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LARGE-TIME BEHAVIOR OF SOLUTIONS TO CAUCHY PROBLEM FOR BIPOLAR EULER-POISSON SYSTEM WITH TIME-DEPENDENT DAMPING IN CRITICAL CASE*

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Abstract. This paper is concerned with the Cauchy problem of a bipolar hydrodynamic model for semiconductor device, a system of one dimensional Euler-Poisson equations with time-dependent damping effect in the critical case. The global existence and uniqueness of the solutions to the Cauchy problem are proved by the technical time-weighted energy method, when the initial perturbation around the constant states are small enough. Particularly, the algebraic time-convergence-rates for the solutions to their constant states are also derived.

Keywords. Euler-Poisson equations; Time-dependent damping; Time-weighted energy method; Asymptotic behavior; Global solutions; Cauchy problem.

AMS subject classifications. 35B40; 35L60; 35L67.

1. Introduction

Mathematical models. To describe the motion of charged particles such as electrons and holes in bipolar semiconductor devices, or positively/negatively charged ions in plasmas, scientists usually employ bipolar hydrodynamic models which are the most popular models [1, 41, 52, 53] for simulations in semiconductor devices, and those simulation results coincide quite well with the corresponding experimental ones, which are presented by the (re-scaled) isentropic Euler-Poisson systems with damping:

$$\begin{cases} n_{1t}+J_{1x}=0, \\ n_{2t}+J_{2x}=0, \\ J_{1t}+\left(\frac{J_1^2}{n_1}+p\right)_x=n_1\Phi_x-\frac{J_1}{\tau}, \\ J_{2t}+\left(\frac{J_2^2}{n_2}+q\right)_x=-n_2\Phi_x-\frac{J_2}{\tau}, \\ \Phi_{xx}=n_1-n_2-D, \end{cases} \tag{1.1}$$

considered in $\mathbb{R}\times(0,T)$ for some positive constant $T>0$. The unknowns $n_1=n_1(x,t)$ and $n_2=n_2(x,t)$ stand for the densities of electrons and holes, $J_1=J_1(x,t)$ and $J_2=J_2(x,t)$ are the current densities for electrons and holes, respectively. $\Phi=\Phi(x,t)$ denotes the electrostatic potential and $p=p(x,t)=\hat{p}(n_1)(x,t)$ and $q=q(x,t)=\hat{q}(n_2)(x,t)$ are the pressure functions for electrons and holes, respectively. The given function $D=D(x)>0$ is the doping profile that is the density of impurities in the semiconductor device. $\tau>0$ is the relaxation-time.

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As mentioned in [32], for different values of the relaxation time τ, the character of the system (1.1) will be essentially different. In fact, when $\tau\to\infty$, the damping effects to the system (1.1) will be vanishing, and the system becomes the pure Euler-Poisson system. While, when $\tau\to 0^+$, the damping effects to the system will extremely enlarge. A mathematical concern for both cases as shown in [32] is to set $\tau=\mu^{-1}(1+t)^\lambda$ for some constants λ and $\mu>0$, where, when $t\to+\infty$, then $\tau=\mu^{-1}(1+t)^\lambda\to\infty$ for $\lambda>0$ and $\tau=\mu^{-1}(1+t)^\lambda\to 0^+$ for $\lambda<0$. Thus, the damping effects in Euler-Poisson system (1.1) become time-gradually-degenerate for $\lambda>0$ and time-gradually-enhancing for $\lambda<0$, and reads as

$$n_{1t}+J_{1x}=0, \tag{1.2}$$
$$n_{2t}+J_{2x}=0, \tag{1.3}$$
$$J_{1t}+\left(\frac{J_1^2}{n_1}+p\right)_x=n_1E-\frac{\mu}{(1+t)^\lambda}J_1, \tag{1.4}$$
$$J_{2t}+\left(\frac{J_2^2}{n_2}+q\right)_x=-n_2E-\frac{\mu}{(1+t)^\lambda}J_2, \tag{1.5}$$
$$E_x=n_1-n_2-D, \tag{1.6}$$

where $E:=\Phi_x$ is the electronic field. The initial condition is chosen to meet

$$(n_1,n_2,J_1,J_2)\,|_{t=0}=(n_{10},n_{20},J_{10},J_{20})(x)\to(\bar n,\bar n,\bar J,\bar J)\quad\text{as}\quad x\to\pm\infty. \tag{1.7}$$

Here $\bar n,\bar J$ are two given constants, for which we assume that $\bar n>0$ and $\bar J\neq 0$, and without loss of generality in what follows we may assume that $\bar J>0$. Technically throughout this paper we also assume

$$\hat p(s)\equiv\hat q(s)\quad\text{in}\quad\mathbb{R}^+, \tag{1.8}$$
$$\hat p(s)>0,\quad \hat p'(s)>0\quad\forall s\in\mathbb{R}^+, \tag{1.9}$$
$$D(x)\equiv 0, \tag{1.10}$$
$$\lambda=1,\quad \mu>2\quad\text{(critical case)}, \tag{1.11}$$

where we used the symbol $\mathbb{R}^+:=(0,+\infty)$.

REMARK 1.1.

(1) The assumption (1.8) is just for the sake of simplicity, of course the case with two different pressures is also interesting and more physical, but more complicated.

(2) The physical meaning of $D\equiv 0$ is explained as follows: One is that the amplitude of doping profile is sufficiently small. Another is we define $\hat E=E+\int_{-\infty}^x D(y)dy$ which is regarded as a fluctuation of the field E. The case for bipolar hydrodynamic models with $D=0$ is also well-studied in [13,21,22,32].

(3) For the critical case of $\lambda=1$ and $\mu>2$, the story comes from what follows. As clearly shown in [3,4,14,33,47,48,50,51] for Euler equations with time-dependent damping:

$$\begin{cases} v_t+u_x=0,\\ u_t+p(v)_x=-\frac{\mu}{(1+t)^\lambda}u, \end{cases}$$

when $0<\lambda<1$ and any $\mu>0$, even the under-damping effect is weaker, but it can still prevent the singularity formation for the solution like shocks, namely, the damped Euler system can possess global-in-time solutions; while, when $\lambda>1$ and any $\mu>0$, the under-damping effect is too weak such that it cannot stop the formation of shocks, and the system really behaves like the pure Euler system to possess shocks, namely, the solutions themselves are bounded but their derivatives will blow up at finite time; moreover, $\lambda=1$ is the critical case for the global existence and blow-up. This was further clarified in [3, 14, 48, 51] that, the solutions globally exist for $\lambda=1$ and $\mu>2$, and they blow up in finite time for $\lambda=1$ and $0<\mu\leq 2$ due to the mechanism of the system. Inspired by the above mentioned studies, here for the bipolar Euler-Poisson equations with time-dependent damping, we are mainly interested in the critical case with $\lambda=1$ and $\mu>2$ for the global existence. The opposite case with $\lambda=1$ and $\mu\leq 2$ for blow-up will be expected in the coming work.

Main results. Let us denote the spatial asymptotic profile of the solutions as follows

$$f(\pm\infty,t)=f_\pm(t), \quad \text{here } f\in \mathcal{A}:=\{n_1,n_2,J_1,J_2,E\}, \tag{1.12}$$

and assume that

$$n_+(t)=n_-(t)=\bar{n}, \tag{1.13}$$
$$J_+(0)=J_-(0)=\bar{J}. \tag{1.14}$$

Define

$$\hat{J}=\bar{J}(1+t)^{-\mu}m_1(x), \tag{1.15}$$
$$\hat{n}=\frac{\bar{J}(1+t)^{-(\mu-1)}}{\mu-1}m_1'(x), \tag{1.16}$$

where

$$m_1(x)=\int_0^x m_0(y)dy$$

and $m_0(x)$ is odd and satisfies

$$m_0\in C_0^\infty(R), \quad m_0(-x)=-m_0(x), \quad \int_0^\infty m_0(y)dy=\int_0^{-\infty} m_0(y)dy=1, \tag{1.17}$$

which implies

$$m_1(-\infty)=m_1(\infty)=1.$$

Let us assume

$$\int_{-\infty}^{\infty}[n_{i0}(x)-\bar{n}(x,0)]dx=0. \tag{1.18}$$

Note from (1.16) and (1.17), we have

$$\int_{-\infty}^{\infty}\hat{n}(x,0)dx=0,$$

which, together with (1.18), gives

$$\int_{-\infty}^{\infty}[n_{i0}(x)-\bar{n}(x,0)-\hat{n}(x,0)]dx=0.$$

Therefore, we may denote the initial perturbations as follows:

$$\phi_{i0}(x):=\int_{-\infty}^{x}[n_{i0}(x)-\bar{n}-\hat{n}(x,0)]dx, \quad \text{and} \quad \psi_{i0}(x):=J_{i0}(x)-\hat{J}(x,0), \quad \text{for } i=1,2. \tag{1.19}$$

We are now able to state our main theorem in this article.

THEOREM 1.1. *For the critical case $\lambda=1$ and $\mu>2$, let (1.18) hold, and let $\phi_{i0}\in H^2(\mathbb{R})$ and $\psi_{i0}\in H^1(\mathbb{R})$ for $i=1,2$. There exists a suitably small number $\delta>0$, when*

$$|\bar{J}|+\|(\phi_{10},\phi_{20})\|_{H^2(\mathbb{R})}+\|(\psi_{10},\psi_{20})\|_{H^1(\mathbb{R})}\le\delta,$$

then the system (1.2)–(1.11) possesses a unique global-in-time solution $(n_1,n_2,J_1,J_2,E)(x,t)$ satisfying, for $l=0,1$,

$$\|\partial_x^l(n_1-\bar{n}-\hat{n},n_2-\bar{n}-\hat{n})(t)\|_{L^2(\mathbb{R})}\le C\delta(1+t)^{-\frac{l+1}{2}}, \tag{1.20}$$

$$\|\partial_x^l(J_1-\hat{J},\ J_2-\hat{J})(t)\|_{L^2(\mathbb{R})}\le C\delta(1+t)^{-\frac{l+1}{2}}, \tag{1.21}$$

$$\|\partial_x^l E(t)\|_{L^2(\mathbb{R})}\le C\delta(1+t)^{-\frac{l+1}{2}}, \tag{1.22}$$

and

$$\|(n_1-\hat{n}-\bar{n},n_2-\hat{n}-\bar{n})(t)\|_{L^\infty(\mathbb{R})}\le C\delta(1+t)^{-\frac{3}{4}}, \tag{1.23}$$

$$\|(J_1-\hat{J},J_2-\hat{J})(t)\|_{L^\infty(\mathbb{R})}\le C\delta(1+t)^{-\frac{3}{4}}, \tag{1.24}$$

$$\|E(t)\|_{L^\infty(\mathbb{R})}\le C\delta(1+t)^{-\frac{3}{4}}. \tag{1.25}$$

The main difficulties in the proof of this theorem are as follows. The first is that the system of several unknowns is of hyperbolic type, we make use of the time-dependent damping term to overcome this difficulty. The second is the decay rates of some terms are not fast enough in their original form. We deal with those terms by using the technique of integration by parts due to the property of the asymptotic profiles that spatial derivatives of those profiles possess the same time-decay rates.

Notations. Throughout this article, C, C_δ, C_ε, $\cdots$ denote positive constants, which are independent of time t, however may depend on parameters (usually small) δ, $\varepsilon,\cdots$, respectively.

For $1\le p<\infty$, $L^p(\mathbb{R})$ are the spaces of measurable functions whose p-powers are integrable on $\mathbb{R}$, with the norm $\|\cdot\|_{L^p(\mathbb{R})}=\left(\int_{\mathbb{R}}|\cdot|^p dx\right)^{1/p}$. For the case that $p=2$, we simply denote $\|\cdot\|_{L^2(\mathbb{R})}$ by $\|\cdot\|$. For $p=\infty$, $L^\infty(\mathbb{R})$ is the space of bounded measurable functions on $\mathbb{R}$, with the norm $\|\cdot\|_{L^\infty(\mathbb{R})}=\text{esssup}_{x\in\mathbb{R}}|\cdot|$. For a nonnegative integer k, $H^k=H^k(\mathbb{R})$ denotes the Hilbert spaces of order k. We write $\|\cdot\|_k$ for the standard norm of $H^k(\mathbb{R})$. In addition, we denote by $C(0,T;H^k(\mathbb{R}))$ (resp. $L^2(0,T;H^k(\mathbb{R}))$) the space of continuous (resp. square integrable) functions on $[0,T]$ with values taken in a Banach space $H^k(\mathbb{R})$.

Literature. There are ever-increasing interests on hydrodynamic models for semiconductor devices, or positively/negatively charged ions in a plasma. We recall mainly the results the following four categories.

(A) For unipolar isentropic and non-isentropic hydrodynamic semiconductor equations, the mathematical modelling [29, 41] played an important role in the course of the study. The papers [5, 11, 12] studied the well-posedness of steady-state solutions and in [15, 16, 23, 35, 37, 43] they studied the stability of the steady-state solutions.

In [2,36,40,49,57], the authors gave the global existence of classical solutions and the entropy weak solutions, respectively. The large-time behavior of solutions was studied in [16,21,30,35].

(B) For bipolar hydrodynamic semiconductor equations, in [17,18], Hsiao and Zhang used the compensated compactness framework to prove the existence of the global entropic weak solutions for the initial-boundary value problem to a bipolar hydrodynamic semiconductor model. Zhu and Hattori discussed the stability of the steady state solutions for an isentropic hydrodynamic model of semiconductors of two species in [58]. Huang and Li [19] considered the large-time behavior of the entropy solutions of the one-dimensional bipolar hydrodynamic model. In addition, a general multidimensional non-isentropic hydrodynamical model for semiconductors with small momentum relaxation time was investigated in [31] but in the n-dimensional case, the bipolar hydrodynamic system of semiconductors is never dealt with.

(C) There are some interesting bipolar hydrodynamic models with time-dependent damping for semiconductors. Physically, the damping term usually causes the dynamical system to process the nonlinear diffusive phenomena. In system (1.2)–(1.6), J_i/τ (i=1,2) are the damping effects which effect the regularity of the solutions. As $\tau\to\infty$ and $\tau\to 0$ respectively, it is a challenge to study the asymptotic behavior of the solutions. In order to see the asymptotic behavior of the solutions, Li-Li-Mei-Zhang take $\tau=(1+t)^{\lambda}$ for some constant λ in [32]. They studied the one-dimensional Euler-Poisson equations of bipolar hydrodynamic model for semiconductor devices with time-dependent damping effect $-(1+t)^{\nu}$ for $-1<\lambda<1$ and proved the existence of a unique global smooth solution for the system. Particularly, $\lambda=\frac{1}{7}$ is the critical point, where the convergence rate is the fastest.

(D) For one-dimensional compressible Euler equations with time-independent damping, there are huge amount of results on this topic. We refer the readers, however not limited, to [20, 24, 38, 39, 42, 44–46]. Particularly, the comprehensive reviews can be found in the recent interesting papers [3, 14, 25–28, 34, 54–56]. For the other topic on Euler-Poisson equations with quantum effect, we refer to [6–10].

The rest of this article is organized as follows. In Section 2, we first construct asymptotic profiles, and formulate a new problem to simplify the original Cauchy problem. Then we employ, in Section 3, the time-weighted energy method to establish *a priori* estimates for the unknowns $E(x,t)$, $n_i(x,t)$ and $J_i(x,t)$, $i=1,2$ and for their derivatives. With the *a priori* estimates, we complete the proof of the main result, i.e., Theorem 1.1.

2. Reformulation of the problem

2.1. Construction of the asymptotic profiles. Recalling the assumption (1.13), we heuristically expect

$$E_-(t)=E(-\infty,t)=0. \tag{2.1}$$

Denoting

$$E_+(t):=E(+\infty,t)=\int_{-\infty}^{+\infty}(n_1(x,t)-n_2(x,t))dx, \tag{2.2}$$

then, as $x\to\pm\infty$, we further expect to reduce the Equations (1.2)–(1.6) to

$$\frac{d}{dt}n_{i\pm}=0, \tag{2.3}$$

$$\frac{d}{dt}J_{i\pm} = (-1)^{i-1}\bar{n}E_{\pm} - \frac{\mu}{1+t}J_{i\pm}, \tag{2.4}$$

$$E(x,t) = \int_{-\infty}^{x} E_x(x,t)dx = \int_{-\infty}^{x} (n_1 - n_2)(x,t)dx. \tag{2.5}$$

Invoking $E_-(\infty,t)=0$, one obtains from (2.4) that

$$\frac{d}{dt}J_{i-} = -\frac{\mu}{1+t}J_{i-},$$
$$J_{i-}(0) = \bar{J},$$

which yields

$$J_{i-}(t) = \bar{J}(1+t)^{-\mu}. \tag{2.6}$$

From (2.4) we derive the equations for the difference and the sum of $J_{1\pm}$ and $J_{2\pm}$ which, respectively, read

$$\frac{d}{dt}(J_{1\pm} - J_{2\pm}) = 2\bar{n}E_{\pm} - \frac{\mu}{1+t}(J_{1\pm} - J_{2\pm}), \tag{2.7}$$

and

$$\frac{d}{dt}(J_{1+} + J_{2+}) = -\frac{\mu}{1+t}(J_{1+} + J_{2+}), \tag{2.8}$$

$$J_{1+}(0) + J_{2+}(0) = 2\bar{J}. \tag{2.9}$$

One obtains from (2.8)–(2.9) that

$$(J_{1+} + J_{2+})(t) = 2\bar{J}(1+t)^{-\mu}. \tag{2.10}$$

Next from (1.6), invoking (1.2)–(1.3) we get

$$E_{xt} = n_{1t} - n_{2t} = -(J_1 - J_2)_x. \tag{2.11}$$

Integrating (2.11) with respect to x over $(-\infty,x)$ and using (2.6) yield

$$\begin{aligned} E_t(x,t) &= -\int_{-\infty}^{x} (J_1 - J_2)_x dx \\ &= -(J_1 - J_2)(x,t) - (J_{1-} - J_{2-})(t) \\ &= -(J_1 - J_2)(x,t). \end{aligned} \tag{2.12}$$

As $x \to +\infty$, (2.12) thus becomes

$$\frac{d}{dt}E_+(t) = -(J_{1+} - J_{2+})(t). \tag{2.13}$$

Inserting (2.13) into (2.7), we arrive at

$$\frac{d^2}{dt^2}E_+(t) + \frac{\mu}{1+t}\frac{d}{dt}E_+(t) - 2\bar{n}E_+(t) = 0, \tag{2.14}$$

$$E_+(0) = \int_{-\infty}^{+\infty} (n_{10} - n_{20})(x)dx = 0, \tag{2.15}$$

$$\frac{d}{dt}E_+(0)=0. \tag{2.16}$$

Solving this initial value problem yields the unique solution, that is

$$E_+(t)=0. \tag{2.17}$$

Now inserting $E_+(t)=0$ into equation (2.4), we obtain

$$J_{i+}(t)=\bar{J}(1+t)^{-\mu}. \tag{2.18}$$

Then

$$n_i(\pm\infty,t)=\bar{n},\quad J_i(\pm\infty,t)=\bar{J}(1+t)^{-\mu},\quad E(\pm\infty,t)=0. \tag{2.19}$$

Remark 2.1. The initial condition $E_+(0)=0$ is obvious due to (2.1), and (2.16) follows from (2.13) and (1.7).

Since J_i is not square-summable in the case that $\bar{J}\neq 0$, we need to construct the asymptotic solution $(\hat{n},\hat{J})$ which guarantees a perturbatuon of J_i summable. We define

$$\hat{J}=\bar{J}(1+t)^{-\mu}m_1(x). \tag{2.20}$$

Then, by solving the following equation

$$\hat{n}_t+\hat{J}_x=0,$$

one gets

$$\hat{n}=\frac{\bar{J}(1+t)^{-(\mu-1)}}{\mu-1}m_0(x). \tag{2.21}$$

Here we choose $m_0(x)$ as

$$m_0\in C_0^\infty(R),\ \ m_0(-x)=-m_0(x),\ \ \int_0^\infty m_0(y)dy=\int_0^{-\infty} m_0(y)dy=1,$$

and define $m_1(x)$ by

$$m_1(x):=\int_0^x m_0(y)dy$$

which implies

$$m_1(-\infty)=m_1(\infty)=1.$$

2.2. The reformulated problem. Now we are able to reformulate problem (1.2)-(1.6). To this end, we introduce new unknowns defined by

$$\phi_i(x,t)=\int_{-\infty}^x (n_i(y,t)-\bar{n}-\hat{n}(y,t))\,dy, \tag{2.22}$$

$$\psi_i(x,t)=J_i(x,t)-\hat{J}(x,t), \tag{2.23}$$

$$E(x,t)=\phi_1(x,t)-\phi_2(x,t), \tag{2.24}$$

here and hereafter the subscript i takes values 1,2. Then we arrive at a new system

$$\begin{cases} \phi_{it}+\psi_i=0, \\ \psi_{it}+a_i\phi_{ixx}+b_i\psi_{ix}+\dfrac{\mu}{1+t}\psi_i+(-1)^i n_i E=F_i, \quad i=1,2, \\ E=\phi_1-\phi_2, \\ (\phi_i,\psi_i)|_{t=0}=(\phi_{i0},\psi_{i0})(x), \quad i=1,2, \end{cases} \tag{2.25}$$

where a_i, b_i, and F_i are defined by

$$a_i:=\hat{p}'(n_i)-\frac{J_i^2}{n_i^2}, \tag{2.26}$$

$$b_i:=\frac{2J_i}{n_i}, \tag{2.27}$$

$$F_i:=\frac{J_i^2\hat{n}_x}{n_i^2}-\frac{2J_i\hat{J}_{ix}}{n_i}-\hat{p}'(n_i)\hat{n}_x. \tag{2.28}$$

For simplicity, we denote

$$\begin{aligned} M(t)^2=\int_0^t\sum_{\ell=0}^{1}\Big(&(1+\tau)^\ell\|\partial_x^\ell(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2 \\ &+(1+\tau)^{\ell-2}\|\partial_x^\ell\|(\phi_1,\phi_2)(\tau)\|^2\Big)d\tau, \end{aligned} \tag{2.29}$$

$$\begin{aligned} N(t)^2=\sup_{0\le\tau\le t}\sum_{\ell=1}^{2}\Big(&(1+\tau)^\ell\|\partial_x^{\ell-1}(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2 \\ &+(1+\tau)^{\ell-2}\|\partial_x^{\ell-1}\|(\phi_1,\phi_2)(\tau)\|^2\Big). \end{aligned} \tag{2.30}$$

THEOREM 2.1. *Assume that $(\phi_{10},\phi_{20},\psi_{10},\psi_{20})\in H^2(\mathbb{R})\times H^2(\mathbb{R})\times H^1(\mathbb{R})\times H^1(\mathbb{R})$. There exists a suitably small number $\delta>0$ such that, when*

$$|\bar{J}|+\|(\phi_{10},\phi_{20})\|_{H^2(\mathbb{R})}+\|(\psi_{10},\psi_{20})\|_{H^1(\mathbb{R})}\le\delta,$$

then there exists a unique global-in-time solution $(\phi_1,\phi_2,\psi_1,\psi_2,E)(x,t)$ to the system (2.25) satisfying

$$N(t)^2+M(t)^2\le C\left(|\bar{J}|+\|(\phi_{10},\phi_{20})\|_2^2+\|(\psi_{10},\psi_{20})\|_1^2\right), \tag{2.31}$$

and

$$\|(\phi_1,\phi_2)(t)\|_{L^\infty(\mathbb{R})}\le C\delta, \tag{2.32}$$

$$\|(\psi_1,\psi_2)(t)\|_{L^\infty(\mathbb{R})}\le C\delta\ (1+t)^{-\frac{3}{4}}, \tag{2.33}$$

$$\|E(t)\|_{L^\infty(\mathbb{R})}\le C\delta\ (1+t)^{-\frac{3}{4}}, \tag{2.34}$$

$$\|(\phi_1,\phi_2)_t(t)\|_{L^\infty(\mathbb{R})}\le C\delta\ (1+t)^{-\frac{3}{4}}, \tag{2.35}$$

$$\|(\phi_1,\phi_2)_x(t)\|_{L^\infty(\mathbb{R})}\le C\delta\ (1+t)^{-\frac{3}{4}}. \tag{2.36}$$

We will prove this theorem by employing a standard procedure: the method of continuation of local solutions. Assuming the initial data are smooth, we thus prove, by using the Banach contracting principle, the existence of smooth solutions. We omit the details for the local existence. The rest of this article is devoted to establishing *a priori* estimates.

Proof of Theorem 1.1. Once Theorem 2.1 is proved, Theorem 1.1 is immediately obtained.

3. A priori estimates

3.1. Basic estimates for ϕ_i, ψ_i and E. In this subsection, we are going to establish estimates for ϕ_i and ψ_i (i=1,2) and their derivatives of first order.

For the initial data that are smooth, thus the solution is smooth too. For the existence of weak solutions to the Cauchy problem with initial data in $H^2\times H^2\times H^1\times H^1$, we apply the density theorem to approximate the original initial data and obtain smooth solutions, then pass the smooth solution to the limit by using *a priori* estimates.

Therefore in this section we assume solutions $(\phi_1,\phi_2,\psi_1,\psi_2,E)$ are smooth and there hold

$$\inf_{x\in\mathbb{R}} n_1>0, \quad \inf_{x\in\mathbb{R}} n_2>0,$$

and

$$\|(\phi_1,\phi_2)\|_2+\|(\psi_1,\psi_2,E)\|_1\le\delta, \tag{3.1}$$

and it follows from the Sobolev embedding theorem that

$$\|(\phi_1,\phi_2)_x\|_{L^\infty(\mathbb{R})}+\|(\psi_1,\psi_2,E)\|_{L^\infty(\mathbb{R})}\le C\delta. \tag{3.2}$$

By simple computations, we have

$$n_i=\bar n+\hat n+\phi_{ix}\ge\frac{\bar n+\hat n}{2}\ge\frac{\bar n}{2}. \tag{3.3}$$

Next we estimate F_i and rewrite it as follows

$$\begin{aligned}
F_i &= \frac{(\psi_i+\hat J)^2\hat n_x}{n_i^2}-\frac{2(\psi_i+\hat J)\hat J_{ix}}{n_i}\\
&\quad -\Big(\hat p'(n_i)-\hat p'(\bar n)-\hat p''(\bar n)(n_i-\bar n)\Big)\hat n_x-\Big(\hat p'(\bar n)+\hat p''(\bar n)(n_i-\bar n)\Big)\hat n_x-\hat J_t-\frac{\mu}{1+t}\hat J\\
&=\left(\frac{(\psi_i^2+2\psi_i\hat J)\hat n_x}{n_i^2}-\frac{2\psi_i\hat J_{ix}}{n_i}-\Big(\hat p'(n_i)-\hat p'(\bar n)-\hat p''(\bar n)(n_i-\bar n)\Big)\hat n_x-\hat p''(\bar n)\phi_{ix}\hat n_x\right)\\
&\quad+\left(\frac{\hat J^2\hat n_x}{n_i^2}-\frac{2\hat J\hat J_{ix}}{n_i}-\hat p''(\bar n)\hat n\hat n_x\right)-\Big(\hat p'(\bar n)\hat n_x\Big)\\
&=: F_{i1}+F_{i2}+F_{i3}.
\end{aligned}\tag{3.4}$$

With the help of the mean value theorem, we treat the term of F_{i1} as follows

$$\hat p'(n_i)-\hat p'(\bar n)-\hat p''(\bar n)(n_i-\bar n)=\frac12\hat p'''(\xi)(n_i-\bar n)^2=\frac12\hat p'''(\xi)(\phi_{ix}+\hat n)^2. \tag{3.5}$$

Here $\xi\in<\bar n,\bar n+\hat n+\phi_{1x}>$, we used the notation $\xi\in<f,g>$ to denote that ξ takes a value between two quantities f and g.

Recalling $m_0 \in C_0^\infty(\mathbb{R})$ and $m_1' = m_0$, one has $\|m_j\|_{L^\infty(\mathbb{R})} \le C$ for $j=0,1$. Thus we obtain

$$
\begin{aligned}
|F_{i1}| &\le C\left((\psi_i^2+|\psi_i\hat{J}|)|\hat{n}_x|+|\psi_i\hat{J}_{ix}|+(\phi_{ix}^2+|\phi_{ix}|+\hat{n}^2)|\hat{n}_x|\right)\\
&\le C\left(\left(\psi_i^2+|\psi_i\hat{J}|+|\phi_{ix}|+\hat{n}^2\right)|\hat{n}_x|+|\psi_i\hat{J}_{ix}|\right)\\
&\le C|\hat{J}|\left((\psi_i^2+|\phi_{ix}|)(1+t)^{-\mu+1}+|\psi_i|(1+t)^{-\mu}+(1+t)^{-3\mu+1}\right) \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
|F_{i2}| &\le C\left(|\hat{J}^2\hat{n}_x|+|\hat{J}\hat{J}_{ix}|+|\hat{J}_t|+\frac{|\hat{J}|}{1+t}+|\hat{n}\hat{n}_x|\right)\\
&\le C|\bar{J}|\left((1+t)^{-3\mu+1}+(1+t)^{-2\mu}+(1+t)^{-\mu-1}+(1+t)^{-2\mu+2}\right)\\
&\le C|\bar{J}|(1+t)^{-\mu_1}, \qquad (3.7)
\end{aligned}
$$

for sufficiently large t, and

$$\mu_1 := \min\{\mu+1, 2\mu-2\} > 2. \qquad (3.8)$$

Finally $F_{i3} := -\hat{p}'(\bar{n})\hat{n}_x$ will be treated late in the process of deriving the basic energy estimates, by using the technique of integration by parts.

Lemma 3.1. *There exist a constant $0 \le \kappa < 1$ and a sufficiently small ε_1 such that if $|\bar{J}|+N(t)^2 < \varepsilon_1$, then there holds*

$$
\begin{aligned}
&(1+t)^{\kappa+1}\left(\|\phi_x\|^2+\|\phi_t\|^2+\|E\|^2\right)+(1+t)^{\kappa-1}\|\phi\|^2\\
&\quad+\int_0^t (1+\tau)^\kappa\left(\|\phi_t\|^2+\|\phi_x\|^2+\|E\|^2\right)+(1+\tau)^{\kappa-2}\|\phi\|^2 d\tau\\
&\le C\left(\|\phi_0\|_1^2+\|\phi_{t0}\|^2+\|E_0\|^2+|\bar{J}|(N(t)^2+N(t)M(t))+N(t)^3\right). \qquad (3.9)
\end{aligned}
$$

Proof. From (2.25), we get

$$\phi_{1tt}-a_1\phi_{1xx}+b_1\phi_{1tx}+\frac{\mu}{1+t}\phi_{1t}+n_1E=\mathcal{F}_1, \qquad (3.10)$$

$$\phi_{2tt}-a_2\phi_{2xx}+b_2\phi_{2tx}+\frac{\mu}{1+t}\phi_{2t}-n_2E=\mathcal{F}_2, \qquad (3.11)$$

here $\mathcal{F}_i = -F_i$.

Let κ, λ be two positive constants which will be determined later. Multiplying (3.10) by $(1+t)^\kappa(\lambda\phi_1+(1+t)\phi_{1t})$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{(1+t)^\kappa}{2}\left(a_1(1+t)\phi_{1x}^2+(1+t)\phi_{1t}^2+2\lambda\phi_1\phi_{1t}+\lambda(\mu-\kappa)(1+t)^{-1}\phi_1^2\right)\right)\\
&\quad+(1+t)^\kappa\left((\mu-\lambda-\frac{\kappa+1}{2})\phi_{1t}^2+(\lambda-\frac{\kappa+1}{2})a_1\phi_{1x}^2\right)\\
&\quad+(1+t)^\kappa\left(-\frac{\lambda}{2}(\mu-\kappa)(\kappa-1)(1+t)^{-2}\phi_1^2+n_1(\lambda\phi_1+(1+t)\phi_{1t})E\right)\\
&=(1+t)^\kappa(\lambda\phi_1+(1+t)\phi_{1t})\mathcal{F}_1+I_1. \qquad (3.12)
\end{aligned}
$$

Here I_1 is given by

$$I_1 := -(1+t)^\kappa (a_{1x}\phi_{1x} + b_1\phi_{1tx})(\lambda\phi_1 + (1+t)\phi_{1t}) + \frac{(1+t)^{\kappa+1}}{2} a_{1t}\phi_{1x}^2. \tag{3.13}$$

In a similar manner, taking $(1+t)^\kappa(\lambda\phi_2 + (1+t)\phi_{2t})$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{(1+t)^\kappa}{2}\left(a_2(1+t)\phi_{2x}^2 + (1+t)\phi_{2t}^2 + 2\lambda\phi_2\phi_{2t} + \lambda(\mu-\kappa)(1+t)^{-1}\phi_2^2\right)\right)\\
&\quad + (1+t)^\kappa\left((\mu - \lambda - \frac{\kappa+1}{2})\phi_{2t}^2 + (\lambda - \frac{\kappa+1}{2})a_2\phi_{2x}^2\right)\\
&\quad + (1+t)^\kappa\left(-\frac{\lambda}{2}(\mu-\kappa)(\kappa-1)(1+t)^{-2}\phi_2^2 - n_2(\lambda\phi_2 + (1+t)\phi_{2t})E\right)\\
&= (1+t)^\kappa(\lambda\phi_2 + (1+t)\phi_{2t})\mathcal{F}_2 + I_2,
\end{aligned} \tag{3.14}$$

where

$$I_2 := -(1+t)^\kappa (a_{2x}\phi_{2x} + b_2\phi_{2tx})(\lambda\phi_2 + (1+t)\phi_{2t}) + \frac{(1+t)^{\kappa+1}}{2} a_{2t}\phi_{2x}^2. \tag{3.15}$$

By adding (3.12) to (3.14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{(1+t)^\kappa}{2}\left((1+t)(a_1\phi_{1x}^2 + a_2\phi_{2x}^2) + (1+t)|\phi_t|^2 + 2\lambda(\phi,\phi_t) + \lambda(\mu-\kappa)(1+t)^{-1}|\phi|^2\right)\right)\\
&\quad + \frac{d}{dt}\left(\frac{n_1(1+t)^{\kappa+1}}{2}E^2\right)\\
&\quad + (1+t)^\kappa\left((\mu - \lambda - \frac{\kappa+1}{2})|\phi_t|^2 + (\lambda - \frac{\kappa+1}{2})(a_1\phi_{1x}^2 + a_2\phi_{2x}^2)\right)\\
&\quad + (1+t)^\kappa\left(-\frac{\lambda}{2}(\mu-\kappa)(\kappa-1)(1+t)^{-2}|\phi|^2 + n_1(\lambda - \frac{\kappa+1}{2})E^2\right)\\
&=: I_0 + I_1 + I_2 + I_3,
\end{aligned} \tag{3.16}$$

where

$$I_0 := (1+t)^\kappa\Big(\lambda(\phi,\mathcal{F}) + (1+t)(\phi_t,\mathcal{F})\Big), \tag{3.17}$$

$$I_3 := \frac{(1+t)^{\kappa+1}}{2} n_{1t}E^2 - (1+t)^\kappa E_x E(\lambda\phi_2 + (1+t)\phi_{2t}). \tag{3.18}$$

Here we used the notations $\phi = (\phi_1,\phi_2)^T$, $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)^T$ which are two-dimensional vectors, and the notation (f,g) denotes the inner product of the functions $f,g \in \mathbb{R}^2$ and correspondingly $|f|$ is the Euclidean norm of 2-d vector f.

It is easy to see that

$$a_1,\ a_2 \geq \underline{a} > 0, \text{ and } n_1 \geq \frac{\bar{n}}{2} =: \underline{n} > 0.$$

Therefore, integrating (3.16) over $\mathbb{R}\times[0,t]$, we obtain

$$\frac{(1+t)^\kappa}{2}\left(\underline{a}(1+t)\|\phi_x\|^2 + (1+t)\|\phi_t\|^2 + 2\lambda\int_{\mathbb{R}}(\phi,\phi_t)dx + \lambda(\mu-\kappa)(1+t)^{-1}\|\phi\|^2\right)$$

$$
\begin{aligned}
&+\frac{(1+t)^{\kappa+1}}{2}\underline{n}\|E\|^2\\
&+\int_0^t(1+\tau)^\kappa\left((\mu-\lambda-\frac{\kappa+1}{2})\|\phi_t\|^2+\underline{a}(\lambda-\frac{\kappa+1}{2})\|\phi_x\|^2\right)d\tau\\
&+\int_0^t(1+\tau)^\kappa\left(\frac{\lambda}{2}(\kappa-\mu)(\kappa-1)(1+\tau)^{-2}\|\phi\|^2+\underline{n}(\lambda-\frac{\kappa+1}{2})\|E\|^2\right)d\tau\\
\le&\frac{1}{2}\left(C\|\phi_{x0}\|^2+\|\phi_{t0}\|^2+2\lambda\int_{\mathbb{R}}(\phi_0,\phi_{t0})dx+\lambda(\mu-\kappa)\|\phi_0\|^2\right)+\frac{1}{2}\underline{n}\|E_0\|^2\\
&+\int_0^t\int_{\mathbb{R}}\Big(I_0+I_1+I_2+I_3\Big)dxd\tau. \qquad (3.19)
\end{aligned}
$$

To get the positivity of the terms in the above inequality, we need:

$$\mu-\kappa>\lambda, \qquad (3.20)$$

$$\mu-\frac{\kappa+1}{2}-\lambda\ge 0, \qquad (3.21)$$

$$\lambda-\frac{\kappa+1}{2}\ge 0, \qquad (3.22)$$

$$(\kappa-\mu)(\kappa-1)\ge 0, \qquad (3.23)$$

which implies

$$0\le\kappa\le 1. \qquad (3.24)$$

However to keep the coefficient of the term $\int_0^t(1+\tau)^\kappa\|\phi\|^2d\tau$ positive, we take

$$0\le\kappa<1. \qquad (3.25)$$

Now we are going to deal with I_j term by term for $j=0,1,2,3$. Let $\hat{I}_0$ (resp. $\hat{\mathcal{F}}$) be the part of I_0 (resp. $\hat{\mathcal{F}}$), in that F_{i3} are dropped. Invoking (2.25) and the estimates (3.6) and (3.7), we have

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}\hat{I}_0dxd\tau\right|\\
\le& C\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(\lambda|(\phi,\hat{\mathcal{F}})|+(1+\tau)|(\phi_\tau,\hat{\mathcal{F}})|\Big)dxd\tau\\
\le& C\int_0^t(1+\tau)^\kappa\left(\lambda\|\phi\|\,\|\hat{\mathcal{F}}\|+(1+\tau)\|\phi_\tau\|\,\|\hat{\mathcal{F}}\|\right)d\tau\\
\le& C|\hat{J}|\int_0^t(1+\tau)^\kappa(\|\phi\|+(1+\tau)\|\phi_\tau\|)\\
&\quad\times\left((\|\psi_i^2\|+\|\phi_{ix}\|)(1+\tau)^{-\mu+1}+\|\psi_i\|(1+\tau)^{-\mu}+(1+\tau)^{-3\mu+1}+(1+\tau)^{-\mu_1}\right)d\tau\\
\le& C|\hat{J}|\int_0^t(1+\tau)^\kappa(\|\phi\|+(1+\tau)\|\phi_\tau\|)\left((\|\phi_\tau\|+\|\phi_x\|)(1+\tau)^{-\mu+1}+(1+\tau)^{-\mu_1}\right)d\tau\\
\le& C|\hat{J}|\int_0^t\left((1+\tau)^{\frac{\kappa}{2}}\|\phi\|\cdot(1+\tau)^{\frac{\kappa+1}{2}}(\|\phi_\tau\|+\|\phi_x\|)(1+\tau)^{-\frac{1}{2}}\right)(1+\tau)^{-\mu+1}d\tau\\
&+C|\hat{J}|\int_0^t(1+\tau)^{\kappa+1}(\|\phi_\tau\|^2+\|\phi_x\|^2)(1+\tau)^{-\mu+1}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C|\hat{J}|\int_0^t [(1+\tau)^{\frac{\kappa}{2}}\|\phi\|(1+\tau)^{\frac{\kappa}{2}}+(1+\tau)^{\frac{\kappa+1}{2}}\|\phi_\tau\|(1+\tau)^{\frac{\kappa+1}{2}}](1+\tau)^{-\mu_1}d\tau\\
&\le C|\hat{J}|N(t)^2\int_0^t \Big((1+\tau)^{-\mu+1-\frac{1}{2}}+(1+\tau)^{-\mu+1}\Big)d\tau+C|\hat{J}|N(t)\int_0^t (1+\tau)^{-\mu_1+\frac{\kappa+1}{2}}d\tau\\
&\le C|\hat{J}|\left(N(t)^2+N(t)\right), \qquad (3.26)
\end{aligned}
$$

as all integrals in (3.24) are finite for $\mu>2$ and $\kappa\le 1$. It remains to deal with I_0 containing F_{i3}. We write

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}}(I_0-\hat{I}_0)dxd\tau &= \int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(\lambda(\phi,\mathcal{F}_3)+(1+\tau)(\phi_\tau,\mathcal{F}_3)\Big)dxd\tau\\
&=\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(\lambda(\phi_x,\tilde{\mathcal{F}}_3)+(1+\tau)(\phi_{\tau x},\tilde{\mathcal{F}}_3)\Big)dxd\tau. \qquad (3.27)
\end{aligned}
$$

Here $\tilde{\mathcal{F}}_3:=\hat{p}'(\bar{n})\hat{n}\mathbf{e}$ and $\mathbf{e}$ is a 2-D vector $\mathbf{e}=(1,1)^T$. Thus we arrive at

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}(I_0-\hat{I}_0)dxd\tau\right|\\
&\le C\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(|\phi_x|+(1+\tau)|\phi_{\tau x}|\Big)\hat{n}dxd\tau\\
&\le C|\bar{J}|\int_0^t(1+\tau)^\kappa\Big(\|\phi_x\|+(1+\tau)\|\phi_{\tau x}\|\Big)(1+\tau)^{-\mu+1}d\tau\\
&\le C|\bar{J}|\int_0^t(1+\tau)^\kappa\Big(\|\phi_x\|+(1+\tau)\|\phi_{\tau x}\|\Big)(1+\tau)^{-\mu+1}d\tau. \qquad (3.28)
\end{aligned}
$$

For the term of I_1, we first calculate

$$
\begin{aligned}
&a_{1x}=\hat{p}'(n_1)(\phi_{1xx}+\hat{n}_{1x})-\frac{2J_1(\psi_{1x}+\hat{J}_{1x})}{n_1^2}+\frac{2J_1^2(\phi_{1xx}+\hat{n}_{1x})}{n_1^3}, \text{ hence}\\
&|a_{1x}|\le C\Big(|\phi_{1xx}|+|\phi_{1tx}|+|\hat{n}_{1x}|+|\hat{J}_{1x}|\Big), \qquad (3.29)\\
&b_1=\frac{2(\psi_1+\hat{J})}{n_1}, \text{ which yields}\\
&|b_1|\le C(|\phi_{1t}|+|\hat{J}|). \qquad (3.30)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}I_1dxd\tau\right|\\
&\le C\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(|a_{1x}\phi_{1x}|+|b_1\phi_{1\tau x}|\Big)\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau\\
&\le C\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(|\phi_{1xx}|+|\phi_{1\tau x}|+|\bar{J}|(1+\tau)^{-\mu+1}\Big)|\phi_{1x}|\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau\\
&\quad+C\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\Big(|\phi_{1\tau}|+|\bar{J}|(1+\tau)^{-\mu}\Big)|\phi_{1\tau x}|\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau. \qquad (3.31)
\end{aligned}
$$

The terms in the above inequality (3.31) may be classified into two types, i.e., one are cubic terms of the form $f\cdot g\cdot h$ where f,g,h are the unknowns ϕ or its derivatives, another one are quadratic terms $f\cdot g\cdot h$ in which f,g are ϕ or its derivatives and h is a small term with suitable time decay rate.

Noting the assumption about the behavior of the unknowns E,ϕ at far state, we are in a position to use interpolation inequality of the following form

$$\|f\|_{L^\infty}\le C\|f\|^{\frac{1}{2}}\|f_x\|^{\frac{1}{2}}, \tag{3.32}$$

where each term in the right-hand side is assumed to be bounded. Using the Hölder inequality, we get easily that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big(|\phi_{1xx}|+|\phi_{1\tau x}|\Big)|\phi_{1x}\phi_1|dxd\tau\\
\le&\int_0^t(1+\tau)^{1}\Big(\|\phi_{1xx}\|+\|\phi_{1\tau x}\|\Big)(1+\tau)^{\frac{3}{4}\kappa-\frac{1}{2}}\|\phi_{1x}\|^{\frac{3}{2}}(1+\tau)^{\frac{\kappa}{4}-\frac{1}{2}}\|\phi_1\|^{\frac{1}{2}}d\tau\\
\le&N(t)\left(\int_0^t(1+\tau)^{(\frac{3}{4}\kappa-\frac{1}{2})\cdot\frac{4}{3}}\|\phi_{1x}\|^2d\tau\right)^{\frac{3}{4}}\left(\int_0^t(1+\tau)^{4(\frac{\kappa}{4}-\frac{1}{2})}\|\phi_1\|^2d\tau\right)^{\frac{1}{4}}\\
\le&N(t)N(t)^{2\cdot\frac{3}{4}}\left(\int_0^t(1+\tau)^{-\frac{5}{3}}d\tau\right)^{\frac{3}{4}}M(t)^{\frac{1}{2}}\\
\le&N(t)^{\frac{5}{2}}M(t)^{\frac{1}{2}}.
\end{aligned}\tag{3.33}$$

Similarly, however a little easier, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\Big(|\phi_{1xx}|+|\phi_{1\tau x}|\Big)|\phi_{1x}||\phi_{1\tau}|dxd\tau\\
\le&C\int_0^t(1+\tau)^{\kappa+1}\Big(\|\phi_{1xx}\|+\|\phi_{1\tau x}\|\Big)\|\phi_{1xx}\|^{\frac{1}{2}}\|\phi_{1x}\|^{\frac{1}{2}}\|\phi_{1\tau}\|d\tau\\
\le&C\int_0^t(1+\tau)^{\frac{3}{2}}\Big(\|\phi_{1xx}\|+\|\phi_{1\tau x}\|\Big)^{\frac{3}{2}}\cdot(1+\tau)^{\frac{\kappa+1}{4}}\|\phi_{1x}\|^{\frac{1}{2}}\\
&\quad\cdot(1+\tau)^{\frac{\kappa+1}{2}}\|\phi_{1\tau}\|\cdot(1+\tau)^{\frac{\kappa+1}{4}-\frac{3}{2}}d\tau\\
\le&CN(t)^3\int_0^t(1+\tau)^{\frac{\kappa+1}{4}-\frac{3}{2}}d\tau\\
\le&CN(t)^3.
\end{aligned}\tag{3.34}$$

Here the last integral in (3.34) is finite due to $\kappa<1$. Also we obtain

$$\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}|\phi_{1\tau}||\phi_{1\tau x}|\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau\le CN(t)^3. \tag{3.35}$$

Now we handle terms of another type. First we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}|\bar J|(1+\tau)^{-\mu+\kappa+1}|\phi_{1x}|\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau\\
\le&|\bar J|\int_0^t(1+\tau)^{-\mu+\kappa+1}\|\phi_{1x}\|\Big(\|\phi_1\|+(1+\tau)\|\phi_{1\tau}\|\Big)d\tau\\
=&|\bar J|\int_0^t\Big((1+\tau)^{-\mu+\frac{1}{2}}\cdot(1+\tau)^{\frac{\kappa+1}{2}}\|\phi_{1x}\|\cdot(1+\tau)^{\frac{\kappa}{2}-1}\|\phi_1\|\\
&\qquad+(1+\tau)^{-\mu+1}\cdot(1+\tau)^{\frac{\kappa+1}{2}}\|\phi_{1x}\|\cdot(1+\tau)^{\frac{\kappa+1}{2}}\|\phi_{1\tau}\|\Big)d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le |\bar{J}|N(t)\left(\int_0^t (1+\tau)^{-2\mu+1}d\tau\right)^{\frac{1}{2}}\left(\int_0^t (1+\tau)^{\kappa-2}\|\phi_1\|^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad +|\bar{J}|N(t)^2\int_0^t (1+\tau)^{-\mu+1}d\tau \\
&\le C|\bar{J}|\Big(N(t)M(t)+N(t)^2\Big). \qquad (3.36)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}|\bar{J}|(1+\tau)^{-\mu})\,|\phi_{1\tau x}|\Big(|\phi_1|+(1+\tau)|\phi_{1\tau}|\Big)dxd\tau \\
&\le C|\bar{J}|\Big(N(t)M(t)+N(t)^2\Big). \qquad (3.37)
\end{aligned}
$$

Therefore (3.31) is estimated as

$$
\left|\int_0^t\int_{\mathbb{R}} I_1 dxd\tau\right| \le C\Big(|\bar{J}|\Big(N(t)M(t)+N(t)^2\Big)+N(t)^3\Big). \qquad (3.38)
$$

The term I_2 is treated in the same manner as that for I_1. It remains to deal with the term containing I_3. We write

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}} I_3 dxd\tau\right| \\
&\le C\int_0^t\int_{\mathbb{R}}\Big((1+\tau)^{\kappa+1}E^2(1+\tau)^{-\mu}+(1+\tau)^{\kappa}|E_xE|(|\phi_2|+(1+\tau)|\phi_{2\tau}|)\Big)dxd\tau \\
&\le C\int_0^t\Big((1+\tau)^{\kappa+1}\|E\|^2(1+\tau)^{-\mu}+(1+\tau)^{\kappa}\|\phi_2\|_{L^\infty}\|E_x\|\,\|E\| \\
&\qquad +(1+\tau)^{\kappa+1}\|E\|_{L^\infty}\|E_x\|\,\|\phi_{2\tau}\|\Big)d\tau. \qquad (3.39)
\end{aligned}
$$

Invoking (3.32) and the Hölder inequality, we hence get

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}} I_3 dxd\tau\right| \\
&\le C|\bar{J}|N(t)^2\int_0^t(1+\tau)^{-\mu}d\tau \\
&\quad +CN(t)\int_0^t\Big((1+\tau)^{\frac{\kappa+\theta}{2}}\|E_x\|\Big)\Big((1+\tau)^{\frac{\kappa+1}{2}}\|E\|\Big)(1+\tau)^{-\frac{\theta+1}{2}}d\tau \\
&\quad +C\int_0^t\Big((1+\tau)^{\frac{1}{4}(\kappa+1)}\|E\|^{\frac{1}{2}}\Big)\Big((1+\tau)^{\frac{3}{4}(\kappa+\theta)}\|E_x\|^{\frac{3}{2}}\Big)\|\phi_{2\tau}\| \\
&\qquad \times\Big((1+\tau)^{\frac{\kappa+1}{2}}\|\phi_{2\tau}\|\Big)(1+\tau)^{-\frac{3\theta+2\kappa-1}{4}}d\tau \\
&\le C|\bar{J}|N(t)^2\int_0^t(1+\tau)^{-\mu}d\tau+CN(t)^3\int_0^t(1+\tau)^{-\frac{\theta+1}{2}}d\tau \\
&\quad +CN^3(t)\int_0^t(1+\tau)^{-\frac{3\theta+2\kappa-1}{4}}d\tau \\
&\le C\left(|\bar{J}|+N(t)\right)N(t)^2. \qquad (3.40)
\end{aligned}
$$

Here $\mu>2$ is used and we suppose that $3\theta+2\kappa-1>4$ which is guaranteed by the conditions $\kappa<1$ and $\theta>1$. Whence all integrals in (3.40) are finite. Therefore, from (3.25)–(3.40) we assert that (3.19) turns out to be

$$\begin{aligned}&(1+t)^{\kappa+1}\left(\|\phi_x\|^2+\|\phi_t\|^2+\|E\|^2\right)+(1+t)^{\kappa-1}\|\phi\|^2\\&\quad+\int_0^t(1+\tau)^{\kappa}\left(\|\phi_x\|^2+\|\phi_t\|^2+\|E\|^2\right)+(1+\tau)^{\kappa-2}\|\phi\|^2d\tau\\&\le C(\|\phi_0\|_1^2+\|\phi_{t0}\|^2+\|E_0\|^2)+C(\delta+N(t))N(t)^2\\&\quad+C\left(|\bar J|\left(N(t)M(t)+N(t)^2\right)+N(t)^3\right).\end{aligned}\tag{3.41}$$

By using the Young inequality we thus complete the proof of Lemma 3.1. ∎

3.2. Estimates for higher derivatives. In this subsection, we will derive estimates for higher derivatives of $E(x,t)$ and $\phi_i(x,t)$, $\psi_i(x,t)$ where the subscript i takes values of $1,2$.

LEMMA 3.2. *There exists a sufficiently small $\varepsilon_2\le\varepsilon_1$ such that if $|\bar J|+N(t)^2<\varepsilon_2$, then there holds*

$$\begin{aligned}&(1+t)^2\left(\|\phi_{xx}\|^2+\|\phi_{xt}\|^2+\|E_x\|^2\right)+\|\phi_x\|^2\\&\quad+\int_0^t(1+\tau)\left(\|\phi_{x\tau}\|^2+\|\phi_{xx}\|^2+\|E_x\|^2\right)+(1+\tau)^{-1}\|\phi_x\|^2d\tau\\&\le C\left(\|\phi_0\|_2^2+\|\phi_{t0}\|_1^2+\|E_0\|_1^2+|\bar J|(N(t)^2+N(t)M(t))+N(t)^3\right).\end{aligned}\tag{3.42}$$

Proof. Multiplying Equation (3.10) by $(1+t)^\kappa\partial_x^2\phi_1$, we have

$$\begin{aligned}&\frac{d}{dt}\left((1+t)^\kappa\phi_{1xx}\phi_{1t}\right)-\kappa(1+t)^{\kappa-1}\phi_{1xx}\phi_{1t}-(1+t)^\kappa\phi_{1xxt}\phi_{1t}-a_1(1+t)^\kappa\phi_{1xx}^2\\&\quad+b_1\phi_{1tx}(1+t)^\kappa\phi_{1xx}+\mu(1+t)^{\kappa-1}\phi_{1xx}\phi_{1t}+n_1(1+t)^\kappa\phi_{1xx}E\\&=(1+t)^\kappa\phi_{1xx}\mathcal{F}_1.\end{aligned}\tag{3.43}$$

Then integrating (3.43) over $\mathbb{R}\times[0,t]$ and applying the technique of integration by parts, we get

$$\begin{aligned}&-\int_{\mathbb{R}}(1+t)^\kappa\phi_{1x}\phi_{1xt}dx-\frac12\int_{\mathbb{R}}(\mu-\kappa)(1+t)^{\kappa-1}\phi_{1x}^2dx+\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\left(\phi_{1x\tau}^2-a_1\phi_{1xx}^2\right)dxd\tau\\&\quad+\int_0^t\int_{\mathbb{R}}\left(\frac12(\kappa-\mu)(\kappa-1)(1+\tau)^{\kappa-2}\phi_{1x}^2+(1+t)^\kappa\left(b_1\phi_{1tx}\phi_{1xx}+n_1\phi_{1xx}E\right)\right)dxd\tau\\&=-\int_{\mathbb{R}}(\phi_{1x}\phi_{1xt})\Big|_{t=0}dx+\frac12\int_{\mathbb{R}}(\mu-\kappa)\phi_{1x}^2|_{t=0}dx+\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\phi_{1xx}\mathcal{F}_1dxd\tau.\end{aligned}\tag{3.44}$$

We multiply (3.10) by $-(1+t)^{\kappa+1}\partial_x^2\phi_{1t}$ and integrate the resulting equation over $\mathbb{R}\times[0,t]$ to obtain

$$\begin{aligned}&\frac12\int_{\mathbb{R}}(1+t)^{\kappa+1}\phi_{1xt}^2dx+\frac12\int_{\mathbb{R}}a_1(1+t)^{\kappa+1}\phi_{1xx}^2dx\\&\quad+(\mu-\frac{\kappa+1}{2})\int_0^t\int_{\mathbb{R}}(1+\tau)^\kappa\phi_{1x\tau}^2dxd\tau\\&\quad-\frac{\kappa+1}{2}\int_0^t\int_{\mathbb{R}}a_1(1+\tau)^\kappa\phi_{1xx}^2dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{1x\tau}(n_1E)_xdxd\tau\end{aligned}$$

$$=\frac{1}{2}\int_{\mathbb{R}}(\phi_{1xt}^2-a_1\phi_{1xx}^2)|_{t=0}dx-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{1xx\tau}F_1dxd\tau+J_1, \tag{3.45}$$

where J_1 is given by

$$J_1:=\frac{\kappa+1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\left(a_{1t}\phi_{1xx}^2-b_{1x}\phi_{1x\tau}^2\right)dxd\tau. \tag{3.46}$$

Multiplying (3.44) by $-\lambda$ and adding the resulting equation to (3.45), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\lambda(\mu-\kappa)(1+t)^{\kappa-1}\phi_{1x}^2dx+\int_{\mathbb{R}}\lambda(1+t)^{\kappa}\phi_{1x}\phi_{1xt}dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}(1+t)^{\kappa+1}\left(\phi_{1xt}^2dx+a_1\phi_{1xx}^2\right)dx\\
&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\left((\mu-\frac{\kappa+1}{2}-\lambda)\phi_{1x\tau}^2+(\lambda-\frac{\kappa+1}{2})a_1\phi_{1xx}^2\right)dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{1x\tau}(n_1E)_xdxd\tau\\
&=ID_1+\frac{\lambda}{2}\int_0^t\int_{\mathbb{R}}(\kappa-\mu)(\kappa-1)(1+\tau)^{\kappa-2}\phi_{1x}^2dxd\tau\\
&\quad+\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big(b_1\phi_{1tx}\phi_{1xx}-\phi_{1x}(n_1E)_x\Big)dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\lambda(1+\tau)^{\kappa}\phi_{1xx}\mathcal{F}_1dxd\tau-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{1xx\tau}\mathcal{F}_1dxd\tau+J_1.
\end{aligned} \tag{3.47}$$

We here used ID_1 to denote the initial data terms.

Similar to the process for deriving (3.47), multiplying Equation (3.11) by $-(1+t)^{\kappa}(\lambda\partial_x^2\phi_2+(1+t)\partial_x^2\phi_{2t})$ and integrating the resultant equation over $\mathbb{R}\times[0,t]$ give

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\lambda(\mu-\kappa)(1+t)^{\kappa-1}\phi_{2x}^2dx+\int_{\mathbb{R}}\lambda(1+t)^{\kappa}\phi_{2x}\phi_{2xt}dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}(1+t)^{\kappa+1}\phi_{2xt}^2dx+\frac{1}{2}\int_{\mathbb{R}}a_2(1+t)^{\kappa+1}\phi_{2xx}^2dx\\
&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\left((\mu-\frac{\kappa+1}{2}-\lambda)\phi_{2x\tau}^2+(\lambda-\frac{\kappa+1}{2})a_2\phi_{2xx}^2\right)dxd\tau\\
&\quad-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{2x\tau}(n_2E)_xdxd\tau\\
&=ID_2+\frac{\lambda}{2}\int_0^t\int_{\mathbb{R}}(\kappa-\mu)(\kappa-1)(1+\tau)^{\kappa-2}\phi_{2x}^2dxd\tau\\
&\quad+\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}(b_2\phi_{2tx}\phi_{2xx}-\phi_{2x}(n_2E)_x)dxd\tau\\
&\quad+\int_0^t\int_{\mathbb{R}}\lambda(1+\tau)^{\kappa}\phi_{2xx}\mathcal{F}_2dxd\tau-\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\phi_{2xx\tau}\mathcal{F}_2dxd\tau+J_2.
\end{aligned} \tag{3.48}$$

Here ID_2 denotes the terms of the initial data and J_2 is given by

$$J_2:=\frac{\kappa+1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa+1}\left(a_{2t}\phi_{2xx}^2-b_{2x}\phi_{2x\tau}^2\right)dxd\tau. \tag{3.49}$$

Taking the sum of (3.47) and (3.48), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\lambda(\mu-\kappa)(1+t)^{\kappa-1}|\phi_x|^2dx+\lambda\int_{\mathbb{R}}(1+t)^{\kappa}(\phi_x,\phi_{xt})dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}(1+t)^{\kappa+1}|\phi_{xt}|^2dx+\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}}a_i(1+t)^{\kappa+1}\phi_{ixx}^2dx\\
&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big[(\mu-\frac{\kappa+1}{2}-\lambda)|\phi_{x\tau}|^2\\
&\quad+\sum_{i=1}^{2}\Big((\lambda-\frac{\kappa+1}{2})a_i\phi_{ixx}^2+(-1)^{i-1}(1+\tau)\phi_{ix\tau}(n_iE)_x\Big)\Big]dxd\tau\\
&=ID+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\lambda(\kappa-\mu)(\kappa-1)(1+\tau)^{\kappa-2}|\phi_x|^2dxd\tau\\
&\quad-\lambda\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}}(-1)^{i-1}(1+\tau)^{\kappa}\phi_{ix}(n_iE)_xdxd\tau+\sum_{i=0}^{2}J_i.
\end{aligned}\tag{3.50}$$

Here $ID:=ID_1+ID_2$ and

$$\begin{aligned}
J_0:=&-\lambda\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}}(-1)^{i-1}(1+\tau)^{\kappa}b_i\phi_{itx}\phi_{ixx}dxd\tau\\
&+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big(\lambda(\phi_{xx},\mathcal{F})+(1+\tau)(\phi_{x\tau},\mathcal{F}_x)\Big)dxd\tau.
\end{aligned}\tag{3.51}$$

Now we treat (3.50) term by term. Invoking $a_i\geq\underline{a}$, one has

$$\sum_{i=1}^{2}a_i\phi_{ixx}^2\geq\underline{a}|\phi_{xx}|^2.\tag{3.52}$$

We rewrite

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}(1+t)^{\kappa+1}\sum_{i=1}^{2}(-1)^{i-1}\phi_{ix\tau}(n_iE)_xdxd\tau\\
&=\frac{1}{2}(1+t)^{\kappa+1}\int_{\mathbb{R}}n_1E_x^2dx-\frac{1}{2}\int_{\mathbb{R}}(n_1E_x^2)|_0dx-\frac{\kappa+1}{2}\int_0^t\int_{\mathbb{R}}(1+t)^{\kappa}n_1E_x^2dxd\tau\\
&\quad-\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+t)^{\kappa+1}\Big(n_{1t}E_x^2-2\phi_{2x\tau}(EE_x)_x\Big)dxd\tau,
\end{aligned}\tag{3.53}$$

and

$$\begin{aligned}
&\lambda\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}}(-1)^{i-1}(1+\tau)^{\kappa}\phi_{ix}(n_iE)_x\Big)dxd\tau\\
&=\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big(E_x(n_1E)_x+\phi_{2x}(EE_x)_x\Big)dxd\tau\\
&=\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}n_1|E_x|^2dxd\tau+\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)^{\kappa}\Big(n_{1x}EE_x+\phi_{2x}(EE_x)_x\Big)dxd\tau.
\end{aligned}\tag{3.54}$$

Now we take

$$\kappa=1,\ \mu-1>\lambda>1,$$

and see that from (3.52)-(3.54) it follows that (3.50) is changed to

$$\begin{aligned}&\frac{1}{2}\lambda(\mu-\kappa)\|\phi_x\|^2+\lambda\int_{\mathbb{R}}(1+t)(\phi_x,\phi_{xt})dx+\frac{1}{2}(1+t)^2\left(\|\phi_{xt}\|^2+\underline{a}\|\phi_{ixx}\|^2+\underline{n}\|E_x\|^2\right)\\&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)\left((\mu-1-\lambda)\|\phi_{x\tau}\|^2+(\lambda-1)\left(\underline{a}\|\phi_{xx}\|^2+\underline{n}\|E_x\|^2\right)\right)dxd\tau\\&\le C\Big(\|\phi_0\|_2^2+\|\psi_0\|_1^2+|\sum_{i=0}^3 J_i|\Big).\end{aligned}\tag{3.55}$$

Here J_3 is defined by

$$\begin{aligned}J_3:=&\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2\Big(n_{1t}E_x^2-2\phi_{2x\tau}(EE_x)_x\Big)dxd\tau\\&+\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)\Big(n_{1x}-\phi_{2xx}\Big)EE_xdxd\tau.\end{aligned}\tag{3.56}$$

We first deal with J_3. Employing the interpolation inequality (3.32) we get

$$\begin{aligned}|J_3|\le& C\int_0^t(1+\tau)^2\Big(|\bar{J}|(1+t)^{-\mu}\|E_x\|^2\\&+(\|\phi_{1x\tau}\|+\|\phi_{2x\tau}\|)\|E_x\|_{L^\infty}\|E_x\|+\|\phi_{2x\tau}\|\|E\|_{L^\infty}\|E_{xx}\|\Big)d\tau\\&+\int_0^t(1+\tau)\Big(|\bar{J}|(1+t)^{-\mu}\|E_x\|\|E\|+\Big(\|\phi_{1xx}\|+\|\phi_{2xx}\|\Big)\|E\|_{L^\infty}\|E_x\|\Big)d\tau\\\le& C\Big(|\bar{J}|N(t)^2+N(t)^3\Big).\end{aligned}\tag{3.57}$$

The term J_2 is handled as follows. Recalling the definition a_i and b_i, one has

$$a_{it}=\left(\hat{p}'(n_i)-\frac{2J_i^2}{n_i^3}\right)(\phi_{itx}+\hat{n}_t)+\frac{2(-\phi_{it}+\hat{J})\left(-\phi_{itt}+\hat{J}_t\right)}{n_i^2},\tag{3.58}$$

$$b_{ix}=\frac{2(-\phi_{itx}+\hat{J}_{ix})}{n_i}-\frac{2(-\phi_{it}+\hat{J})(\phi_{ixx}+\hat{n}_x)}{n_i^2}.\tag{3.59}$$

Using (3.10) and (3.11), we can deal with the term ϕ_{itt} as follows

$$|\phi_{itt}|\le C(|\phi_{ixx}|+|\phi_{itx}|+(1+t)^{-1}|\phi_{it}|+|E|+|\mathcal{F}_i|).\tag{3.60}$$

Combining this with (3.6)-(3.7), we thus obtain

$$\begin{aligned}|J_2|\le& C\int_0^t\int_{\mathbb{R}}(1+\tau)^2\left(|\phi_{2tx}|+|\hat{n}_t|+(|\phi_{2t}|+|\hat{J}|)(|\phi_{2tt}|+|\hat{J}_t|)\right)\phi_{2xx}^2dxd\tau\\&+C\int_0^t\int_{\mathbb{R}}(1+\tau)^2\left(|\phi_{2tx}|+|\hat{J}_x|+(|\phi_{2t}|+|\hat{J}|)(|\phi_{2xx}|+|\hat{n}_x|)\right)\phi_{2x\tau}^2dxd\tau\\\le& C(|\bar{J}|+N(t)^3).\end{aligned}\tag{3.61}$$

J_1 can be estimated in the same way as that for J_2. So it remains to deal with J_0. To this end, we compute $\partial_x F_i$:

$$F_{ix} = \left(\frac{(\psi_i^2 + 2\psi_i \hat{J})\hat{n}_x}{n_i^2} - \frac{2\psi_i \hat{J}_{ix}}{n_i} - \Big(\hat{p}'(n_i) - \hat{p}'(\bar{n}) - \hat{p}''(\bar{n})(n_i - \bar{n})\Big)\hat{n}_x - \hat{p}''(\bar{n})\phi_{ix}\hat{n}_x \right)_x$$
$$+ \left(\frac{\hat{J}^2 \hat{n}_x}{n_i^2} - \frac{2\hat{J}\hat{J}_{ix}}{n_i} - \hat{J}_t - \frac{\mu}{1+t}\hat{J} - \hat{p}''(\bar{n})\hat{n}\hat{n}_x \right)_x - \Big(\hat{p}'(\bar{n})\hat{n}_x\Big)_x. \tag{3.62}$$

Here

$$\Big(\hat{p}'(n_i) - \hat{p}'(\bar{n}) - \hat{p}''(\bar{n})(n_i - \bar{n})\Big)_x = (\hat{p}''(n_i) - \hat{p}''(\bar{n}))n_{ix}$$
$$= \hat{p}'''(\xi_i)(\phi_{ix} + \hat{n})(\phi_{ixx} + \hat{n}_x). \tag{3.63}$$

Further we recall the definition of $\hat{n}$ and $\hat{J}$, and denote their time-decay rates by

$$r(\hat{n}) := |\bar{J}|(1+t)^{-\mu+1}, \ r(\hat{J}) := |\bar{J}|(1+t)^{-\mu}, \text{ and } r(\hat{J}) \le r(\hat{n}),$$

respectively. It is easy to see that for any $i \in \mathbb{N}$ there hold

$$r(\partial_x^i f) = O(1) r(f), \tag{3.64}$$
$$r(\Pi_{i=1}^m \partial_x^{m_i} f) = O(1)(r(f))^m, \tag{3.65}$$
$$(r(f))^m \le r(f). \tag{3.66}$$

for all $f \in \{\hat{n}, \hat{J}\}$, $m_i \in \mathbb{N} \cup \{0\}$, and $\sum_{i=1}^m m_i = m$ where we assume that $m \in \mathbb{N}$. Hence

$$|F_{ix} - F_{i3x}| \le C\Big((|\psi_i| + |\hat{J}|)|\psi_{ix}\hat{n}_x| + |\psi_i \hat{J}_x \hat{n}_x|$$
$$+ (|\psi_i| + |\hat{J}|)|\psi_i \hat{n}_x n_{ix}| + |\psi_{ix}\hat{J}_x| + |\psi_i \hat{J}_{xx}| + |\psi_i \hat{J}_x n_{ix}|$$
$$+ |(n_i - \bar{n})n_{ix}\hat{n}_x| + |n_i - \bar{n}|^2 |\hat{n}_{xx}| + |\phi_{ixx}\hat{n}_x| + |\phi_{ix}\hat{n}_{xx}| \Big)$$
$$+ C\Big(|\hat{J}\hat{J}_x \hat{n}_x| + |\hat{J}^2 \hat{n}_{xx}| + |\hat{J}^2 \hat{n}_x n_{ix}| + \hat{J}_x^2$$
$$+ |\hat{J}\hat{J}_{xx}| + |\hat{J}\hat{J}_x n_{ix}| + |(\hat{n}\hat{n}_x)_x| + |\hat{n}_{xx}| \Big)$$
$$\le C\Big(|\phi_{itx}|(r(\hat{n}) + r(\hat{J})) + |\phi_{it}| r(\hat{J}) r(\hat{n}) + (|\phi_{it}| + |\phi_{ix}| + |\phi_{ixx}|) r(\hat{n})$$
$$+ |\phi_{it}\phi_{ixx}| r(\hat{J}) + r(\hat{n})^2 \chi_{\mathrm{supp}(\hat{n})} \Big). \tag{3.67}$$

Here χ_A denotes the characteristic function of the set A and $\mathrm{supp}(\hat{n})$ is the support of the function $\hat{n}$. For the term containing F_{i3x}, we have

$$\int_0^t \int_{\mathbb{R}} (1+\tau)^2 \phi_{x\tau} F_{i3x} dx d\tau$$
$$= -\int_0^t \int_{\mathbb{R}} (1+\tau)^2 \phi_{xx\tau} F_{i3} dx d\tau$$
$$= -\int_0^t \frac{d}{d\tau} \int_{\mathbb{R}} (1+\tau)^2 \phi_{xx} F_{i3} dx d\tau + \int_0^t \int_{\mathbb{R}} 2(1+\tau)\phi_{xx} F_{i3} dx d\tau + \int_0^t \int_{\mathbb{R}} (1+\tau)^2 \phi_{xx} F_{i3\tau} dx d\tau$$
$$= -\int_{\mathbb{R}} (1+\tau)^2 \phi_{xx} F_{i3} dx \Big|_0^t + \int_0^t \int_{\mathbb{R}} 2(1+\tau)\phi_{xx} F_{i3} dx d\tau$$

$$+\int_0^t\int_{\mathbb{R}}(1+\tau)^2\phi_{xx}F_{i3\tau}dxd\tau. \tag{3.68}$$

It is easy to show that

$$\begin{aligned}\left|\int_{\mathbb{R}}(1+t)^2\phi_{xx}F_{i3}dx\right| &\le C\int_{\mathbb{R}}(1+t)^2|\phi_{xx}F_{i3}|dx\\ &\le C|\bar{J}|\sup_{0\le\tau\le t}((1+\tau)\|\phi_{xx}(\tau)\|)\cdot(1+t)^{-\mu+2}\\ &\le C\delta N(t).\end{aligned} \tag{3.69}$$

For $\mu>2$,

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}2(1+\tau)\phi_{xx}F_{i3}dxd\tau\right| &\le C\int_0^t\Big((1+\tau)\|\phi_{xx}\|\Big)\|F_{i3}\|d\tau\\ &\le C\int_0^t\sup_{0\le\tau\le t}\Big((1+\tau)\|\phi_{xx}\|\Big)|\bar{J}|(1+\tau)^{-\mu+1}d\tau\\ &\le C\delta N(t),\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}}(1+\tau)^2\phi_{xx}F_{i3\tau}dxd\tau\right| &\le C\int_0^t\Big((1+\tau)\|\phi_{xx}\|\Big)\|F_{i3\tau}\|d\tau\\ &\le C\int_0^t\sup_{0\le\tau\le t}\Big((1+\tau)\|\phi_{xx}\|\Big)\Big((1+\tau)|\bar{J}|(1+\tau)^{-\mu}\Big)d\tau\\ &\le C\delta N(t).\end{aligned} \tag{3.71}$$

Then it follows from (3.67)-(3.71) that

$$\begin{aligned}|J_0| \le\, & C\int_0^t\int_{\mathbb{R}}(1+\tau)\sum_{i=1}^2|b_i\phi_{itx}\phi_{ixx}|dxd\tau\\ &+C\int_0^t\int_{\mathbb{R}}(1+\tau)\Big|\lambda(\phi_{xx},\mathcal{F})+(1+\tau)(\phi_{x\tau},\mathcal{F}_x)\Big|dxd\tau\\ \le\, & C(|\bar{J}|+N(t))N(t)^2+C|\bar{J}|N(t).\end{aligned} \tag{3.72}$$

Thus, combining (3.50)-(3.51), we complete the proof of the lemma. ◻

From Lemma 3.1 and Lemma 3.2, we immediately obtain the following *a priori* estimates.

LEMMA 3.3 (A priori estimates). *It holds that*

$$\begin{aligned}&\sum_{\ell=1}^2\left((1+\tau)^\ell\|\partial_x^{\ell-1}(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2+(1+\tau)^{\ell-2}\|\partial_x^{\ell-1}\|(\phi_1,\phi_2)(\tau)\|^2\right)\\ &\quad+\int_0^t\sum_{\ell=0}^1\left((1+\tau)^\ell\|\partial_x^\ell(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2+(1+\tau)^{\ell-2}\|\partial_x^\ell\|(\phi_1,\phi_2)(\tau)\|^2\right)d\tau\\ &\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\end{aligned} \tag{3.73}$$

and

$$\|(\phi_1,\phi_2)(t)\|_{L^\infty(\mathbb{R})}\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2), \tag{3.74}$$

$$\|(\psi_1,\psi_2)(t)\|_{L^\infty(\mathbb{R})} \le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ (1+t)^{-\frac{3}{4}}, \tag{3.75}$$

$$\|E(t)\|_{L^\infty(\mathbb{R})} \le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ (1+t)^{-\frac{3}{4}}, \tag{3.76}$$

$$\|(\phi_1,\phi_2)_t(t)\|_{L^\infty(\mathbb{R})} \le C(|\bar{J}|+\|\phi_0\|_2^2)\ \ (1+t)^{-\frac{3}{4}}, \tag{3.77}$$

$$\|(\phi_1,\phi_2)_x(t)\|_{L^\infty(\mathbb{R})} \le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ (1+t)^{-\frac{3}{4}}, \tag{3.78}$$

provided with $|\bar{J}|+N(t)<\varepsilon_3$ *with some* $0<\varepsilon_3\le\varepsilon_2$*, where* ε_2 *is given in Lemma 3.2.*

Proof. From Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
&\sum_{\ell=1}^{2}\big((1+\tau)^{\ell}\|\partial_x^{\ell-1}(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2+(1+\tau)^{\ell-2}\|\partial_x^{\ell-1}\|(\phi_1,\phi_2)(\tau)\|^2\big)\\
&\quad+\int_0^t\sum_{\ell=0}^{1}\big((1+\tau)^{\ell}\|\partial_x^{\ell}(\phi_{1x},\phi_{2x},\psi_1,\psi_2,E)(\tau)\|^2+(1+\tau)^{\ell-2}\|\partial_x^{\ell}\|(\phi_1,\phi_2)(\tau)\|^2\big)\,d\tau\\
&\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\\
&\quad+C(\|\phi_0\|_2^2+\|\phi_{t0}\|_1^2+\|E_0\|_1^2+|\bar{J}|(N(t)^2+N(t)M(t))+N(t)^3),
\end{aligned}\tag{3.79}$$

that is,

$$(1-C|\bar{J}|-CN(t))N^2(t)+(1-C|\bar{J}|)M^2(t)\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2),$$

which implies the *a priori* estimates

$$N^2(t)+M^2(t)\ll C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2),$$

provided with

$$|\bar{J}|+N(t)\ll 1.$$

Furthermore, by the interpolation inequality (3.32), we have

$$\begin{aligned}
\|\phi_1(t)\|_{L^\infty(\mathbb{R})}&\le C\|\phi_1(t)\|^{\frac{1}{2}}\|\phi_{1x}(t)\|^{\frac{1}{2}}\\
&\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ \Big((1+t)^{-\frac{1}{2}}(1+t)^{\frac{1}{2}}\Big)^{\frac{1}{2}}\\
&\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2),
\end{aligned}\tag{3.80}$$

and

$$\begin{aligned}
\|\phi_{1x}(t)\|_{L^\infty(\mathbb{R})}&\le C\|\phi_{1x}(t)\|^{\frac{1}{2}}\|\phi_{1xx}(t)\|^{\frac{1}{2}}\\
&\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ \Big((1+t)^{-\frac{1}{2}}(1+t)^{-1}\Big)^{\frac{1}{2}}\\
&\le C(|\bar{J}|+\|\phi_0\|_2^2+\|\psi_0\|_1^2)\ \ (1+t)^{-\frac{3}{4}}.
\end{aligned}\tag{3.81}$$

For the rate of ψ_i, we use the equation $\phi_t=-\psi_i$. The other decay properties are obtained in a similar way.

Therefore the proof is complete. ∎

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