frac{\phi_{zzz}^2}{U^{3-3m}}\eta^2 + \frac{CN(T)}{\sqrt{L}} \int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^{3-m}} + \frac{CTN^3(T)}{\sqrt{L}}. \tag{4.85}$$

Substituting (4.85) into (4.72) and passing the limit $L \to +\infty$, we get

$$\begin{aligned}
&\frac{1}{2}\int \frac{\phi_{zz}^2}{U^{2(1-m)}} + \iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} \\
&\leq \frac{1}{2}\int \frac{\phi_{0zz}^2}{U^{2(1-m)}} + 4(1-m)\left| \iint_0^t \frac{U_z}{U^{4-3m}}\phi_{zz}\phi_{zzz} \right| \\
&\quad + (1-m)\left| \iint_0^t \left(\frac{(2-m)U_z^2}{U^{5-3m}} - \frac{U_{zz}}{U^{4-3m}} \right)\phi_z\phi_{zzz} \right| \\
&\quad + 2(1-m)^2\left| \iint_0^t \left(\frac{(2-m)U_z^3}{U^{6-3m}} - \frac{U_zU_{zz}}{U^{5-3m}} \right)\phi_z\phi_{zz} \right| \\
&\quad + \iint_0^t \left(\left| g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}} \right)_z \right| + \left| g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^{2(1-m)}} \right)_z \right| \right) \\
&\quad + \left| \iint_0^t F_z\left(\frac{\phi_{zz}}{U^{2(1-m)}} \right)_z \right| + \left| \frac{1}{m}\iint_0^t G_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}} \right)_z \right|.
\end{aligned} \tag{4.86}$$

We next estimate the terms on right hand side of (4.86). By virtue of Corollary 3.1 and the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}
&4(1-m)\left| \iint_0^t \frac{U_z}{U^{4-3m}}\phi_{zz}\phi_{zzz} \right| + (1-m)\left| \iint_0^t \left(\frac{(2-m)U_z^2}{U^{5-3m}} - \frac{U_{zz}}{U^{4-3m}} \right)\phi_z\phi_{zzz} \right| \\
&\leq C\left(\iint_0^t \frac{|\phi_{zz}\phi_{zzz}|}{U^{2-2m}} + \iint_0^t \frac{|\phi_z\phi_{zzz}|}{U^{1-m}} \right) \\
&\leq \frac{1}{4}\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} + C\left(\iint_0^t \frac{\phi_{zz}^2}{U^{1-m}} + \iint_0^t \frac{\phi_z^2}{U^{m-1}} \right),
\end{aligned} \tag{4.87}$$

and

$$2(1-m)^2\left|\iint_0^t \left(\frac{(2-m)U_z^3}{U^{6-3m}}-\frac{U_zU_{zz}}{U^{5-3m}}\right)\phi_z\phi_{zz}\right|\le C\left(\iint_0^t \phi_{zz}^2+\iint_0^t \phi_z^2\right). \tag{4.88}$$

Similarly, thanks to (2.6), (3.10), (3.11) and $f\in C^{\max\{(k_\pm+1),\ 3\}}(\mathbb{R})$, it holds that

$$\begin{aligned}
&\iint_0^t \left(\left|g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\right|+\left|g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\right|\right)\\
&=\iint_0^t (|g'(U)\phi_{zz}|+|g''(U)U_z\phi_z|)\left|\frac{\phi_{zzz}}{U^{2(1-m)}}+\frac{2(m-1)U_z}{U^{3-2m}}\phi_{zz}\right|\\
&\le C\left(\iint_0^t \frac{|\phi_{zz}\phi_{zzz}|}{U^{2(1-m)}}+\iint_0^t \frac{\phi_{zz}^2}{U^{1-m}}+\iint_0^t \frac{|\phi_z\phi_{zzz}|}{U^{-m}}+\iint_0^t |\phi_z\phi_{zz}|\right)\\
&\le \frac{1}{2}\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}}+C\left(\iint_0^t \frac{\phi_{zz}^2}{U^{1-m}}+\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}}+\iint_0^t \frac{\phi_z^2}{U^{m-3}}\right).
\end{aligned} \tag{4.89}$$

Noting that $\|\phi_z(\cdot,t)/U\|_{L^\infty}\le CN(t)$, by (4.74), one obtains

$$\begin{aligned}
\left|\iint_0^t F_z\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\right| &= \left|\iint_0^t F_z\left(\frac{\phi_{zzz}}{U^{2(1-m)}}+\frac{2(m-1)U_z}{U^{3-2m}}\phi_{zz}\right)\right|\\
&\le C\iint_0^t \left(U^{2-m}\phi_z^2+|\phi_z||\phi_{zz}|\right)\left(\frac{|\phi_{zzz}|}{U^{2(1-m)}}+\frac{|\phi_{zz}|}{U^{1-m}}\right)\\
&\le CN(t)\iint_0^t \left(\frac{|\phi_z||\phi_{zzz}|}{U^{-1-m}}+|\phi_z||\phi_{zz}|+\frac{|\phi_{zz}||\phi_{zzz}|}{U^{1-2m}}+\frac{\phi_{zz}^2}{U^{-m}}\right)\\
&\le CN(t)\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}}+C\left(\iint_0^t \frac{\phi_{zz}^2}{U^{-m}}+\iint_0^t \frac{\phi_z^2}{U^{m-5}}\right).
\end{aligned} \tag{4.90}$$

Moreover, in view of (4.75), the last term on the right hand side of (4.86) can be estimated as

$$\left|\frac{1}{m}\iint_0^t G_{zz}\left(\frac{\phi_{zz}}{U^{2(1-m)}}\right)_z\right|=\left|\frac{1}{m}\iint_0^t G_{zz}\left(\frac{\phi_{zzz}}{U^{2(1-m)}}+\frac{2(m-1)U_z}{U^{3-2m}}\phi_{zz}\right)\right|$$

$$
\begin{aligned}
&\leq C \iint_0^t \frac{\phi_{zz}^2|\phi_{zzz}|}{U^{4-3m}} + C\iint_0^t \frac{|\phi_{zz}|^3}{U^{3-2m}} + C\iint_0^t \left(\frac{\phi_z^2|\phi_{zzz}|}{U^{2-m}}\right.\\
&\quad \left. + \frac{\phi_z^2|\phi_{zz}|}{U} + \frac{|\phi_z|\phi_{zz}^2}{U^{2-m}} + \frac{|\phi_z|\phi_{zzz}^2}{U^{4-3m}} + \frac{|\phi_z||\phi_{zz}||\phi_{zzz}|}{U^{3-2m}}\right)\\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}
$$

Utilizing $\|\phi_{zz}(\cdot,t)/U^{1-m}\|_{L^2} \leq CN(t)$ and Hölder's inequality, we get

$$
\begin{aligned}
I_1 &\leq C\int_0^t \left\|\frac{\phi_{zz}}{U^{\frac{3-m}{2}}}\right\|_{L^\infty} \left\|\frac{\phi_{zzz}}{U^{\frac{3-3m}{2}}}\right\|_{L^2} \left\|\frac{\phi_{zz}}{U^{1-m}}\right\|_{L^2}\\
&\leq CN(t)\int_0^t \left(\left\|\frac{\phi_{zz}}{U^{\frac{3-m}{2}}}\right\|_{L^2} + \left\|\frac{\phi_{zzz}}{U^{\frac{3-m}{2}}}\right\|_{L^2}\right)\left\|\frac{\phi_{zzz}}{U^{\frac{3-3m}{2}}}\right\|_{L^2}\\
&\leq CN(t)\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} + C\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}},
\end{aligned}
$$

and

$$
I_2 \leq C\int_0^t \left\|\frac{\phi_{zz}}{U^{\frac{1-m}{2}}}\right\|_{L^\infty}\left\|\frac{\phi_{zz}}{U^{\frac{3-m}{2}}}\right\|_{L^2}\left\|\frac{\phi_{zz}}{U^{1-m}}\right\|_{L^2} \leq CN(t)\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} + C\iint_0^t \frac{\phi_{zz}^2}{U^{3-m}},
$$

where we have applied $\|\phi_{zz}(\cdot,t)/U^{1-m}\|_{L^2} \leq CN(t)$ and $\frac{1}{U^{1-m}} \leq \frac{C}{U^{3-m}} \leq \frac{C}{U^{3-3m}}$ in the last inequality. Furthermore, we utilize $\|\phi_z(\cdot,t)/U\|_{L^\infty} \leq CN(t)$ and the Cauchy-Schwarz inequality to get

$$
\begin{aligned}
I_3 &\leq CN(t)\iint_0^t \left(\frac{|\phi_z||\phi_{zzz}|}{U^{1-m}} + |\phi_z||\phi_{zz}| + \frac{\phi_{zz}^2}{U^{1-m}} + \frac{\phi_{zzz}^2}{U^{3-3m}} + \frac{|\phi_{zz}||\phi_{zzz}|}{U^{2-2m}}\right)\\
&\leq CN(t)\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} + C\iint_0^t \left(\frac{\phi_z^2}{U^{m-1}} + \frac{\phi_{zz}^2}{U^{1-m}}\right).
\end{aligned}
\tag{4.91}
$$

Noting that $m < 0$, as in the proof of (4.69), one can show that

$$
\frac{1}{U^{1-m}} \leq \frac{C}{U^{-m}} \leq \frac{C}{U^{3-m}}, \quad \forall z \in (-\infty, +\infty) \tag{4.92}
$$

and

$$
\frac{1}{U^{m-5}} \leq \frac{C}{U^{m-3}} \leq \frac{C}{U^{m-1}} \leq \frac{Cw_1(U)}{U^{1-m}} = Cw_3(U), \quad \forall z \in (-\infty, +\infty). \tag{4.93}
$$

Therefore, by (4.92) and (4.93), combining (4.87)-(4.91) with (4.31) and (4.54) gives rise to

$$\frac{1}{2}\int \frac{\phi_{zz}^2}{U^{2(1-m)}} + \left(\frac{1}{4} - CN(t)\right)\iint_0^t \frac{\phi_{zzz}^2}{U^{3-3m}} \le \frac{1}{2}\int \frac{\phi_{0zz}^2}{U^{2(1-m)}} + C\iint_0^t \left(\frac{\phi_{zz}^2}{U^{3-m}} + \frac{w_1(U)}{U^{1-m}}\phi_z^2\right)$$
$$\le C\left(\int \frac{\phi_{0zz}^2}{U^{2(1-m)}} + \int \frac{\phi_{0z}^2}{U^2} + \int w_1(U)\phi_0^2\right).$$

Thus (4.70) is derived if $CN(t) \le 1/8$ and we finish the proof of Lemma 4.4. □

Proof of Proposition 4.1. The desired estimate (4.22) follows from (4.31), (4.54) and (4.70). □

Proof of Theorem 4.1. The a priori estimate (4.22) guarantees that if $\|\phi_0\|_{w_1} + \|\phi_{0z}\|_{w_2} + \|\phi_{0zz}\|_{w_4}$ is small enough, then $N(t)$ is small for all $t > 0$. Therefore, applying the standard extension argument, we obtain the global well-posedness of equation (4.10)-(4.13) in $X(0,\infty)$. It remains to establish the convergence

$$\sup_{z\in\mathbb{R}} |\phi_z(z,t)| \to 0 \quad \text{as } t \to +\infty. \tag{4.94}$$

Denote $p(t) \triangleq \|\phi_z(\cdot,t)\|^2$. Utilizing the established estimate (4.22) for all $t \in [0,\infty)$, and $w_3(U) = w_1(U)/U^{1-m} \ge 1$, we have

$$\int_0^\infty \|\phi_z(\cdot,\tau)\|^2 \mathrm{d}\tau \le \int_0^\infty \|\phi_z(\cdot,\tau)\|_{w_3}^2 \,\mathrm{d}\tau \le CN^2(0) \le C,$$

which implies $p(t) \in L^1(0,\infty)$. We next show that $p'(t) \in L^1(0,\infty)$. By the equation (4.71), a direct calculation gives

$$\begin{aligned}
\int_0^\infty |p'(\tau)|\mathrm{d}\tau &= \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\|\phi_z(\cdot,\tau)\|^2\right| \mathrm{d}\tau \\
&= 2\int_0^\infty \left|\int \phi_z\phi_{zt}\right| \mathrm{d}\tau \\
&= \int_0^\infty \left|\int \phi_z\left(-g''(U)U_z\phi_z - g'(U)\phi_{zz} + \left(\frac{\phi_z}{U^{1-m}}\right)_{zz} + F_z + \frac{1}{m}G_{zz}\right)\right|.
\end{aligned} \tag{4.95}$$

We get from (3.10), (3.11) and Cauchy-Schwarz inequality that

$$\begin{aligned}\int \phi_z \left(\frac{\phi_z}{U^{1-m}}\right)_{zz} &= -\int \phi_{zz}\left(\frac{\phi_z}{U^{1-m}}\right)_z \\ &= -\int \phi_{zz}\left[\frac{\phi_{zz}}{U^{1-m}} - (m-1)\frac{U_z}{U^{2-m}}\phi_z\right] \\ &\le C\int\left(\frac{\phi_{zz}^2}{U^{1-m}} + \phi_{zz}^2 + \phi_z^2\right).\end{aligned}$$

Moreover, in view of $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t)$, (4.28) and (4.29), we have

$$\begin{aligned}\frac{1}{m}\int \phi_z G_{zz} = -\frac{1}{m}\int \phi_{zz}G_z \le \int \phi_{zz}\left(\frac{|U_z|}{U^{3-m}}\phi_z^2 + \frac{|\phi_z||\phi_{zz}|}{U^{2-m}}\right) \\ \le CN(t)\int\left(|\phi_z\phi_{zz}| + \frac{\phi_{zz}^2}{U^{1-m}}\right) \\ \le C\int\left(\frac{\phi_{zz}^2}{U^{1-m}} + \phi_{zz}^2 + \phi_z^2\right),\end{aligned}$$

and

$$\begin{aligned}\int \phi_z F_z \le \int \phi_z\left(|U_z|\phi_z^2 + |\phi_z||\phi_{zz}|\right) \\ \le C\int\left(\phi_{zz}^2 + \phi_z^2\right).\end{aligned} \tag{4.96}$$

Therefore, combining (4.95)-(4.96) and (4.22), one obtains

$$\int_0^\infty |p'(\tau)|\mathrm{d}\tau \le C\iint_0^\infty\left(\frac{\phi_{zz}^2}{U^{1-m}} + \phi_{zz}^2 + \phi_z^2\right) \le C. \tag{4.97}$$

Thus,

$$\|\phi_z(\cdot,t)\|^2 \to 0, \qquad \text{as } t \to +\infty.$$

Now for all $z \in \mathbb{R}$, we get

$$\begin{aligned}\phi_z^2(z,t) = 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \leqslant 2\left(\int_{-\infty}^{\infty}\phi_z^2 dy\right)^{1/2}\left(\int_{-\infty}^{\infty}\phi_{zz}^2 dy\right)^{1/2} \\ \leqslant C\,\|\phi_z(\cdot,t)\| \to 0, \qquad \text{as } t \to +\infty,\end{aligned}$$

where we have used the boundedness of $\|\phi_{zz}\|$. Therefore, (4.94) is proved and the proof of Theorem 4.1 is complete. □

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Data availability

No data was used for the research described in the article.

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