SIAM J. MATH. ANAL. © 2023 Society for Industrial and Applied Mathematics
Vol. 55, No. 2, pp. 1048–1099

OPTIMAL DECAY RATES OF THE COMPRESSIBLE EULER EQUATIONS WITH TIME-DEPENDENT DAMPING IN $\mathbb{R}^n$: (II) OVERDAMPING CASE*

SHANMING JI† AND MING MEI‡

Abstract. This paper is concerned with the large time behavior of the multidimensional compressible Euler equations with time-dependent overdamping of the form $-\frac{\mu}{(1+t)^\lambda}\rho\boldsymbol{u}$ in $\mathbb{R}^n$, where $n\geq 2$, $\mu>0$, and $\lambda\in[-1,0)$. This continues our previous work dealing with the underdamping case for $\lambda\in[0,1)$. We show the optimal decay estimates of the solutions such that for $\lambda\in(-1,0)$ and $n\geq 2$, $\|\rho-1\|_{L^2(\mathbb{R}^n)}\approx(1+t)^{-\frac{1+\lambda}{4}n}$ and $\|\boldsymbol{u}\|_{L^2(\mathbb{R}^n)}\approx(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}$, which indicates that a stronger damping gives rise to solutions decaying optimally slower. For the critical case of $\lambda=-1$, we prove the optimal logarithmical decay of the perturbation of density for the damped Euler equations such that $\|\rho-1\|_{L^2(\mathbb{R}^n)}\approx|\ln(e+t)|^{-\frac{n}{4}}$ and $\|\boldsymbol{u}\|_{L^2(\mathbb{R}^n)}\approx(1+t)^{-1}\cdot|\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}$ for $n\geq 7$. The overdamping effect reduces the decay rates of the solutions to be slow, which causes us some technical difficulty in obtaining the optimal decay rates by the Fourier analysis method and the Green function method. Here, we propose a new idea to overcome such a difficulty by artfully combining the Green function method and the time-weighted energy method.

Key words. Euler equation, time-dependent damping, optimal decay rates, overdamping

MSC codes. 35Q31, 76N10, 35B40

DOI. 10.1137/21M144476X

1. Introduction.

1.1. Modeling equations and research background. We consider the following multidimensional compressible Euler equations with time-dependent damping:

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0,\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p(\rho)=-\dfrac{\mu}{(1+t)^\lambda}\rho\boldsymbol{u},\\ \rho|_{t=0}=\rho_0(x):=1+\tilde\rho_0(x),\quad \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x),\end{cases}\tag{1.1}$$

where $x\in\mathbb{R}^n$, $n\geq 2$, $\mu>0$, $\lambda\in[-1,0)$. Here, the unknown functions $\rho(t,x)$ and $\boldsymbol{u}(t,x)$ represent the density and velocity of the fluid, and the pressure $p(\rho)=\frac{1}{\gamma}\rho^\gamma$ with $\gamma>1$. The initial data satisfy

$$\rho_0(x)\to 1,\quad\text{i.e.,}\quad \tilde\rho_0(x)\to 0,\quad\text{and}\quad \boldsymbol{u}_0(x)\to\boldsymbol{0},\quad\text{as } |x|\to\infty.\tag{1.2}$$

*Received by the editors September 7, 2021; accepted for publication September 22, 2022; published electronically April 26, 2023.
https://doi.org/10.1137/21M144476X
Funding: This work was done when the first author visited McGill University, supported by the China Scholarship Council (CSC) for the senior visiting scholar program. The research of the first author was supported by NSFC grants 12271178 and 12171166, CSC grant 201906155021, Guangdong Basic and Applied Basic Research Foundation grant 2021A1515010367, Guangzhou Basic and Applied Basic Research Foundation grant 202102021050, and the Fundamental Research Funds for the Central Universities, 2022ZYGXZR032. The research of the second author was supported in part by NSERC grant RGPIN-2022-03374.

†School of Mathematics, South China University of Technology, Guangzhou 510641, People's Republic of China (jism@scut.edu.cn).
‡Corresponding author. Department of Mathematics, Champlain College, Saint-Lambert J4P 3P2, PQ, Canada, and Department of Mathematics and Statistics, McGill University, Montreal H3A 2K6, PQ, Canada (ming.mei@mcgill.ca).

The underdamping case of $\lambda \in [0,1)$ is considered in the first part [23] of our series of studies, where we showed that weaker damping leads to faster decays. Here in this paper, we focus on the overdamping case of $\lambda \in [-1,0)$ and we prove that stronger damping gives rise to optimally slower decays.

As we mentioned in the first part [23] of this series of studies, the damping effect plays a key role in the structure of solutions to the compressible Euler equations. Without the damping effect, the solutions of Euler equations usually possess singularity-like shock waves and exhibit blow-up for their gradients [6, 7, 10, 12, 24, 34, 39]. When the Euler system of equations has the damping effect, the structure of the solutions becomes more complicated and varies according to the size of the damping effect, and of course, the study is more challenging. When $\lambda = 0$ and $\mu > 0$, the regular case of damping effect in the form of $-\mu\rho\boldsymbol{u}$, once the initial data and their gradients are small enough, the damping effect can prevent the formation of shocks for the damped Euler equations [38] and makes the solutions behave time-asymptotically as the so-called diffusion waves for the corresponding nonlinear diffusion (porous media) equations [19, 29, 30, 32, 33], while once the gradients of the initial data are bigger, the blow-up phenomena for the solutions of Euler equations with regular damping still occur [26, 46]. When $\lambda > 0$ and $\mu > 0$, the damping effect $-\frac{\mu}{(1+t)^\lambda}\rho\boldsymbol{u}$ becomes weaker as λ increases—the so-called underdamping case. Here, for $\lambda \in (0,1)$ and $\mu > 0$, once the initial data and their gradients are small enough, the weak damping effect can still guarantee the global existence of the solutions for the Euler equations with underdamping [8, 11, 17, 18, 25, 35, 40], while the solutions will blow up at finite time when the gradients of the initial data are big [8]. However, when $\lambda > 1$ with $\mu > 0$, the damping effect is too weak, and the Euler system with such a weak damping essentially behaves like the pure Euler system so that the singularity of shocks cannot be avoided, no matter how smooth and small the initial data are [8, 17, 18, 36, 41]. Such blow-up phenomena in this super underdamping case of $\lambda > 1$ are determined by the mechanism of the dynamic system itself, rather than the selection of the initial data [8]. When $\lambda = 1$, this is the critical case, where the solutions globally exist for $\mu > 3 - n$ as shown in [17, 18] (see also [8, 14, 36, 41] for the one-dimensional case) and blow-up occurs for $\mu \le 3 - n$ as studied in [17, 18].

For the global solutions of the dynamic system of partial differential equations, one of the fundamental problems from both mathematical and physic points of view is to investigate the asymptotic behavior at large time. For the time-dependent damped Euler equations (1.1), when $\lambda = 0$, the optimal decay rates were technically obtained by Sideris, Thomas, and Wang [38] when the initial data are in certain Sobolev space and by Tan and others [42, 43] in some Besov spaces. For $\lambda \in (0,1)$, the methods for deriving the decay estimates of the solutions adopted in the previous studies for the $\lambda = 0$ case in [38, 42, 43] cannot be directly applied, due to the complexity of the damping effect involving the time t. In our study [23], we apply the technical Fourier analysis to derive the optimal decay estimates for the linearized system which can be formally expressed by the implicit Green functions, then use the weighted energy method with some new developments to obtain the optimal decay rates of the solutions for the nonlinear Euler equations with time-dependent underdamping:

$$\|\partial_x^\alpha(\rho-1)\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)},$$
$$\|\partial_x^\alpha \boldsymbol{u}\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)-\frac{1-\lambda}{2}}, \quad \lambda \in [0,1).$$

The new point observed in [23] is that, for $\lambda \in [0,1)$, the weaker underdamping effect makes the faster decay of the solutions, namely, the decay of the solutions at $\lambda = 0$ is weakest, while the decays of the solutions around $\lambda = 1^-$ are much faster.

However, for $\lambda < 0$, the so-called overdamping case, the relevant study for the damped Euler equations, almost nothing is known, to the best of our knowledge. This will be the main concern of the present paper. We consider the case for $\lambda \in [-1,0)$ and $\mu > 0$. First of all, we focus on the case of $\lambda \in (-1,0)$ and show the optimal decay of the implicit Green functions by using Fourier analysis for the high frequency part and the low frequency part, respectively, and further obtain the optimal decay estimates for the solutions to the nonlinear Euler equations with time-dependent overdamping (1.1) by the Green function method with some restriction on λ. That is,

$$\begin{aligned}&\|\partial_x^\alpha(\rho-1)\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)},\\ &\|\partial_x^\alpha \boldsymbol{u}\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)-\frac{1-\lambda}{2}}, \qquad \lambda\in\left(-\frac{n}{n+2},0\right),\end{aligned}$$

but we have to restrict $\lambda \in (-\frac{n}{n+2},0)$ due to the bad effect of the overdamping. In fact, from the above decay estimates, we realize that the overdamping effect for $\lambda\in(-1,0)$ makes the decay of the solutions become slower and slower, as $\lambda\to-1^+$. Namely, the strongest overdamping at $\lambda=-1^+$ reduces the solution decay the most. Just because of this, we cannot close the high-order decay estimates for all $\lambda\in(-1,0)$ by the Green function method and have to leave the case of $\lambda\in(-1,-\frac{n}{n+2}]$ open. In order to delete such a gap for $\lambda\in(-1,-\frac{n}{n+2}]$, we propose a new technique, which is an artful combination of the Green function method and the time-weighted energy method. The Green function method cannot perfectly treat the high-order decay estimates for λ near -1, and the time-weighted energy method is also short in deriving the optimal decay estimates, but it is very efficient to treat the high-order estimates. Hence we try to combine these two methods together to get the optimal decay estimates for all $\lambda\in(-1,0)$. In fact, the procedure to cleverly combine both existing methods is also technical as we know. Thus, we can finally prove the optimal decay estimates for all $\lambda\in(-1,0)$ as follows:

$$\|\rho(t,x)-1\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{4}n}, \quad \|\boldsymbol{u}(t,x)\|_{L^2(\mathbb{R}^n)} \approx (1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}, \quad \lambda\in(-1,0).$$

Second, we consider the critical case of $\lambda=-1$, the most interesting but also the most difficult case. We further show the optimal decay rates as follows:

$$\|\rho(t,x)-1\|_{L^2(\mathbb{R}^n)}\approx|\ln(e+t)|^{-\frac{n}{4}}, \quad \|\boldsymbol{u}(t,x)\|_{L^2(\mathbb{R}^n)}\approx(1+t)^{-1}\cdot|\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}, \quad \lambda=-1.$$

But we have to restrict the space dimension $n\geq 7$ for technical reasons.

For the other topics with vacuum for the damped Euler equations, we refer to the significant works [13, 15, 20, 21, 22, 28]. For the recent study of shock singularities, we refer to the important contributions [1, 2, 3, 9, 27, 31]. For the linear wave equations with time-dependent damping, we refer to the pioneering studies by Wirth in [47, 48, 49]. For the time-dependent damped Klein–Gordon equations, we refer to the interesting results by Burq, Raugel, and Schlag in [4, 5].

1.2. Transformation of equations and notation. In order to study the system (1.1), we switch it to a symmetric system. Let $v=\frac{2}{\gamma-1}(\sqrt{p'(\rho)}-1)=\frac{2}{\gamma-1}(\rho^{\frac{\gamma-1}{2}}-1)$ and $\varpi=\frac{\gamma-1}{2}$. Then $(v,\boldsymbol{u})$ satisfies the symmetric system

$$\begin{cases}\partial_t v+\nabla\cdot\boldsymbol{u}=-\boldsymbol{u}\cdot\nabla v-\varpi v\nabla\cdot\boldsymbol{u},\\ \partial_t\boldsymbol{u}+\nabla v+\dfrac{\mu}{(1+t)^\lambda}\boldsymbol{u}=-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\varpi v\nabla v,\\ v|_{t=0}=v_0(x),\quad \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x),\end{cases}\tag{1.3}$$

where $v_0(x)=\frac{2}{\gamma-1}((1+\tilde{\rho}_0(x))^{\frac{\gamma-1}{2}}-1)$, which behaves like $\tilde{\rho}_0(x)$ if the initial perturbation is small.

Notation. We denote $D_t=-i\partial_t$, and $\hat{v}(\xi)=\mathscr{F}(v)$ the n-dimensional Fourier transform of a function $v(x)$. We use $H^s=H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, to denote Sobolev spaces and $L^p=L^p(\mathbb{R}^n)$, $1\le p\le\infty$, to denote the L^p spaces. The spatial derivatives ∂_x^α stand for $\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$ with nonnegative multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ (the order of α is denoted by $|\alpha|=\sum_{j=1}^{j=n}\alpha_j$) and $\partial_x^{|\alpha|}$ stands for all the spatial partial derivatives of order $|\alpha|$. The pseudodifferential operator Λ is defined by $\Lambda^s v:=\mathscr{F}^{-1}(|\xi|^s\hat{v}(\xi))$ for $s\in\mathbb{R}$. The norm $\|v\|_X^l$ stands for the $\|\cdot\|_X$ norm of the low frequency part $v^l:=\mathscr{F}^{-1}(\chi(\xi)\hat{v}(\xi))$ of v, while $\|v\|_X^h$ stands for the $\|\cdot\|_X$ norm of the high frequency part $v^h:=\mathscr{F}^{-1}((1-\chi(\xi))\hat{v}(\xi))$ of v, where $0\le\chi(\xi)\le 1$ is a smooth cut-off function supported in $B_{2R}(0)$ and $\chi(\xi)\equiv 1$ on $B_R(0)$ for a given $R>0$.

Throughout this paper, we denote $b(t)=\frac{\mu}{(1+t)^\lambda}$ with $\mu>0$ and $\lambda\in[-1,0)$ and we let C (or C_j with $j=1,2,\dots$) denote some positive universal constants (may depend on n, λ, μ, γ, and α). We use $f\lesssim g$ or $g\gtrsim f$ if $f\le Cg$, and denote $f\approx g$ if $f\lesssim g$ and $g\gtrsim f$. For simplicity, we define $\|(f,g)\|_X:=\|f\|_X+\|g\|_X$ and $\int f:=\int_{\mathbb{R}^n}f(x)dx$. The norm $\|\cdot\|_{L^2}$ will be simplified as $\|\cdot\|$ without confusion. For a matrix $A=(A_{j,k})$, the norm $\|A\|_{\max}:=\max_{j,k}|A_{j,k}|$ is the maximum absolute value of all its elements. We define the following time decay function:

$$\Gamma(t,s):=\begin{cases}[1+(1+t)^{1+\lambda}-(1+s)^{1+\lambda}]^{-\frac{1}{2}}, & \lambda\in(-1,0),\\ \Big[1+\ln\Big(\dfrac{1+t}{1+s}\Big)\Big]^{-\frac{1}{2}}, & \lambda=-1.\end{cases}\tag{1.4}$$

1.3. Main results. For the overdamping case with $\lambda\in[-1,0)$, our main results for the global existence and uniqueness of the solutions as well as the optimal decay esitmates are stated as follows.

THEOREM 1.1 (optimal L^2 decay estimates of nonlinear Euler system). *For the dimension $n\ge 2$ and $\lambda\in(-\frac{n}{n+2},0)$, there exists a constant $\varepsilon_0>0$ such that the solution $(v,\boldsymbol{u})$ of the nonlinear system* (1.3) *corresponding to initial data $(v_0,\boldsymbol{u}_0)$ with small energy $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+3}}\le\varepsilon_0$ exists time-globally and satisfies*

$$\begin{cases}\|\partial_x^\alpha v\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}|\alpha|}, & 0\le|\alpha|\le[\frac{n}{2}]+1,\\ \|\partial_x^\alpha\boldsymbol{u}\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}(|\alpha|+1)+\lambda}, & 0\le|\alpha|\le[\frac{n}{2}],\\ \|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+3}}\lesssim 1.\end{cases}\tag{1.5}$$

The first two decay estimates in (1.5) *(i.e., the decay estimates on $\|\partial_x^\alpha v\|$ with $0\le|\alpha|\le[\frac{n}{2}]+1$ and $\|\partial_x^\alpha\boldsymbol{u}\|$ with $0\le|\alpha|\le[\frac{n}{2}]$) are optimal and consistent with the linearized hyperbolic system.*

THEOREM 1.2 (optimal L^q decay estimates of nonlinear Euler system). *For $n\ge 2$, $q\in[2,\infty]$, $k\ge 3+[\gamma_{2,q}]$ with $\gamma_{2,q}:=n(1/2-1/q)$, and $\lambda\in(-\frac{n}{n+2},0)$, let $(v,\boldsymbol{u})$ be the solution to the nonlinear system* (1.3) *corresponding to initial data $(v_0,\boldsymbol{u}_0)$ with small energy such that $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+k}}\le\varepsilon_0$, where $\varepsilon_0>0$ is a small constant only depending on n,q,k and the constants γ,μ,λ in the system. Then $(v,\boldsymbol{u})\in L^\infty(0,+\infty;H^{[\frac{n}{2}]+k})$ and satisfies*

$$\begin{cases}\|\partial_x^\alpha v\|_{L^q}\lesssim(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}|\alpha|}, & 0\le|\alpha|\le 1,\\ \|\boldsymbol{u}\|_{L^q}\lesssim(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1-\lambda}{2}},\end{cases}\tag{1.6}$$

where $\gamma_{1,q}=n(1-1/q)$. The decay estimates in (1.6) *are optimal.*

Remark 1.3. The above optimal L^2 and L^q decays are formulated by means of the technical Fourier analysis and the Green function method. The restriction of $\lambda \in (-\frac{n}{n+2}, 0)$ comes from the following two main difficulties caused by the overdamping:

(i) The optimal decay of $\|\partial_x^\alpha v\|$ for the linearized hyperbolic system of (1.3) is slow,

$$\begin{cases} \|\partial_x^\alpha \mathcal{G}_{11}(t,0)v_0\| \approx (1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}|\alpha|}, & \lambda \in (-1,0), \\ \|\partial_x^\alpha \mathcal{G}_{11}(t,0)v_0\| \approx |\ln(e+t)|^{-\frac{n}{4}-\frac{|\alpha|}{2}}, & \lambda = -1, \end{cases} \tag{1.7}$$

where $\mathcal{G}(t,s)$ is the Green matrix (see (2.2)). One should be careful in calculating the estimates of $\int_0^t \mathcal{G}(t,s)Q(s)ds$ involving general nonlinear terms $Q(t)$.

(ii) The overdamping $b(t)$ causes trouble in the estimates on $b(t)\partial_x^k \boldsymbol{u} \cdot \partial_x^{k+1} v$, which is crucial for the high-order energy estimates on $\|\partial_x^{k+1} v\|$ in the closure of the a priori assumption.

Remark 1.4. The solutions to the linearized hyperbolic system of (1.3) decay optimally slower for the overdamping case. We may understand it as follows: when the overdamping is stronger as $\lambda \in [-1,0)$, the high frequencies decay faster as $e^{-C(1+t)^{1-\lambda}}$ (superexponential), while the low frequencies decay slower as

$$\begin{cases} e^{-C|\xi|^2(1+t)^{1+\lambda}} & \text{for } \lambda \in (-1,0), \\ e^{-C|\xi|^2 \ln(e+t)} & \text{for critical } \lambda = -1, \end{cases}$$

and on the whole the solutions decay slower.

In order to formulate the decay estimates for all $\lambda \in (-1,0)$ and especially for the critical case of $\lambda = -1$, we develop a time-weighted iteration scheme, which is the combined time-weighted energy estimates and Green functions we build up in the above, to close the decay estimates.

THEOREM 1.5 (optimal decay estimates for $\lambda \in (-1,0)$). *For $n \geq 2$, $N \geq [\frac{n}{2}]+2$, and $\lambda \in (-1,0)$, there exists a constant $\varepsilon_0 > 0$ such that the solution $(v, \boldsymbol{u})$ of the nonlinear system* (1.3) *corresponding to small initial data $\|(v_0, \boldsymbol{u}_0)\|_{L^1 \cap H^N} \leq \varepsilon_0$ exists globally and satisfies*

$$\begin{cases} \|v(t,\cdot)\| \lesssim (1+t)^{-\frac{1+\lambda}{4}n}, \\ \|\boldsymbol{u}(t,\cdot)\| \lesssim (1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}. \end{cases} \tag{1.8}$$

The above decay estimates are optimal and consistent with the linearized hyperbolic system.

Remark 1.6. Theorem 1.1 shows the optimal decay rates of all derivatives of solutions $\|\partial_x^\alpha v\|$ with $0 \leq |\alpha| \leq [\frac{n}{2}]+1$ and $\|\partial_x^\alpha \boldsymbol{u}\|$ with $0 \leq |\alpha| \leq [\frac{n}{2}]$, but λ is restricted in $(-\frac{n}{n+2}, 0)$. Based on Theorem 1.1, applying the new developed time-weighted energy method, we further improve the optimal decay rates of $\|(v,\boldsymbol{u})\|$ in Theorem 1.5 for all $\lambda \in (-1,0)$ and $n \geq 2$. But for the optimal decay rates to the derivatives of the solutions as $\lambda \in (-1, -\frac{n}{n+2})$, they still remain open.

THEOREM 1.7 (optimal logarithmic decays for the critical case of $\lambda = -1$). *For $n \geq 7$, $\lambda = -1$, and $N \geq [\frac{n}{2}]+2$, there exists a constant $\varepsilon_0 > 0$ such that the solution $(v,\boldsymbol{u})$ of the nonlinear system* (1.3) *corresponding to small initial data $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^N} \leq \varepsilon_0$ exists globally and satisfies*

$$\begin{cases} \|v(t,\cdot)\| \lesssim |\ln(e+t)|^{-\frac{n}{4}}, \\ \|\boldsymbol{u}(t,\cdot)\| \lesssim (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}. \end{cases} \tag{1.9}$$

The above decay estimates are optimal and consistent with the linearized hyperbolic system.

Remark 1.8. For the critical $\lambda=-1$, the optimal decay of $\|v\|$ of the nonlinear Euler system (1.3) is powers of the logarithmic function, i.e., $|\ln(e+t)|^{-\frac{n}{4}}$, which differs from the classical algebraical decays. To the best of our knowledge, this is the first result that shows the optimal logarithmical decays of the damped Euler equations.

All the above decay estimates are valid for the Euler equation (1.1).

Corollary 1.9. *For $n\ge 2$ and $\lambda\in(-\frac{n}{n+2},0)$, there exists a constant $\varepsilon_0>0$ such that the solution $(\rho,\boldsymbol{u})$ of the nonlinear system* (1.1) *corresponding to initial data $(\rho_0,\boldsymbol{u}_0)$ with small energy $\|(\rho_0-1,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+3}}\le\varepsilon_0$ exists globally and satisfies*

$$\begin{cases}\|\partial_x^\alpha(\rho-1)\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}|\alpha|}, & 0\le|\alpha|\le[\frac{n}{2}]+1,\\ \|\partial_x^\alpha\boldsymbol{u}\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}(|\alpha|+1)+\lambda}, & 0\le|\alpha|\le[\frac{n}{2}],\\ \|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+3}}\lesssim 1.\end{cases}\tag{1.10}$$

The first two decay estimates in (1.10) *(i.e., the decay estimates on $\|\partial_x^\alpha(\rho-1)\|$ with $0\le|\alpha|\le[\frac{n}{2}]+1$ and $\|\partial_x^\alpha\boldsymbol{u}\|$ with $0\le|\alpha|\le[\frac{n}{2}]$) are optimal.*

For $n\ge 2$, $q\in[2,\infty]$, $k\ge 3+[\gamma_{2,q}]$ with $\gamma_{2,q}:=n(1/2-1/q)$, and $\lambda\in(-\frac{n}{n+2},0)$, let $(\rho,\boldsymbol{u})$ be the solution to the nonlinear system (1.3) *corresponding to initial data $(\rho_0,\boldsymbol{u}_0)$ with small energy such that $\|(\rho_0-1,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+k}}\le\varepsilon_0$, where $\varepsilon_0>0$ is a small constant only depending on n,q,k and the constants γ,μ,λ in the system. Then $(\rho-1,\boldsymbol{u})\in L^\infty(0,+\infty;H^{[\frac{n}{2}]+k})$ and satisfies*

$$\begin{cases}\|\partial_x^\alpha(\rho-1)\|_{L^q}\lesssim(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}|\alpha|}, & 0\le|\alpha|\le 1,\\ \|\boldsymbol{u}\|_{L^q}\lesssim(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1-\lambda}{2}},\end{cases}\tag{1.11}$$

where $\gamma_{1,q}=n(1-1/q)$. The decay estimates in (1.11) *are optimal.*

Corollary 1.10. *For $n\ge 2$, $N\ge[\frac{n}{2}]+2$, and $\lambda\in(-1,0)$, there exists a constant $\varepsilon_0>0$ such that the solution $(\rho,\boldsymbol{u})$ of the nonlinear system* (1.3) *corresponding to small initial data $\|(\rho_0-1,\boldsymbol{u}_0)\|_{L^1\cap H^N}\le\varepsilon_0$ exists globally and satisfies*

$$\begin{cases}\|\rho(t,x)-1\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n},\\ \|\boldsymbol{u}(t,x)\|\lesssim(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}.\end{cases}$$

The above decay estimates are optimal.

For $n\ge 7$, $N\ge[\frac{n}{2}]+2$, and $\lambda=-1$, there exists a constant $\varepsilon_0>0$ such that the solution $(\rho,\boldsymbol{u})$ of the nonlinear system (1.3) *corresponding to small initial data $\|(\rho_0-1,\boldsymbol{u}_0)\|_{L^1\cap H^N}\le\varepsilon_0$ exists globally and satisfies*

$$\begin{cases}\|\rho(t,x)-1\|\lesssim|\ln(e+t)|^{-\frac{n}{4}},\\ \|\boldsymbol{u}(t,x)\|\lesssim(1+t)^{-1}\cdot|\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}.\end{cases}$$

The above decay estimates are optimal.

The paper is organized as follows. We first leave the optimal decay estimates of the time-dependent damped wave equations and the linearized system (2.4) to the appendix. In section 2 we formulate the optimal decay rates of the solutions with high-order derivatives up to $[\frac{n}{2}]$th order for the nonlinear system (1.3) with $\lambda\in(-\frac{n}{n+2},0)$.

In section 3, by developing a new approach combining the Green function method with the time-weighted energy method, we further improve the optimal decay rates of $\|(v,\boldsymbol{u})\|$ for all $\lambda \in (-1,0)$. Finally, the critical case of $\lambda=-1$ with optimal logarithmic decays is considered in section 4.

2. Green function method. In this section we apply the technical Fourier analysis and the Green function method to the study of the asymptotic behavior of nonlinear system (1.3). We rewrite (1.3) as

$$\partial_t \begin{pmatrix} v \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -\frac{\mu}{(1+t)^\lambda} \end{pmatrix} \begin{pmatrix} v \\ \boldsymbol{u} \end{pmatrix} + \begin{pmatrix} -\boldsymbol{u}\cdot\nabla v - \varpi v\nabla\cdot\boldsymbol{u} \\ -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \varpi v\nabla v \end{pmatrix}, \tag{2.1}$$

and the solution can be expressed, by the Duhamel principle, as follows:

$$\begin{pmatrix} v(t,x) \\ \boldsymbol{u}(t,x) \end{pmatrix} = \mathcal{G}(t,0)\begin{pmatrix} v(0,x) \\ \boldsymbol{u}(0,x) \end{pmatrix} + \int_0^t \mathcal{G}(t,s)Q(s,x)ds, \tag{2.2}$$

where

$$Q(s,x) = \begin{pmatrix} Q_1(s,x) \\ Q_2(s,x) \end{pmatrix} := \begin{pmatrix} -\boldsymbol{u}\cdot\nabla v - \varpi v\nabla\cdot\boldsymbol{u} \\ -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \varpi v\nabla v \end{pmatrix}, \quad \mathcal{G}(t,s) = \begin{pmatrix} \mathcal{G}_{11}(t,s) & \mathcal{G}_{12}(t,s) \\ \mathcal{G}_{21}(t,s) & \mathcal{G}_{22}(t,s) \end{pmatrix}.$$

The Green matrix $\mathcal{G}(t,s)$ represents the evolution of the linear system starting from time s to t. It should be noted that $\mathcal{G}(t,s) \neq \mathcal{G}(t-s,0)$ since the time-asymptotically growing damping $\frac{\mu}{(1+t)^\lambda}$ on (s,t) is completely different from the damping on $(0,t-s)$. Moreover, there is no explicit (matrix exponential type) expression of the Green matrix $\mathcal{G}(t,s)$ due to the time-dependent coefficient $b(t)$. In fact, the abstract expression of $\mathcal{G}(t,s)$ based on the Peano–Baker formula (see Proposition A.3 in [47], for example) is

$$\mathcal{G}(t,s) = I + \sum_{k=1}^{\infty}\int_s^t \mathcal{A}(t_1,\xi)\int_s^{t_1}\mathcal{A}(t_2,\xi)\cdots\int_s^{t_{k-1}}\mathcal{A}(t_k,\xi)dt_k\cdots dt_2dt_1,$$

with the noncommutative ($\mathcal{A}(t,\xi)\mathcal{A}(s,\xi)\neq\mathcal{A}(s,\xi)\mathcal{A}(t,\xi)$ for general $s\neq t$) matrix

$$\mathcal{A}(t,\xi) := \begin{pmatrix} 0 & -i\xi^{\mathrm{T}} \\ -i\xi & -\frac{\mu}{(1+t)^\lambda}I_{n\times n} \end{pmatrix},$$

where $(\cdot)^{\mathrm{T}}$ is the transpose of a vector. The exact time decay estimates of $\mathcal{G}(t,s)$ are shown in Theorem A.5 in the appendix, where we write the Green function of time and space $\mathcal{G}(t,s;x,\xi)$ as $\mathcal{G}(t,s)$ for the sake of simplicity. Here and hereafter, in order to emphasize the effect of time t for a given function $v(t,x)$, we often simply write $v(t)$ instead of $v(t,x)$ if there is no confusion.

The linearized system of (1.3) (or (2.1)) is

$$\begin{cases} \partial_t v + \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\boldsymbol{u} + \nabla v + \dfrac{\mu}{(1+t)^\lambda}\boldsymbol{u} = 0, \\ v|_{t=0} = v_0(x), \quad \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(x). \end{cases} \tag{2.3}$$

Let $u := \Lambda^{-1}\nabla\cdot\boldsymbol{u}$ and $\boldsymbol{w} := \Lambda^{-1}\mathrm{curl}\,\boldsymbol{u}$ (with $(\mathrm{curl}\,\boldsymbol{u})_j^k := \partial_{x_j}u^k - \partial_{x_k}u^j$ for $\boldsymbol{u} = (u^1,\ldots,u^n)$) (see [43], for example), where the pseudodifferential operator Λ is defined by $\Lambda^s v := \mathscr{F}^{-1}(|\xi|^s\hat{v}(\xi))$ for $s\in\mathbb{R}$. Then the linearized system (2.3) is equivalent to

$$
\begin{cases}
\partial_t v + \Lambda u = 0, \\
\partial_t u - \Lambda v + \dfrac{\mu}{(1+t)^\lambda} u = 0, \\
\partial_t \boldsymbol{w} + \dfrac{\mu}{(1+t)^\lambda} \boldsymbol{w} = 0, \\
v|_{t=0} = v_0(x), \quad u|_{t=0} = u_0(x), \quad \boldsymbol{w}|_{t=0} = \boldsymbol{w}_0(x),
\end{cases}
\tag{2.4}
$$

where $u_0(x) = \Lambda^{-1}\nabla \cdot \boldsymbol{u}_0(x)$ and $\boldsymbol{w}_0(x) = \Lambda^{-1}\mathrm{curl}\, \boldsymbol{u}_0(x)$. We note that the estimates on $(v, \boldsymbol{u})$ are equivalent to the estimates on $(v, u, \boldsymbol{w})$. From $(2.4)_3$, the vorticity $\boldsymbol{w}(t,x)$ of the linearized system decays to zero superexponentially (as $\boldsymbol{w}_0(x)e^{-\mu(1+t)^{1-\lambda}/(1-\lambda)}$ with $\lambda \in [-1,0)$), which is faster than any algebraical decays. So we only consider the first two equations of (2.4).

In order to formulate the optimal decay rates of the linearized system (2.4), we consider the following two kinds of wave equations with time-dependent damping:

$$
\begin{cases}
\partial_t^2 v - \Delta v + \dfrac{\mu}{(1+t)^\lambda} \partial_t v = 0, \quad x \in \mathbb{R}^n, \\
v|_{t=0} = v_1(x), \quad \partial_t v|_{t=0} = v_2(x),
\end{cases}
\tag{2.5}
$$

and

$$
\begin{cases}
\partial_t^2 u - \Delta u + \partial_t \left(\dfrac{\mu}{(1+t)^\lambda} u \right) = 0, \quad x \in \mathbb{R}^n, \\
u|_{t=0} = u_1(x), \quad \partial_t u|_{t=0} = u_2(x),
\end{cases}
\tag{2.6}
$$

which are satisfied by the solutions $v(t,x)$ and $u(t,x)$ of (2.4), respectively.

We show that the optimal decay rate of $u(t,x)$ in the damped wave equation (2.6) is faster than the optimal decay rate of $v(t,x)$ in the wave equation (2.5), and further we prove that $u(t,x)$ in the damped linear system (2.4) decays optimally faster than all the damped wave equations (2.5) and (2.6). Therefore, there are cancellations between the evolution of initial data if we regard $u(t,x)$ in the linear system (2.4) as a solution of the wave equation (2.6) with initial data $u_1(x) = u_0(x)$ and $u_2(x) = \Lambda v_0(x) - \mu u_0(x)$.

The optimal decay estimates of the time-dependent damped linearized system (2.4), together with the optimal decays of the wave equations (2.5) and (2.6), are proved in the appendix (Theorems A.1 and A.5) by means of the technique of Fourier analysis.

Compared with the underdamping case $\lambda \in [0,1)$ in [23], here the overdamping case $\lambda \in [-1,0)$ gives rise to two main difficulties in the decay estimates of the nonlinear system:

(i) $\|\partial_x^\alpha v\|$ decays slowly since $\|\mathcal{G}_{11}(t,0)v_0\| \approx (1+t)^{-\frac{1+\lambda}{4}n}$ for $\lambda \in (-1,0)$ and even worse $\|\mathcal{G}_{11}(t,0)v_0\| \approx |\ln(e+t)|^{-\frac{n}{4}}$ for $\lambda = -1$. One should be careful in calculating the estimates on $\int_0^t \mathcal{G}(t,s)Q(s)ds$.

(ii) The high-order energy estimates on $\|\partial_x^{[n/2]+3}(v, \boldsymbol{u})\|$ are generally deduced through the energy method, but the estimate on $\|\partial_x^{[n/2]+3} v\|$ needs the estimate

$$
\int b(t) \partial_x^{[n/2]+2} \boldsymbol{u}(t) \cdot \nabla \partial_x^{[n/2]+2} v(t),
$$

where the overdamping coefficient $b(t) = \frac{\mu}{(1+t)^\lambda} = \mu(1+t)^{|\lambda|}$ for $\lambda \in [-1,0)$ is growing and causes trouble for λ near -1.

2.1. High-order energy estimates with overdamping. For the closure of the decay estimates of nonlinear system (1.3), we need to formulate high-order energy estimates. Note that the overdamping coefficient $b(t)=\frac{\mu}{(1+t)^\lambda}$ is growing for $\lambda\in[-1,0)$.

LEMMA 2.1. *Let* $(v_0,\boldsymbol{u}_0)\in H^{[\frac{n}{2}]+k}$ *with* $k\geq 2$, *and* $(v,\boldsymbol{u})(x,t)$ *be the solutions of the nonlinear system* (1.3) *for* $t\in[0,T]$ *with a positive number* T *and satisfy*

$$\|(v(t),\boldsymbol{u}(t))\|_{H^{[\frac{n}{2}]+2}}\leq \delta_0\frac{1}{b(t)}, \tag{2.7}$$

where $\delta_0>0$ *is a small number. Then it holds that*

$$\begin{aligned}\|(v,\boldsymbol{u})\|^2_{H^{[\frac{n}{2}]+k}}+\int_0^t\left(\frac{1}{b(s)}\|\nabla v(s)\|^2_{H^{[\frac{n}{2}]+k-1}}+b(s)\|\boldsymbol{u}(s)\|^2_{H^{[\frac{n}{2}]+k}}\right)ds\\ \lesssim\|(v_0,\boldsymbol{u}_0)\|^2_{H^{[\frac{n}{2}]+k}},\quad t\in[0,T].\end{aligned} \tag{2.8}$$

Proof. The case of time-independent damping (i.e., $\lambda=0$) is proved in [42], and the underdamping case $\lambda\in(0,1)$ is proved in [23]. But, different from the previous studies, for the overdamping case with $\lambda\in[-1,0)$, here the main difficulty lies in the absence of uniform upper bound of the overdamping coefficient. We divide the proof into four steps.

Step 1. For $0\leq j\leq[\frac{n}{2}]+k-1$, we have

$$\frac{d}{dt}\|\partial_x^j(v,\boldsymbol{u})\|^2+b(t)\|\partial_x^j\boldsymbol{u}\|^2\lesssim\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2+\|\partial_x^j\boldsymbol{u}\|^2). \tag{2.9}$$

This can be proved by applying ∂_x^j to (1.3) and then multiplying the resultant equations by $\partial_x^j(v,\boldsymbol{u})$, summing them up, and integrating it with respect to x over $\mathbb{R}^n$. Here we omit the details.

Step 2. By applying ∂_x^{j+1} to (1.3) with $0\leq j\leq[\frac{n}{2}]+k-1$, and multiplying the resultant equations by $\partial_x^{j+1}(v,\boldsymbol{u})$, and summing them up and integrating it over $\mathbb{R}^n$, we have

$$\frac{d}{dt}\|\partial_x^{j+1}(v,\boldsymbol{u})\|^2+b(t)\|\partial_x^{j+1}\boldsymbol{u}\|^2\lesssim\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2+\|\partial_x^{j+1}\boldsymbol{u}\|^2). \tag{2.10}$$

Step 3. For $0\leq j\leq[\frac{n}{2}]+k-1$, we can obtain

$$\begin{aligned}\frac{d}{dt}\int\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^jv+\|\partial_x^{j+1}v\|^2\lesssim b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\|+\|\partial_x^{j+1}\boldsymbol{u}\|^2\\ +\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2+\|\partial_x^{j+1}\boldsymbol{u}\|^2).\end{aligned} \tag{2.11}$$

In fact, this can be proved by applying ∂_x^j to $(1.3)_2$ and multiplying it by $\partial_x^{j+1}v$ (specifically, $\nabla\partial_x^jv$), utilizing $(1.3)_1$ to deal with the mixed space-time derivative term $\int\partial_x^j\partial_t\boldsymbol{u}\cdot\partial_x^{j+1}v$, that is,

$$\|\partial_x^j\nabla v\|^2+\int\partial_t(\partial_x^j\boldsymbol{u})\cdot\nabla\partial_x^jv+\int b(t)\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^jv=\int\partial_x^jQ_2\cdot\nabla\partial_x^jv,$$

and

$$\begin{aligned}\int\partial_t(\partial_x^j\boldsymbol{u})\cdot\nabla\partial_x^jv=&\frac{d}{dt}\int\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^jv+\int\partial_x^j(\nabla\cdot\boldsymbol{u})\cdot\partial_t\partial_x^jv\\ =&\frac{d}{dt}\int\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^jv-\|\partial_x^j(\nabla\cdot\boldsymbol{u})\|^2+\int\partial_x^j(\nabla\cdot\boldsymbol{u})\cdot\partial_x^jQ_1.\end{aligned}$$

Applying Cauchy's inequality to (2.11), we then arrive at

$$\frac{d}{dt}\int \partial_x^j \boldsymbol{u}\cdot\nabla\partial_x^j v + \|\partial_x^{j+1}v\|^2 \lesssim b^2(t)\|\partial_x^j\boldsymbol{u}\|^2 + \|\partial_x^{j+1}\boldsymbol{u}\|^2 + \|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2 + \|\partial_x^{j+1}\boldsymbol{u}\|^2). \tag{2.12}$$

Next, we multiply (2.12) by $\frac{1}{b(t)}$, for $0\le j\le[\frac{n}{2}]+k-1$, to have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{b(t)}\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right) + \frac{1}{b(t)}\|\partial_x^{j+1}v\|^2\\
&\quad\lesssim \frac{|b'(t)|}{b^2(t)}\int\left|\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right| + b(t)\|\partial_x^j\boldsymbol{u}\|^2 + \frac{1}{b(t)}\|\partial_x^{j+1}\boldsymbol{u}\|^2\\
&\qquad + \frac{1}{b(t)}\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2 + \|\partial_x^{j+1}\boldsymbol{u}\|^2)\\
&\quad\lesssim \varepsilon_1\frac{1}{b(t)}\|\partial_x^{j+1}v\|^2 + b(t)\|\partial_x^j\boldsymbol{u}\|^2 + \frac{1}{b(t)}\|\partial_x^{j+1}\boldsymbol{u}\|^2\\
&\qquad + \frac{1}{b(t)}\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2 + \|\partial_x^{j+1}\boldsymbol{u}\|^2),
\end{aligned}$$

where $\varepsilon_1>0$ is a small number. Therefore, for $0\le j\le[\frac{n}{2}]+k-1$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{b(t)}\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right) + \frac{1}{b(t)}\|\partial_x^{j+1}v\|^2\\
&\quad\lesssim b(t)\|\partial_x^j\boldsymbol{u}\|^2 + \frac{1}{b(t)}\|\partial_x^{j+1}\boldsymbol{u}\|^2 + \frac{1}{b(t)}\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot(\|\partial_x^{j+1}v\|^2 + \|\partial_x^{j+1}\boldsymbol{u}\|^2).
\end{aligned}\tag{2.13}$$

Step 4. Multiplying (2.13) by a small number $\varepsilon_2>0$, summing it up with (2.9) and (2.10), we have

$$\begin{aligned}
&\frac{d}{dt}\|(v,\boldsymbol{u})\|^2_{H^{[\frac{n}{2}]+k}} + \frac{d}{dt}\left(\varepsilon_2\sum_{j=0}^{[n/2]+k-1}\frac{1}{b(t)}\int\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right)\\
&\quad + \frac{1}{b(t)}\|\nabla v\|^2_{H^{[\frac{n}{2}]+k-1}} + b(t)\|\boldsymbol{u}\|^2_{H^{[\frac{n}{2}]+k}}\le 0,
\end{aligned}$$

provided with the a priori assumption (2.7). Let us choose $\varepsilon_2>0$ to be small such that

$$\left|\varepsilon_2\sum_{j=0}^{[n/2]+k-1}\frac{1}{b(t)}\int\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right|\le\frac{1}{2}\|(v,\boldsymbol{u})\|^2_{H^{[\frac{n}{2}]+k}};$$

then we obtain (2.8). The proof is completed. ∎

The trickiest part lies in the treatment of $b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\|$ in (2.11), where $\|\partial_x^{j+1}v\|^2$ is the only good term, and therefore $b^2(t)\|\partial_x^j\boldsymbol{u}\|^2$ arises (if Cauchy's inequality is applied) and grows faster than $b(t)\|\partial_x^j\boldsymbol{u}\|^2$ in (2.9). This is the reason for the a priori assumption (2.7). We can prove that (2.7) is satisfied for λ near zero. However, the decay estimates required in (2.7) are not true for $\lambda\in[-1,0)$ near -1, especially for the case $\lambda=-1$. In fact,

$$\|\partial_x^\alpha\mathcal{G}_{11}(t,0)v_0\|\approx|\ln(e+t)|^{-\frac{n}{4}-\frac{|\alpha|}{2}}\ \text{ for } \lambda=-1,$$

and the decay condition $\|v(t)\|_{H^{[n/2]+2}}\le\delta_0(1+t)^{-1}$ in (2.7) is not valid.

We can relax the decay condition of high-order estimates in (2.7) to a wider range of λ. The crucial point is to avoid the decay conditions of $\|v(t)\|_{H^{[n/2]+2}}$. For application, we prove the following inequality, which can be regarded as a generalized Grönwall's inequality with relaxation.

LEMMA 2.2 (Grönwall's inequality with relaxation). *Assume that $\omega(t)$, $g(t)$, and $H(t)$ are nonnegative functions, $C_2 \geq C_1 > 0$, $\theta \in (0,1)$, $\eta > 0$, all are constants, and $F(t)$ satisfies (note that $F(t)$ is not necessarily nonnegative)*

$$C_1 H(t) - g(t) \leq F(t) \leq C_2 H(t) + g(t), \tag{2.14}$$

and the differential inequality

$$\frac{d}{dt}F(t) + \eta F(t) \leq \omega(t) H^{\theta}(t) + g(t) \quad \forall t > 0; \tag{2.15}$$

then

$$F(t) \lesssim \max\{F(0), \sup_{s\in(0,t)} ((\omega(s)/\eta)^{\frac{1}{1-\theta}} + g(s)(1+1/\eta))\}. \tag{2.16}$$

and

$$H(t) \lesssim \max\{F(0), \sup_{s\in(0,t)} ((\omega(s)/\eta)^{\frac{1}{1-\theta}} + g(s)(1+1/\eta))\}. \tag{2.17}$$

Furthermore, if $\omega(t)$ and $g(t)$ are monotonically decreasing, then

$$\begin{aligned} F(t) \lesssim\, & F(0)e^{-\frac{\eta}{2}t} + \left(\frac{1}{\eta^{1/(1-\theta)}}\omega^{\frac{1}{1-\theta}}(0) + \left(1+\frac{1}{\eta}\right)g(0)\right)e^{-\frac{\eta}{8}t} \\ & + \frac{1}{\eta^{1/(1-\theta)}}\omega^{\frac{1}{1-\theta}}\left(\frac{t}{2}\right) + \left(1+\frac{1}{\eta}\right)g\left(\frac{t}{2}\right) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} H(t) \lesssim\, & F(0)e^{-\frac{\eta}{2}t} + \left(\frac{1}{\eta^{1/(1-\theta)}}\omega^{\frac{1}{1-\theta}}(0) + \left(1+\frac{1}{\eta}\right)g(0)\right)e^{-\frac{\eta}{8}t} \\ & + \frac{1}{\eta^{1/(1-\theta)}}\omega^{\frac{1}{1-\theta}}\left(\frac{t}{2}\right) + \left(1+\frac{1}{\eta}\right)g\left(\frac{t}{2}\right). \end{aligned} \tag{2.19}$$

Proof. We may assume that $C_1 = \frac{1}{2}$, $C_2 = 2$, and $\theta = \frac{1}{2}$. The other situation follows similarly. For any $t > 0$, if $F(t) > F(0)$, then two cases happen: (i) $F(t) = \sup_{s\in(0,t)} F(s)$ such that $F'(t) \geq 0$; (ii) there exists a number $s \in (0,t)$ such that $F'(s) = 0$ and $F(s) > F(t)$. In both cases, we can find a number $s \in (0,t]$ such that $F'(s) \geq 0$ and $F(s) \geq F(t)$. Therefore, according to the differential inequality (2.15), we have

$$\eta F(s) \leq \omega(s)H^{\frac{1}{2}}(s) + g(s) \leq \frac{1}{\eta}\omega^2(s) + \frac{\eta}{4}H(s) + g(s) \leq \frac{1}{\eta}\omega^2(s) + \frac{1}{2}\eta F(s) + \left(\frac{\eta}{2}+1\right)g(t),$$

which implies

$$F(t) \leq F(s) \lesssim \omega^2(s)/\eta^2 + g(s)(1+1/\eta).$$

This immediately guarantees (2.16). On the other hand, (2.14) implies

$$H(t) \lesssim F(t) + g(t).$$

This together with (2.16) proves (2.17).

If $\omega(t)$ and $g(t)$ are monotonically decreasing, then according to (2.15) and Young's inequality

$$\begin{aligned}\frac{d}{dt}F(t)+\eta F(t) &\le \omega(t)H^{\theta}(t)+g(t)\\ &\le \frac{1}{2(C_1\eta)^{\theta/(1-\theta)}}\omega^{1/(1-\theta)}(t)+\frac{C_1}{2}\eta H(t)+g(t)\\ &\le \frac{1}{2(C_1\eta)^{\theta/(1-\theta)}}\omega^{1/(1-\theta)}(t)+\frac{1}{2}\eta F(t)+\left(\frac{\eta}{2}+1\right)g(t),\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}(e^{\frac{\eta}{2}t}F(t)) &= e^{\frac{\eta}{2}t}\left(\frac{d}{dt}F(t)+\frac{\eta}{2}F(t)\right)\\ &\le e^{\frac{\eta}{2}t}\left(\frac{1}{2(C_1\eta)^{\theta/(1-\theta)}}\omega^{1/(1-\theta)}(t)+\left(\frac{\eta}{2}+1\right)g(t)\right)\\ &\lesssim e^{\frac{\eta}{2}t}(\omega^{1/(1-\theta)}(t)+g(t)),\end{aligned}$$

where we have slightly abused the notation "$\lesssim$" such that the inequality depends on η and θ and the dependence is clear. Integrating it with respect to t over $(0,t)$ gives

$$\begin{aligned}F(t) &\lesssim F(0)e^{-\frac{\eta}{2}t}+\int_0^t e^{-\frac{\eta}{2}(t-s)}\Big(\omega^{1/(1-\theta)}(s)+g(s)\Big)ds\\ &\lesssim F(0)e^{-\frac{\eta}{2}t}+\int_0^{\frac{t}{2}} e^{-\frac{\eta}{4}t}\Big(\omega^{1/(1-\theta)}(0)+g(0)\Big)ds\\ &\quad+\int_{\frac{t}{2}}^t e^{-\frac{\eta}{2}(t-s)}\left((\omega^{1/(1-\theta)}\left(\frac{t}{2}\right)+g\left(\frac{t}{2}\right)\right)ds\\ &\lesssim F(0)e^{-\frac{\eta}{2}t}+\Big(\omega^{1/(1-\theta)}(0)+g(0)\Big)e^{-\frac{\eta}{8}t}+\omega^{1/(1-\theta)}\left(\frac{t}{2}\right)+g\left(\frac{t}{2}\right),\end{aligned}$$

since $te^{-\frac{\eta}{4}t}\lesssim e^{-\frac{\eta}{8}t}$. Thus, (2.18) and (2.19) are immediately obtained. The proof is completed. ∎

We modify the high-order estimates Lemma 2.1 such that $\|v(t)\|_{H^{[\frac{n}{2}]+2}}$ does not necessarily decay as fast as $\frac{1}{b(t)}$. The key ingredient is to avoid the estimate on $\|v(t)\|$ such that Step 1 in the proof of Lemma 2.1 is excluded.

LEMMA 2.3. *Let $(v_0,\boldsymbol{u}_0)\in H^{[\frac{n}{2}]+k}$ with $k\ge 2$, and let $(v,\boldsymbol{u})(x,t)$ be the solutions of the nonlinear system* (1.3) *for $t\in[0,T]$ with a positive number T, and satisfy*

$$\|(v(t),\boldsymbol{u}(t))\|_{H^{[\frac{n}{2}]+2}}\le\delta_0,\quad \|\boldsymbol{u}(t)\|_{H^{[\frac{n}{2}]+k-1}}\le\delta_0\frac{\omega(t)}{b(t)},\quad t\in[0,T], \tag{2.20}$$

where $\delta_0>0$ is a small number and $\omega(t)$ is a nonnegative decreasing function. Then it holds that

$$\|\nabla(v,\boldsymbol{u})\|^2_{H^{[\frac{n}{2}]+k-1}}\lesssim\|\nabla(v_0,\boldsymbol{u}_0)\|^2_{H^{[\frac{n}{2}]+k-1}}+\delta_0^2\cdot\omega^2(t/2),\quad t\in[0,T]. \tag{2.21}$$

Proof. According to the estimates (2.10) and (2.11) in Step 2 and Step 3 of the proof of Lemma 2.1, for $0\le j\le[\frac{n}{2}]+k-1$, we have

$$\frac{d}{dt}\|\partial_x^{j+1}(v,\boldsymbol{u})\|^2+b(t)\|\partial_x^{j+1}\boldsymbol{u}\|^2\lesssim\|(v,\boldsymbol{u})\|_{H^{[\frac{n}{2}]+2}}\cdot\|\partial_x^{j+1}v\|^2 \tag{2.22}$$

and

$$\frac{d}{dt}\int \partial_x^j \boldsymbol{u}\cdot\nabla\partial_x^j v + \|\partial_x^{j+1}v\|^2 \lesssim b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\| + \|\partial_x^{j+1}\boldsymbol{u}\|^2, \tag{2.23}$$

where we have used the a priori assumption (2.20) such that $\|(v(t),\boldsymbol{u}(t))\|_{H^{[\frac{n}{2}]+2}} \le \delta_0$ with a small δ_0. Multiplying (2.23) by a small number $\tilde{\varepsilon}_1$ (only depending on the dimension n) such that

$$\tilde{\varepsilon}_1\int \left|\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right| \le \frac{1}{2}\|\partial_x^{j+1}v\|^2 + \|\partial_x^j\boldsymbol{u}\|^2,$$

and making addition of $\tilde{\varepsilon}_1\cdot$(2.23)+(2.22), then we have

(2.24)

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_x^{j+1}(v,\boldsymbol{u})\|^2 + \tilde{\varepsilon}_1\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right) + b(t)\|\partial_x^{j+1}\boldsymbol{u}\|^2 + \tilde{\varepsilon}_1\|\partial_x^{j+1}v\|^2 + \frac{\tilde{\varepsilon}_1^2}{2}\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\\
&\quad\lesssim \tilde{\varepsilon}_1 b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\| + \frac{\tilde{\varepsilon}_1^2}{2}\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v + \delta_0\|\partial_x^{j+1}v\|^2 + \tilde{\varepsilon}_1\|\partial_x^{j+1}\boldsymbol{u}\|^2\\
&\quad\lesssim \tilde{\varepsilon}_1 b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\| + \tilde{\varepsilon}_2\tilde{\varepsilon}_1\|\partial_x^{j+1}v\|^2 + \frac{\tilde{\varepsilon}_1^3}{\tilde{\varepsilon}_2}\|\partial_x^j\boldsymbol{u}\|^2 + \delta_0\|\partial_x^{j+1}v\|^2 + \tilde{\varepsilon}_1\|\partial_x^{j+1}\boldsymbol{u}\|^2,
\end{aligned}$$

where $\tilde{\varepsilon}_2 > 0$ is another small number (only dependent on n) such that $\tilde{\varepsilon}_2\tilde{\varepsilon}_1\|\partial_x^{j+1}v\|^2$ is dominated by $\frac{1}{4}\tilde{\varepsilon}_1\|\partial_x^{j+1}v\|^2$. Noticing that $b(t)$ is growing, $\tilde{\varepsilon}_1$ can be chosen small enough, and δ_0 is small, too, we rewrite (2.24) into

(2.25)

$$\begin{aligned}
&\frac{d}{dt}\left(\|\partial_x^{j+1}(v,\boldsymbol{u})\|^2 + \tilde{\varepsilon}_1\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\right) + \frac{\tilde{\varepsilon}_1}{2}\|\partial_x^{j+1}\boldsymbol{u}\|^2 + \frac{\tilde{\varepsilon}_1}{2}\|\partial_x^{j+1}v\|^2 + \frac{\tilde{\varepsilon}_1^2}{2}\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v\\
&\quad\lesssim \tilde{\varepsilon}_1 b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}v\| + \frac{\tilde{\varepsilon}_1^3}{\varepsilon_2}\|\partial_x^j\boldsymbol{u}\|^2\\
&\quad\lesssim b(t)\|\partial_x^j\boldsymbol{u}\|\cdot\|\partial_x^{j+1}(v,\boldsymbol{u})\| + \|\partial_x^j\boldsymbol{u}\|^2.
\end{aligned}$$

Let

$$F(t) := \|\partial_x^{j+1}(v,\boldsymbol{u})\|^2 + \tilde{\varepsilon}_1\int \partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^j v, \quad H(t) := \|\partial_x^{j+1}(v,\boldsymbol{u})\|^2, \quad g(t) := \|\partial_x^j\boldsymbol{u}\|^2,$$

then

$$\frac{1}{2}H(t) - g(t) \le F(t) \le 2H(t) + g(t)$$

and

$$\frac{d}{dt}F(t) + \frac{\tilde{\varepsilon}_1}{2}F(t) \lesssim \delta_0\omega(t)H^{\frac{1}{2}}(t) + g(t),$$

with $\delta_0\omega(t) = b(t)\|\partial_x^j\boldsymbol{u}\|$ decreasing, provided the a priori assumption (2.20). Applying the generalized Grönwall's inequality with relaxation in Lemma 2.2, we have

$$H(t) = \|\partial_x^{j+1}(v,\boldsymbol{u})\|^2 \lesssim F(0)e^{-\frac{\varepsilon_1}{4}t} + \left(\delta_0^2\omega^2(0) + g(0)\right)e^{-\frac{\varepsilon_1}{16}t} + \delta_0^2\omega^2(t/2) + g(t/2).$$

The proof is completed. ∎

2.2. Optimal L^2 decay estimates. We start with the optimal L^1–L^2 decay estimates of the nonlinear system (1.3) for the overdamping case of $\lambda \in [-1, 0)$.

LEMMA 2.4. *For $\lambda \in [-1,0)$ and $t \geq s \geq T_0$ ($T_0 \geq 0$ is a universal constant only depending on the constants λ and μ), then there hold*

$$
\begin{aligned}
&\|\partial_x^\alpha \mathcal{G}_{11}(t,s)\phi(x)\| \lesssim \Gamma^{\frac{n}{2}+|\alpha|}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right),\\
&\|\partial_x^\alpha \mathcal{G}_{12}(t,s)\phi(x)\| \lesssim (1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right),\\
&\|\partial_x^\alpha \mathcal{G}_{21}(t,s)\phi(x)\| \lesssim (1+t)^\lambda\cdot\Gamma^{\frac{n}{2}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right),\\
&\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| \lesssim \left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\frac{n}{2}+|\alpha|}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right).
\end{aligned}
\tag{2.26}
$$

Furthermore,

$$
\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| \lesssim (1+t)^\lambda(1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+|\alpha|+2}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+1}\phi\|^h\right),
\tag{2.27}
$$

$$
\begin{aligned}
\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| \lesssim &\left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\frac{n}{2}+|\alpha|}(t,s)\\
&\cdot\left((1+s)^{2\lambda}\cdot\Gamma^2(t,s) + \frac{1}{(1+s)^{\lambda-1}} + C_\kappa\Gamma^\kappa(t,s)\right)\\
&\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right),
\end{aligned}
\tag{2.28}
$$

where $\kappa \geq 2$ can be chosen arbitrarily large and $C_\kappa > 0$ is a constant depending on κ.

Proof. These estimates are simple conclusions of Theorem A.5 in the appendix. ∎

LEMMA 2.5. *For $\beta > 0$, $\gamma > 0$, and $\lambda \in (-1,0)$, there holds*

$$
\int_0^t \Gamma^\beta(t,s)\cdot(1+s)^{-\gamma}ds \lesssim \begin{cases} (1+t)^{-\min\{\frac{1+\lambda}{2}\beta,\gamma\}}, & \max\{\frac{1+\lambda}{2}\beta,\gamma\} > 1,\\ (1+t)^{-\min\{\frac{1+\lambda}{2}\beta,\gamma\}}\cdot\ln(e+t), & \max\{\frac{1+\lambda}{2}\beta,\gamma\} = 1,\\ (1+t)^{-\gamma-\frac{1+\lambda}{2}\beta+1}, & \max\{\frac{1+\lambda}{2}\beta,\gamma\} < 1. \end{cases}
\tag{2.29}
$$

Proof. This can be proved by dividing the interval of integration into $(0, \frac{t}{2})$ and $(\frac{t}{2}, 2)$. For details, see the first part of our series of studies [23, Lemma 4.2], for example. ∎

We are now going to prove the optimal L^1–L^2 decay rates in Theorem 1.1 for the nonlinear system (1.3).

Proof of Theorem 1.1. The outline of the proof is similar to that of the underdamping case $\lambda \in [0, 1)$ in Theorem 1.5, as we show in [23]. But the details are totally different.

Suppose that the local solution $(v, \boldsymbol{u})$ exists for $t \in (0, T)$. Since we are concerned with the large time behavior, we may assume that the constant $T_0 = 0$ in Lemma 2.4. Denote the weighted energy function by

$$
\begin{aligned}
E_n(\tilde t) := \sup_{t\in(0,\tilde t)} \Bigg\{ & \sum_{0\le|\alpha|\le[n/2]+1} (1+t)^{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}|\alpha|}\|\partial_x^\alpha v\|,\\
& \sum_{0\le|\alpha|\le[n/2]} (1+t)^{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}(|\alpha|+1)-\lambda}\|\partial_x^\alpha \boldsymbol{u}\|,\\
& \sum_{|\alpha|=[n/2]+1} (1+t)^{\frac{1+\lambda}{4}n+\omega_{|\alpha|}}\|\partial_x^\alpha \boldsymbol{u}\|, \sum_{|\alpha|=[n/2]+2} (1+t)^{\frac{1+\lambda}{4}n+\theta_{|\alpha|}}\|\partial_x^\alpha v\|,\\
& \sum_{|\alpha|=[n/2]+2} (1+t)^{\frac{1+\lambda}{4}n+\omega_{|\alpha|}}\|\partial_x^\alpha \boldsymbol{u}\|, \sum_{|\alpha|=[n/2]+3} \|\partial_x^\alpha (v,\boldsymbol{u})\| \Bigg\},
\end{aligned}
$$

where $\omega_{[n/2]+1}$, $\omega_{[n/2]+2}$, and $\theta_{[n/2]+2}$ are constants depending on n and λ, and $\tilde t\in(0,T)$. We claim that under the smallness of the initial data, $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+3}}\le\varepsilon_0$, there holds

$$
E_n(\tilde t)\lesssim\delta_0 \quad \forall\tilde t\in(0,T), \tag{2.30}
$$

where $\varepsilon_0>0$ and $\delta_0>0$ are some small numbers to be determined later.

The global existence and the a priori assumption (2.30) are proved through the following three steps. For the sake of simplicity, we take the case $n=3$ for an example. Other cases with $n\ge2$ follow similarly.

Step 1*: Basic energy decay estimates.* According to the Duhamel principle (2.2) and the decay estimates of the Green matrix $\mathcal{G}(t,s)$ in Lemma 2.4, we have

$$
\begin{aligned}
\|v(t)\| &\lesssim \|\mathcal{G}_{11}(t,0)v_0\|+\|\mathcal{G}_{12}(t,0)\boldsymbol{u}_0\|+\int_0^t\|\mathcal{G}_{11}(t,s)Q_1(s)\|ds+\int_0^t\|\mathcal{G}_{12}(t,s)Q_2(s)\|ds\\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n}+\int_0^t\Gamma^{\frac{n}{2}}(t,s)\cdot(\|Q_1(s)\|_{L^1}^l+\|Q_1(s)\|^h)ds\\
&\quad+\int_0^t(1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(\|Q_2(s)\|_{L^1}^l+\|Q_2(s)\|^h)ds\\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n}+E_n^2(t)\int_0^t\Gamma^{\frac{n}{2}}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-1}ds\\
&\quad+E_n^2(t)\int_0^t(1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\frac{1+\lambda}{2}}ds\\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n}+E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n},
\end{aligned}
$$

where we have used Lemma 2.5 (note that $\frac{1+\lambda}{2}n+\frac{1+\lambda}{2}-\lambda>1$ for all $n\ge2$ and $\lambda\in(-1,0)$) and the following decay estimates on $\|Q(s)\|_{L^1}$ and $\|Q(s)\|$ (here and after, we use $D^j:=\partial_x^j$ and we may also write $\boldsymbol{u}$ as u for simplicity):

$$
\|Q_1(s)\|_{L^1}\lesssim\|uDv\|_{L^1}+\|vDu\|_{L^1}\lesssim\|u\|\|Dv\|+\|v\|\|Du\|\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-1},
$$

$$
\|Q_2(s)\|_{L^1}\lesssim\|uDu\|_{L^1}+\|vDv\|_{L^1}\lesssim\|u\|\|Du\|+\|v\|\|Dv\|\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\frac{1+\lambda}{2}}.
$$

For $n=3$, we have

$$\begin{aligned}
\|u(s)\|_{L^\infty} &\lesssim \|Du\|^{\frac12}\|D^2u\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac12(1+\omega_2)},\\
\|v(s)\|_{L^\infty} &\lesssim \|Dv\|^{\frac12}\|D^2v\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac34(1+\lambda)},\\
\|Du(s)\|_{L^\infty} &\lesssim \|D^2u\|^{\frac12}\|D^3u\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac12(\omega_2+\omega_3)},\\
\|Dv(s)\|_{L^\infty} &\lesssim \|D^2v\|^{\frac12}\|D^3v\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac12(1+\lambda+\theta_3)},\\
\|D^2u(s)\|_{L^\infty} &\lesssim \|D^3u\|^{\frac12}\|D^4u\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{8}n-\frac12\omega_3},\\
\|D^2v(s)\|_{L^\infty} &\lesssim \|D^3v\|^{\frac12}\|D^4v\|^{\frac12} \lesssim E_n(s)(1+s)^{-\frac{1+\lambda}{8}n-\frac12\theta_3}
\end{aligned}$$

and

$$\begin{aligned}
\|Q_1(s)\| &\lesssim \|uDv\|+\|vDu\| \lesssim \|u\|_{L^\infty}\|Dv\|+\|v\|_{L^\infty}\|Du\|\\
&\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\frac{1+\lambda}{2}-\frac12(1+\omega_2)},\\
\|Q_2(s)\| &\lesssim \|uDu\|+\|vDv\| \lesssim \|u\|_{L^\infty}\|Du\|+\|v\|_{L^\infty}\|Dv\|\\
&\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\frac54(1+\lambda)},\\
\|DQ_1(s)\| &\lesssim \|DuDv\|+\|uD^2v\|+\|vD^2u\| \lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{11}},\\
\|DQ_2(s)\| &\lesssim \|uD^2u\|+\|DuDu\|+\|vD^2v\|+\|DvDv\| \lesssim E^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{12}},
\end{aligned}$$

where

$$\begin{aligned}
\theta_{11} &= \min\left\{1+\frac{1+\lambda}{2}+\frac{\theta_3}{2}, 1+\lambda+\frac12(1+\omega_2), 1+\frac34(1+\lambda)\right\},\\
\theta_{12} &= \min\left\{\omega_2+\frac12(1+\omega_2), 1+\frac12(\omega_2+\omega_3), 1+\lambda+\frac34(1+\lambda), 1+\lambda+\frac{\theta_3}{2}\right\}.
\end{aligned}$$

Using the above estimates, we have

$$\begin{aligned}
&\|Dv(t)\|\\
&\quad\lesssim \|D\mathcal{G}_{11}(t,0)v_0\|+\|D\mathcal{G}_{12}(t,0)u_0\|+\int_0^t \|D\mathcal{G}_{11}(t,s)Q_1(s)\|ds\\
&\qquad+\int_0^t \|D\mathcal{G}_{12}(t,s)Q_2(s)\|ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}}+\int_0^t \Gamma^{\frac n2+1}(t,s)\cdot(\|Q_1(s)\|_{L^1}+\|DQ_1(s)\|)ds\\
&\qquad+\int_0^t (1+s)^\lambda\cdot\Gamma^{\frac n2+2}(t,s)\cdot(\|Q_2(s)\|_{L^1}+\|DQ_2(s)\|)ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}}+E_n^2(t)\int_0^t \Gamma^{\frac n2+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{1,\theta_{11}\}}ds\\
&\qquad+E_n^2(t)\int_0^t (1+s)^\lambda\cdot\Gamma^{\frac n2+2}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\frac{1+\lambda}{2}}ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}}+E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}},
\end{aligned}$$

provided that $\frac{1+\lambda}{2}n+\min\{1,\theta_{11}\}>1$. Similarly, we also have

$$\begin{aligned}
&\|D^2v(t)\| \\
&\quad\lesssim \|D^2\mathcal{G}_{11}(t,0)v_0\|+\|D^2\mathcal{G}_{12}(t,0)u_0\|+\int_0^t\|D^2\mathcal{G}_{11}(t,s)Q_1(s)\|ds \\
&\qquad+\int_0^t\|D^2\mathcal{G}_{12}(t,s)Q_2(s)\|ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-(1+\lambda)}+\int_0^t\Gamma^{\frac{n}{2}+2}(t,s)\cdot(\|Q_1(s)\|_{L^1}+\|D^2Q_1(s)\|)ds \\
&\qquad+\int_0^t(1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot(\|Q_2(s)\|_{L^1}+\|D^2Q_2(s)\|)ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-(1+\lambda)}+E_n^2(t)\int_0^t\Gamma^{\frac{n}{2}+2}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{1,\theta_{21}\}}ds \\
&\qquad+E_n^2(t)\int_0^t(1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{\frac{1+\lambda}{2},\theta_{22}\}}ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-(1+\lambda)}+E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-(1+\lambda)},
\end{aligned}$$

provided that

$$\begin{cases}\frac{1+\lambda}{2}n+\min\{1,\theta_{21}\}>1, & \frac{1+\lambda}{2}n+\min\{1,\theta_{21}\}\geq\frac{1+\lambda}{4}n+(1+\lambda),\\ \frac{1+\lambda}{2}n+\min\{\frac{1+\lambda}{2},\theta_{22}\}-\lambda>1, & \frac{1+\lambda}{2}n+\min\{\frac{1+\lambda}{2},\theta_{22}\}-\lambda\geq\frac{1+\lambda}{4}n+(1+\lambda),\end{cases}\tag{2.31}$$

where we have also used the following estimates:

$$\begin{aligned}
&\|D^2Q_1(s)\|\lesssim\|uD^3v\|+\|DuD^2v\|+\|DvD^2u\|+\|vD^3u\|\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{21}},\\
&\|D^2Q_2(s)\|\lesssim\|uD^3u\|+\|DuD^2u\|+\|vD^3v\|+\|DvD^2v\|\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{22}}
\end{aligned}$$

with

$$\begin{aligned}
&\theta_{21}=\min\left\{\frac{1}{2}(1+\omega_2)+\theta_3,1+\lambda+\frac{1}{2}(\omega_2+\omega_3),1+\frac{1}{2}(1+\lambda+\theta_3),\omega_3+\frac{3}{4}(1+\lambda)\right\},\\
&\theta_{22}=\min\left\{\frac{1}{2}(1+\omega_2)+\omega_3,\omega_2+\frac{1}{2}(\omega_2+\omega_3),\theta_3+\frac{3}{4}(1+\lambda),1+\lambda+\frac{1}{2}(1+\lambda+\omega_3)\right\}.
\end{aligned}$$

The decay estimates on $\|\partial_x^\alpha v\|$ for $0\leq|\alpha|\leq[\frac{n}{2}]+1$ are based on the optimal decay estimates on $\|\partial_x^\alpha G_{11}(t,s)\|$ and $\|\partial_x^\alpha G_{12}(t,s)\|$ in (2.26). However, the estimates on $\|\partial_x^\alpha G_{21}(t,s)\|$ and $\|\partial_x^\alpha G_{22}(t,s)\|$ in (2.26) are insufficient for the optimal decay estimates on $\|\partial_x^\alpha \boldsymbol{u}\|$ for $0\leq|\alpha|\leq[\frac{n}{2}]$. In fact, we use the optimal decay estimates in (2.27) to show the decay estimates on $\|\partial_x^\alpha \boldsymbol{u}\|$ for $0\leq|\alpha|\leq[\frac{n}{2}]$ in a similar way as $\|\partial_x^\alpha v\|$ for $1\leq|\alpha|\leq[\frac{n}{2}]+1$. One can check that the condition on the estimate of $\|\partial_x^k \boldsymbol{u}\|$ for $0\leq k\leq[\frac{n}{2}]$ is equivalent to the condition on the estimate of $\|\partial_x^{k+1}v\|$. For example,

$$
\begin{aligned}
\|\boldsymbol{u}(t)\| &\lesssim \|\mathcal{G}_{21}(t,0)v_0\| + \|\mathcal{G}_{22}(t,0)\boldsymbol{u}_0\| + \int_0^t \|\mathcal{G}_{21}(t,s)Q_1(s)\|ds + \int_0^t \|\mathcal{G}_{22}(t,s)Q_2(s)\|ds \\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n} + \int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(\|Q_1(s)\|_{L^1}^l + \|DQ_1(s)\|^h)ds \\
&\quad + \int_0^t (1+t)^{\lambda}(1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot(\|Q_2(s)\|_{L^1}^l + \|DQ_2(s)\|^h)ds \\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}} + E_n^2(t)\int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-1}ds \\
&\quad + E_n^2(t)\int_0^t (1+t)^{\lambda}(1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\frac{1+\lambda}{2}}ds \\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}} + E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}.
\end{aligned}
$$

Further, we use the decay estimates in (2.28) to show the decay estimates on $\|\partial_x^\alpha \boldsymbol{u}\|$ for $[\frac{n}{2}]+1 \le |\alpha| \le [\frac{n}{2}]+2$ since the regularity required in (2.28) is one order lower than that in (2.27). We note that in this case the condition on the estimate of $\|\partial_x^k \boldsymbol{u}\|$ for $[\frac{n}{2}]+1 \le k \le [\frac{n}{2}]+2$ is similar to the condition on the estimate of $\|\partial_x^k v\|$. We have

$$
\begin{aligned}
\|D^3Q_1(s)\| &\lesssim \|uD^4v\| + \|DuD^3v\| + \|D^2uD^2v\| + \|DvD^3u\| + \|vD^4u\| \\
&\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{4}n-\theta_{31}}, \\
\|D^3Q_2(s)\| &\lesssim \|uD^4u\| + \cdots + \|D^2uD^2u\| + \|vD^4v\| + \cdots + \|D^2vD^2v\| \\
&\lesssim E_n^2(s)(1+s)^{-\frac{1+\lambda}{4}n-\theta_{32}}
\end{aligned}
$$

with

$$
\begin{aligned}
\theta_{31} = \min\Bigg\{&\frac{1}{2}(1+\omega_2), \frac{1}{2}(\omega_2+\omega_3) + \frac{1+\lambda}{4}n + \theta_3, \frac{1+\lambda}{8}n + 1 + \frac{\theta_3}{2}, \\
&\frac{1+\lambda}{4}n + \frac{1}{2}(1+\lambda+\theta_3) + \omega_3, \frac{3}{4}(1+\lambda)\Bigg\}, \\
\theta_{32} = \min\Bigg\{&\frac{1}{2}(1+\omega_2), \frac{1}{2}(\omega_2+\omega_3) + \frac{1+\lambda}{4}n + \omega_3, \frac{1+\lambda}{8}n + 1 + \frac{\omega_3}{2}, \\
&\frac{3}{4}(1+\lambda), \frac{1+\lambda}{4}n + \frac{1}{2}(1+\lambda+\theta_3) + \theta_3, \frac{1+\lambda}{8}n + 1 + \lambda + \frac{\theta_3}{2}\Bigg\}.
\end{aligned}
$$

Therefore, we arrive at

$$
\begin{aligned}
\|D^2u(t)\| &\lesssim \|D^2\mathcal{G}_{21}(t,0)v_0\| + \|D^2\mathcal{G}_{22}(t,0)u_0\| + \int_0^t \|D^2\mathcal{G}_{21}(t,s)Q_1(s)\|ds \\
&\quad + \int_0^t \|D^2\mathcal{G}_{22}(t,s)Q_2(s)\|ds \\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-(1+\lambda)+\lambda} + \int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot(\|Q_1(s)\|_{L^1} \\
&\quad + \|D^2Q_1(s)\|)ds \\
&\quad + \int_0^t \Big(\frac{1+t}{1+s}\Big)^{\lambda}\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot\Big((1+s)^{2\lambda}\cdot\Gamma^2(t,s) + \frac{1}{(1+s)^{\lambda-1}} + C_\kappa\Gamma^\kappa(t,s)\Big) \\
&\quad \cdot(\|Q_2(s)\|_{L^1} + \|D^2Q_2(s)\|)ds
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-1} + E_n^2(t)\int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{1,\theta_{21}\}}ds\\
&\quad + E_n^2(t)\int_0^t \Big(\frac{1+t}{1+s}\Big)^{\lambda}\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot\bigg((1+s)^{2\lambda}\cdot\Gamma^2(t,s)+\frac{1}{(1+s)^{\lambda-1}}+C_\kappa\Gamma^{\kappa}(t,s)\bigg)\\
&\quad \cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{\frac{1+\lambda}{2},\theta_{22}\}}ds\\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\omega_2} + E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-\omega_2},
\end{aligned}
$$

provided that

$$
\begin{cases}
\frac{1+\lambda}{2}n+\min\{1,\theta_{21}\}-\lambda \geq \frac{1+\lambda}{4}n+\omega_2,\\
\frac{1+\lambda}{2}n+\min\{\frac{1+\lambda}{2},\theta_{22}\} \geq \frac{1+\lambda}{4}n+\omega_2.
\end{cases} \tag{2.32}
$$

Furthermore, we similarly have

$$
\begin{aligned}
&\|D^3u(t)\|\\
&\quad\lesssim \|D^3\mathcal{G}_{21}(t,0)v_0\| + \|D^3\mathcal{G}_{22}(t,0)u_0\| + \int_0^t \|D^3\mathcal{G}_{21}(t,s)Q_1(s)\|ds\\
&\qquad + \int_0^t \|D^3\mathcal{G}_{22}(t,s)Q_2(s)\|ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{3}{2}(1+\lambda)+\lambda}\\
&\qquad + \int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+4}(t,s)\cdot(\|Q_1(s)\|_{L^1}+\|D^3Q_1(s)\|)ds\\
&\qquad + \int_0^t \Big(\frac{1+t}{1+s}\Big)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot\bigg((1+s)^{2\lambda}\cdot\Gamma^2(t,s)+\frac{1}{(1+s)^{\lambda-1}}+C_\kappa\Gamma^{\kappa}(t,s)\bigg)\\
&\qquad\quad \cdot(\|Q_2(s)\|_{L^1}+\|D^3Q_2(s)\|)ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{3}{2}(1+\lambda)+\lambda}\\
&\qquad + E_n^2(t)\int_0^t (1+t)^{\lambda}\cdot\Gamma^{\frac{n}{2}+4}(t,s)\cdot(1+s)^{-\min\{\frac{1+\lambda}{2}n+1,\frac{1+\lambda}{4}n+\theta_{31}\}}ds\\
&\qquad + E_n^2(t)\int_0^t \Big(\frac{1+t}{1+s}\Big)^{\lambda}\cdot\Gamma^{\frac{n}{2}+3}(t,s)\cdot\bigg((1+s)^{2\lambda}\cdot\Gamma^2(t,s)+\frac{1}{(1+s)^{\lambda-1}}+C_\kappa\Gamma^{\kappa}(t,s)\bigg)\\
&\qquad\quad \cdot(1+s)^{-\min\{\frac{1+\lambda}{2}n+\frac{1+\lambda}{2},\frac{1+\lambda}{4}n+\theta_{32}\}}ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\omega_3} + E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-\omega_3},
\end{aligned}
$$

provided that

$$
\begin{cases}
\frac{3}{2}(1+\lambda)-\lambda \geq \omega_3,\\
\frac{1+\lambda}{4}n+\theta_{31}-\lambda \geq \frac{1+\lambda}{4}n+\omega_3,\\
\frac{1+\lambda}{4}n+\theta_{32} \geq \frac{1+\lambda}{4}n+\omega_3.
\end{cases} \tag{2.33}
$$

Hence, the estimate on $\|D^3 v\|$ is

$$
\begin{aligned}
&\|D^3 v(t)\| \\
&\quad\lesssim \|D^3\mathcal{G}_{11}(t,0)v_0\| + \|D^3\mathcal{G}_{12}(t,0)u_0\| + \int_0^t \|D^3\mathcal{G}_{11}(t,s)Q_1(s)\|ds \\
&\qquad + \int_0^t \|D^3\mathcal{G}_{12}(t,s)Q_2(s)\|ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{3}{2}(1+\lambda)} \\
&\qquad + \int_0^t \Gamma^{\frac{n}{2}+3}(t,s)\cdot(\|Q_1(s)\|_{L^1} + \|D^3 Q_1(s)\|)ds \\
&\qquad + \int_0^t (1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+4}(t,s)\cdot(\|Q_2(s)\|_{L^1} + \|D^3 Q_2(s)\|)ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\frac{3}{2}(1+\lambda)} \\
&\qquad + E_n^2(t)\int_0^t \Gamma^{\frac{n}{2}+3}(t,s)\cdot(1+s)^{-\min\left\{\frac{1+\lambda}{2}n+1,\frac{1+\lambda}{4}n+\theta_{31}\right\}}ds \\
&\qquad + E_n^2(t)\int_0^t (1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+4}(t,s)\cdot(1+s)^{-\min\{\frac{1+\lambda}{2}n+\frac{1+\lambda}{2},\frac{1+\lambda}{4}n+\theta_{32}\}}ds \\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n-\theta_3} + E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n-\theta_3},
\end{aligned}
$$

under the condition

$$
\begin{cases}
\frac{3}{2}(1+\lambda)\ge\theta_3, \\
\frac{1+\lambda}{4}n+\theta_{31}+\frac{1+\lambda}{2}(\frac{n}{2}+3)-1\ge\frac{1+\lambda}{4}n+\theta_3, \\
\frac{1+\lambda}{4}n+\theta_{32}-\lambda>1.
\end{cases}
\tag{2.34}
$$

Combining the above conditions together, we fix $\theta_3=-\frac{1+\lambda}{3}$, $\omega_3=\frac{1+\lambda}{4}\cdot 3$, and $\omega_2=\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}$ for the case $n=3$. We note that the restriction on ω_3 is $(2.33)_3$ such that $\omega_3\le\theta_{32}$ and $\frac{1+\lambda}{4}n+\theta_{32}$ is the decay rate of $\|D^3Q_2(s)\|$, where the worst term (decaying slowest) is $\|vD^4v\|$ restricted by $\|v\|_{L^\infty}$. For general dimension n, we have

$$
\begin{aligned}
&\|D^{[\frac{n}{2}]+2}Q_2(s)\| \lesssim \|vD^{[\frac{n}{2}]+3}v(s)\| \lesssim \|v\|_{L^\infty}\cdot E_n(s) \\
&\quad\lesssim \begin{cases}
\|D^{[\frac{n}{2}]}v\|^{\frac{1}{2}}\cdot\|D^{[\frac{n}{2}]+1}v\|^{\frac{1}{2}}\cdot E_n(s) & \text{for odd } n, \\
\|D^{[\frac{n}{2}]-1}v\|^{\frac{1}{2}}\cdot\|D^{[\frac{n}{2}]+1}v\|^{\frac{1}{2}}\cdot E_n(s) & \text{for even } n,
\end{cases} \\
&\quad\lesssim \begin{cases}
E_n^2(s)\cdot\left((1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}[\frac{n}{2}]}\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}([\frac{n}{2}]+1)}\right)^{\frac{1}{2}} & \text{for odd } n, \\
E_n^2(s)\cdot\left((1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}([\frac{n}{2}]-1)}\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{2}([\frac{n}{2}]+1)}\right)^{\frac{1}{2}} & \text{for even } n,
\end{cases} \\
&\quad\approx E_n^2(s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda}{4}n}.
\end{aligned}
$$

Therefore, it suffices to take $\omega_{[\frac{n}{2}]+2}=\frac{1+\lambda}{4}n$ for general dimension of n. The condition $(2.31)_1$, which is necessary for the optimal decay of $\|D^2v\|$, is $\frac{1+\lambda}{2}n+\omega_3+\frac{1+\lambda}{4}\cdot 3>1$ for $n=3$ and is $\frac{1+\lambda}{2}n+\frac{1+\lambda}{4}n+\frac{1+\lambda}{4}n>1$ for general $n\ge 2$. That is, $(1+\lambda)n>1$, which is equivalent to $\lambda\in(-\frac{n-1}{n},0)$. The condition $\lambda\in(-\frac{n}{n+2},0)$ is stronger than $\lambda\in(-\frac{n-1}{n},0)$.

Step 2: High-order energy estimates. We note that the condition (2.7) in Lemma 2.1 under the a priori assumption (2.30) is $\frac{1+\lambda}{4}n>-\lambda$, which is true for $\lambda\in(-\frac{n}{n+4},0)$

and is false for $\lambda\in(-1,-\frac{n}{n+4})$. Fortunately, the condition (2.20) in Lemma 2.3 under the a priori assumption (2.30) is

$$\frac{1+\lambda}{4}n+\min\left\{\frac{1+\lambda}{2}-\lambda,\omega_{[n/2]+2}\right\}=\min\left\{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}-\lambda,\frac{1+\lambda}{2}n\right\}>-\lambda,$$

which is true for all $\lambda\in(-\frac{n}{n+2},0)$. Therefore, we can apply the high-order energy estimates of Lemma 2.2 to get

$$\|\nabla(v,\boldsymbol{u})\|^2_{H^{[\frac{n}{2}]+2}}\lesssim\|\nabla(v_0,\boldsymbol{u}_0)\|^2_{H^{[\frac{n}{2}]+2}}+\delta_0^2\omega^2(t), \tag{2.35}$$

where $\delta_0\omega(t)=(1+t)^{-\frac{1+\lambda}{2}n-\lambda}$ decays to zero.

Step 3*: Closure of the a priori estimate* (2.30). We now combine the above estimates and choose $\varepsilon_0>0$ and $\delta_0>0$ to be sufficiently small such that

$$C(\varepsilon_0+\delta_0^2+\delta_0\omega(t))\le\delta_0,$$

where $C>0$ is a universal constant. It suffices to choose $C\delta_0\le 1/4$, and $C\varepsilon_0=\delta_0/2$, and to consider the problem starting form t_0 such that $C\omega(t_0)\le 1/4$ since $\omega(t)$ decays to zero. We see that the a priori estimate (2.30) holds for all the time $t\in(0,+\infty)$.

Finally, we show that those estimates ($\|\partial_x^\alpha v\|$ with $0\le|\alpha|\le[\frac{n}{2}]+1$ and $\|\partial_x^\alpha\boldsymbol{u}\|$ with $0\le|\alpha|\le[\frac{n}{2}]$) are optimal. We take the estimate on $\|v\|$ for an example. According to the optimal decay estimates in Lemma 2.4 and the energy estimates in Step 1 before, we choose the initial data $(v_0,\boldsymbol{u}_0)$ such that $\|\mathcal{G}_{11}(t,0)v_0\|$ decays optimally; then we have

$$\|v(t)\|\gtrsim\|\mathcal{G}_{11}(t,0)v_0\|-\|\mathcal{G}_{12}(t,0)\boldsymbol{u}_0\|-\int_0^t\|\mathcal{G}_{11}(t,s)Q_1(s)\|ds-\int_0^t\|\mathcal{G}_{12}(t,s)Q_2(s)\|ds,$$

where $\|\mathcal{G}_{12}(t,0)\boldsymbol{u}_0\|$ decays faster than $\|\mathcal{G}_{11}(t,0)v_0\|$, and the energy $\int_0^t\|\mathcal{G}_{11}(t,s)Q_1(s)\|ds+\int_0^t\|\mathcal{G}_{12}(t,s)Q_2(s)\|ds$ decays no slower than $\|\mathcal{G}_{11}(t,0)v_0\|$. We note that $Q_1(t,x)$ and $Q_2(t,x)$ are quadratic, and we rescale the initial data as $(\varepsilon_1v_0,\varepsilon_1\boldsymbol{u}_0)$ with $\varepsilon_1>0$ sufficiently small such that neither $\int_0^t\|\mathcal{G}_{11}(t,s)Q_1(s)\|ds$ nor $\int_0^t\|\mathcal{G}_{12}(t,s)Q_2(s)\|ds$ is comparable with $\|\mathcal{G}_{11}(t,0)v_0\|$. In fact, according to the proof in Step 1, we have

$$\|\mathcal{G}_{11}(t,0)\varepsilon_1v_0\|\approx\varepsilon_0(1+t)^{-\frac{1+\lambda}{4}n}$$

and

$$\begin{aligned}&\int_0^t\|\mathcal{G}_{11}(t,s)Q_1(s)\|ds+\int_0^t\|\mathcal{G}_{12}(t,s)Q_2(s)\|ds\\&\qquad\lesssim E_n^2(t)(1+t)^{-\frac{1+\lambda}{4}n}\lesssim\delta_0^2(1+t)^{-\frac{1+\lambda}{4}n}\lesssim\varepsilon_0^2(1+t)^{-\frac{1+\lambda}{4}n},\end{aligned}$$

even though they are nonlinear. Therefore, $\|v(t)\|$ decays in the same order as $\|\mathcal{G}_{11}(t,0)v_0\|$. The proof is completed. ∎

2.3. Optimal L^q decay estimates. We now turn to the L^1–L^q decay estimates of the nonlinear system (1.3).

LEMMA 2.6. *For $q \in [2,\infty]$ and $\lambda \in (-1,0)$, then*

$$\|\partial_x^\alpha \mathcal{G}_{11}(t,s)\phi(x)\|_{L^q} \lesssim \Gamma^{\gamma_{1,q}+|\alpha|}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+\omega_{2,q}}\phi\|^h\right),$$

$$\|\partial_x^\alpha \mathcal{G}_{12}(t,s)\phi(x)\|_{L^q} \lesssim (1+s)^\lambda\cdot\Gamma^{\gamma_{1,q}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+\omega_{2,q}}\phi\|^h\right),$$

$$\|\partial_x^\alpha \mathcal{G}_{21}(t,s)\phi(x)\|_{L^q} \lesssim (1+t)^\lambda\cdot\Gamma^{\gamma_{1,q}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+\omega_{2,q}}\phi\|^h\right),$$

$$\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\|_{L^q} \lesssim \left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\gamma_{1,q}+|\alpha|}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+\omega_{2,q}}\phi\|^h\right),$$

where $\gamma_{1,q} := n(1-1/q)$, and $\omega_{2,q} > \gamma_{2,q} := n(1/2-1/q)$. Furthermore, it holds that

$$\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\|_{L^q} \lesssim (1+t)^\lambda(1+s)^\lambda\cdot\Gamma^{\gamma_{1,q}+|\alpha|+2}(t,s)\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+1+\omega_{2,q}}\phi\|^h\right),$$

$$\begin{aligned}\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| &\lesssim \left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\gamma_{1,q}+|\alpha|}(t,s)\\ &\quad\cdot\left((1+s)^{2\lambda}\cdot\Gamma^2(t,s) + \frac{1}{(1+s)^{\lambda-1}} + C_\kappa\Gamma^\kappa(t,s)\right)\\ &\quad\cdot\left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+1+\omega_{2,q}}\phi\|^h\right),\end{aligned}$$

where $\kappa \geq 2$ can be chosen arbitrarily large and $C_\kappa > 0$ is a constant depending on κ.

Proof. These estimates are conclusions of Theorem A.5 in the appendix. ∎

We prove the optimal L^q decay estimates Theorem 1.2 of the nonlinear system (1.3).

Proof of Theorem 1.2. Since $\lambda \in (-\frac{n}{n+2}, 0)$ satisfies the condition in Theorem 1.1, we see that the a priori assumption (2.30) in the proof of Theorem 1.1 is valid, which is based on the smallness of the initial data $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+3}} \leq \varepsilon_0$. Here under the stronger condition $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+k}} \leq \varepsilon_0$, we can enforce the decay estimates as follows. Denote the new weighted energy function by

$$\begin{aligned}F_n(\tilde{t}) := \sup_{t\in(0,\tilde{t})}\Bigg\{ &\sum_{0\leq|\alpha|\leq[n/2]+1}(1+t)^{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}|\alpha|}\|\partial_x^\alpha v\|,\\ &\sum_{0\leq|\alpha|\leq[n/2]}(1+t)^{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}(|\alpha|+1)-\lambda}\|\partial_x^\alpha \boldsymbol{u}\|,\\ &\sum_{|\alpha|=[n/2]+1}(1+t)^{\frac{1+\lambda}{4}n+\omega_{|\alpha|}}\|\partial_x^\alpha \boldsymbol{u}\|,\\ &\sum_{[n/2]+2\leq|\alpha|\leq[n/2]+k-1}(1+t)^{\frac{1+\lambda}{4}n+\theta_{|\alpha|}}\|\partial_x^\alpha v\|,\\ &\sum_{[n/2]+2\leq|\alpha|\leq[n/2]+k-1}(1+t)^{\frac{1+\lambda}{4}n+\omega_{|\alpha|}}\|\partial_x^\alpha \boldsymbol{u}\|, \sum_{|\alpha|=[n/2]+k}\|\partial_x^\alpha(v,\boldsymbol{u})\|\Bigg\},\end{aligned}$$

where $\omega_{|\alpha|}$ and $\theta_{|\alpha|}$ are constants depending on n and λ. We claim that under the small initial data condition $\|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^{[\frac{n}{2}]+k}} \leq \varepsilon_0$, there holds

$$F_n(\tilde{t}) \lesssim \delta_0 \quad \forall \tilde{t} \in (0,T), \tag{2.36}$$

where $\varepsilon_0 > 0$ and $\delta_0 > 0$ are small constants to be determined.

We take, for example, the case $n=3$ again. Note that for $n=3$, $\gamma_{2,q}=3(1/2-1/q)\leq 3/2<2$. We take $k=3+[\gamma_{2,q}]=4$ and $\omega_{2,q}=2>\gamma_{2,q}$. We prove the estimate on $\|\partial_x^\alpha v\|_{L^q}$ with $|\alpha|=1$ in (1.6). According to the Duhamel principle (2.2) and the L^1–L^q decay estimates of the Green matrix in Lemma 2.6, we have

$$\begin{aligned}
&\|Dv(t)\|_{L^q}\\
&\quad\lesssim \|D\mathcal{G}_{11}(t,0)v_0\|_{L^q}+\|D\mathcal{G}_{12}(t,0)u_0\|_{L^q}\\
&\qquad+\int_0^t\|D\mathcal{G}_{11}(t,s)Q_1(s)\|_{L^q}ds+\int_0^t\|D\mathcal{G}_{12}(t,s)Q_2(s)\|_{L^q}ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}}+\int_0^t\Gamma^{\gamma_{1,q}+1}(t,s)\cdot(\|Q_1(s)\|_{L^1}+\|D^{1+\omega_{2,q}}Q_1(s)\|)ds\\
&\quad+\int_0^t(1+s)^{\lambda}\cdot\Gamma^{\gamma_{1,q}+2}(t,s)\cdot(\|Q_2(s)\|_{L^1}+\|D^{1+\omega_{2,q}}Q_2(s)\|)ds\\
&\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}}+F_n^2(t)\int_0^t\Gamma^{\gamma_{1,q}+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{1,\theta_{31}\}}ds\\
&\qquad+F_n^2(t)\int_0^t(1+s)^{\lambda}\cdot\Gamma^{\gamma_{1,q}+2}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{2}n-\min\{\frac{1+\lambda}{2},\theta_{32}\}}ds\\
&\quad\lesssim \varepsilon_0(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}}+F_n^2(t)(1+t)^{-\frac{1+\lambda}{2}\gamma_{1,q}-\frac{1+\lambda}{2}}
\end{aligned}$$

with $\omega_{2,q}=2>\gamma_{2,q}$ provided that

(2.37)
$$\begin{cases}\frac{1+\lambda}{2}n+\min\{1,\theta_{31}\}\geq\frac{1+\lambda}{2}\gamma_{1,q}+\frac{1+\lambda}{2}, & \frac{1+\lambda}{2}n+\min\{1,\theta_{31}\}>1,\\ \frac{1+\lambda}{2}n+\min\{\frac{1+\lambda}{2},\theta_{32}\}-\lambda\geq\frac{1+\lambda}{2}\gamma_{1,q}+\frac{1+\lambda}{2}, & \frac{1+\lambda}{2}n+\min\{\frac{1+\lambda}{2},\theta_{32}\}-\lambda>1.\end{cases}$$

Here, $\frac{1+\lambda}{2}n+\theta_{31}$ and $\frac{1+\lambda}{2}n+\theta_{32}$ are the decay rates of $\|D^3Q_1\|$ and $\|D^3Q_2\|$ under the a priori assumption (2.36) (which is stronger than (2.30)) such that

$$\begin{aligned}
\|D^3Q_1(s)\|&\lesssim\|uD^4v\|+\|DuD^3v\|+\|D^2uD^2v\|+\|DvD^3u\|+\|vD^4u\|\\
&\lesssim F_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{31}},\\
\|D^3Q_2(s)\|&\lesssim\|uD^4u\|+\cdots+\|D^2uD^2u\|+\|vD^4v\|+\cdots+\|D^2vD^2v\|\\
&\lesssim F_n^2(s)(1+s)^{-\frac{1+\lambda}{2}n-\theta_{32}}
\end{aligned}$$

with

$$\begin{aligned}
\theta_{31}=\min\Big\{&\frac{1}{2}(1+\omega_2)+\theta_4,\frac{1}{2}(\omega_2+\omega_3)+\theta_3,\omega_2+\frac{\theta_3+\theta_4}{2},\\
&\frac{1}{2}(1+\lambda+\theta_3)+\omega_3,\frac{3}{4}(1+\lambda)+\omega_4\Big\},\\
\theta_{32}=\min\Big\{&\frac{1}{2}(1+\omega_2)+\omega_4,\frac{1}{2}(\omega_2+\omega_3)+\omega_3,\omega_2+\frac{1}{2}(\omega_3+\omega_4),\\
&\frac{3}{4}(1+\lambda)+\theta_4,\frac{1}{2}(1+\lambda+\theta_3)+\theta_3,1+\lambda+\frac{1}{2}(\theta_3+\theta_4)\Big\}.
\end{aligned}$$

Similar to the proof of Theorem 1.1, where $\|D^3Q\|$ decays at the same rate as $\|(v,u)\|_{L^\infty}$ since the energy $\|D^4(v,u)\|$ is only bounded, here $\|D^4Q\|$ decays at the same rate as $\|(v,u)\|_{L^\infty}$ due to the boundedness of $\|D^5(v,u)\|$ and we take $\omega_4=\frac{1+\lambda}{4}n$. Then (2.37) is equivalent to $\frac{1+\lambda}{2}n+\frac{1+\lambda}{4}n+\frac{1+\lambda}{4}n>1$, that is, $(1+\lambda)n>1$. The condition $\lambda>-\frac{n}{n+2}$ is stronger than $(1+\lambda)n>1$ for all $n\geq 2$.

The high-order energy estimate is similar to Step 2 in the proof of Theorem 1.1, where the restriction is the condition (2.20) in Lemma 2.3. Now it reads as

$$\frac{1+\lambda}{4}n+\min\left\{\frac{1+\lambda}{2}-\lambda,\omega_{[n/2]+k-2}\right\}=\min\left\{\frac{1+\lambda}{4}n+\frac{1+\lambda}{2}-\lambda,\frac{1+\lambda}{2}n\right\}>-\lambda$$

under the a priori assumption (2.36) with $\omega_{[n/2]+k-2}=\frac{1+\lambda}{4}n$. It suffices to set $\lambda\in(-\frac{n}{n+2},0)$. ∎

3. Time-weighted iteration scheme.

In this section we develop a new technique which is the artful combination of the time-weighted energy method and the Green function method to formulate the decay estimates of the overdamped Euler equation. As shown in the above section, the Green function method is powerful in the optimal decay estimates of the low-order energies but may have some trouble for the high-order energies. Meanwhile, the classical weighted energy method is suitable for high-order energy estimates but the decay rates are generally not optimal. Therefore, we combine these two methods together.

Denote as before $b(t)=\frac{\mu}{(1+t)^\lambda}$ with $\mu>0$ and $\lambda\in[-1,0)$, and

$$Q_1(t,x)=-\boldsymbol{u}\cdot\nabla v-\varpi v\nabla\cdot\boldsymbol{u},\qquad Q_2(t,x)=-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\varpi v\nabla v.$$

We may write $\boldsymbol{u}$ as u in the proof of this section for convenience. Rewrite the nonlinear system into nonlinear wave equations

$$\partial_t^2v-\Delta v+b(t)\cdot\partial_tv=\partial_tQ_1+b(t)\cdot Q_1-\nabla\cdot Q_2 \tag{3.1}$$

and

$$\partial_t^2\boldsymbol{u}-\Delta\boldsymbol{u}+\partial_t(b(t)\cdot\boldsymbol{u})=\partial_tQ_2-\nabla Q_1. \tag{3.2}$$

3.1. Time-weighted energy estimates.

The main idea of the time-weighted iteration scheme is to sacrifice the decay estimates of the low-order energies (i.e., $\|\partial_t^j\partial_x^k(v,\boldsymbol{u})\|$ with $j=0,1$ and $k+j=m\geq 0$) for better decay rates of high-order energies (i.e., $\|\partial_t^j\partial_x^k(v,\boldsymbol{u})\|$ with $j=0,1$ and $k+j=m+1$) in the time-weighted energy estimates, and the optimal decay rates of the basic energy $\|(v,\boldsymbol{u})\|$ are closed through the Green function method, where those better decays of high-order energies are necessary.

We have the following time-weighted energy estimates for $\lambda\in(-1,0)$ (the critical case of $\lambda=-1$ will be treated separately in the next section). Note that we replace the small negative constant in the classical time-weighted energy method by a small positive constant δ, such that the high-order energies are decaying better but the estimates on the low-order energies are absent.

LEMMA 3.1. *For any nonnegative integer k, $\lambda\in(-1,0)$, $\delta\in(0,\frac{1+\lambda}{4})$, and $|\alpha|=k$, there hold*

(3.3)
$$\frac{d}{dt}\int E^v(\partial_t\partial_x^\alpha v,\nabla\partial_x^\alpha v,\partial_x^\alpha v)+\int\left[(1+t)^{1+\delta}|\partial_t\partial_x^\alpha v|^2+(1+t)^{\lambda+\delta}|\nabla\partial_x^\alpha v|^2\right]$$
$$\lesssim\int(1+t)^{\delta-1}(\partial_x^\alpha v)^2+\int\partial_x^\alpha(\partial_tQ_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot((1+t)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v+\mu_1(1+t)^{\lambda+\delta}\partial_x^\alpha v)$$

and

(3.4)
$$\frac{d}{dt}\int E^u(\partial_t\partial_x^\alpha \boldsymbol{u},\nabla\partial_x^\alpha \boldsymbol{u},\partial_x^\alpha \boldsymbol{u})+\int\left[(1+t)^{1-2\lambda+\delta}|\partial_t\partial_x^\alpha \boldsymbol{u}|^2+(1+t)^{-\lambda+\delta}|\nabla\partial_x^\alpha \boldsymbol{u}|^2\right]$$
$$\lesssim\int(1+t)^{-2\lambda+\delta-1}|\partial_x^\alpha \boldsymbol{u}|^2+\int\partial_x^\alpha(\partial_tQ_2-\nabla Q_1)\cdot((1+t)^{1-\lambda+\delta}\partial_t\partial_x^\alpha \boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\partial_x^\alpha \boldsymbol{u}),$$

where $\mu_1>0$ and $\mu_2>0$ are constants and

$$E^v(\partial_t\partial_x^\alpha v,\nabla\partial_x^\alpha v,\partial_x^\alpha v)\approx(1+t)^{1+\lambda+\delta}(|\partial_t\partial_x^\alpha v|^2+|\nabla\partial_x^\alpha v|^2)+(1+t)^{\delta}(\partial_x^\alpha v)^2,$$
$$E^u(\partial_t\partial_x^\alpha \boldsymbol{u},\nabla\partial_x^\alpha \boldsymbol{u},\partial_x^\alpha \boldsymbol{u})\approx(1+t)^{1-\lambda+\delta}(|\partial_t\partial_x^\alpha \boldsymbol{u}|^2+|\nabla\partial_x^\alpha \boldsymbol{u}|^2)+(1+t)^{-2\lambda+\delta}|\partial_x^\alpha \boldsymbol{u}|^2.$$

Proof. Multiplying (3.1) by $(1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v$ with $\delta\in(0,\frac{1+\lambda}{4})$ and $\mu_1>0$, we have (similar to Proposition A.1 in Appendix A of [45])

$$\begin{aligned}\frac{d}{dt}\int&\big[(1+t)^{1+\lambda+\delta}(|\partial_t v|^2+|\nabla v|^2)+2\mu_1(1+t)^{\lambda+\delta}v\partial_t v\\&+(\mu_1b(t)(1+t)^{\lambda+\delta}-(\lambda+\delta)\mu_1(1+t)^{\lambda+\delta-1})v^2\big]\\&+\int\big[(-(1+\lambda+\delta)(1+t)^{\lambda+\delta}+2b(t)(1+t)^{1+\lambda+\delta}-2\mu_1(1+t)^{\lambda+\delta})|\partial_t v|^2\\&+(-(1+\lambda+\delta)(1+t)^{\lambda+\delta}+2\mu_1(1+t)^{\lambda+\delta})|\nabla v|^2\big]\\&+\int((\lambda+\delta)(\lambda+\delta-1)\mu_1(1+t)^{\lambda+\delta-2}-\partial_t(\mu_1b(t)(1+t)^{\lambda+\delta}))v^2\\=&\,2\int(\partial_tQ_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot\left((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\right),\end{aligned}$$

which can be simplified as

$$\frac{d}{dt}\int E^v(\partial_t v,\nabla v,v)+\int\left[(1+t)^{1+\delta}|\partial_t v|^2+(2\mu_1-(1+\lambda+\delta))(1+t)^{\lambda+\delta}|\nabla v|^2\right]$$
$$\lesssim\int(1+t)^{\delta-1}v^2+\int(\partial_tQ_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot\left((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\right),$$

where

$$\begin{aligned}E^v(\partial_t v,\nabla v,v):=&\,(1+t)^{1+\lambda+\delta}(|\partial_t v|^2+|\nabla v|^2)+2\mu_1(1+t)^{\lambda+\delta}v\partial_t v\\&+(\mu_1b(t)(1+t)^{\lambda+\delta}-(\lambda+\delta)\mu_1(1+t)^{\lambda+\delta-1})v^2\\\approx&\,(1+t)^{1+\lambda+\delta}(|\partial_t v|^2+|\nabla v|^2)+(1+t)^{\delta}v^2.\end{aligned}$$

Here we fix μ_1 such that $\mu_1\geq 1+\lambda+\delta>0$.

Next, multiplying (3.2) by $(1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}$ with $\delta\in(0,\frac{1+\lambda}{4})$ and $\mu_2>0$, we have

$$\begin{aligned}&\frac{d}{dt}\int\big[(1+t)^{1-\lambda+\delta}(|\partial_t\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+2\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\\&\quad+(\mu_2b(t)(1+t)^{-\lambda+\delta}-(-\lambda+\delta)\mu_2(1+t)^{-\lambda+\delta-1}+b'(t)\cdot(1+t)^{1-\lambda+\delta})|\boldsymbol{u}|^2\big]\\&\quad+\int\big[(-(1-\lambda+\delta)(1+t)^{-\lambda+\delta}+2b(t)(1+t)^{1-\lambda+\delta}-2\mu_2(1+t)^{-\lambda+\delta})|\partial_t\boldsymbol{u}|^2\\&\quad+(-(1-\lambda+\delta)(1+t)^{-\lambda+\delta}+2\mu_2(1+t)^{-\lambda+\delta})|\nabla\boldsymbol{u}|^2\big]\\&\quad+\int((-\lambda+\delta)(-\lambda+\delta-1)\mu_2(1+t)^{-\lambda+\delta-2}-\partial_t(\mu_2b(t)(1+t)^{-\lambda+\delta})\\&\quad+2\mu_2b'(t)(1+t)^{-\lambda+\delta}-\partial_t(b'(t)(1+t)^{1-\lambda+\delta}))|\boldsymbol{u}|^2\\&=2\int(\partial_tQ_2-\nabla Q_1)\cdot((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}).\end{aligned}$$

We simplify the above equality as

$$\begin{aligned}&\frac{d}{dt}\int E^u(\partial_t\boldsymbol{u},\nabla\boldsymbol{u},\boldsymbol{u})+\int\big[(1+t)^{1-2\lambda+\delta}|\partial_t\boldsymbol{u}|^2+(2\mu_2-(1-\lambda+\delta))(1+t)^{-\lambda+\delta}|\nabla\boldsymbol{u}|^2\big]\\&\lesssim\int((\mu_2-\lambda)\delta+2\lambda^2)(1+t)^{-2\lambda+\delta-1}|\boldsymbol{u}|^2\\&\quad+\int(\partial_tQ_2-\nabla Q_1)\cdot((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}),\end{aligned}$$

where

$$\begin{aligned}E^u(\partial_t\boldsymbol{u},\nabla\boldsymbol{u},\boldsymbol{u})&:=(1+t)^{1-\lambda+\delta}(|\partial_t\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+2\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\\&\quad+(\mu_2b(t)(1+t)^{-\lambda+\delta}-(-\lambda+\delta)\mu_2(1+t)^{-\lambda+\delta-1}+b'(t)\\&\quad\cdot(1+t)^{1-\lambda+\delta})|\boldsymbol{u}|^2\\&\approx(1+t)^{1-\lambda+\delta}(|\partial_t\boldsymbol{u}|^2+|\nabla\boldsymbol{u}|^2)+(1+t)^{-2\lambda+\delta}|\boldsymbol{u}|^2.\end{aligned}$$

We choose $\mu_2>0$ such that $\mu_2\geq 1-\lambda+\delta$. Thus, the proof for the case of $k=0$ is completed.

Differentiating ∂_x^α (3.1) and ∂_x^α (3.2), and multiplying the resulting equations by $(1+t)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v+\mu_1(1+t)^{\lambda+\delta}\partial_x^\alpha v$ and $(1+t)^{1-\lambda+\delta}\partial_t\partial_x^\alpha\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\partial_x^\alpha\boldsymbol{u}$, respectively, we can prove (3.3) and (3.4) in a similar procedure. The details are omitted. ◻

Remark 3.2. Compared with the multiplier method developed by Todorova and Yordanov [45] for the wave equation with variable coefficients ($b(t)=\frac{\mu}{(1+t)^\lambda}$ replaced by $\frac{\mu}{(1+|x|)^\alpha}$ with $\alpha\in(0,1)$) and the weighted energy method employed by Pan [37] for the wave equation with underdamping with $\lambda\in(0,1)$, here for overdamping with $\lambda\in[-1,0)$ we take the weights only dependent on time. The reason is that for the overdamping case, the simple weights depending on time can take advantage of the time-asymptotically growing overdamping, which turns out to be sufficient for the closure of the decay estimates for all $\lambda\in(-1,0)$.

Remark 3.3. The energy estimates (3.3) and (3.4) are deduced by rewriting both v and $\boldsymbol{u}$ as solutions to time-dependent damped nonlinear wave equations. This differs from the approach in [37] for the underdamping case, where the estimates of $\boldsymbol{u}$ are formulated according to $(1.3)_2$. Here for the overdamping case we cannot apply the above procedure in [37] since the estimates on $\|\partial_x^k \boldsymbol{u}\|$ depend on at least one of $\|\partial_x^{k+1} v\|$ and $\|\partial_x^{k+1}\boldsymbol{u}\|$, and other efforts should be made for the closure of the weighted energy estimates.

We define the following time-weighted energy functions for $N \geq [\frac{n}{2}]+2$ and $0 \leq k \leq N-1$,

$$\Phi_{k+1}(T) := \sup_{t\in(0,T)} \left\{ \sum_{|\alpha|=k} \left[(1+t)^{1+\lambda+\delta} \int (|\partial_t \partial_x^\alpha v|^2 + |\nabla \partial_x^\alpha v|^2) \right.\right. \tag{3.5}$$

$$\left.\left. + (1+t)^{1-\lambda+\delta} \int (|\partial_t \partial_x^\alpha \boldsymbol{u}|^2 + |\nabla \partial_x^\alpha \boldsymbol{u}|^2) \right] \right\}^{\frac{1}{2}} \tag{3.6}$$

and

$$\begin{aligned} \Psi_{k+1}(T) := \sup_{t\in(0,T)} \Bigg\{ \sum_{|\alpha|=k} \Bigg[&\int \Big((1+t)^{1+\delta} |\partial_t \partial_x^\alpha v|^2 + (1+t)^{\lambda+\delta} |\nabla \partial_x^\alpha v|^2 \Big) \\ &+ \int \left[(1+t)^{1-2\lambda+\delta} |\partial_t \partial_x^\alpha \boldsymbol{u}|^2 + (1+t)^{-\lambda+\delta} |\nabla \partial_x^\alpha \boldsymbol{u}|^2 \right] \Bigg] \Bigg\}^{\frac{1}{2}}, \end{aligned} \tag{3.7}$$

which satisfies $\Psi_{k+1}^2(t) \geq (1+t)^{-1} \cdot \Phi_{k+1}^2(t)$. We may assume that $\Phi_{k+1}(T) \geq \Phi_k(T)$ for all $k \geq 1$ and T. Otherwise, we can modify the definition of $\Phi_{k+1}(T)$. The energy function $\Phi_{k+1}(T)$ is defined according to the time-weighted energy estimates in Lemma 3.1, but the decay estimates on $\|v\|$ and $\|\boldsymbol{u}\|$ are absent and insufficient for the closure of the energy estimates. Additionally, we define the following weighted energy function:

$$\Psi_0(T) := \sup_{t\in(0,T)} \left\{ (1+t)^{\frac{1+\lambda}{4}n} \|v\|, (1+t)^{\frac{1+\lambda}{4}n + \frac{1-\lambda}{2}} \|\boldsymbol{u}\| \right\}. \tag{3.8}$$

The energy estimates in $\Psi_0(T)$ will be closed through the Green function method instead of the time-weighted energy method. There holds

$$\|(v_0, \boldsymbol{u}_0)\|_{H^N} \approx \sum_{k=1}^N \Phi_k(0) + \Psi_0(0) \approx \Phi_N(0) + \Psi_0(0). \tag{3.9}$$

According to the Sobolev embedding theorem, we have

$$\begin{aligned} &(1+t)^{\frac{1+\lambda+\delta}{2}} \|\partial_x^j v\|_{L^\infty} + (1+t)^{\frac{1-\lambda+\delta}{2}} \|\partial_x^j \boldsymbol{u}\|_{L^\infty} \\ &\quad \lesssim \max_{1\leq k\leq [\frac{n}{2}]+2} \Phi_k(t) \lesssim \Phi_N(t), \quad 0 \leq j \leq 1, n \geq 3, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &(1+t)^{\frac{1+\lambda}{2}+\frac{\delta}{4}} \|v\|_{L^\infty} + (1+t)^{\frac{1-\lambda}{2}+\frac{1+\lambda+\delta}{4}} \|\boldsymbol{u}\|_{L^\infty} + (1+t)^{\frac{1+\lambda+\delta}{2}} \|\partial_x v\|_{L^\infty} \\ &\quad + (1+t)^{\frac{1-\lambda+\delta}{2}} \|\partial_x \boldsymbol{u}\|_{L^\infty} \\ &\lesssim \max_{1\leq k\leq [\frac{n}{2}]+2} \Phi_k(t) + \Psi_0(t) \lesssim \Phi_N(t) + \Psi_0(t), \quad n = 2. \end{aligned} \tag{3.11}$$

We have the following iteration scheme based on Lemma 3.1.

LEMMA 3.4 (time-weighted iteration scheme). *For $\lambda \in (-1,0)$ and $\delta \in (0, \frac{1+\lambda}{4})$, there holds*

$$\begin{aligned}\Phi_1^2(t)+\int_0^t \Psi_1^2(s)ds \lesssim \Phi_1^2(0) + \int_0^t (1+s)^{-1-\frac{1+\lambda}{2}n+\delta} \cdot \Psi_0^2(s)ds \\ + \int_0^t \int (\partial_t Q_1 + b(s)\cdot Q_1 - \nabla\cdot Q_2)\cdot((1+s)^{1+\lambda+\delta}\partial_t v + \mu_1(1+s)^{\lambda+\delta} v)ds \\ + \int_0^t \int (\partial_t Q_2 - \nabla Q_1)\cdot((1+s)^{1-\lambda+\delta}\partial_t \boldsymbol{u} + \mu_2(1+s)^{-\lambda+\delta}\boldsymbol{u})ds.\end{aligned} \tag{3.12}$$

For any integer $k \geq 1$, there holds

$$\begin{aligned}\Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s)ds \lesssim \Phi_{k+1}^2(0) + \int_0^t (1+s)^{-1-\lambda}\cdot \Psi_k^2(s)ds \\ + \sum_{|\alpha|=k}\int_0^t \int \partial_x^\alpha(\partial_t Q_1 + b(s)\cdot Q_1 - \nabla\cdot Q_2)\cdot((1+s)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v + \mu_1(1+s)^{\lambda+\delta}\partial_x^\alpha v)ds \\ + \sum_{|\alpha|=k}\int_0^t \int \partial_x^\alpha(\partial_t Q_2 - \nabla Q_1)\cdot((1+s)^{1-\lambda+\delta}\partial_t\partial_x^\alpha \boldsymbol{u} + \mu_2(1+s)^{-\lambda+\delta}\partial_x^\alpha\boldsymbol{u})ds.\end{aligned} \tag{3.13}$$

Proof. This is a simple conclusion of Lemma 3.1 with the notation $\Phi_k(t)$, $\Psi_k(t)$, and $\Psi_0(t)$ defined by (3.5), (3.7), and (3.8). ∎

3.2. A priori estimates involving inhomogeneous terms. We estimate the inhomogeneous terms in the inequalities (3.3) and (3.4) in Lemma 3.1. We first consider the case of $k=0$ and in order to extend the proof to a general case of $k>0$ we should avoid directly using the energy estimates of the second order derivatives (such as $\|\partial_t\nabla v\|$) in $\Phi_k(t)$, since that would be $(k+2)$th order derivatives for general $k>0$ and cause trouble in the closure of the weighted energy estimates.

LEMMA 3.5. *There holds, for $\lambda \in (-1,0)$ and $\delta \in (0, \frac{1+\lambda}{4})$, that*

$$\begin{aligned}\int (\partial_t Q_1 + b(t)\cdot Q_1 - \nabla\cdot Q_2)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v + \mu_1(1+t)^{\lambda+\delta}v\big) \\ + \int(\partial_t Q_2 - \nabla Q_1)\cdot((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u} + \mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u}) \\ \lesssim \partial_t J_1(t) + (\Psi_0(t)+\Phi_N(t))\cdot\Psi_1^2(t) + \Phi_N(t)\cdot\Psi_0^2(t)\cdot(1+t)^{-1-\frac{1+\lambda}{4}},\end{aligned}$$

provided that $\|v\|_{L^\infty} \leq \frac{1}{\gamma-1}$ (which is valid under the a priori assumption $\Phi_N(t) + \Psi_0(t) \leq \delta_0$ with a small constant δ_0), where

$$J_1(t) \lesssim \|v\|_{L^\infty}\cdot\Phi_1^2(t).$$

Proof. The estimates of the two integrals are separated into two steps.

Step I. We first estimate the term involving $b(t)\cdot Q_1$ as follows:

$$\begin{aligned}
&\int b(t)\cdot Q_1\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad\lesssim\int(|\boldsymbol{u}\cdot\nabla v|+|v\nabla\cdot\boldsymbol{u}|)\cdot((1+t)^{1+\delta}|\partial_t v|+(1+t)^{\delta}|v|)\\
&\quad\lesssim\int(\|\nabla\boldsymbol{u}\|_{L^\infty}\cdot|v|+\|\nabla v\|_{L^\infty}\cdot|\boldsymbol{u}|)\cdot((1+t)^{1+\delta}|\partial_t v|+(1+t)^{\delta}|v|)\\
&\quad\lesssim\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}(1+t)^{1+\delta}\int|v||\partial_t v|+\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}(1+t)^{\delta}\int v^2\\
&\qquad+\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}(1+t)^{1+\delta}\int|\boldsymbol{u}||\partial_t v|+\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}(1+t)^{\delta}\int|\boldsymbol{u}||v|\\
&\quad\lesssim\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}(1+t)^{1+\delta}\cdot\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}}\cdot\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}\\
&\qquad+\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}(1+t)^{\delta}\cdot\Psi_0^2(t)(1+t)^{-\frac{1+\lambda}{2}n}\\
&\qquad+\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}(1+t)^{1+\delta}\cdot\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}}\cdot\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}\\
&\qquad+\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}(1+t)^{\delta}\cdot\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}\cdot\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}\\
&\quad\lesssim\Phi_N(t)\Psi_1^2(t)+\Phi_N(t)\Psi_0^2(t)(1+t)^{\lambda-\frac{1+\lambda}{2}n}+\Phi_N(t)\Psi_0^2(t)(1+t)^{-\frac{1-\lambda}{2}-\frac{1+\lambda}{2}n+\frac{\delta}{2}}\\
&\qquad+\Phi_N(t)\Psi_0^2(t)(1+t)^{-1-\frac{1+\lambda}{2}n}+\Phi_N(t)\Psi_0^2(t)(1+t)^{-1-\frac{1+\lambda}{2}n+\frac{\delta}{2}},
\end{aligned}$$

where

$$\begin{cases}\frac{1-\lambda}{2}+\frac{1+\lambda}{2}n-\frac{\delta}{2}\ge\frac{1-\lambda}{2}+\frac{1+\lambda}{2}+\frac{1+\lambda}{2}-\frac{\delta}{2}\ge 1+\frac{1+\lambda}{4},\\ 1+\frac{1+\lambda}{2}n\ge 1+\frac{1+\lambda}{4},\\ 1+\frac{1+\lambda}{2}n-\frac{\delta}{2}\ge 1+\frac{1+\lambda}{4},\end{cases}$$

for all $n\ge 2$ and $\lambda\in(-1,0)$, and

$$-\lambda+\frac{1+\lambda}{2}n=1+\frac{1+\lambda}{2}(n-2)>1+\frac{1+\lambda}{4}$$

for $n\ge 3$. For the case of $n=2$, we modify the above estimate (replacing the inequality (3.10) by (3.11)) as

$$\begin{aligned}
&\int|v\nabla\cdot\boldsymbol{u}|\cdot(1+t)^{1+\delta}|\partial_t v|\\
&\quad\lesssim\int\|v\|_{L^\infty}|\nabla\cdot\boldsymbol{u}|\cdot(1+t)^{1+\delta}|\partial_t v|\\
&\quad\lesssim(\Psi_0(t)+\Phi_N(t))(1+t)^{-\frac{1+\lambda}{2}-\frac{\delta}{4}}\cdot(1+t)^{1+\delta}\cdot\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}}\cdot\Psi_1(t)(1+t)^{-\frac{-\lambda+\delta}{2}}\\
&\quad=(\Psi_0(t)+\Phi_N(t))\Psi_1^2(t)(1+t)^{-\frac{\delta}{4}}.
\end{aligned}$$

Next, we calculate the term involving $\partial_t Q_1$ as follows:

$$\begin{aligned}
&\int\partial_t Q_1\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad=\int(-\partial_t\boldsymbol{u}\cdot\nabla v-\varpi\partial_t v\nabla\cdot\boldsymbol{u})\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\qquad+\int(-\boldsymbol{u}\cdot\nabla\partial_t v)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\qquad+\int(-\varpi v\nabla\cdot\partial_t\boldsymbol{u})\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad=:I_{11}+I_{12}+I_{13}.
\end{aligned}$$

We have

$$\begin{aligned}
I_{11} &\lesssim (\|\partial_t \boldsymbol{u}\| \cdot \|\nabla v\|_{L^\infty} + \|\partial_t v\| \cdot \|\nabla\cdot \boldsymbol{u}\|_{L^\infty}) \cdot ((1+t)^{1+\lambda+\delta}\|\partial_t v\| + (1+t)^{\lambda+\delta}\|v\|) \\
&\lesssim (\Psi_1(t)(1+t)^{-\frac{1-2\lambda+\delta}{2}} \cdot \Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}} \\
&\qquad + \Psi_1(t)(1+t)^{-\frac{1+\delta}{2}} \cdot \Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}) \\
&\qquad \cdot ((1+t)^{1+\lambda+\delta}\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}} + (1+t)^{\lambda+\delta}\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}) \\
&\lesssim \Phi_N(t)\Psi_1^2(t)(1+t)^{-\frac{1+\delta}{2}+\frac{3}{2}\lambda} + \Phi_N(t)\Psi_1^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda} \\
&\qquad + \Phi_N(t)\Psi_0^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda}.
\end{aligned}$$

The crucial point in the estimates of I_{12} and I_{13} is to avoid the direct estimates on $\nabla\partial_t v$ and $\nabla\cdot\partial_t \boldsymbol{u}$ through integration by parts such that

$$\begin{aligned}
I_{12} &= -\frac{1}{2}(1+t)^{1+\lambda+\delta}\int \boldsymbol{u}\cdot\nabla(\partial_t v)^2 - \mu_1(1+t)^{\lambda+\delta}\int v\boldsymbol{u}\cdot\nabla\partial_t v \\
&= \frac{1}{2}(1+t)^{1+\lambda+\delta}\int(\nabla\cdot\boldsymbol{u})(\partial_t v)^2 + \mu_1(1+t)^{\lambda+\delta}\int \partial_t v\cdot(\boldsymbol{u}\cdot\nabla v + v\nabla\cdot\boldsymbol{u}) \\
&\lesssim (1+t)^{1+\lambda+\delta}\|\nabla\cdot\boldsymbol{u}\|_{L^\infty}\|\partial_t v\|^2 + (1+t)^{\lambda+\delta}\|\partial_t v\|\cdot(\|\boldsymbol{u}\|\cdot\|\nabla v\|_{L^\infty} + \|v\|\cdot\|\nabla\cdot\boldsymbol{u}\|_{L^\infty}) \\
&\lesssim (1+t)^{1+\lambda+\delta}\cdot\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}\cdot\Psi_1^2(t)(1+t)^{-(1+\delta)} \\
&\quad + (1+t)^{\lambda+\delta}\cdot\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}}\cdot\Bigg(\Phi_0(t)(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}\cdot\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}} \\
&\quad + \Phi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}\cdot\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}\Bigg) \\
&\lesssim \Phi_N(t)\Psi_1^2(t)\cdot(1+t)^{-\frac{1+\lambda+\delta}{2}} + \Phi_N(t)\Psi_1^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda} \\
&\quad + \Phi_N(t)\Psi_0^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda}
\end{aligned}$$

and

$$I_{13} = -(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\nabla\cdot\partial_t\boldsymbol{u} - \mu_1(1+t)^{\lambda+\delta}\int\varpi v^2\nabla\cdot\partial_t\boldsymbol{u} =: I_{13}^1 + I_{13}^2,$$

where

$$\begin{aligned}
I_{13}^2 &= \mu_1(1+t)^{\lambda+\delta}\int\varpi\nabla(v^2)\cdot\partial_t\boldsymbol{u} \\
&\lesssim (1+t)^{\lambda+\delta}\|v\|\cdot\|\nabla v\|_{L^\infty}\cdot\|\partial_t\boldsymbol{u}\| \\
&\lesssim (1+t)^{\lambda+\delta}\cdot\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}\cdot\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}\cdot\Psi_1(t)(1+t)^{-\frac{1-2\lambda+\delta}{2}} \\
&\lesssim \Phi_N(t)\Psi_1^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda} + \Phi_N(t)\Psi_0^2(t)(1+t)^{-1-\frac{1+\lambda}{4}n+\frac{3}{2}\lambda}.
\end{aligned}$$

The treatment of $\nabla\cdot\partial_t\boldsymbol{u}$ in I_{13}^1 is to rewrite $(1.3)_1$ into

$$\nabla\cdot\boldsymbol{u} = -\frac{\partial_t v + \boldsymbol{u}\cdot\nabla v}{1+\varpi v} \tag{3.14}$$

with $1+\varpi v \geq 1/2$ since $\|v\|_{L^\infty} \leq 1/(\gamma-1)$ and then

$$\nabla\cdot\partial_t\boldsymbol{u} = -\frac{\partial_t^2 v + \partial_t\boldsymbol{u}\cdot\nabla v + \boldsymbol{u}\cdot\nabla\partial_t v}{1+\varpi v} + \frac{\varpi(\partial_t v + \boldsymbol{u}\cdot\nabla v)\partial_t v}{(1+\varpi v)^2},$$

similar to the proof of Lemma 2.2 in [37] but the trickiest parts and details are different. Therefore,

$$\begin{aligned}I_{13}^1 &= -(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\nabla\cdot\partial_t \boldsymbol{u}\\ &= (1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\partial_t^2 v+\partial_t\boldsymbol{u}\cdot\nabla v+\boldsymbol{u}\cdot\nabla\partial_t v}{1+\varpi v}\\ &\quad -(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\varpi(\partial_t v+\boldsymbol{u}\cdot\nabla v)\partial_t v}{(1+\varpi v)^2}\\ &=: I_{13}^{11}+I_{13}^{12},\end{aligned}$$

where

$$\begin{aligned}I_{13}^{12} &\lesssim (1+t)^{1+\lambda+\delta}\|v\|_{L^\infty}\|\partial_t v\|\cdot(\|\partial_t v\|_{L^\infty}+\|\boldsymbol{u}\|_{L^\infty}\|\nabla v\|_{L^\infty})\|\partial_t v\|\\ &\lesssim (1+t)^{1+\lambda+\delta}\cdot\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}\cdot\Psi_1^2(t)(1+t)^{-(1+\delta)}\\ &\lesssim \Phi_N(t)\Psi_1^2(t)\cdot(1+t)^{-\frac{1-\lambda+\delta}{2}}.\end{aligned}$$

The estimate on I_{13}^{11} is

$$\begin{aligned}I_{13}^{11} &= (1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\partial_t^2 v}{1+\varpi v}+(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\partial_t\boldsymbol{u}\cdot\nabla v}{1+\varpi v}\\ &\quad +(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\boldsymbol{u}\cdot\nabla\partial_t v}{1+\varpi v}\\ &= (1+t)^{1+\lambda+\delta}\frac{1}{2}\int \varpi v\cdot\frac{\partial_t(\partial_t v)^2}{1+\varpi v}+(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\partial_t\boldsymbol{u}\cdot\nabla v}{1+\varpi v}\\ &\quad +(1+t)^{1+\lambda+\delta}\frac{1}{2}\int \varpi v\cdot\frac{\boldsymbol{u}\cdot\nabla(\partial_t v)^2}{1+\varpi v}\\ &= \partial_t\left((1+t)^{1+\lambda+\delta}\frac{1}{2}\int \varpi v\cdot\frac{(\partial_t v)^2}{1+\varpi v}\right)-\frac{1}{2}\int(\partial_t v)^2\cdot\partial_t\left(\frac{\varpi v}{1+\varpi v}\cdot(1+t)^{1+\lambda+\delta}\right)\\ &\quad +(1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot\frac{\partial_t\boldsymbol{u}\cdot\nabla v}{1+\varpi v}-(1+t)^{1+\lambda+\delta}\frac{1}{2}\int(\partial_t v)^2\cdot\left(\nabla\cdot\frac{\varpi v\boldsymbol{u}}{1+\varpi v}\right)\\ &=: \partial_t J_1(t)+\tilde{I}_{13}^{11},\end{aligned}$$

where

(3.15)
$$J_1(t) := (1+t)^{1+\lambda+\delta}\frac{1}{2}\int \varpi v\cdot\frac{(\partial_t v)^2}{1+\varpi v}\lesssim (1+t)^{1+\lambda+\delta}\|v\|_{L^\infty}\|\partial_t v\|^2\lesssim \|v\|_{L^\infty}\cdot\Phi_1^2(t).$$

We see that $\tilde{I}_{13}^{11}$ are integrals only involving first order derivatives and can be estimated in a similar way as I_{11}. This completes the proof of the estimates involving $\partial_t Q_1$.

We now consider the term involving $-\nabla\cdot Q_2$ such that

$$\begin{aligned}
&\int(-\nabla\cdot Q_2)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&=\int\left(\sum_{j=1}^n\sum_{k=1}^n\partial_{x_k}u^j\cdot\partial_{x_j}u^k+\varpi\nabla v\cdot\nabla v\right)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad+\int(\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u})\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad+\int\varpi v(\nabla\cdot\nabla v)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&=: I_{21}+I_{22}+I_{23}.
\end{aligned}$$

Similar to I_{11},

$$\begin{aligned}
I_{21}&\lesssim(\|\nabla\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|+\|\nabla v\|_{L^\infty}\|\nabla v\|)\cdot((1+t)^{1+\lambda+\delta}\|\partial_t v\|+(1+t)^{\lambda+\delta}\|v\|)\\
&\lesssim\Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}\cdot\Psi_1(t)(1+t)^{-\frac{\lambda+\delta}{2}}\\
&\quad\cdot\Big((1+t)^{1+\lambda+\delta}\Psi_1(t)(1+t)^{-\frac{1+\delta}{2}}+(1+t)^{\lambda+\delta}\Psi_0(t)(1+t)^{-\frac{1+\lambda}{4}n}\Big)\\
&\lesssim\Phi_N(t)\Psi_1^2(t)\cdot(1+t)^{-\frac{\delta}{2}}+\Phi_N(t)\Psi_1^2(t)+\Phi_N(t)\Psi_0^2(t)\cdot(1+t)^{-1-\frac{1+\lambda}{2}n}.
\end{aligned}$$

Integrating by parts implies that

$$\begin{aligned}
I_{22}&=\int(\boldsymbol{u}\cdot\nabla)(\nabla\cdot\boldsymbol{u})\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&=-\int(\nabla\cdot\boldsymbol{u})^2\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&\quad-\int(\nabla\cdot\boldsymbol{u})\boldsymbol{u}\cdot((1+t)^{1+\lambda+\delta}\nabla\partial_t v+\mu_1(1+t)^{\lambda+\delta}\nabla v)
\end{aligned}$$

and

$$\begin{aligned}
I_{23}&=\int\varpi v(\nabla\cdot\nabla v)\cdot\big((1+t)^{1+\lambda+\delta}\partial_t v+\mu_1(1+t)^{\lambda+\delta}v\big)\\
&=-\int\varpi|\nabla v|^2\cdot(1+t)^{1+\lambda+\delta}\partial_t v-\int\varpi v\nabla v\cdot(1+t)^{1+\lambda+\delta}\nabla\partial_t v\\
&\quad-2\int\varpi|\nabla v|^2\cdot\mu_1(1+t)^{\lambda+\delta}v.
\end{aligned}$$

All the above integrals not involving second order derivatives in I_{22} and I_{23} can be estimated as I_{21}, except for

$$\begin{aligned}
I_{22}^1&:=-\int(\nabla\cdot\boldsymbol{u})\boldsymbol{u}\cdot(1+t)^{1+\lambda+\delta}\nabla\partial_t v,\\
I_{23}^1&:=-\int\varpi v\nabla v\cdot(1+t)^{1+\lambda+\delta}\nabla\partial_t v,
\end{aligned}$$

which need to be treated in the same procedure as I_{13}^1. Specifically, we have according to (3.14),

$$
\begin{aligned}
I_{22}^1 :&= -\int (\nabla\cdot \boldsymbol{u})\boldsymbol{u}\cdot (1+t)^{1+\lambda+\delta}\nabla\partial_t v\\
&= \int \frac{\partial_t v}{1+\varpi v}\boldsymbol{u}\cdot (1+t)^{1+\lambda+\delta}\nabla\partial_t v+\cdots\\
&= \frac{1}{2}\int \frac{\boldsymbol{u}}{1+\varpi v}\cdot (1+t)^{1+\lambda+\delta}\nabla(\partial_t v)^2+\cdots\\
&= -\frac{1}{2}\int \left(\nabla\cdot \frac{\boldsymbol{u}}{1+\varpi v}\right)\cdot (1+t)^{1+\lambda+\delta}(\partial_t v)^2+\cdots,
\end{aligned}
$$

where we only write down the cubic terms involving second order derivatives and the integral in the last equality only involves first order derivatives. According to (3.1) and integration by parts

$$
I_{23}^1 = \int \varpi v\Delta v\cdot (1+t)^{1+\lambda+\delta}\partial_t v+\cdots = (1+t)^{1+\lambda+\delta}\int \varpi v\partial_t v\cdot \partial_t^2 v+\cdots
$$

whose trickiest part is the same as I_{13}^{11} in I_{13}^1. This completes the proof of the estimates involving $-\nabla\cdot Q_2$.

Step II. We turn to show the estimates of the second integral of this lemma. We may only focus on the terms involving second order derivatives since the estimates on the others are similar to those in the first step of this proof. We have

$$
\begin{aligned}
&\int (\partial_t Q_2-\nabla Q_1)\cdot ((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u})\\
&\quad= \int (-(\boldsymbol{u}\cdot\nabla)\partial_t\boldsymbol{u})\cdot ((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u})\\
&\qquad+\int (-\varpi v\nabla\partial_t v)\cdot ((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u})\\
&\qquad+\int ((\boldsymbol{u}\cdot\nabla)\nabla v)\cdot ((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u})\\
&\qquad+\int (\varpi v\nabla(\nabla\cdot\boldsymbol{u}))\cdot ((1+t)^{1-\lambda+\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\boldsymbol{u})+\cdots\\
&\quad= I_{31}+I_{32}+I_{33}+I_{34}+\cdots.
\end{aligned}
$$

We proceed as before such that

$$
\begin{aligned}
I_{31} &= -\frac{1}{2}\int (1+t)^{1-\lambda+\delta}(\boldsymbol{u}\cdot\nabla)|\partial_t\boldsymbol{u}|^2-\int \mu_2(1+t)^{-\lambda+\delta}|\boldsymbol{u}|^2(\nabla\cdot\partial_t\boldsymbol{u})\\
&= \frac{1}{2}\int (1+t)^{1-\lambda+\delta}(\nabla\cdot\boldsymbol{u})|\partial_t\boldsymbol{u}|^2+\int \mu_2(1+t)^{-\lambda+\delta}(\nabla|\boldsymbol{u}|^2))\cdot\partial_t\boldsymbol{u}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{32} &= -(1+t)^{1-\lambda+\delta}\int (\varpi v\nabla\partial_t v)\cdot\partial_t\boldsymbol{u}+\mu_2(1+t)^{-\lambda+\delta}\int \varpi\partial_t v\cdot(\nabla\cdot(v\boldsymbol{u}))\\
&= (1+t)^{1-\lambda+\delta}\int \varpi v\partial_t v(\nabla\cdot\partial_t\boldsymbol{u})+\cdots,
\end{aligned}
\tag{3.16}
$$

where the integral in the last inequality of (3.16) is in the same form as I_{13}^1 but the signs are opposite (such that this one is a good term) and the time-weight is stronger. It suffices to modify the definition of $J_1(t)$ in (3.15) by adding a negative integral, which does not affect the inequality $J_1(t)\lesssim \|v\|_{L^\infty}\cdot\Phi_1^2(t)$ in (3.15). We also have

$$
\begin{aligned}
I_{33} &= -(1+t)^{1-\lambda+\delta}\int(\nabla\cdot\boldsymbol{u})\nabla v\cdot\partial_t\boldsymbol{u}-(1+t)^{1-\lambda+\delta}\int(\boldsymbol{u}\cdot\nabla v)\cdot\partial_t(\nabla\cdot\boldsymbol{u})\\
&\quad-\int\mu_2(1+t)^{-\lambda+\delta}\nabla|\boldsymbol{u}|^2\cdot\nabla v\\
&= -(1+t)^{1-\lambda+\delta}\int(\boldsymbol{u}\cdot\nabla v)\cdot\partial_t(\nabla\cdot\boldsymbol{u})+\cdots\\
&= (1+t)^{1-\lambda+\delta}\int(\boldsymbol{u}\cdot\partial_t\boldsymbol{u})\cdot\partial_t(\nabla\cdot\boldsymbol{u})+\cdots\\
&= (1+t)^{1-\lambda+\delta}\int\boldsymbol{u}\cdot\Big(\nabla\cdot(\partial_t\boldsymbol{u}\otimes\partial_t\boldsymbol{u})-\frac{1}{2}\nabla|\partial_t\boldsymbol{u}|^2\Big)+\cdots\\
&= -(1+t)^{1-\lambda+\delta}\int(\partial_t\boldsymbol{u}\otimes\partial_t\boldsymbol{u})\odot(\nabla\boldsymbol{u})+(1+t)^{1-\lambda+\delta}\frac{1}{2}\int|\partial_t\boldsymbol{u}|^2(\nabla\cdot\boldsymbol{u})+\cdots,
\end{aligned}
$$

where "$\odot$" denotes the summation of all the elementwise products of two matrices and we have used the following identity for a general vector-valued function $\boldsymbol{\varphi}$ (we take $\boldsymbol{\varphi}=\partial_t\boldsymbol{u}$):

$$
(\nabla\cdot\boldsymbol{\varphi})\boldsymbol{\varphi}=\nabla\cdot(\boldsymbol{\varphi}\otimes\boldsymbol{\varphi})-\frac{1}{2}\nabla|\boldsymbol{\varphi}|^2. \tag{3.17}
$$

The last integral I_{34} is estimated as

$$
\begin{aligned}
I_{34} &= -(1+t)^{1-\lambda+\delta}\int\varpi(\nabla\cdot\boldsymbol{u})\nabla v\cdot\partial_t\boldsymbol{u}-(1+t)^{1-\lambda+\delta}\int\varpi v(\nabla\cdot\boldsymbol{u})\cdot\partial_t(\nabla\cdot\boldsymbol{u})\\
&\quad-\mu_2(1+t)^{-\lambda+\delta}\int\varpi(\nabla\cdot\boldsymbol{u}))(\nabla\cdot(v\boldsymbol{u}))+\cdots\\
&= -(1+t)^{1-\lambda+\delta}\int\varpi v(\nabla\cdot\boldsymbol{u})\cdot\partial_t(\nabla\cdot\boldsymbol{u})+\cdots\\
&= (1+t)^{1-\lambda+\delta}\int\varpi v\left(\frac{\partial_t v}{1+\varpi v}\right)\cdot\partial_t(\nabla\cdot\boldsymbol{u})+\cdots
\end{aligned}
$$

according to (3.14) similar to the treatment of I_{13}^1. Here the integral in the last inequality of the estimate of I_{34} is of the opposite sign compared with I_{13}^1 and hence is a good term. The proof is completed. ∎

Remark 3.6. From the decay estimates in the proof of Lemma 3.5, we see that the inhomogeneous terms involving $b(t)\cdot Q_1$ and the terms involving $v\nabla v$ in Q_2 decay slowest since $b(t)$ is time-asymptotically growing and v decays slower than $\boldsymbol{u}$.

For general integer $k\geq 1$, we proceed similarly to deduce the time-weighted energy estimates. The following "tame" product estimate is needed.

LEMMA 3.7 (see [16, 44]). *For $1<p<\infty$, $s\geq 0$, there holds*

$$
\|uv\|_{W^{s,p}}\lesssim\|u\|_{L^\infty}\|v\|_{W^{s,p}}+\|v\|_{L^\infty}\|u\|_{W^{s,p}}
$$

for functions u and v in $L^\infty\cap W^{s,p}$.

LEMMA 3.8. *There holds, for any integer $k\geq 1$, $\lambda\in(-1,0)$, that $\delta\in(0,\frac{1+\lambda}{4})$, and $|\alpha|=k$,*

$$\int \partial_x^\alpha(\partial_t Q_1 + b(t)\cdot Q_1 - \nabla\cdot Q_2)\cdot((1+t)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v + \mu_1(1+t)^{\lambda+\delta}\partial_x^\alpha v)$$
$$+ \int \partial_x^\alpha(\partial_t Q_2 - \nabla Q_1)\cdot((1+t)^{1-\lambda+\delta}\partial_t\partial_x^\alpha \boldsymbol{u} + \mu_2(1+t)^{-\lambda+\delta}\partial_x^\alpha \boldsymbol{u})$$
$$\lesssim \partial_t J_{k+1}(t) + (\Psi_0(t)+\Phi_N(t))\cdot\Psi_{k+1}^2(t)(1+t)^{-\frac{\delta}{4}}$$
$$+ (\Psi_0(t)+\Phi_N(t))\cdot\Psi_k^2(t)(1+t)^{-(1+\lambda+\frac{\delta}{4})},$$

under the assumption that $\|v\|_{L^\infty} \le \frac{1}{\gamma-1}$, *where*

$$J_{k+1}(t) \lesssim \|v\|_{L^\infty}\cdot\Phi_{k+1}^2(t).$$

Proof. For $|\alpha| = k \ge 1$ and $n \ge 3$, we have

$$\int b(t)\cdot\partial_x^\alpha Q_1\cdot((1+t)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v + \mu_1(1+t)^{\lambda+\delta}\partial_x^\alpha v)$$
$$\lesssim \int \left(|\boldsymbol{u}\cdot\nabla\partial_x^\alpha v| + \sum_{j=1}^{k}|\partial_x^j\boldsymbol{u}\cdot\nabla\partial_x^{k-j}v| + |v\nabla\cdot\partial_x^\alpha\boldsymbol{u}|\right)$$
$$\cdot\left((1+t)^{1+\delta}|\partial_t\partial_x^\alpha v| + (1+t)^\delta|\partial_x^\alpha v|\right)$$
$$\lesssim ((1+t)^{1+\delta}\|\partial_t\partial_x^\alpha v\| + (1+t)^\delta\|\partial_x^\alpha v\|)\cdot\left(\|\boldsymbol{u}\|_{L^\infty}\cdot\|\nabla\partial_x^\alpha v\| + \|v\|_{L^\infty}\cdot\|\nabla\partial_x^\alpha\boldsymbol{u}\|\right)$$
$$\lesssim \left((1+t)^{1+\delta}\Psi_{k+1}(t)(1+t)^{-\frac{1+\delta}{2}} + (1+t)^\delta\Psi_k(t)(1+t)^{-\frac{\lambda+\delta}{2}}\right)$$
$$\cdot\Bigg(\Phi_N(t)(1+t)^{-\frac{1-\lambda+\delta}{2}}\cdot\Psi_{k+1}(t)(1+t)^{-\frac{\lambda+\delta}{2}}$$
$$+ \Phi_N(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}\cdot\Psi_{k+1}(t)(1+t)^{-\frac{-\lambda+\delta}{2}}\Bigg)$$
$$\lesssim \Phi_N(t)\Psi_{k+1}(t)\left(\Psi_{k+1}(t)(1+t)^{-\frac{\delta}{2}} + \Psi_k(t)(1+t)^{-\frac{1+\lambda+\delta}{2}}\right)$$
$$\lesssim \Phi_N(t)\Psi_{k+1}^2(t)(1+t)^{-\frac{\delta}{2}} + \Phi_N(t)\Psi_k^2(t)(1+t)^{-\left(1+\lambda+\frac{\delta}{2}\right)},$$

where we have used (3.10) and Lemma 3.7. The case of $n = 2$ follows similarly according to (3.11) as follows:

$$\int b(t)\cdot\partial_x^\alpha Q_1\cdot\left((1+t)^{1+\lambda+\delta}\partial_t\partial_x^\alpha v + \mu_1(1+t)^{\lambda+\delta}\partial_x^\alpha v\right)$$
$$\lesssim \left((1+t)^{1+\delta}\|\partial_t\partial_x^\alpha v\| + (1+t)^\delta\|\partial_x^\alpha v\|\right)\cdot(\|\boldsymbol{u}\|_{L^\infty}\cdot\|\nabla\partial_x^\alpha v\| + \|v\|_{L^\infty}\cdot\|\nabla\partial_x^\alpha\boldsymbol{u}\|)$$
$$\lesssim \left((1+t)^{1+\delta}\Psi_{k+1}(t)(1+t)^{-\frac{1+\delta}{2}} + (1+t)^\delta\Psi_k(t)(1+t)^{-\frac{\lambda+\delta}{2}}\right)$$
$$\cdot\Big((\Psi_0(t)+\Phi_N(t))(1+t)^{-\frac{1-\lambda}{2}-\frac{1+\lambda+\delta}{4}}\cdot\Psi_{k+1}(t)(1+t)^{-\frac{\lambda+\delta}{2}}$$
$$+ (\Psi_0(t)+\Phi_N(t))(1+t)^{-\frac{1+\lambda}{2}-\frac{\delta}{4}}\cdot\Psi_{k+1}(t)(1+t)^{-\frac{-\lambda+\delta}{2}}\Big)$$
$$\lesssim (\Psi_0(t)+\Phi_N(t))\Psi_{k+1}(t)\left(\Psi_{k+1}(t)(1+t)^{-\frac{\delta}{4}} + \Psi_k(t)(1+t)^{-\frac{1+\lambda}{2}-\frac{\delta}{4}}\right)$$
$$\lesssim (\Psi_0(t)+\Phi_N(t))\cdot\Psi_{k+1}^2(t)(1+t)^{-\frac{\delta}{4}} + (\Psi_0(t)+\Phi_N(t))\cdot\Psi_k^2(t)(1+t)^{-(1+\lambda+\frac{\delta}{4})}.$$

The other integrals are treated in the same procedure as in the proof of Lemma 3.5, where all the terms involving the $(k+2)$th order derivatives are estimated through integration by parts such that $\Psi_{k+2}(t)$ is not needed. ∎

3.3. Closure through Green function method. We employ the Green function method to deduce the basic energy estimates in $\Psi_0(t)$.

LEMMA 3.9. *There holds, for* $\lambda \in (-1,0)$, *that*

$$\begin{aligned}\|v\| &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap L^2}\cdot(1+t)^{-\frac{1+\lambda}{4}n} + \Psi_0(t)\Phi_N(t)\cdot(1+t)^{-\frac{1+\lambda}{4}n},\\ \|\boldsymbol{u}\| &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1}\cdot(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}} + (\Phi_N(t)+\Psi_0(t))\Phi_N(t)\\ &\quad\cdot(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}.\end{aligned}$$

Proof. The proof is similar to that of Theorem 1.1 but the a priori assumptions are different. According to the Duhamel principle (2.2) and the decay estimates of the Green matrix $\mathcal{G}(t,s)$ in Lemma 2.4, we have

$$\begin{aligned}\|v(t)\| &\lesssim \|\mathcal{G}_{11}(t,0)v_0\| + \|\mathcal{G}_{12}(t,0)\boldsymbol{u}_0\| + \int_0^t \|\mathcal{G}_{11}(t,s)Q_1(s)\|ds + \int_0^t \|\mathcal{G}_{12}(t,s)Q_2(s)\|ds\\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap L^2}\cdot(1+t)^{-\frac{1+\lambda}{4}n} + \int_0^t \Gamma^{\frac{n}{2}}(t,s)\cdot(\|Q_1(s)\|^l_{L^1}+\|Q_1(s)\|^h)ds\\ &\quad+\int_0^t (1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(\|Q_2(s)\|^l_{L^1}+\|Q_2(s)\|^h)ds\\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap L^2}\cdot(1+t)^{-\frac{1+\lambda}{4}n} + \Psi_0(t)\Phi_N(t)\int_0^t \Gamma^{\frac{n}{2}}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda+\delta}{2}}ds\\ &\quad+\Psi_0(t)\Phi_N(t)\int_0^t (1+s)^{\lambda}\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda+\delta}{2}}ds\\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap L^2}\cdot(1+t)^{-\frac{1+\lambda}{4}n} + \Psi_0(t)\Phi_N(t)\cdot(1+t)^{-\frac{1+\lambda}{4}n},\end{aligned}$$

where we have used Lemma 2.5 (note that

$$\begin{cases}\frac{1+\lambda}{4}n+\frac{1-\lambda+\delta}{2} \geq \frac{1+\lambda}{2}+\frac{1-\lambda+\delta}{2} = 1+\frac{\delta}{2} > 1,\\ \frac{1+\lambda}{4}n+\frac{1+\lambda+\delta}{2}-\lambda \geq \frac{1+\lambda}{2}+\frac{1+\lambda+\delta}{2}-\lambda = 1+\frac{\delta}{2} > 1\end{cases} \tag{3.18}$$

for all $n \geq 2$ and $\lambda \in (-1,0)$) and the following decay estimates on $\|Q(s)\|_{L^1}$ and $\|Q(s)\|$ (here and after, we use $D^j := \partial_x^j$):

$$\begin{aligned}\|Q_1(s)\|_{L^1} &\lesssim \|uDv\|_{L^1} + \|vDu\|_{L^1} \lesssim \|u\|\|Dv\| + \|v\|\|Du\|\\ &\lesssim \Psi_0(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}\cdot\Phi_N(s)(1+s)^{-\frac{1+\lambda+\delta}{2}}\\ &\quad+\Psi_0(s)(1+s)^{-\frac{1+\lambda}{4}n}\cdot\Phi_N(s)(1+s)^{-\frac{1-\lambda+\delta}{2}}\\ &\lesssim \Psi_0(s)\Phi_N(s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda+\delta}{2}},\\ \|Q_2(s)\|_{L^1} &\lesssim \|uDu\|_{L^1} + \|vDv\|_{L^1} \lesssim \|u\|\|Du\| + \|v\|\|Dv\|\\ &\lesssim \Psi_0(s)(1+s)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}\cdot\Phi_N(s)(1+s)^{-\frac{1-\lambda+\delta}{2}}\\ &\quad+\Psi_0(s)(1+s)^{-\frac{1+\lambda}{4}n}\cdot\Phi_N(s)(1+s)^{-\frac{1+\lambda+\delta}{2}}\\ &\lesssim \Psi_0(s)\Phi_N(s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1+\lambda+\delta}{2}}.\end{aligned}$$

The decay estimates on $\|Q_1\|$ and $\|Q_2\|$ are at least at the same rates as $\|Q_1\|_{L^1}$ and $\|Q_2\|_{L^1}$ since the estimates on $\|Dv\|_{L^\infty}$ and $\|D\boldsymbol{u}\|_{L^\infty}$ decay at the same rates as $\|Dv\|$ and $\|D\boldsymbol{u}\|$ according to (3.10) and (3.11).

In order to deduce the optimal decay estimate on $\|\boldsymbol{u}\|$ we need to utilize the optimal decay estimate on $\mathcal{G}_{22}$ in (2.27), which needs $\|DQ_2\|$, instead of (2.26), which only needs $\|Q_2\|$. We see that

$$\begin{aligned}\|DQ_2(s)\| &\lesssim \|uD^2u\| + \|DuDu\| + \|vD^2v\| + \|DvDv\| \\ &\lesssim \|u\|_{L^\infty}\|D^2u\| + \|Du\|_{L^\infty}\|Du\| + \|v\|_{L^\infty}\|D^2v\| + \|Dv\|_{L^\infty}\|Dv\| \\ &\lesssim \begin{cases} \Phi_N^2(s)(1+s)^{-(1-\lambda+\delta)} + \Phi_N^2(s)(1+s)^{-(1+\lambda+\delta)}, & n\ge 3,\\ (\Phi_N(s)+\Psi_0(s))(1+s)^{-\frac{1-\lambda}{2}-\frac{1+\lambda+\delta}{4}}\cdot\Phi_N(s)(1+s)^{-\frac{1-\lambda+\delta}{2}} & \\ \quad +\Phi_N^2(s)(1+s)^{-(1-\lambda+\delta)} + \Phi_N^2(s)(1+s)^{-(1+\lambda+\delta)} & \\ \quad +(\Phi_N(s)+\Psi_0(s))(1+s)^{-\frac{1+\lambda}{2}-\frac{\delta}{4}}\cdot\Phi_N(s)(1+s)^{-\frac{1+\lambda+\delta}{2}}, & n=2,\end{cases} \\ &\lesssim \begin{cases} \Phi_N^2(s)(1+s)^{-(1+\lambda+\delta)}, & n\ge 3,\\ (\Phi_N(s)+\Psi_0(s))\Phi_N(s)\cdot(1+s)^{-(1+\lambda+\frac{3}{4}\delta)}, & n=2,\end{cases} \\ &\lesssim (\Phi_N(s)+\Psi_0(s))\Phi_N(s)\cdot(1+s)^{-(1+\lambda+\frac{3}{4}\delta)}, \quad n\ge 2,\end{aligned}$$

according to (3.10) and (3.11). Therefore, we have

$$\begin{aligned}\|\boldsymbol{u}(t)\| &\lesssim \|\mathcal{G}_{21}(t,0)v_0\| + \|\mathcal{G}_{22}(t,0)\boldsymbol{u}_0\| + \int_0^t \|\mathcal{G}_{21}(t,s)Q_1(s)\|ds + \int_0^t \|\mathcal{G}_{22}(t,s)Q_2(s)\|ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1}\cdot(1+t)^{-\frac{1+\lambda}{4}n} + \int_0^t (1+t)^\lambda\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(\|Q_1(s)\|_{L^1}^l + \|Q_1(s)\|^h)ds \\ &\quad + \int_0^t (1+t)^\lambda(1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot(\|Q_2(s)\|_{L^1}^l + \|DQ_2(s)\|^h)ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1}\cdot(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}} \\ &\quad + \Psi_0(t)\Phi_N(t)\int_0^t (1+t)^\lambda\cdot\Gamma^{\frac{n}{2}+1}(t,s)\cdot(1+s)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda+\delta}{2}}ds \\ &\quad + (\Phi_N(t)+\Psi_0(t))\Phi_N(t)\int_0^t (1+t)^\lambda(1+s)^\lambda\cdot\Gamma^{\frac{n}{2}+2}(t,s)\cdot(1+s)^{-(1+\lambda+\frac{3}{4}\delta)}ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1}\cdot(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}} + (\Phi_N(t)+\Psi_0(t))\Phi_N(t)\cdot(1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}},\end{aligned}$$

since

$$\begin{cases} \frac{1+\lambda}{4}n + \frac{1-\lambda+\delta}{2} - \lambda \ge \frac{1+\lambda}{2} + \frac{1-\lambda+\delta}{2} - \lambda = 1-\lambda+\frac{\delta}{2} > 1,\\ \frac{1+\lambda}{4}n + \frac{1-\lambda+\delta}{2} - \lambda \ge \frac{1+\lambda}{4}n + \frac{1-\lambda}{2},\\ 1+\lambda+\frac{3}{4}\delta - 2\lambda = 1-\lambda+\frac{3}{4}\delta > 1,\\ 1+\lambda+\frac{3}{4}\delta - 2\lambda \ge \frac{1+\lambda}{4}n + \frac{1-\lambda}{2},\end{cases} \tag{3.19}$$

for all $n\ge 2$ and $\lambda\in(-1,0)$, except that the last inequality in (3.19) is not true for the case of $\frac{1+\lambda}{2}n > 1-\lambda+\frac{3}{2}\delta$. Fortunately, this case has already been proved in Theorem 1.1 by means of the Green function method (for $\lambda\in(-\frac{n}{n+2},0)$, i.e., $\frac{1+\lambda}{2}n > -\lambda$, which covers the exceptional case here). The proof is completed. ∎

Remark 3.10. Introduction of the positive constant δ plays an important role in the closure of the optimal decay estimate of $\|v\|$ (especially for the case of $n = 2$) according to the condition (3.18) in the proof of Lemma 3.9.

We combine the above time-weighted iteration scheme and Green function method to close the decay estimates for $\lambda \in (-1,0)$.

PROPOSITION 3.11. *For $n \geq 2$, $N \geq [\frac{n}{2}]+2$ and $\lambda \in (-1,0)$, there exists a constant $\varepsilon_0 > 0$ such that the solution $(v, \boldsymbol{u})$ of the nonlinear system (1.3) corresponding to small initial data $\|(v_0, \boldsymbol{u}_0)\|_{L^1 \cap H^N} \leq \varepsilon_0$ exists globally and satisfies*

$$\begin{cases} \|v(t)\| \lesssim (1+t)^{-\frac{1+\lambda}{4}n}, \\ \|\boldsymbol{u}(t)\| \lesssim (1+t)^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}}. \end{cases} \tag{3.20}$$

The above decay rates are optimal and consistent with the optimal decay rates of the linearized hyperbolic system.

Proof. We claim that the a priori decay estimate

$$\Phi_N(t) + \Psi_0(t) \leq \delta_0 \tag{3.21}$$

holds for all the time $t > 0$, under the small energy assumption of initial data $\|(v_0, \boldsymbol{u}_0)\|_{L^1 \cap H^N} \leq \varepsilon_0$, where ε_0 and δ_0 are positive constants to be determined. In fact, Lemma 3.9 tells us that

$$\Psi_0(T) \leq \sup_{t \in (0,T)} \left\{ (1+t)^{\frac{1+\lambda}{4}n} \|v\|, (1+t)^{\frac{1+\lambda}{4}n+\frac{1-\lambda}{2}} \|\boldsymbol{u}\| \right\} \lesssim \varepsilon_0 + \delta_0^2. \tag{3.22}$$

Substituting the estimates of inhomogeneous terms in Lemmas 3.5 and 3.8 into the time-weighted iteration scheme (3.12) and (3.13) in Lemma 3.4, we have for integer $0 \leq k \leq N-1$ that

$$\begin{aligned} \Phi_1^2(t) + \int_0^t \Psi_1^2(s)ds &\lesssim \Phi_1^2(0) + J_1(t) + \int_0^t (1+s)^{-1-\frac{1+\lambda}{2}n+\delta} \cdot \Psi_0^2(s)ds \\ &\quad + \delta_0 \int_0^t \Psi_1^2(s)ds + \delta_0 \int_0^t (1+s)^{-1-\frac{1+\lambda}{4}} \cdot \Psi_0^2(s)ds, \\ \Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s)ds &\lesssim \Phi_{k+1}^2(0) + J_{k+1}(t) + \int_0^t (1+s)^{-1-\lambda} \cdot \Psi_k^2(s)ds \\ &\quad + \delta_0 \int_0^t \Psi_{k+1}^2(s)ds + \delta_0 \int_0^t (1+s)^{-1-\lambda} \cdot \Psi_k^2(s)ds, \end{aligned}$$

where

$$\begin{aligned} J_1(t) &\lesssim \|v\|_{L^\infty} \cdot \Phi_1^2(t) \lesssim (\Phi_N(t) + \Psi_0(t)) \cdot \Phi_1^2(t) \lesssim \delta_0 \Phi_1^2(t), \\ J_{k+1}(t) &\lesssim \|v\|_{L^\infty} \cdot \Phi_{k+1}^2(t) \lesssim (\Phi_N(t) + \Psi_0(t)) \Phi_{k+1}^2(t) \lesssim \delta_0 \Phi_{k+1}^2(t). \end{aligned}$$

We note that δ_0 and ε_0 are small such that the above inequalities can be simplified as

$$\Phi_1^2(t) + \int_0^t \Psi_1^2(s)ds \lesssim \varepsilon_0^2 + \Psi_0^2(t) \int_0^t (1+s)^{-1-\frac{1+\lambda}{4}} ds, \tag{3.23}$$

$$\Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s)ds \lesssim \varepsilon_0^2 + \int_0^t (1+s)^{-1-\lambda} \cdot \Psi_k^2(s)ds. \tag{3.24}$$

Multiplying (3.24) by small positive constants for $0 \leq k \leq N-1$, summing the resulting inequalities up together with (3.23), we have

$$\sum_{1 \leq j \leq N} \Phi_j^2(t) \lesssim \varepsilon_0^2 + \Psi_0^2(t) \int_0^t (1+s)^{-1-\frac{1+\lambda}{4}} ds \lesssim \varepsilon_0^2 + (\varepsilon_0 + \delta_0^2)^2,$$

according to the estimate (3.22). Therefore,

$$\Phi_N(t) + \Psi_0(t) \lesssim \varepsilon_0 + \delta_0^2 \leq \delta_0$$

for positive constants ε_0 and δ_0 small enough.

We can show that the decay estimates (3.20) are optimal in a similar way as the proof of Theorem 1.1, just replacing the estimates on $\|v\|$ and $\|\boldsymbol{u}\|$ by those in Lemma 3.9. The proof is completed. ∎

Proof of Theorem 1.5. It is immediately proved from Proposition 3.11. ∎

4. Critical case of $\boldsymbol{\lambda = -1}$: Optimal logarithmic decays. This section is devoted to the critical case of $\lambda = -1$. We show the optimal decay estimates such that $\|v(t)\|$ decays as powers of $\ln(e+t)$, that is, $\|v(t)\| \approx |\ln(e+t)|^{-\frac{n}{4}}$.

We start with the optimal decay estimates of the Green matrix for the critical case of $\lambda = -1$, which are special cases of Lemma 2.4. Here we write it down for the sake of convenience.

LEMMA 4.1. *For $\lambda = -1$, there hold*

(4.1)
$$\begin{aligned}
\|\partial_x^\alpha \mathcal{G}_{11}(t,s)\phi(x)\| &\lesssim \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+|\alpha|)} \left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right), \\
\|\partial_x^\alpha \mathcal{G}_{12}(t,s)\phi(x)\| &\lesssim (1+s)^{-1} \cdot \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+|\alpha|+1)} \left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right), \\
\|\partial_x^\alpha \mathcal{G}_{21}(t,s)\phi(x)\| &\lesssim (1+t)^{-1} \cdot \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+|\alpha|+1)} \left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right), \\
\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| &\lesssim \left(\frac{1+t}{1+s}\right)^{-1} \cdot \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+|\alpha|)} \left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|}\phi\|^h\right).
\end{aligned}$$

Moreover,

(4.2)
$$\begin{aligned}
&\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| \\
&\quad \lesssim (1+t)^{-1}(1+s)^{-1} \cdot \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+|\alpha|+2)} \left(\|\phi\|_{L^1}^l + \|\partial_x^{|\alpha|+1}\phi\|^h\right).
\end{aligned}$$

Proof. These estimates are simple conclusions of Theorem A.5 in the appendix. ∎

The following time decay estimate of the "convolution" type integral of two critical time decay functions involving a logarithm plays an essential role in the Green function method for $\lambda = -1$.

LEMMA 4.2 (logarithmic time decay functions). *For $\beta > 0$ and $\gamma > 1$, there holds (we may assume that $t \geq 1$)*

(4.3)
$$\int_0^t \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\beta} (1+s)^{-1} |\ln(e+s)|^{-\gamma} ds \approx \begin{cases} |\ln(e+t)|^{-\min\{\beta,\gamma-1\}}, & \gamma > 1, \\ \ln(\ln(e^e+t)), & \gamma = 1, \\ |\ln(e+t)|^{1-\gamma}, & \gamma < 1. \end{cases}$$

Proof. For $\gamma \le 1$, we have

$$\int_0^t \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds \lesssim \int_0^t (e+s)^{-1}|\ln(e+s)|^{-\gamma}ds \lesssim \begin{cases} \ln(\ln(e^e+t)), & \gamma=1, \\ |\ln(e+t)|^{1-\gamma}, & \gamma<1, \end{cases}$$

and

$$\begin{aligned} &\int_0^t \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\gtrsim \int_{\frac{t}{2}}^t \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\approx \int_{\frac{t}{2}}^t (e+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\approx \begin{cases} \ln(\ln(e^e+t)), & \gamma=1, \\ |\ln(e+t)|^{1-\gamma}, & \gamma<1. \end{cases} \end{aligned}$$

For $\gamma>1$, we calculate the integral divided into $(0,t^\varepsilon)$ and (t^ε,t), where $\varepsilon\in(0,1)$ is a small constant to be determined, as follows:

$$\begin{aligned} &\int_{t^\varepsilon}^t \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\lesssim \int_{t^\varepsilon}^t (e+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\approx |\ln(e+t^\varepsilon)|^{-(\gamma-1)} \approx |\varepsilon\ln(e+t)|^{-(\gamma-1)} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\int_0^{t^\varepsilon} \Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &\approx \int_0^{t^\varepsilon} \Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta}(e+s)^{-1}|\ln(e+s)|^{-\gamma}ds \\ &= \int_0^{t^\varepsilon} \Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta} d\Big(\frac{-1}{\gamma-1}|\ln(e+s)|^{-(\gamma-1)}\Big) \\ &= \Bigg[\frac{-1}{\gamma-1}|\ln(e+s)|^{-(\gamma-1)}\Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta}\Bigg]_0^{t^\varepsilon} \\ &\quad + \int_0^{t^\varepsilon} \frac{\beta}{\gamma-1}|\ln(e+s)|^{-(\gamma-1)}(e+s)^{-1}\Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta-1}ds. \end{aligned} \tag{4.5}$$

Now we fix $\varepsilon>0$ to be sufficiently small such that

$$\begin{aligned} &\int_0^{t^\varepsilon} \frac{\beta}{\gamma-1}|\ln(e+s)|^{-(\gamma-1)}(e+s)^{-1}\Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta-1}ds \\ &\le \frac12\int_0^{t^\varepsilon} \Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta}(e+s)^{-1}|\ln(e+s)|^{-\gamma}ds, \end{aligned}$$

one of whose sufficient conditions is

$$\frac{\beta}{\gamma-1}|\ln(e+s)|\Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-1}\le\frac{1}{2}\quad\forall s\in(0,t^{\varepsilon}).$$

It suffices to take $\frac{\beta}{\gamma-1}\cdot\frac{\varepsilon}{1-\varepsilon}\le\frac{1}{2}$, which is true for a small $\varepsilon\in(0,1)$. Now (4.5) reads as

$$\begin{aligned}&\int_0^{t^{\varepsilon}}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds\\&\quad\approx\Big[\frac{-1}{\gamma-1}|\ln(e+s)|^{-(\gamma-1)}\Big(1+\ln\Big(\frac{e+t}{e+s}\Big)\Big)^{-\beta}\Big]_0^{t^{\varepsilon}}\\&\quad\approx|\ln(e+t)|^{-\beta}-|\ln(e+t)|^{-(\gamma-1)-\beta}\approx|\ln(e+t)|^{-\beta}.\end{aligned}$$

On the other hand, we can improve (4.4) as

$$\int_{t^{\varepsilon}}^{t}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds\approx|\ln(e+t)|^{-(\gamma-1)},$$

since

$$\begin{aligned}&\int_{t^{\varepsilon}}^{t}\Big(1+\ln\Big(\frac{1+t}{1+s}\Big)\Big)^{-\beta}(1+s)^{-1}|\ln(e+s)|^{-\gamma}ds\\&\quad\gtrsim\int_{\frac{t}{2}}^{t}(e+s)^{-1}|\ln(e+s)|^{-\gamma}ds\\&\quad\approx|\ln(e+t/2)|^{-(\gamma-1)}\approx|\ln(e+t)|^{-(\gamma-1)}.\end{aligned}$$

The proof is completed. ∎

We apply the time-weighted iteration scheme developed in section 3 to the critical case of $\lambda=-1$.

LEMMA 4.3. *For any nonnegative integer k, $\lambda=-1$, $\delta\in(0,\frac{n}{2})$, and $|\alpha|=k$, there hold*

$$\begin{aligned}&\frac{d}{dt}\int E^{v}(\partial_t\partial_x^{\alpha}v,\nabla\partial_x^{\alpha}v,\partial_x^{\alpha}v)\\&\qquad+\int\big[(1+t)\cdot|\ln(e+t)|^{\delta+1}|\partial_t\partial_x^{\alpha}v|^2+(1+t)^{-1}\cdot|\ln(e+t)|^{\delta}|\nabla\partial_x^{\alpha}v|^2\big]\\&\quad\lesssim\int(1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1}(\partial_x^{\alpha}v)^2\\&\qquad+\int\partial_x^{\alpha}(\partial_tQ_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot(|\ln(e+t)|^{\delta+1}\partial_t\partial_x^{\alpha}v\\&\qquad+\mu_1(1+t)^{-1}\cdot|\ln(e+t)|^{\delta}\partial_x^{\alpha}v)\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}&\frac{d}{dt}\int E^{u}(\partial_t\partial_x^{\alpha}\boldsymbol{u},\nabla\partial_x^{\alpha}\boldsymbol{u},\partial_x^{\alpha}\boldsymbol{u})\\&\qquad+\int\big[(1+t)^3\cdot|\ln(e+t)|^{\delta}|\partial_t\partial_x^{\alpha}\boldsymbol{u}|^2+(1+t)\cdot|\ln(e+t)|^{\delta}|\nabla\partial_x^{\alpha}\boldsymbol{u}|^2\big]\\&\quad\lesssim\int(1+t)\cdot|\ln(e+t)|^{\delta}|\partial_x^{\alpha}\boldsymbol{u}|^2\\&\qquad+\int\partial_x^{\alpha}(\partial_tQ_2-\nabla Q_1)\cdot((1+t)^2\cdot|\ln(e+t)|^{\delta}\partial_t\partial_x^{\alpha}\boldsymbol{u}+\mu_2(1+t)\cdot|\ln(e+t)|^{\delta}\partial_x^{\alpha}\boldsymbol{u}),\end{aligned}\tag{4.7}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are constants and

$$E^v(\partial_t\partial_x^\alpha v, \nabla\partial_x^\alpha v, \partial_x^\alpha v) \approx |\ln(e+t)|^{\delta+1}(|\partial_t\partial_x^\alpha v|^2 + |\nabla\partial_x^\alpha v|^2) + |\ln(e+t)|^\delta(\partial_x^\alpha v)^2,$$

$$E^u(\partial_t\partial_x^\alpha \boldsymbol{u}, \nabla\partial_x^\alpha \boldsymbol{u}, \partial_x^\alpha \boldsymbol{u}) \approx (1+t)^2 \cdot |\ln(e+t)|^\delta(|\partial_t\partial_x^\alpha \boldsymbol{u}|^2 + |\nabla\partial_x^\alpha \boldsymbol{u}|^2) + (1+t)^2 \cdot |\ln(e+t)|^\delta|\partial_x^\alpha \boldsymbol{u}|^2.$$

Proof. This is proved by multiplying (3.1) by

$$|\ln(e+t)|^{\delta+1}\partial_t\partial_x^\alpha v + \mu_1(1+t)^{-1} \cdot |\ln(e+t)|^\delta\partial_x^\alpha v$$

and multiplying (3.2) by

$$(1+t)^2 \cdot |\ln(e+t)|^\delta\partial_t\partial_x^\alpha \boldsymbol{u} + \mu_2(1+t) \cdot |\ln(e+t)|^\delta\partial_x^\alpha \boldsymbol{u}$$

with $\delta \in (0, \frac{n}{2})$ and $\mu_1, \mu_2 > 0$. We note that the time-weight of $\partial_t\partial_x^\alpha v$ is $|\ln(e+t)|^{\delta+1}$ instead of $|\ln(e+t)|^\delta$. The reason is that the time-weights are chosen such that

$$\partial_t(|\ln(e+t)|^{\delta+1}) \approx (1+t)^{-1} \cdot |\ln(e+t)|^\delta$$

and

$$\partial_t((1+t)^2 \cdot |\ln(e+t)|^\delta) \approx (1+t) \cdot |\ln(e+t)|^\delta.$$

The rest of the proof is similar to Lemma 3.1. We omit the details. ∎

We define the following time-weighted energies for the critical case of $\lambda = -1$, $N \ge [\frac{n}{2}] + 2$, and $0 \le k \le N-1$:

$$\Phi_{k+1}(T) := \sup_{t\in(0,T)} \left\{ \sum_{|\alpha|=k} \left[|\ln(e+t)|^{\delta+1} \int (|\partial_t\partial_x^\alpha v|^2 + |\nabla\partial_x^\alpha v|^2) + (1+t)^2 \cdot |\ln(e+t)|^\delta \int (|\partial_t\partial_x^\alpha \boldsymbol{u}|^2 + |\nabla\partial_x^\alpha \boldsymbol{u}|^2) \right] \right\}^{\frac{1}{2}} \tag{4.8}$$

and

$$\Psi_{k+1}(T) := \sup_{t\in(0,T)} \left\{ \sum_{|\alpha|=k} \left[\int \left[(1+t) \cdot |\ln(e+t)|^{\delta+1}|\partial_t\partial_x^\alpha v|^2 + (1+t)^{-1} \cdot |\ln(e+t)|^\delta|\nabla\partial_x^\alpha v|^2\right] + \int \left[(1+t)^3 \cdot |\ln(e+t)|^\delta|\partial_t\partial_x^\alpha \boldsymbol{u}|^2 + (1+t) \cdot |\ln(e+t)|^\delta|\nabla\partial_x^\alpha \boldsymbol{u}|^2\right] \right] \right\}^{\frac{1}{2}}. \tag{4.9}$$

We may assume that $\Phi_{k+1}(T) \ge \Phi_k(T)$ for all $k \ge 1$ and T. Similar to the case of $\lambda \in (-1, 0)$, here for $\lambda = -1$ the energy $\Phi_{k+1}(T)$ is defined according to the time-weighted energy estimates in Lemma 4.3, but the decay estimates on $\|v\|$ and $\|\boldsymbol{u}\|$ are absent. Therefore, we define the following weighted energy:

$$\Psi_0(T) := \sup_{t\in(0,T)} \left\{ |\ln(e+t)|^{\frac{n}{4}}\|v\|, (1+t) \cdot |\ln(e+t)|^{\frac{n}{4}+\frac{1}{2}}\|\boldsymbol{u}\| \right\}. \tag{4.10}$$

The energy estimates in $\Psi_0(T)$ will be closed through the Green function method instead of the time-weighted energy method. There still holds

$$\|(v_0, \boldsymbol{u}_0)\|_{H^N} \approx \sum_{k=1}^{N} \Phi_k(0) + \Psi_0(0) \approx \Phi_N(0) + \Psi_0(0). \tag{4.11}$$

According to the Sobolev embedding theorem, we have

$$\begin{aligned} &|\ln(e+t)|^{\frac{\delta+1}{2}} \|\partial_x^j v\|_{L^\infty} + (1+t) \cdot |\ln(e+t)|^{\frac{\delta}{2}} \|\partial_x^j \boldsymbol{u}\|_{L^\infty} \\ &\lesssim \max_{1 \le k \le [\frac{n}{2}]+2} \Phi_k(t) \lesssim \Phi_N(t), \qquad 0 \le j \le 1, n \ge 3, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &|\ln(e+t)|^{\frac{1}{2}+\frac{\delta}{4}} \|v\|_{L^\infty} + (1+t) \cdot |\ln(e+t)|^{\frac{1}{2}+\frac{\delta}{4}} \|\boldsymbol{u}\|_{L^\infty} \\ &+ |\ln(e+t)|^{\frac{\delta+1}{2}} \|\partial_x v\|_{L^\infty} + (1+t) \cdot |\ln(e+t)|^{\frac{\delta}{2}} \|\partial_x \boldsymbol{u}\|_{L^\infty} \\ &\lesssim \max_{1 \le k \le [\frac{n}{2}]+2} \Phi_k(t) + \Psi_0(t) \lesssim \Phi_N(t) + \Psi_0(t), \qquad n = 2. \end{aligned} \tag{4.13}$$

We have the following iteration scheme based on Lemma 4.3 for the critical case of $\lambda = -1$.

LEMMA 4.4 (time-weighted iteration scheme). *For $\lambda = -1$ and $\delta \in (0, \frac{n}{2})$, there holds*

$$\begin{aligned} &\Phi_1^2(t) + \int_0^t \Psi_1^2(s) ds \\ &\lesssim \Phi_1^2(0) + \int_0^t (1+s)^{-1} \cdot |\ln(e+s)|^{\delta - 1 - \frac{n}{2}} \cdot \Psi_0^2(s) ds \\ &\quad + \int_0^t \int (\partial_t Q_1 + b(s) \cdot Q_1 - \nabla \cdot Q_2) \cdot (|\ln(e+s)|^{\delta+1} \partial_t v \\ &\quad + \mu_1 (1+s)^{-1} \cdot |\ln(e+s)|^{\delta} v) ds \\ &\quad + \int_0^t \int (\partial_t Q_2 - \nabla Q_1) \cdot ((1+s)^2 \cdot |\ln(e+s)|^{\delta} \partial_t \boldsymbol{u} + \mu_2 (1+s) \cdot |\ln(e+s)|^{\delta} \boldsymbol{u}) ds, \end{aligned} \tag{4.14}$$

and for any integer $k \ge 1$, there holds

$$\begin{aligned} \Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s) ds &\lesssim \Phi_{k+1}^2(0) + \int_0^t \Psi_k^2(s) ds \\ &+ \sum_{|\alpha|=k} \int_0^t \int \partial_x^\alpha (\partial_t Q_1 + b(s) \cdot Q_1 - \nabla \cdot Q_2) \\ &\cdot (|\ln(e+s)|^{\delta+1} \partial_t \partial_x^\alpha v + \mu_1 (1+s)^{-1} \cdot |\ln(e+s)|^{\delta} \partial_x^\alpha v) ds \\ &+ \sum_{|\alpha|=k} \int_0^t \int \partial_x^\alpha (\partial_t Q_2 - \nabla Q_1) \cdot ((1+s)^2 \cdot |\ln(e+s)|^{\delta} \partial_t \partial_x^\alpha \boldsymbol{u} \\ &+ \mu_2 (1+s) \cdot |\ln(e+s)|^{\delta} \partial_x^\alpha \boldsymbol{u}) ds. \end{aligned} \tag{4.15}$$

Proof. These are conclusions of Lemma 4.3 with the notation $\Phi_k(t)$, $\Psi_k(t)$, and $\Psi_0(t)$ defined by (4.8), (4.9), and (4.10). We note that

$$\begin{aligned}\int (1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1}|v|^2 &\lesssim (1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1-\frac{n}{2}}\cdot\|v\|^2\cdot|\ln(e+t)|^{\frac{n}{2}}\\ &\lesssim (1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1-\frac{n}{2}}\cdot\Psi_0^2(t),\\ \int (1+t)\cdot|\ln(e+t)|^{\delta}|\boldsymbol{u}|^2 &\lesssim (1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1-\frac{n}{2}}\\ &\qquad\cdot\|\partial_x^\alpha\boldsymbol{u}\|^2\cdot(1+t)^2\cdot|\ln(e+t)|^{1+\frac{n}{2}}\\ &\lesssim (1+t)^{-1}\cdot|\ln(e+t)|^{\delta-1-\frac{n}{2}}\cdot\Psi_0^2(t).\end{aligned}$$

The proof is completed. □

The inhomogeneous terms in the inequalities (4.6) and (4.7) in Lemma 4.3 are estimated in a similar way as Lemmas 3.5 and 3.8.

LEMMA 4.5. *There holds, for $\lambda=-1$ and $\delta\in(0,\frac{n}{2})$, that*

$$\begin{aligned}&\int(\partial_t Q_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot(|\ln(e+t)|^{\delta+1}\partial_t v+\mu_1(1+t)^{-1}\cdot|\ln(e+t)|^{\delta}v)\\ &\quad+\int(\partial_t Q_2-\nabla Q_1)\cdot((1+t)^2\cdot|\ln(e+t)|^{\delta}\partial_t\boldsymbol{u}+\mu_2(1+t)\cdot|\ln(e+t)|^{\delta}\boldsymbol{u})\\ &\quad\lesssim\partial_t J_1(t)+(\Psi_0(t)+\Phi_N(t))\cdot\Psi_1^2(t)+\Phi_N(t)\cdot\Psi_0^2(t)\cdot(1+t)^{-1}\cdot|\ln(e+t)|^{-\frac{n}{4}},\end{aligned}$$

provided that $\|v\|_{L^\infty}\le\frac{1}{\gamma-1}$ (which is valid under the a priori assumption $\Phi_N(t)+\Psi_0(t)\le\delta_0$ with a small constant δ_0), where

$$J_1(t)\lesssim\|v\|_{L^\infty}\cdot\Phi_1^2(t).$$

Proof. Noticing that the only difference between this lemma and Lemma 3.5 is the time-weights, we can prove the above decay estimates in the same way as before. Here we omit the details. □

LEMMA 4.6. *There holds, for integer $k\ge1$, $\lambda=-1$, $\delta\in(0,\frac{n}{2})$, and $|\alpha|=k$,*

$$\begin{aligned}&\int\partial_x^\alpha(\partial_t Q_1+b(t)\cdot Q_1-\nabla\cdot Q_2)\cdot(|\ln(e+t)|^{\delta+1}\partial_t\partial_x^\alpha v+\mu_1(1+t)^{-1}\cdot|\ln(e+t)|^{\delta}\partial_x^\alpha v)\\ &\quad+\int\partial_x^\alpha(\partial_t Q_2-\nabla Q_1)\cdot((1+t)^2\cdot|\ln(e+t)|^{\delta}\partial_t\partial_x^\alpha\boldsymbol{u}+\mu_2(1+t)\cdot|\ln(e+t)|^{\delta}\partial_x^\alpha\boldsymbol{u})\\ &\lesssim\partial_t J_{k+1}(t)+(\Psi_0(t)+\Phi_N(t))\cdot\Psi_{k+1}^2(t)\cdot|\ln(e+t)|^{-\frac{\delta}{4}}\\ &\quad+(\Psi_0(t)+\Phi_N(t))\cdot\Psi_k^2(t)\cdot|\ln(e+t)|^{-\frac{\delta}{4}},\end{aligned}$$

under the assumption that $\|v\|_{L^\infty}\le\frac{1}{\gamma-1}$, where

$$J_{k+1}(t)\lesssim\|v\|_{L^\infty}\cdot\Phi_{k+1}^2(t).$$

Proof. This is proved in a similar way as Lemma 3.8 since the differences only lie in the time-weights. □

The basic energy decay estimates in $\Psi_0(t)$ are deduced by means of the Green function method.

LEMMA 4.7. *There hold for $\lambda = -1$ and $n \geq 7$ that*

$$\|v\| \lesssim \|(v_0, \boldsymbol{u}_0)\|_{L^1\cap L^2} \cdot |\ln(e+t)|^{-\frac{n}{4}} + \Psi_0(t)\Phi_N(t) \cdot |\ln(e+t)|^{-\frac{n}{4}},$$
$$\|\boldsymbol{u}\| \lesssim \|(v_0, \boldsymbol{u}_0)\|_{L^1\cap H^1} \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}} + (\Phi_N(t) + \Psi_0(t))\Phi_N(t) \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}.$$

Proof. According to the Duhamel principle (2.2) and the decay estimates of the Green matrix $\mathcal{G}(t,s)$ in Lemma 4.1, we have

$$\begin{aligned}
\|v(t)\| &\lesssim \|\mathcal{G}_{11}(t,0)v_0\| + \|\mathcal{G}_{12}(t,0)\boldsymbol{u}_0\| + \int_0^t \|\mathcal{G}_{11}(t,s)Q_1(s)\|ds + \int_0^t \|\mathcal{G}_{12}(t,s)Q_2(s)\|ds \\
&\lesssim \|(v_0, \boldsymbol{u}_0)\|_{L^1\cap L^2} \cdot |\ln(e+t)|^{-\frac{n}{4}} + \int_0^t \Gamma^{\frac{n}{2}}(t,s) \cdot (\|Q_1(s)\|^l_{L^1} + \|Q_1(s)\|^h)ds \\
&\quad + \int_0^t (1+s)^{-1} \cdot \Gamma^{\frac{n}{2}+1}(t,s) \cdot (\|Q_2(s)\|^l_{L^1} + \|Q_2(s)\|^h)ds \\
&\lesssim \|(v_0, \boldsymbol{u}_0)\|_{L^1\cap L^2} \cdot |\ln(e+t)|^{-\frac{n}{4}} \\
&\quad + \Psi_0(t)\Phi_N(t) \int_0^t \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{n}{4}} \cdot (1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{\delta}{2}} ds \\
&\quad + \Psi_0(t)\Phi_N(t) \int_0^t (1+s)^{-1} \cdot \left(1 + \ln\left(\frac{1+t}{1+s}\right)\right)^{-\frac{1}{2}(\frac{n}{2}+1)} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{\delta+1}{2}} ds \\
&\lesssim \|(v_0, \boldsymbol{u}_0)\|_{L^1\cap L^2} \cdot |\ln(e+t)|^{-\frac{n}{4}} + \Psi_0(t)\Phi_N(t) \cdot |\ln(e+t)|^{-\frac{n}{4}},
\end{aligned}$$

where we have used Lemma 4.2 (note that

$$\begin{cases} \frac{n}{4} + \frac{\delta}{2} > 1, \quad \frac{n}{4} + \frac{\delta}{2} - 1 \geq \frac{n}{4}, \\ \frac{n}{4} + \frac{\delta+1}{2} > 1, \quad \frac{n}{4} + \frac{\delta+1}{2} - 1 \geq \frac{n}{4}, \end{cases} \tag{4.16}$$

for $n \geq 5$ and $\delta \in (2, \frac{n}{2})$) and the following decay estimates on $\|Q(s)\|_{L^1}$ and $\|Q(s)\|$ (we use $D^j := \partial_x^j$):

$$\begin{aligned}
\|Q_1(s)\|_{L^1} &\lesssim \|uDv\|_{L^1} + \|vDu\|_{L^1} \lesssim \|u\|\|Dv\| + \|v\|\|Du\| \\
&\lesssim \Psi_0(s)(1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{1}{2}} \cdot \Phi_N(s)|\ln(e+s)|^{-\frac{\delta+1}{2}} \\
&\quad + \Psi_0(s)|\ln(e+s)|^{-\frac{n}{4}} \cdot \Phi_N(s)(1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{\delta}{2}} \\
&\lesssim \Psi_0(s)\Phi_N(s) \cdot (1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{\delta}{2}}, \\
\|Q_2(s)\|_{L^1} &\lesssim \|uDu\|_{L^1} + \|vDv\|_{L^1} \lesssim \|u\|\|Du\| + \|v\|\|Dv\| \\
&\lesssim \Psi_0(s)(1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{1}{2}} \cdot \Phi_N(s)(1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{\delta}{2}} \\
&\quad + \Psi_0(s)|\ln(e+s)|^{-\frac{n}{4}} \cdot \Phi_N(s)|\ln(e+s)|^{-\frac{\delta+1}{2}} \\
&\lesssim \Psi_0(s)\Phi_N(s) \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{\delta+1}{2}}.
\end{aligned}$$

The decay estimates on $\|Q_1\|$ and $\|Q_2\|$ are at least at the same rates as $\|Q_1\|_{L^1}$ and $\|Q_2\|_{L^1}$ since the estimates on $\|Dv\|_{L^\infty}$ and $\|D\boldsymbol{u}\|_{L^\infty}$ decay at the same rates as $\|Dv\|$ and $\|D\boldsymbol{u}\|$ according to (4.12).

We estimate $\|DQ_2\|$ for $n \geq 3$ as

$$\begin{aligned}
\|DQ_2(s)\| &\lesssim \|uD^2u\| + \|DuDu\| + \|vD^2v\| + \|DvDv\| \\
&\lesssim \|u\|_{L^\infty}\|D^2u\| + \|Du\|_{L^\infty}\|Du\| + \|v\|_{L^\infty}\|D^2v\| + \|Dv\|_{L^\infty}\|Dv\| \\
&\lesssim \Phi_N^2(s)(1+s)^{-2} \cdot |\ln(e+s)|^{-\delta} + \Phi_N^2(s) \cdot |\ln(e+s)|^{-\delta-1} \\
&\lesssim \Phi_N^2(s) \cdot |\ln(e+s)|^{-\delta-1},
\end{aligned}$$

according to (4.12). Therefore, we have

$$\begin{aligned}\|\boldsymbol{u}(t)\| &\lesssim \|\mathcal{G}_{21}(t,0)v_0\| + \|\mathcal{G}_{22}(t,0)\boldsymbol{u}_0\| + \int_0^t \|\mathcal{G}_{21}(t,s)Q_1(s)\|ds + \int_0^t \|\mathcal{G}_{22}(t,s)Q_2(s)\|ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1} \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}} \\ &\quad + \int_0^t (1+t)^{-1} \cdot \Gamma^{\frac{n}{2}+1}(t,s) \cdot (\|Q_1(s)\|_{L^1}^l + \|Q_1(s)\|^h)ds \\ &\quad + \int_0^t (1+t)^{-1}(1+s)^{-1} \cdot \Gamma^{\frac{n}{2}+2}(t,s) \cdot (\|Q_2(s)\|_{L^1}^l + \|DQ_2(s)\|^h)ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1} \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}} \\ &\quad + \Psi_0(t)\Phi_N(t) \int_0^t (1+t)^{-1} \cdot \Gamma^{\frac{n}{2}+1}(t,s) \cdot (1+s)^{-1} \cdot |\ln(e+s)|^{-\frac{n}{4}-\frac{\delta}{2}}ds \\ &\quad + (\Phi_N(t)+\Psi_0(t))\Phi_N(t) \int_0^t (1+t)^{-1}(1+s)^{-1} \cdot \Gamma^{\frac{n}{2}+2}(t,s) \cdot |\ln(e+s)|^{-\delta-1}ds \\ &\lesssim \|(v_0,\boldsymbol{u}_0)\|_{L^1\cap H^1} \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}} \\ &\quad + (\Phi_N(t)+\Psi_0(t))\Phi_N(t) \cdot (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}},\end{aligned}$$

since

$$\begin{cases} \frac{n}{4}+\frac{\delta}{2} > 1, & \frac{n}{4}+\frac{\delta}{2}-1 \geq \frac{n}{4}+\frac{1}{2}, \\ \delta+1 > 1, & \delta+1-1 \geq \frac{n}{4}+\frac{1}{2}, \end{cases} \tag{4.17}$$

for $n \geq 7$ and $\delta \in (3, \frac{n}{2})$. The proof is completed. ◻

Remark 4.8. The restriction of $n \geq 7$ comes from the imperfect decay estimate of $\|Q_1\|_1$, which lays a barrier on the decay estimates of $\|(v,\boldsymbol{u})\|$. From the view of the optimal decay estimates of the linearized hyperbolic system, it is supposed that both $\|\boldsymbol{u}\partial_x v\|$ and $\|v\partial_x \boldsymbol{u}\|$ decay as $(1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{2}-1}$. We note that here in the proof of Lemma 4.6, the estimate on $\|\boldsymbol{u}\partial_x v\|$ decays as $(1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}-\frac{\delta+1}{2}}$, which is close to the expected optimal decays since $\delta \in (0, \frac{n}{2})$, while the estimate on $\|v\partial_x \boldsymbol{u}\|$ decays at $(1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{\delta}{2}}$, which has at least a gap of $|\ln(e+t)|^{-1}$ decay to the expected optimal decays.

We combine the above time-weighted iteration scheme and Green function method to close the decay estimates for $\lambda = -1$.

PROPOSITION 4.9. *For $n \geq 7$, $N \geq [\frac{n}{2}] + 2$, and $\lambda = -1$, there exists a constant $\varepsilon_0 > 0$ such that the solution $(v, \boldsymbol{u})$ of the nonlinear system* (1.3) *corresponding to small initial data $\|(v_0, \boldsymbol{u}_0)\|_{L^1\cap H^N} \leq \varepsilon_0$ exists globally and satisfies*

$$\begin{cases} \|v(t)\| \lesssim |\ln(e+t)|^{-\frac{n}{4}}, \\ \|\boldsymbol{u}(t)\| \lesssim (1+t)^{-1} \cdot |\ln(e+t)|^{-\frac{n}{4}-\frac{1}{2}}. \end{cases} \tag{4.18}$$

The above decay rates are optimal and consistent with the optimal decay rates of the linearized hyperbolic system.

Proof. The outline of this proof is similar to Proposition 3.11 for the case of $\lambda \in (-1, 0)$. We claim that the a priori decay estimate

$$\Phi_N(t) + \Psi_0(t) \leq \delta_0 \tag{4.19}$$

holds for all the time $t > 0$, under the small energy assumption of initial data $\|(v_0, \boldsymbol{u}_0)\|_{L^1\cap H^N} \le \varepsilon_0$, where ε_0 and δ_0 are positive constants to be determined. Lemma 4.7 tells us that for $n \ge 7$

$$\Psi_0(T) \le \sup_{t\in(0,T)} \Big\{ |\ln(e+t)|^{\frac{n}{4}} \|v\|, (1+t)\cdot|\ln(e+t)|^{\frac{n}{4}+\frac{1}{2}}\|\boldsymbol{u}\| \Big\} \lesssim \varepsilon_0 + \delta_0^2. \tag{4.20}$$

According to the time-weighted iteration scheme (4.14) and (4.15) in Lemma 4.4 and the estimates of inhomogeneous terms in Lemmas 4.5 and 4.6, we have for integer $0 \le k \le N-1$ that

$$\begin{aligned}
&\Phi_1^2(t) + \int_0^t \Psi_1^2(s)ds \\
&\quad \lesssim \Phi_1^2(0) + J_1(t) + \int_0^t (1+s)^{-1}\cdot|\ln(e+s)|^{\delta-1-\frac{n}{2}}\cdot\Psi_0^2(s)ds \\
&\qquad + \delta_0 \int_0^t \Psi_1^2(s)ds + \delta_0 \int_0^t (1+s)^{-1}\cdot|\ln(e+s)|^{-\frac{n}{4}}\cdot\Psi_0^2(s)ds, \\
&\Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s)ds \\
&\quad \lesssim \Phi_{k+1}^2(0) + J_{k+1}(t) + \int_0^t \Psi_k^2(s)ds + \delta_0\int_0^t \Psi_{k+1}^2(s)ds + \delta_0 \int_0^t \Psi_k^2(s)ds,
\end{aligned}$$

where

$$\begin{aligned}
J_1(t) &\lesssim \|v\|_{L^\infty}\cdot\Phi_1^2(t) \lesssim (\Phi_N(t)+\Psi_0(t))\cdot\Phi_1^2(t) \lesssim \delta_0\Phi_1^2(t), \\
J_{k+1}(t) &\lesssim \|v\|_{L^\infty}\cdot\Phi_{k+1}^2(t) \lesssim (\Phi_N(t)+\Psi_0(t))\Phi_{k+1}^2(t) \lesssim \delta_0\Phi_{k+1}^2(t).
\end{aligned}$$

We simplify the above inequalities as (note that δ_0 and ε_0 are small)

$$\Phi_1^2(t) + \int_0^t \Psi_1^2(s)ds \lesssim \varepsilon_0^2 + \Psi_0^2(t)\int_0^t (1+s)^{-1}\cdot|\ln(e+s)|^{\max\{\delta-1-\frac{n}{2},-\frac{n}{4}\}}ds, \tag{4.21}$$

$$\Phi_{k+1}^2(t) + \int_0^t \Psi_{k+1}^2(s)ds \lesssim \varepsilon_0^2 + \int_0^t (1+s)^{-1-\lambda}\cdot\Psi_k^2(s)ds. \tag{4.22}$$

Multiplying (4.22) by small positive constants for $0 \le k \le N-1$, summing the resulting inequalities up together with (4.21), we have

$$\sum_{1\le j\le N} \Phi_j^2(t) \lesssim \varepsilon_0^2 + \Psi_0^2(t)\int_0^t (1+s)^{-1}\cdot|\ln(e+s)|^{\max\{\delta-1-\frac{n}{2},-\frac{n}{4}\}}ds \lesssim \varepsilon_0^2 + (\varepsilon_0+\delta_0^2)^2,$$

according to the estimate (4.20) and $\max\{\delta-1-\frac{n}{2},-\frac{n}{4}\} < -1$ for $n \ge 5$. Therefore,

$$\Phi_N(t) + \Psi_0(t) \lesssim \varepsilon_0 + \delta_0^2 \le \delta_0$$

for positive constants ε_0 and δ_0 small enough.

The optimal property of the decay estimates (4.18) follows from the estimates on $\|v\|$ and $\|\boldsymbol{u}\|$ in Lemma 4.7 through a similar procedure as in the proof of Theorem 1.1. ∎

Proof of Theorem 1.7. The critical case of $\lambda = -1$ is proved in Proposition 4.9. ∎

Appendix A. Time-dependent damped linear system. The optimal decay estimates of the time-dependent damped wave equations (2.5) and (2.6) with overdamping $\lambda \in [-1, 0)$ are formulated in a procedure similar to the underdamping case $\lambda \in [0, 1)$ in [23], but modifications should be made. Here we omit the details for the sake of simplicity. The frequency analysis implies the following optimal decay estimates of the wave equations (2.5) and (2.6). Note that the time decay function $\Gamma(t, s)$ is defined in (1.4).

THEOREM A.1 (optimal decay rates of linear wave equations). *Let $v(t,x)$ and $u(t,x)$ be the solutions of the Cauchy problems* (2.5) *and* (2.6) *corresponding to given initial data $(v(s,x), \partial_t v(s,x))$ and $(u(s,x), \partial_t u(s,x))$ starting from the time s, respectively. For $q \in [2,\infty]$, $1 \le p, r \le 2$, and $\lambda \in [-1, 0)$, we have*

(A.1)
$$\begin{aligned}\|\partial_x^\alpha v\|_{L^q} \lesssim\, & \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\Big(\big\|v(s,\cdot)\big\|_{L^p}^l + \big\|\partial_x^{|\alpha|+\omega_{r,q}} v(s,\cdot)\big\|_{L^r}^h\Big)\\ &+ (1+s)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\Big(\big\|\partial_t v(s,\cdot)\big\|_{L^p}^l + \big\|\partial_x^{|\alpha|-1+\omega_{r,q}}\partial_t v(s,\cdot)\big\|_{L^r}^h\Big)\end{aligned}$$

and

(A.2)
$$\begin{aligned}\|\partial_x^\alpha u\|_{L^q} \lesssim\, & \Big(\frac{1+t}{1+s}\Big)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\Big(\big\|u(s,\cdot)\big\|_{L^p}^l + \big\|\partial_x^{|\alpha|+\omega_{r,q}} u(s,\cdot)\big\|_{L^r}^h\Big)\\ &+ (1+t)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\Big(\big\|\partial_t u(s,\cdot)\big\|_{L^p}^l + \big\|\partial_x^{|\alpha|-1+\omega_{r,q}}\partial_t u(s,\cdot)\big\|_{L^r}^h\Big),\end{aligned}$$

where $\gamma_{p,q} := n(1/p - 1/q)$, and $\omega_{r,q} > \gamma_{r,q}$ for $(r,q) \ne (2,2)$ and $\omega_{2,2} = 0$.

The decay estimates (A.1) *and* (A.2) *are optimal for all $t \ge s \ge 0$. Moreover, there exists a $T_0 \ge 0$ such that the decay estimates* (A.1) *and* (A.2) *are element-by-element optimal for all $\frac{t}{2} \ge s \ge T_0$.*

Remark A.2. The decay estimate (A.1) for $s = 0$ was first proved by Wirth [48] by developing a perfect diagonalization method. For the application to nonlinear systems, we need to consider the evolution of initial data starting from any $s \ge 0$ to $t \ge s$ since the damping is time-dependent. One of the main difficulties caused by the time-dependent damping is that the evolution of the initial data starting from $s \ge 0$ to $t \ge s$ is completely different from that starting from 0 to $t - s$.

Remark A.3. The two Cauchy problems (2.5) and (2.6) decay with different rates. We note that the function

$$\varphi(t,x) := \begin{cases} \dfrac{1}{(1+t)^{\frac{1+\lambda}{2}n}} e^{-\frac{\mu(1+\lambda)|x|^2}{4(1+t)^{1+\lambda}}}, & \lambda \in (-1, 0),\\[2ex] \dfrac{1}{|\ln(e+t)|^{\frac{n}{2}}} e^{-\frac{\mu|x|^2}{4\ln(e+t)}}, & \lambda = -1,\end{cases}$$

which satisfies $\frac{\mu}{(1+t)^\lambda}\partial_t\varphi = \Delta\varphi$, is an asymptotic profile of (2.5), while the function $\psi(t,x) := \varphi(t,x)/(\frac{\mu}{(1+t)^\lambda})$, which satisfies $\partial_t(\frac{\mu}{(1+t)^\lambda}\psi) = \Delta\psi$, is a good asymptotic profile of (2.6), and $\psi(t,x)$ decays faster than $\varphi(t,x)$.

Proof of Theorem A.1. The estimate (A.1) for $s = 0$ was proved by Wirth [48]. Here we focus on the influence of s and show that $u(t,x)$ decays optimally faster than $v(t,x)$. The results are proved through the same procedure as Proposition 2.1 in [23] according to the optimal decay estimates on the Fourier multiplies. ∎

We next show the optimal decay estimates of the linear hyperbolic system (2.4).

THEOREM A.4 (optimal decay rates of linear hyperbolic system). *Let* $(v(t,x), u(t,x))$ *be the solution of the linear hyperbolic system* (2.4) *(the third equation of* $\boldsymbol{w}(t,x)$ *is neglected as it decays superexponentially) corresponding to the initial data* $(v(s,x),u(s,x))$ *starting from time* s*. There exists a universal constant* $T_0 \geq 0$ *such that for* $q \in [2,\infty]$, $1 \leq p,r \leq 2$, $\lambda \in [-1,0)$, *and* $t \geq s \geq T_0$, *we have*

$$\begin{aligned}\|\partial_x^\alpha v\|_{L^q} \lesssim{}& \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h\right)\\ &+ (1+s)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right)\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}\|\partial_x^\alpha u\|_{L^q} \lesssim{}& \left(\frac{1+t}{1+s}\right)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right)\\ &+ (1+t)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h\right),\end{aligned} \tag{A.4}$$

where $\gamma_{p,q} := n(1/p-1/q)$, *and* $\omega_{r,q} > \gamma_{r,q}$ *for* $(r,q) \neq (2,2)$ *and* $\omega_{2,2} = 0$.

Moreover, $u(t,x)$ *decays faster than* (A.4) *if we assume one-order-higher regularity as follows:*

$$\begin{aligned}\|\partial_x^\alpha u\|_{L^q} \lesssim{}& (1+t)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+1+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h\right)\\ &+ (1+t)^\lambda(1+s)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+2}(t,s)\cdot\left(\left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+1+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right).\end{aligned} \tag{A.5}$$

The decay estimate (A.4) *is improved by cancellation without one-order-higher regularity as follows:*

$$\begin{aligned}&\|\partial_x^\alpha u(t,\cdot)\|_{L^q}\\ &\quad\lesssim (1+t)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h\right)\\ &\qquad+ (1+t)^\lambda(1+s)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+2}(t,s)\cdot\left(\left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right)\\ &\qquad+ \left(\frac{1+t}{1+s}\right)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\frac{1}{(1+s)^{1-\lambda}} + e^{-\varepsilon_u((1+t)^{1-\lambda}-(1+s)^{1-\lambda})}\right)\\ &\qquad\cdot\left(\left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right),\end{aligned} \tag{A.6}$$

where $\varepsilon_u > 0$ *is a constant.*

The decay estimate (A.3) *is element-by-element optimal for all* $\frac{t}{2} \geq s \geq T_0$*; the decay estimate* (A.5) *is optimal with respect to* $v(s,x)$ *for all* $\frac{t}{2} \geq s \geq T_0$*; the decay estimates* (A.3) *and* (A.5) *are optimal for all* $t \geq s \geq 0$ *such that*

$$\begin{aligned}\|\partial_x^\alpha v\|_{L^q} \approx{}& \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h\right.\\ &\left.+ \left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right)\end{aligned}$$

and

$$\begin{aligned}\|\partial_x^\alpha u\|_{L^q} \approx{}& (1+t)^\lambda \cdot \Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\left\|v(s,\cdot)\right\|_{L^p}^l\right.\\ &\left.+ \left\|\partial_x^{|\alpha|+1+\omega_{r,q}}v(s,\cdot)\right\|_{L^r}^h + \left\|u(s,\cdot)\right\|_{L^p}^l + \left\|\partial_x^{|\alpha|+1+\omega_{r,q}}u(s,\cdot)\right\|_{L^r}^h\right)\end{aligned}$$

for some nontrivial initial data.

Proof. The optimal decay estimates (A.3) and (A.5) are proved in a similar way as Proposition 2.2 in [23]. The proof of the decay estimate (A.6) improved by cancellation is similar to Proposition 2.3 in [23]. ∎

Theorem A.4 implies the optimal decay estimates of the Green matrix $\mathcal{G}(t,s)$ in (2.2).

THEOREM A.5. *For $q\in[2,\infty]$, $1\le p,r\le 2$, $t\ge s\ge T_0$ (T_0 is the universal constant in Theorem A.4), and $\lambda\in[-1,0)$, we have*

$$\|\partial_x^\alpha \mathcal{G}_{11}(t,s)\phi(x)\|_{L^q} \lesssim \Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+\omega_{r,q}}\phi\|^h_{L^r}\right),$$
$$\|\partial_x^\alpha \mathcal{G}_{12}(t,s)\phi(x)\|_{L^q} \lesssim (1+s)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+\omega_{r,q}}\phi\|^h_{L^r}\right),$$
$$\|\partial_x^\alpha \mathcal{G}_{21}(t,s)\phi(x)\|_{L^q} \lesssim (1+t)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|+1}(t,s)\cdot\left(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+\omega_{r,q}}\phi\|^h_{L^r}\right),$$
$$\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\|_{L^q} \lesssim \left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\cdot\left(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+\omega_{r,q}}\phi\|^h_{L^r}\right),$$

where $\gamma_{p,q}:=n(1/p-1/q)$, and $\omega_{r,q}>\gamma_{r,q}$ for $(r,q)\neq(2,2)$ and $\omega_{2,2}=0$. Furthermore,

$$\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\|_{L^q} \lesssim (1+t)^\lambda(1+s)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|+2}(t,s)\cdot(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+1+\omega_{r,q}}\phi\|^h_{L^r}),$$
$$\begin{aligned}\|\partial_x^\alpha \mathcal{G}_{22}(t,s)\phi(x)\| \lesssim{}& \left(\frac{1+t}{1+s}\right)^\lambda\cdot\Gamma^{\gamma_{p,q}+|\alpha|}(t,s)\\ &\cdot\left((1+s)^{2\lambda}\cdot\Gamma^2(t,s)+\frac{1}{(1+s)^{\lambda-1}}+C_\kappa\Gamma^\kappa(t,s)\right)\\ &\cdot(\|\phi\|^l_{L^p}+\|\partial_x^{|\alpha|+1+\omega_{r,q}}\phi\|^h_{L^r}),\end{aligned}$$

where $\kappa\ge 2$ can be chosen arbitrarily large and $C_\kappa>0$ is a constant depending on κ.

Proof. These estimates are conclusions of Theorem A.4. The last estimate is proved according to (A.6) since the superexponential function decays faster than any algebraical decays. ∎

Remark A.6. The decay estimate (A.4) for u in the linear system (2.4) derived from the optimal decay estimate (A.2) in Theorem A.1 is not optimal here since the initial data $u(0,x)=u_0(x)$ and $\partial_t u(0,x)=\Lambda v_0(x)-\mu u_0(x)$ are not independent. Cancellation occurs and the decay rate increases as in (A.5). However, the estimate (A.4) is still of importance in the decay estimates of the nonlinear system (1.3) since the regularity required is one order lower than that in the estimate (A.5).

Acknowledgments. The first author would like to express his sincere thanks for the hospitality of McGill University and the China Scholarship Council.

REFERENCES

[1] S. ALINHAC, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. Math., 149 (1999), pp. 97–127.
[2] S. ALINHAC, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions.* II, Acta Math., 182 (1999), pp. 1–23.
[3] T. BUCKMASTER, S. SHKOLLER, AND V. VICOL, *Formation of shocks for 2D isentropic compressible Euler*, Comm. Pure Appl. Math., 75 (2022), pp. 2069–2120.
[4] R. BURQ, G. RAUGEL, AND W. SCHLAG, *Long time dynamics for damped Klein-Gordon equations*, Ann. Sci. Éc. Norm. Supér., 50 (2015), pp. 1447–1498.
[5] R. BURQ, G. RAUGEL, AND W. SCHLAG, *Long Time Dynamics for Weakly Damped Nonlinear Klein-Gordon Equations*, https://arxiv.org/abs/1801.06735v1, 2018.

[6] G.-Q. Chen, C. Dafermos, M. Slemrod, and D. Wang, *On two-dimensional sonic-subsonic flow*, Comm. Math. Phys., 271 (2007), pp. 635–647.
[7] G. Chen, R. Pan, and S. Zhu, *Singularity formation for the compressible Euler equations*, SIAM J. Math. Anal., 49 (2017), pp. 2591–2614.
[8] S. G. Chen, H. Li, J. Li, M. Mei, and K. Zhang, *Global and blow-up solutions to compressible Euler equations with time-dependent damping*, J. Differential Equations, 268 (2020), pp. 5035–5077.
[9] D. Christodoulou, *The Formation of Shocks in* 3*-Dimensional Fluids*, EMS Monogr. Math., European Mathematical Society, Zürich, 2007.
[10] R. Courant and O. K. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag, New York, 1948.
[11] H.-B. Cui, H.-Y. Yin, J.-S. Zhang, and C.-J. Zhu, *Convergence to nonlinear diffusion waves for solutions of Euler equations with time-depending damping*, J. Differential Equations, 264 (2018), pp. 4564–4602.
[12] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 3rd ed., Springer-Verlag, New York, 2010.
[13] S. Geng and F. Huang, *L^1-convergence rates to the Barenblatt solution for the damped compressible Euler equations*, J. Differential Equations, 266 (2019), pp. 7890–7908.
[14] S. Geng, Y. Lin, and M. Mei, *Asymptotic behavior of solutions to Euler equations with time-dependent damping in critical case*, SIAM J. Math. Anal., 52 (2020), pp. 1463–1488.
[15] Y. Geng, Y. Li, D. Wang, and R. Xu, *Well-posedness of non-isentropic Euler equations with physical vacuum*, Interfaces Free Bound., 21 (2019), pp. 231–266.
[16] Y. Guo and B. Pausader, *Global smooth ion dynamics in the Euler-Poisson system*, Comm. Math. Phys., 303 (2011), pp. 89–125.
[17] F. Hou, I. Witt, and H. C. Yin, *Global existence and blowup of smooth solutions of 3-D potential equations with time-dependent damping*, Pacific J. Math., 292 (2018), pp. 389–426.
[18] F. Hou and H. C. Yin, *On the global existence and blowup of smooth solutions to the multi-dimensional compressible Euler equations with time-depending damping*, Nonlinearity, 30 (2017), pp. 2485–2517.
[19] L. Hsiao and T.-P. Liu, *Convergence to diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Phys., 143 (1992), pp. 599–605.
[20] F. M. Huang, P. Marcati, and R. H. Pan, *Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum*, Arch. Ration. Mech. Anal., 176 (2005), pp. 1–24.
[21] F. M. Huang and R. H. Pan, *Convergence rate for compressible Euler equations with damping and vacuum*, Arch. Ration. Mech. Anal., 166 (2003), pp. 359–376.
[22] F. M. Huang, R. Pan, and Z. Wang, *L^1 convergence to the Barenblatt solution for compressible Euler equations with damping*, Arch. Ration. Mech. Anal., 200 (2011), pp. 665–689.
[23] S. Ji and M. Mei, *Optimal decay rates of the compressible Euler equations with time-dependent damping in $\mathbb{R}^n$:* (I) *Under-damping case*, J. Nonlinear Sci., 33 (2023), Article 7.
[24] P. D. Lax, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, J. Math. Phys., 5 (1964), pp. 611–614.
[25] H. Li, J. Li, M. Mei, and K. Zhang, *Convergence to nonlinear diffusion waves for solutions of p-system with time-dependent damping*, J. Math. Anal. Appl., 456 (2017), pp. 849–871.
[26] H.-L. Li and X. Wang, *Formation of singularities of spherically symmetric solutions to the 3D compressible Euler equations and Euler-Poisson equations*, Nonlinear Differential Equations Appl., 25 (2018), pp. 1–15.
[27] J. Luk and J. Speck, *Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity*, Invent. Math., 214 (2018), pp. 1–169.
[28] T. Luo and H. H. Zeng, *Global existence of smooth solutions and convergence to Barenblatt solutions for the physical vacuum free boundary problem of compressible Euler equations with damping*, Comm. Pure Appl. Math., 69 (2016), pp. 1354–1396.
[29] P. Marcati and A. Milani, *The one-dimensional Darcy's law as the limit of a compressible Euler flow*, J. Differential Equations, 84 (1990), pp. 129–147.
[30] M. Mei, *Best asymptotic profile for hyperbolic p-system with damping*, SIAM J. Math. Anal., 42 (2010), pp. 1–23.
[31] S. Miao and P. Yu, *On the formation of shocks for quasilinear wave equations*, Invent. Math., 207 (2017), pp. 697–831.
[32] K. Nishihara, *Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping*, J. Differential Equations, 131 (1996), pp. 171–188.
[33] K. Nishihara, W. K. Wang, and T. Yang, *L_p-convergence rates to nonlinear diffusion waves for p-system with damping*, J. Differential Equations, 161 (2000), pp. 191–218.

[34] R. Pan and Y. Zhu, *Singularity formation for one dimensional full Euler equations*, J. Differential Equations, 261 (2016), pp. 7132–7144.

[35] X. Pan, *Blow up of solutions to* 1*-d Euler equations with time-dependent damping*, J. Math. Anal. Appl., 442 (2016), pp. 435–445.

[36] X. Pan, *Global existence of solutions to* 1*-d Euler equations with time-dependent damping*, Nonlinear Anal., 132 (2016), pp. 327–336.

[37] X. Pan, *Global existence and asymptotic behavior of solutions to the Euler equations with time-dependent damping*, Appl. Anal., 2020), pp. 1–30.

[38] T. Sideris, B. Thomases, and D. Wang, *Long time behavior of solutions to the* 3*D compressible Euler equations with damping*, Comm. Partial Differential Equations, 28 (2003), pp. 795–816.

[39] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1982.

[40] Y. Sugiyama, *Singularity formation for the* 1*D compressible Euler equations with variable damping coefficient*, Nonlinear Anal., 170 (2018), pp. 70–87.

[41] Y. Sugiyama, *Remark on global existence of solutions to the* 1*D compressible Euler equation with time-dependent damping*, in The Role of Metrics in the Theory of Partial Differential Equations, Adv. Stud. Pure Math. 85, Mathematical Society of Japan, Tokyo, 2020, pp. 379–389.

[42] Z. Tan and Y. Wang, *Global solution and large-time behavior of the* 3*D compressible Euler equations with damping*, J. Differential Equations, 254 (2013), pp. 1686–1704.

[43] Z. Tan and G. Wu, *Large time behavior of solutions for compressible Euler equations with damping in* $\mathbb{R}^3$, J. Differential Equations, 252 (2012), pp. 1546–1561.

[44] T. Tao, *Nonlinear Dispersive Equations, Local and Global Analysis*, CBMS Reg. Conf. Ser. Math. 106, AMS, Providence, RI, 2006.

[45] G. Todorova and B. Yordanov, *Weighted* L^2*-estimates for dissipative wave equations with variable coefficients*, J. Differential Equations, 246 (2009), pp. 4497–4518.

[46] D. Wang and G.-Q. Chen, *Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation*, J. Differential Equations, 144 (1998), pp. 44–65.

[47] J. Wirth, *Wave equations with time-dependent dissipation* I. *Non-effective dissipation*, J. Differential Equations, 222 (2006), pp. 487–514.

[48] J. Wirth, *Wave equations with time-dependent dissipation* II. *Effective dissipation*, J. Differential Equations, 232 (2007), pp. 74–103.

[49] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Methods Appl. Sci., 27 (2004), pp. 101–124.