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Structural stability of subsonic solutions to a steady hydrodynamic model for semiconductors: From the perspective of boundary data

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ABSTRACT

In this paper, we are concerned with a one-dimensional isothermal steady hydrodynamic model for semiconductors driven by boundary data. In the purely subsonic setting, we obtain the existence, uniqueness and structural stability of purely subsonic solutions. Moreover, when the boundary data range from the subsonic region to the sonic line, we further study the degenerate problem from the perspective of boundary data, and prove that there exists a unique interior subsonic solution to the degenerate problem. As a byproduct, we also establish the structural stability between purely subsonic solution and interior subsonic solution in a relatively weak sense. These results provide us with a completely new perspective to understand the singularity caused by the boundary degeneracy. A number of numerical simulations are also carried out, which confirm our theoretical results.

1. Introduction

In 1990, Degond et al. [1] first studied the steady hydrodynamic (HD) model for semiconductors, represented by Euler-Poisson equations. Its isothermal version in one dimension reads

$$\begin{cases} J = \text{const.}, \\ \left(T - \dfrac{J^2}{n^2}\right) n_x = nE - \dfrac{J}{\tau}, \\ E_x = n - b(x), \quad x \in (0,1), \end{cases} \tag{1.1}$$

prescribing boundary data

$$n(0) = n_l, \quad n(1) = n_r, \tag{1.2}$$

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where $n(x), J, E(x)$ denote the electron density, current density, electric field; the constant $T > 0$ is the lattice temperature; the given function $b \in C[0,1]$ is the doping profile, which satisfies $b(x) > 0$; the positive constant $\tau > 0$ is the momentum relaxation time; boundary data n_l, n_r are positive constants.

In order to make clear the issues we are interested in, we introduce some terms from gas dynamics. We call $c := \sqrt{T}$ the *speed of sound*. The state of steady flow is referred to as *subsonic*, *sonic* or *supersonic* provided the velocity J/n satisfies

$$\frac{J}{n} < c, \quad \frac{J}{n} = c \quad \text{or} \quad \frac{J}{n} > c.$$

We say a solution to the HD model (1.1) is *purely subsonic* if $n(x)$ is subsonic at both interior points and end points of the device interval $[0,1]$. We say a solution is *interior subsonic* if $n(x)$ is subsonic at interior points but sonic at the boundary.

In the purely subsonic setting, Degond et al. [1] obtained the existence and uniqueness of steady states to the isentropic HD model from the current-driven perspective. Thanks to the current–voltage relationship, Nishibata et al. [2] rebuilt the same well-posedness as [1] from the voltage-driven viewpoint.

In the case of sonic boundary data, Li et al. [3] recently studied the isothermal HD model (assuming $T = J = 1$ for simplicity)

$$\begin{cases} \left(1 - \dfrac{1}{n^2}\right) n_x = nE - \dfrac{1}{\tau}, \\ E_x = n - b(x), \quad x \in (0,1), \end{cases} \tag{1.3}$$

subject to sonic boundary conditions

$$n(0) = n(1) = 1. \tag{1.4}$$

Due to the boundary degeneracy, a series of unexpected results was observed in [3]. One of them is the existence and uniqueness of interior subsonic solutions $(n, E)(x)$ to the boundary value problem (1.3) and (1.4), under the restriction of subsonic doping $b(x) > 1$. The key point of their proof is that they constructed a family of current-driven approximate solutions $\{n_j(x)\}_{0<j<1}$. Once $n(x)$ was known, the $E(x)$ could be defined as

$$E(x) = \frac{(n+1)[(n-1)^2]_x}{2n^3} + \frac{1}{\tau n}. \tag{1.5}$$

However, it is more natural, from the point of view of semiconductor physics, to construct a family of boundary-data-driven approximate solutions. Namely, for any constant $\delta > 0$, we are supposed to consider the following approximate problem:

$$\begin{cases} \left(1 - \dfrac{1}{n_\delta^2}\right) n_{\delta,x} = n_\delta E_\delta - \dfrac{1}{\tau}, \\ E_{\delta,x} = n_\delta - b(x), \quad x \in (0,1), \end{cases} \tag{1.6}$$

with subsonic boundary data

$$n_\delta(0) = n_\delta(1) = 1 + \delta. \tag{1.7}$$

For notational convenience, we set

$$\underline{b} := \inf_{x \in [0,1]} b(x), \quad \bar{b} := \sup_{x \in [0,1]} b(x).$$

We now state the main results of this paper.

Theorem 1.1 (*Structural Stability Between Purely Subsonic Solutions*)**.** *Assume that the doping profile $b \in C[0,1]$ and $\underline{b} > 1$. Then for any $\delta > 0$ the subsonic boundary value problem* (1.6) and (1.7) *has a unique solution $(n_\delta, E_\delta) \in C^1[0,1] \times C^1[0,1]$ satisfying*

$$1 < \min\{1+\delta, \underline{b}\} \le n_\delta(x) \le \max\{1+\delta, \bar{b}\}, \quad \forall x \in [0,1]. \tag{1.8}$$

Furthermore, for arbitrary two parameters $\delta_1, \delta_2 > 0$*, there exists a positive constant* $C = C(\delta_{min}, \underline{b}, \bar{b}, \tau)$ *independent of* $|\delta_1 - \delta_2|$ *such that*

$$\|n_{\delta_1} - n_{\delta_2}\|_{C^1[0,1]} + \|E_{\delta_1} - E_{\delta_2}\|_{C^1[0,1]} \leq C|\delta_1 - \delta_2|, \tag{1.9}$$

where $\delta_{min} = \min\{\delta_1, \delta_2\}$*. In particular, if* $\delta_1 > \delta_2$*, then*

$$n_{\delta_1}(x) \geq n_{\delta_2}(x), \quad \forall x \in [0,1]. \tag{1.10}$$

Remark 1.1. The uniqueness result in the above theorem holds without the hypothesis of $\delta \gg 1$. As δ_{min} tends to zero, the singularity of solutions will occur, which is reflected in the generic constant of structural stability estimate (1.9). In fact, from the proof of Theorem 1.1 in Section 2, it is easy to see that $C(\delta_{min}, \underline{b}, \bar{b}, \tau) \to +\infty$ as $\delta_{min} \to 0^+$. Instead, if $\delta_{min} > \underline{b} - 1$, then this constant $C = C(\underline{b}, \bar{b}, \tau)$ will also be independent of δ_{min}. These essential phenomena are confirmed by numerical simulations as well.

Based on Theorem 1.1, we can rebuild the same existence and uniqueness result as in [3] by using a compactness argument. The corresponding result is displayed in the following proposition. The strategy of proof permits us to view this known result from a completely different perspective, see Section 3.

Proposition 1.2 (*Existence and Uniqueness of Interior Subsonic Solutions [3]*)**.** *Let the assumptions of Theorem 1.1 hold. Then the sonic boundary value problem (1.3)* and *(1.4) admits a unique solution* $(n, E) \in C[0,1] \times C^1[0,1]$ *which satisfies*

$$1 + \beta \sin(\pi x) \leq n(x) \leq \bar{b}, \quad \forall x \in [0,1], \tag{1.11}$$

where $\beta = \beta(\tau, \underline{b}) > 0$ *is a constant.*

As a byproduct of the alternative proof of the known result in Proposition 1.2, we can show the structural stability in pointwise sense, see (1.12). In particular, the lower bound in (1.11) ensures us the locally structural stability, see (1.13).

Theorem 1.3 (*Structural Stability Between Purely and Interior Subsonic Solutions*)**.** *Let* $\{n_\delta(x)\}_{\delta>0}$ *be the family of purely subsonic solutions in Theorem 1.1, and* $n(x)$ *be the interior subsonic solution with the sonic boundary (1.4) in Proposition 1.2. Then the pointwise structural stability holds, that is, for any fixed* $x \in [0,1]$,

$$n(x) \leq n_\delta(x), \quad \text{and} \ \lim_{\delta \to 0^+} (n_\delta - n)(x) = 0. \tag{1.12}$$

Moreover, the structural stability in locally strong topology also holds. Namely, for arbitrary constant $\varepsilon \in (0, 1/2)$,

$$\lim_{\delta \to 0^+} \|n_\delta - n\|_{C[\varepsilon, 1-\varepsilon]} = 0. \tag{1.13}$$

Remark 1.2. In view of Eqs. $(1.3)_2$ and $(1.6)_2$, we have $E_{\delta,x} - E_x = n_\delta - n$. So, it is easy to see that $E_{\delta,x} - E_x$ also satisfies the structural stability results (1.12) and (1.13). This conclusion is also numerically observed. Unfortunately, one cannot directly get the structural stability between $(n_{\delta,x}, E_\delta)$ and (n_x, E) by letting δ_{min} tend to zero in (1.9), see Remark 1.1. Actually, as $\delta \to 0^+$ in the BVP (2.1), the ellipticity of the quasilinear equation $(2.1)_1$ will be degenerate at the boundary. Therefore, we are unable to establish the uniform estimate of $n_{\delta,x}(x)$ with respect to the parameter $\delta > 0$, even in $L^2(0,1)$-norm. This difficulty leads to the failure of establishing the structural stability between $(n_{\delta,x}, E_\delta)$ and (n_x, E). However, the numerical simulations seemingly show that $(n_{\delta,x}, E_\delta)$ will, to some extent, converge to (n_x, E). Inspired by Feng et al. [4], we presume that a certain weight should be introduced to control the singularities caused by setting $\delta \to 0^+$. To completely figure out this problem will be more complicated than the work in [4], and we shall work on it in our future study.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we give a unified proof for both Proposition 1.2 and Theorem 1.3 in the same time. Section 4 provides the reader with some numerical simulations in order to better understand our theoretical results.

2. Proof of Theorem 1.1

Firstly, we recast the system (1.6) by differentiating and dividing by n_δ. We obtain the following second order boundary value problem (BVP), parametrized by $\delta \in \mathbb{R}_+$:

$$\text{(subsonic BVP)} \quad \begin{cases} Q(n_\delta) = 0, & x \in (0,1), \\ n_\delta(0) = n_\delta(1) = 1 + \delta, \end{cases} \tag{2.1}$$

where the quasilinear differential operator Q is defined as

$$Q(\rho) := \left[\left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x + \frac{1}{\tau\rho}\right]_x - (\rho - b(x)). \tag{2.2}$$

Once n_δ is known from the subsonic BVP (2.1), E_δ can be easily computed by the equation $(1.6)_1$ as follows:

$$E_\delta(x) = F(n_\delta)n_{\delta,x} + \frac{1}{\tau n_\delta}, \quad \forall x \in [0,1], \tag{2.3}$$

where the function F is defined as

$$F(\rho) := \frac{\rho^2 - 1}{\rho^3}, \quad \forall \rho \in (1, +\infty). \tag{2.4}$$

So, for the regular solution, the BVP (1.6) & (1.7) is equivalent to the BVP (2.1) together with the formula (2.3).

In much the same way as in [1], one can prove the following existence result with ease, thus we omit the details. The key idea of the proof is the Schauder Fixed Point Theorem along with the maximum principle of linear elliptic equations.

Lemma 2.1. *Suppose that $b \in C[0,1]$ and $\underline{b} > 1$. Let $\delta \in \mathbb{R}_+$. Then the BVP (2.1) has a solution $n_\delta \in C^1[0,1]$ satisfying the lower and upper bounds*

$$\min\{1+\delta, \underline{b}\} \le n_\delta(x) \le \max\{1+\delta, \bar{b}\}, \quad \forall x \in [0,1]. \tag{2.5}$$

It remains for us to prove the uniqueness result for any $\delta > 0$. To this end, we have to establish the following comparison principle for the operator Q in (2.2). Henceforth, we say a function ρ satisfies $Q(\rho) \ge 0$ $(= 0, \le 0)$ in $(0,1)$ if

$$Q(\rho, \varphi) = \int_0^1 \left\{\left[\left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x + \frac{1}{\tau\rho}\right]\varphi_x - (b(x) - \rho)\varphi\right\} dx \le 0 \ (= 0, \ \ge 0) \tag{2.6}$$

for all non-negative $\varphi \in H_0^1(0,1)$.

Lemma 2.2 (*Comparison Principle*)**.** *Let $1 < u, v \in C^1[0,1]$ satisfy $Q(u) \ge 0$ in $(0,1)$, $Q(v) \le 0$ in $(0,1)$ and $u \le v$ at both endpoints. Then $u \le v$ in $(0,1)$.*

Proof. The basic idea is attributed to [5], we adapt it to our use. So, we include the full proof here. Let us define

$$A(y,z) := \left(\frac{1}{y} - \frac{1}{y^3}\right)z + \frac{1}{\tau y}, \quad w := u - v,$$

$$u_t := v + tw, \quad \forall t \in [0,1].$$

Then we have

$$\begin{aligned} 0 &\geq Q(u,\varphi) - Q(v,\varphi) \\ &= \int_0^1 \Big(A(u_t, u_{t,x})|_{t=0}^{t=1} \ \varphi_x + w\varphi \Big) dx \\ &= \int_0^1 \Big[\big(a(x) w_x + c(x) w \big) \varphi_x + w\varphi \Big] dx \\ &= L(w,\varphi) \end{aligned} \tag{2.7}$$

for all non-negative $\varphi \in H_0^1(0,1)$, and L is the linear differential operator given by

$$L(w) := \big(a(x) w_x + c(x) w \big)_x - w, \tag{2.8}$$

where

$$a(x) := \int_0^1 A_z(u_t, u_{t,x}) dt, \quad c(x) := \int_0^1 A_y(u_t, u_{t,x}) dt.$$

Since $1 < u, v \in C^1[0,1]$, then we know $1 < u_t \in C^1[0,1]$ for all $t \in [0,1]$, and thereby there exist positive constants λ, Λ such that

$$a(x) \geq \lambda > 0, \quad |a(x)|, |c(x)| \leq \Lambda, \quad \forall x \in [0,1],$$

and therefore L is uniformly elliptic in $[0,1]$ with bounded coefficients. For $\alpha > 0$, take $\varphi = \frac{w^+}{w^+ + \alpha} \in H_0^1(0,1)$, where $w^+ := \max\{0, w\}$. From (2.7), we have

$$\lambda \int_0^1 \left| \left[\ln \left(1 + \frac{w^+}{\alpha} \right) \right]_x \right|^2 dx \leq \Lambda \int_0^1 \left| \left[\ln \left(1 + \frac{w^+}{\alpha} \right) \right]_x \right| dx, \tag{2.9}$$

which in turn implies

$$\int_0^1 \left| \ln \left(1 + \frac{w^+}{\alpha} \right) \right|^2 dx \leq C(\lambda, \Lambda) < +\infty. \tag{2.10}$$

Letting $\alpha \to 0^+$, we see that w^+ must vanish in $(0,1)$, that is $u \leq v$ in $(0,1)$. □

Proof of Theorem 1.1. Apparently, it follows from Lemmas 2.1 and 2.2 that the problem (1.6) and (1.7) has a unique subsonic solution (n_δ, E_δ) for any $\delta > 0$. By Lemma 2.2 again, we can also observe the following monotonicity relation: if $\delta_1 > \delta_2 > 0$, then

$$n_{\delta_1}(x) \geq n_{\delta_2}(x), \quad \forall x \in [0,1]. \tag{2.11}$$

For arbitrary two positive constants δ_1, δ_2, without loss of generality, we assume $\delta_1 > \delta_2$. Thus, $\delta_{min} = \delta_2$. We also adopt throughout the convention that if the generic constant C depends on δ, then it must tend to $+\infty$ as $\delta \to 0^+$.

Before setting about the structural stability, it is convenient to have the estimates for the BVP (2.1) at our disposal:

$$\|n_{\delta,x}\|_{C[0,1]} \leq C(\delta, \underline{b}, \bar{b}, \tau), \quad \forall \delta > 0, \tag{2.12}$$

$$|(n_{\delta_1} - n_{\delta_2})_x|(0) \leq C(\delta_2, \tau)(\delta_1 - \delta_2), \quad \forall \delta_1 > \delta_2 > 0, \tag{2.13}$$

which are guaranteed by using the higher regularity and boundary gradient estimate for quasilinear elliptic equations (see [5,6]). In view of (2.3), (2.5), (2.12) and (2.13), it is appropriate at this juncture to calculate following estimates:

$$|E_\delta(x)| = \left| F(n_\delta) n_{\delta,x} + \frac{1}{\tau n_\delta} \right|$$

$$\begin{aligned}
&\le (n_\delta^2-1)|n_{\delta,x}| + \frac{1}{\tau}\\
&\le C(\delta,\underline{b},\bar{b},\tau), \quad \forall x\in[0,1], \quad \forall \delta>0, \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
|E_{\delta_1}-E_{\delta_2}|(0) =& \left|\left(F(n_{\delta_1})n_{\delta_1,x}+\frac{1}{\tau n_{\delta_1}}\right)-\left(F(n_{\delta_2})n_{\delta_2,x}+\frac{1}{\tau n_{\delta_2}}\right)\right|(0)\\
\le& \left[\left(\frac{1}{n_{\delta_1}n_{\delta_2}^3}+\frac{1}{n_{\delta_1}^2n_{\delta_2}^2}+\frac{1}{n_{\delta_1}^3n_{\delta_2}}\right)|n_{\delta_1,x}|+\frac{1}{\tau n_{\delta_1}n_{\delta_2}}\right](n_{\delta_1}-n_{\delta_2})(0)\\
&+F(n_{\delta_2})|(n_{\delta_1}-n_{\delta_2})_x|(0)\\
\le& \left(3|n_{\delta_1,x}|(0)+\frac{1}{\tau}\right)(\delta_1-\delta_2)+F(1+\delta_2)|(n_{\delta_1}-n_{\delta_2})_x|(0)\\
\le& C(\delta_2,\underline{b},\bar{b},\tau)(\delta_1-\delta_2), \quad \forall \delta_1>\delta_2>0. \qquad (2.15)
\end{aligned}$$

Based on estimates (2.14) and (2.15), we can now discuss the structural stability. From (1.6) and (1.7), taking the difference of working equations for both n_{δ_1} and n_{δ_2}, we have

$$\begin{cases}
(n_{\delta_1}-n_{\delta_2})_x = \left[f(n_{\delta_1})E_{\delta_1}-\frac{1}{\tau}g(n_{\delta_1})\right]-\left[f(n_{\delta_2})E_{\delta_2}-\frac{1}{\tau}g(n_{\delta_2})\right],\\
(E_{\delta_1}-E_{\delta_2})_x = n_{\delta_1}-n_{\delta_2}, \quad x\in(0,1),\\
(n_{\delta_1}-n_{\delta_2})(0) = (n_{\delta_1}-n_{\delta_2})(1) = \delta_1-\delta_2>0,
\end{cases} \qquad (2.16)$$

where

$$f(\rho) := \frac{\rho^3}{\rho^2-1}, \quad g(\rho) := \frac{\rho^2}{\rho^2-1}, \quad \forall \rho\in(1,+\infty), \qquad (2.17)$$

and thereby

$$f'(\rho) = \frac{\rho^2(\rho^2-3)}{(\rho^2-1)^2}, \quad g'(\rho) = -\frac{2\rho}{(\rho^2-1)^2}. \qquad (2.18)$$

Multiplying through $(2.16)_1$ by $n_{\delta_1}-n_{\delta_2}$, and $(2.16)_2$ by $E_{\delta_1}-E_{\delta_2}$, and adding these two resultant identities, we compute by using the mean-value theorem of differentials and Cauchy's inequality that

$$\begin{aligned}
&\left[(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\right]_x\\
\le& \left(2|f'(\xi)\parallel E_{\delta_1}|+\frac{2}{\tau}|g'(\eta)|+f(n_{\delta_2})\right)(n_{\delta_1}-n_{\delta_2})^2+f(n_{\delta_2})(E_{\delta_1}-E_{\delta_2})^2\\
&+(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\\
\le& \left(2|f'(1+\delta_2)\parallel E_{\delta_1}|+\frac{2}{\tau}|g'(1+\delta_2)|+f(1+\delta_2)\right)(n_{\delta_1}-n_{\delta_2})^2+f(1+\delta_2)(E_{\delta_1}-E_{\delta_2})^2\\
&+(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\\
\le& C(\delta_2,\underline{b},\bar{b},\tau)\left[(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\right], \quad \forall x\in(0,1), \qquad (2.19)
\end{aligned}$$

where we have used the estimate (2.14) for $\delta=\delta_1(>\delta_2)$. Applying Gronwall's inequality to (2.19), we obtain

$$\begin{aligned}
\left[(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\right](x) \le& \exp\left(\int_0^x C(\delta_2,\underline{b},\bar{b},\tau)dy\right)\left[(n_{\delta_1}-n_{\delta_2})^2+(E_{\delta_1}-E_{\delta_2})^2\right](0)\\
\le& C(\delta_2,\underline{b},\bar{b},\tau)(\delta_1-\delta_2)^2, \quad \forall x\in[0,1], \qquad (2.20)
\end{aligned}$$

where we have used the estimate (2.15). The estimate (2.20) further implies, for any $x \in [0,1]$,

$$|n_{\delta_1} - n_{\delta_2}|(x) + \left|(n_{\delta_1} - n_{\delta_2})_x\right|(x) + |E_{\delta_1} - E_{\delta_2}|(x) + \left|(E_{\delta_1} - E_{\delta_2})_x\right|(x) \le C(\delta_2, \underline{b}, \bar{b}, \tau)(\delta_1 - \delta_2), \quad (2.21)$$

where we have used the estimate (2.14), Eqs. $(2.16)_1$ and $(2.16)_2$ again. An inspection of the above arguments shows that the generic constant $C = C(\delta_2, \underline{b}, \bar{b}, \tau)$ in (2.21) has the properties: (1) C is independent of $\delta_1 - \delta_2$; (2) if $\delta_2 > \underline{b} - 1$, then C is also independent of δ_2; (3) if $\delta_2 \le \underline{b} - 1$, then C depends on δ_2 and tends to $+\infty$ as $\delta_2 \to 0^+$. Thus the proof of the theorem is complete. □

3. Proof of Theorem 1.3

Actually, we find that a family of $\{n_\delta(x)\}_{\delta>0}$ can also be used to approximate the interior subsonic solution $n(x)$. Now, we start off by proving Proposition 1.2. For this reason, multiplying through $(1.6)_1$ by $\frac{[(n_\delta - 1 - \delta)^2]_x}{n_\delta}$, integrating the resultant over $[0,1]$ together with $(1.6)_2$, integration by parts gives

$$\int_0^1 \frac{(n_\delta + 1)\left\{\left[(n_\delta - 1 - \delta)^2\right]_x\right\}^2}{2n_\delta^3} dx \le \int_0^1 (b - n_\delta)(n_\delta - 1 - \delta)^2 \, dx. \quad (3.1)$$

Applying Young's inequality and Poincaré's inequality to the right-hand side of (3.1), we have

$$\left(\frac{1}{\bar{b}}\right)^3 \int_0^1 \left\{\left[(n_\delta - 1 - \delta)^2\right]_x\right\}^2 dx \le \frac{\mu}{2} \int_0^1 \left\{\left[(n_\delta - 1 - \delta)^2\right]_x\right\}^2 dx + \frac{1}{2\mu}\bar{b}^2, \quad (3.2)$$

where we have also used the estimate (1.8) by letting $0 < \delta \le \bar{b} - 1$. Taking $\mu = \bar{b}^{-3}$, there exists a positive constant $C = C(\bar{b})$ which is independent of δ such that

$$\left\|(n_\delta - 1 - \delta)^2\right\|_{H^1(0,1)} \le C, \quad \forall \delta \in (0, \bar{b} - 1]. \quad (3.3)$$

Consequently, by the compact imbedding of $H^1(0,1)$ into $C[0,1]$, there is a subsequence $\{n_{\delta_k}\}$ and a function $n(x)$ such that

$$\begin{aligned} (n_{\delta_k} - 1 - \delta_k)^2 &\rightharpoonup (n-1)^2 \quad \text{weakly in } H^1(0,1) \text{ as } \delta_k \to 0, \\ (n_{\delta_k} - 1 - \delta_k)^2 &\to (n-1)^2 \quad \text{strongly in } C[0,1] \text{ as } \delta_k \to 0. \end{aligned} \quad (3.4)$$

Obviously, the function $n(x)$ exactly satisfies the problem (1.3) and (1.4) in the weak sense, for the weak form, we refer the readers to [3]. Thus, the function $n(x)$ is the unique interior subsonic solution which has already been known in [3]. Due to the uniqueness of $n(x)$, it is easy to see that the strong convergence result in (3.4) can be improved to the following

$$\lim_{\delta \to 0^+} \|N_\delta - N\|_{C[0,1]} = 0, \quad (3.5)$$

where

$$N_\delta(x) := (n_\delta - 1 - \delta)^2(x), \quad N(x) := (n-1)^2(x), \quad \forall x \in [0,1]. \quad (3.6)$$

To complete the proof of Proposition 1.2, we still need to establish the lower bound of the interior subsonic solution $n(x)$. To this end, for $0 < \delta \le (\underline{b} - 1)/2$, we set

$$q(x) := (1 + \delta) + \beta \sin(\pi x), \quad \forall x \in [0,1], \quad (3.7)$$

where the small positive constant β will be determined later. From (2.2), we calculate by a priori letting $\beta \ll 1$ that

$$\begin{aligned} Q(q) &= \left[\left(\frac{1}{q} - \frac{1}{q^3}\right) q_x + \frac{1}{\tau q}\right]_x - (q - b(x)) \\ &\geq - \left| q_{xx} - \frac{1}{\tau q^2} q_x - q \right| + \underline{b} \\ &\geq - \left(\pi^2 + \frac{\pi}{\tau} + 1\right)\beta + (\underline{b} - 1 - \delta) \\ &\geq - \left(\pi^2 + \frac{\pi}{\tau} + 1\right)\beta + \frac{\underline{b} - 1}{2}. \end{aligned} \tag{3.8}$$

Therefore, there exists a small positive constant $\beta = \beta(\tau, \underline{b})$ such that $Q(q) \geq 0$ in $(0, 1)$, thereby implying by Lemma 2.2 again

$$(1 + \delta) + \beta \sin(\pi x) \leq n_\delta(x) \leq \bar{b}, \quad \forall x \in [0, 1], \quad \delta \in \left(0, \frac{\underline{b} - 1}{2}\right]. \tag{3.9}$$

Next, we compute

$$\begin{aligned} 0 \leq |n_\delta - n|(x) &\leq \left| N_\delta^{1/2} + 1 + \delta - N^{1/2} - 1 \right|(x) \\ &= \left| \frac{N_\delta - N}{N_\delta^{1/2} + N^{1/2}} \right|(x) + \delta \\ &\leq \frac{1}{\beta \sin(\pi x)} |N_\delta - N|(x) + \delta, \quad \forall x \in (0, 1). \end{aligned} \tag{3.10}$$

Letting $\delta \to 0^+$, we have

$$\lim_{\delta \to 0^+} |n_\delta - n|(x) = 0, \quad \forall x \in [0, 1]. \tag{3.11}$$

Combining (2.11), (3.9) and (3.11), we obtain

$$1 + \beta \sin(\pi x) \leq n(x) \leq n_\delta(x), \quad \lim_{\delta \to 0^+} (n_\delta - n)(x) = 0, \quad \forall x \in [0, 1]. \tag{3.12}$$

Moreover, for arbitrary constant $\varepsilon \in (0, 1/2)$, note that $0 < \beta \sin(\pi\varepsilon) \leq \beta \sin(\pi x)$ for all $x \in [\varepsilon, 1 - \varepsilon]$, then it follows from (3.10) that

$$0 \leq \|n_\delta - n\|_{C[\varepsilon, 1-\varepsilon]} \leq \frac{1}{\beta \sin(\pi\varepsilon)} \|N_\delta - N\|_{C[0,1]} + \delta \tag{3.13}$$

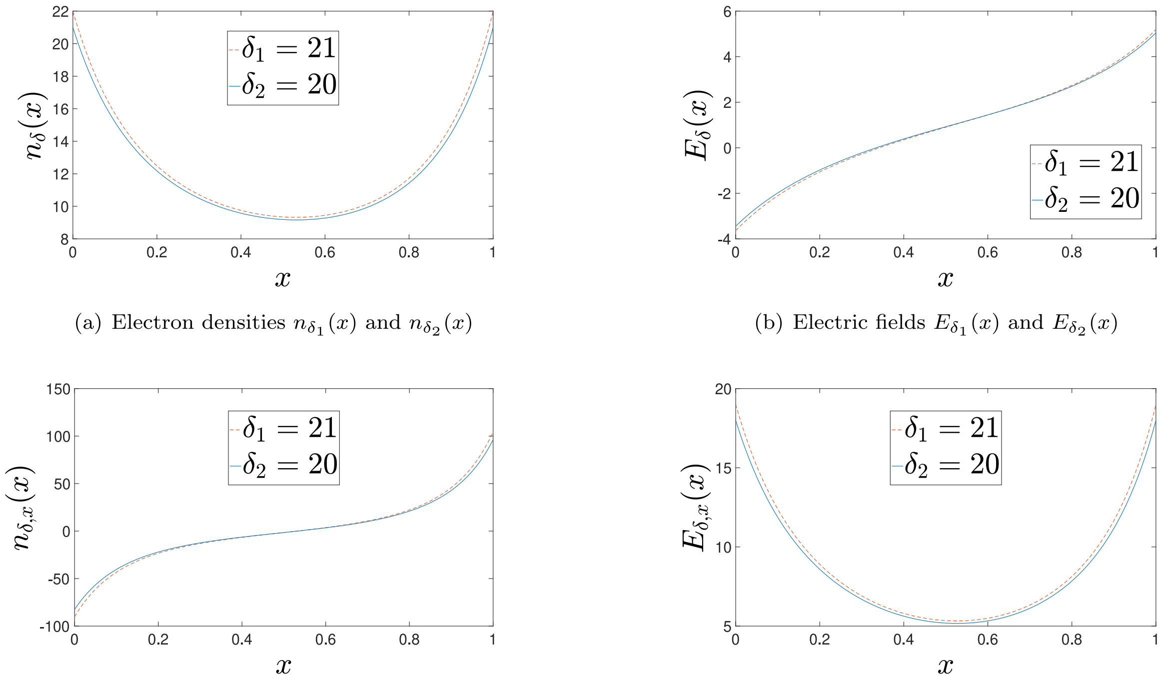
Letting $\delta \to 0^+$ in the above inequality, from (3.5), we get

$$\lim_{\delta \to 0^+} \|n_\delta - n\|_{C[\varepsilon, 1-\varepsilon]} = 0. \tag{3.14}$$

From all the analyses above, we simultaneously prove Proposition 1.2 and Theorem 1.3. □

4. Numerical simulations

In this section, we present numerical simulations to confirm our theoretical results. The computational interval is $[0, 1]$ with 100,000 uniform mesh points. We use the `bvp5c` solver in `MATLAB` to numerically study a specific case of the BVP (1.6) and (1.7), where we choose the `initial guess` as $[1 + \delta; 1/\tau]$ for $(n_\delta, E_\delta)(0)$. The choice is reasonable because we have known that in boundary degenerate case (i.e. $\delta = 0$) the electric



(a) Electron densities $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

(b) Electric fields $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

(c) First derivatives of $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

(d) First derivatives of $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

Fig. 1. Structural stability between $(n_{\delta_1}, E_{\delta_1})(x)$ and $(n_{\delta_2}, E_{\delta_2})(x)$ in the large (L) scale $(21 = \delta_1 > \delta_2 = 20)$.

field $E(x)$ satisfies $E(0) = 1/\tau$ and $E(1) < 1/\tau$ (see [3]). More precisely, we take the value $\tau = 0.1$ and let the doping profile be the function $b(x) = 3 + \sin(\pi x)$. So, the example in use is the following, for any $\delta > 0$,

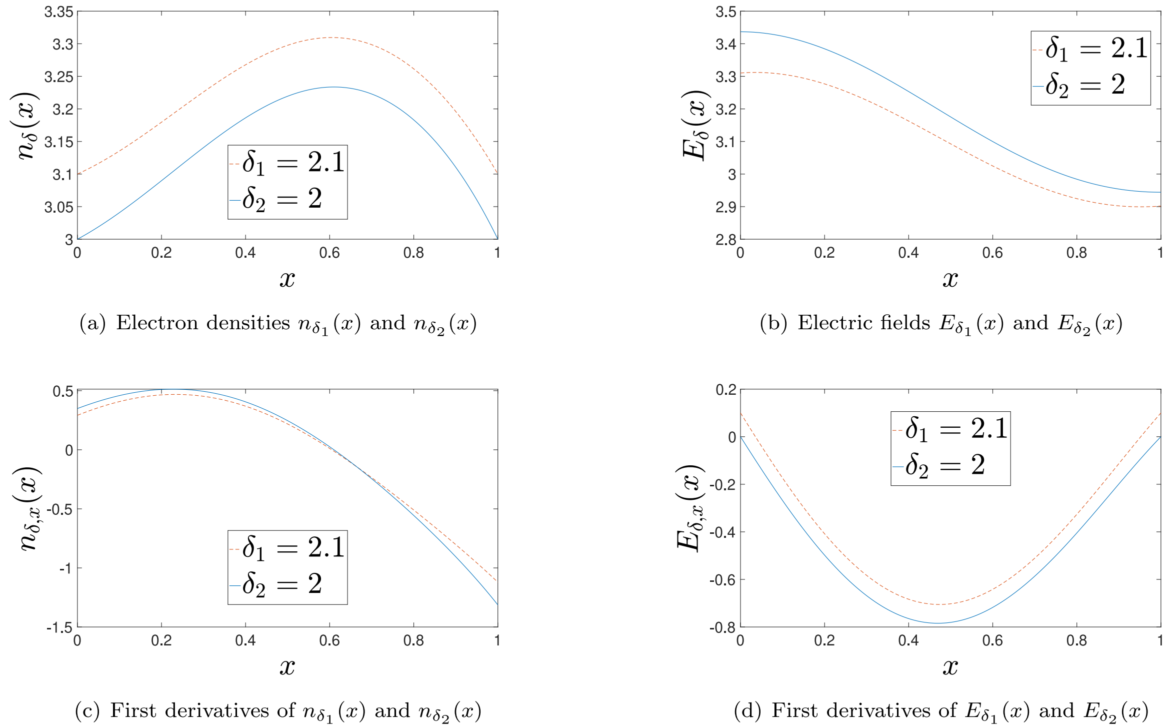
$$\begin{cases} n_{\delta,x} = \dfrac{n_\delta^3}{n_\delta^2 - 1} E_\delta - \dfrac{10 n_\delta^2}{n_\delta^2 - 1}, \\ E_{\delta,x} = n_\delta - (3 + \sin(\pi x)), \quad x \in (0,1), \\ n_\delta(0) = n_\delta(1) = 1 + \delta. \end{cases} \tag{4.1}$$

Note that Theorem 1.1 is valid for any $\delta_{min} > 0$. Therefore, in what follows, we shall numerically substantiate the theoretical results (1.8), (1.9) and (1.10) by checking four typical scales of the boundary perturbation parameter $\delta_{min} > 0$:

1. Large (L) scale: $\delta_{min} > \underline{b} - 1$;
2. Medium (M) scale: $\delta_{min} = \underline{b} - 1$;
3. Small (S) scale: $\delta_{min} < \underline{b} - 1$;
4. Extra small (XS) scale: δ_{min} is very close to zero.

The threshold $\underline{b} - 1$ equals 2 in our example here.

First of all, we restrict ourselves to the large scale: $\delta_1 = 21$ and $\delta_2 = 20$. The corresponding numerical simulations are displayed in Fig. 1. To begin, we compare the components of solutions $(n_{\delta_1}, E_{\delta_1})(x)$ and $(n_{\delta_2}, E_{\delta_2})(x)$ in Fig. 1(a) and Fig. 1(b), respectively. Our observation reveals that $(n_{\delta_1}, E_{\delta_1})(x)$ is adjacent to $(n_{\delta_2}, E_{\delta_2})(x)$, and the differences $(n_{\delta_1} - n_{\delta_2})(x)$ and $(E_{\delta_1} - E_{\delta_2})(x)$ can be controlled by the difference $\delta_1 - \delta_2 = 1$. Particularly, Fig. 1(a) also confirms the boundedness estimate (1.8) and monotonicity relation (1.10). Subsequently, we examine the graphs of the first derivatives of solutions: Fig. 1(c) shows that $n_{\delta_1,x}(x)$ is near to $n_{\delta_2,x}(x)$, and the difference $(n_{\delta_1} - n_{\delta_2})_x(x)$ can be controlled by $10(\delta_1 - \delta_2)$; the difference $(E_{\delta_1} - E_{\delta_2})_x(x)$, depicted in Fig. 1(d), exhibits the same pattern as the difference $(n_{\delta_1} - n_{\delta_2})(x)$, which is



(a) Electron densities $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

(b) Electric fields $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

(c) First derivatives of $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

(d) First derivatives of $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

Fig. 2. Structural stability between $(n_{\delta_1}, E_{\delta_1})(x)$ and $(n_{\delta_2}, E_{\delta_2})(x)$ in the medium (M) scale $(2.1 = \delta_1 > \delta_2 = 2)$.

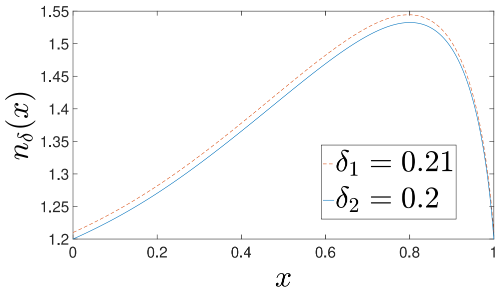
evident upon comparing Fig. 1(a) and Fig. 1(d). Our numerical findings are closely aligned with Theorem 1.1, validating the structural stability estimate (1.9) for the large scale.

In addition, we inspect the medium scale: $\delta_1 = 2.1$ and $\delta_2 = 2$. The numerical results are presented in Fig. 2. We follow the same procedure as before, comparing the components of solutions in Fig. 2(a) and Fig. 2(b), and their first derivatives in Fig. 2(c) and Fig. 2(d). Specifically, we can see that the differences $(n_{\delta_1} - n_{\delta_2})(x)$, $(E_{\delta_1} - E_{\delta_2})(x)$, $(n_{\delta_1} - n_{\delta_2})_x(x)$ and $(E_{\delta_1} - E_{\delta_2})_x(x)$ are dominated by $2(\delta_1 - \delta_2)$. Our observation in the medium scale is also consistent with (1.8), (1.9) and (1.10) in Theorem 1.1.

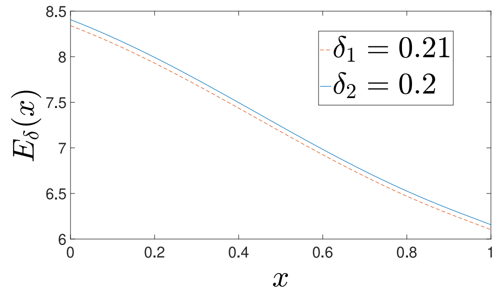
Following the same procedure as before, we proceed to numerically analyze the small scale with $\delta_1 = 0.21$ and $\delta_2 = 0.2$, as depicted in Fig. 3. We can easily affirm in Fig. 3(a) that (1.8) and (1.10) in Theorem 1.1 still hold true for the small scale. It is demonstrated visually in Figs. 3(a)∼ 3(d) that $(n_{\delta_1}, E_{\delta_1}, n_{\delta_1,x}, E_{\delta_1,x})(x)$ is in close proximity to $(n_{\delta_2}, E_{\delta_2}, n_{\delta_2,x}, E_{\delta_2,x})(x)$. Furthermore, we extend our numerical analysis to the extra small scale with $\delta_1 = 0.051$ and $\delta_2 = 0.05$, as presented in Fig. 4. We observe in Fig. 4(a) that (1.8) and (1.10) in Theorem 1.1 are still valid even for the extra small scale. And then Fig. 4(b) shows that the graph of $E_{\delta_1}(x)$ is almost overlapped with the one of $E_{\delta_2}(x)$. The same phenomenon also occurs between $n_{\delta_1,x}(x)$ and $n_{\delta_2,x}(x)$ in Fig. 4(c), and between $E_{\delta_1,x}(x)$ and $E_{\delta_2,x}(x)$ in Fig. 4(d). Therefore, the structural stability estimate (1.9) in Theorem 1.1 is true for both the small scale and the extra small scale.

Of concern is numerical verification of the behavior of the estimate constant $C = C(\delta_2, \underline{b}, \overline{b}, \tau)$ in (1.9). To this end, we also simulate the quantities $\frac{(n_{\delta_1}-n_{\delta_2})(x)}{\delta_1-\delta_2}$, $\frac{(E_{\delta_1}-E_{\delta_2})(x)}{\delta_1-\delta_2}$, $\frac{(n_{\delta_1}-n_{\delta_2})_x(x)}{\delta_1-\delta_2}$ and $\frac{(E_{\delta_1}-E_{\delta_2})_x(x)}{\delta_1-\delta_2}$ in four different scales (L, M, S and XS), see Fig. 5. Actually, Figs. 5(a)∼5(d) together explain the intrinsic phenomena mentioned in Remark 1.1.

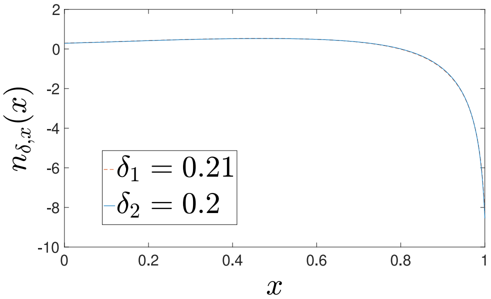
In order to confirm Theorem 1.3 numerically, we have to make a comparison of $(n_{\delta_2}, E_{\delta_2})(x)$ for all four scales, that is $\delta_2 = 20$, $\delta_2 = 2$, $\delta_2 = 0.2$ and $\delta_2 = 0.05$, as illustrated in Fig. 6. According to Theorem 1.3, we must be able to observe a tendency of convergence of the family $\{(n_{\delta_2}, E_{\delta_2,x})\}_{\delta_2>0}$ as $\delta_2 \to 0^+$. In fact, the tendency of convergence depicted in Fig. 6(a) and Fig. 6(d) coincides with Theorem 1.3, which is not



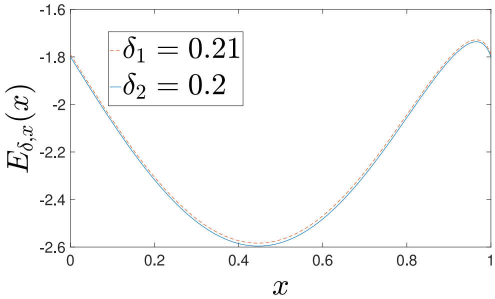
(a) Electron densities $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$



(b) Electric fields $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

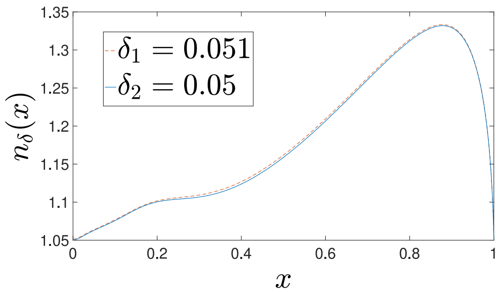


(c) First derivatives of $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

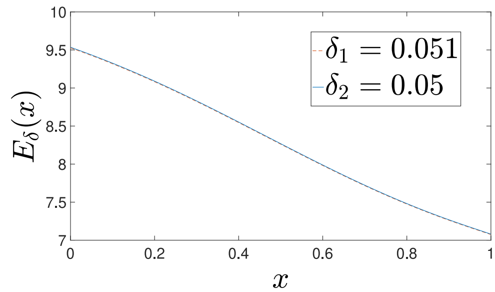


(d) First derivatives of $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

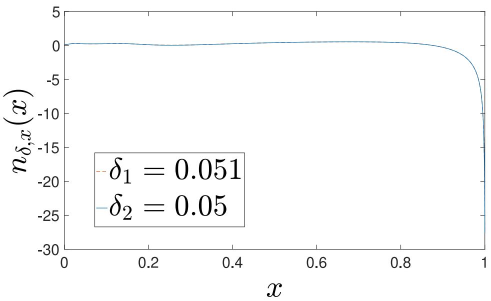
Fig. 3. Structural stability between $(n_{\delta_1}, E_{\delta_1})(x)$ and $(n_{\delta_2}, E_{\delta_2})(x)$ in the small (S) scale $(0.21 = \delta_1 > \delta_2 = 0.2)$.



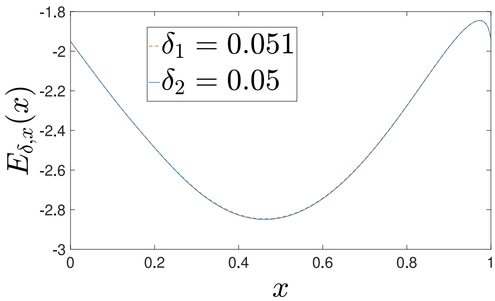
(a) Electron densities $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$



(b) Electric fields $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

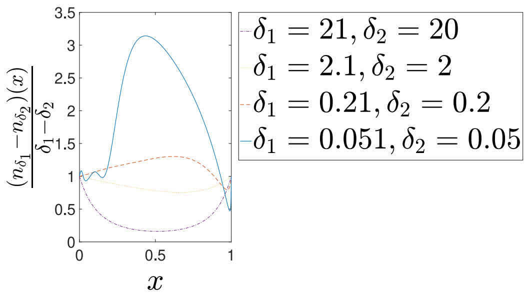


(c) First derivatives of $n_{\delta_1}(x)$ and $n_{\delta_2}(x)$

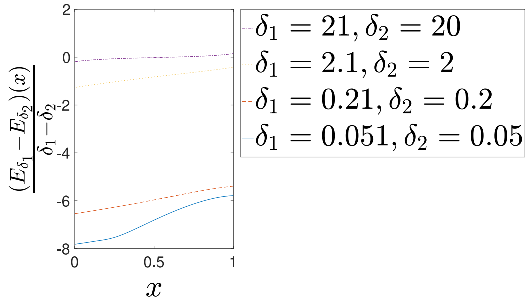


(d) First derivatives of $E_{\delta_1}(x)$ and $E_{\delta_2}(x)$

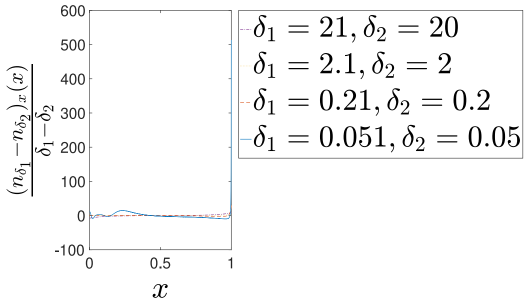
Fig. 4. Structural stability between $(n_{\delta_1}, E_{\delta_1})(x)$ and $(n_{\delta_2}, E_{\delta_2})(x)$ in the extra small (XS) scale $(0.051 = \delta_1 > \delta_2 = 0.05)$.



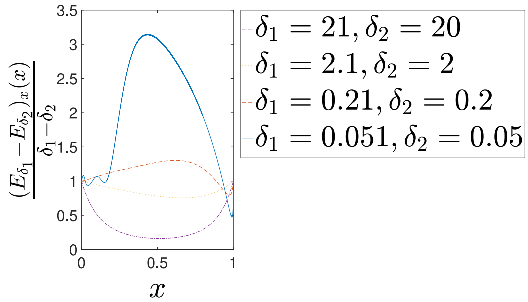
(a) $\frac{(n_{\delta_1}-n_{\delta_2})(x)}{\delta_1-\delta_2}$ in L, M, S and XS scale



(b) $\frac{(E_{\delta_1}-E_{\delta_2})(x)}{\delta_1-\delta_2}$ in L, M, S and XS scale

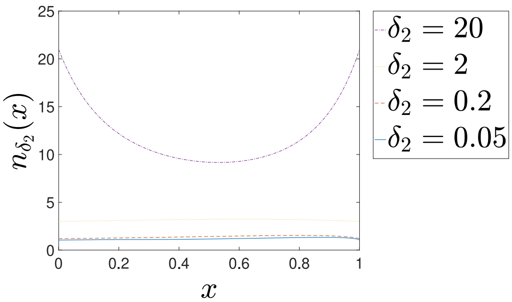


(c) $\frac{(n_{\delta_1}-n_{\delta_2})_x(x)}{\delta_1-\delta_2}$ in L, M, S and XS scale

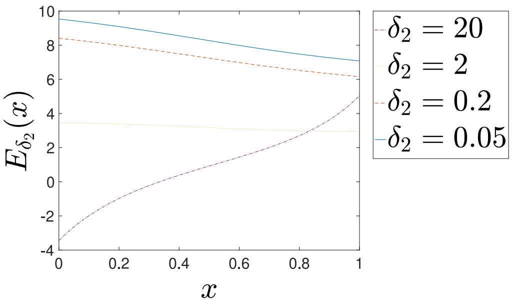


(d) $\frac{(E_{\delta_1}-E_{\delta_2})_x(x)}{\delta_1-\delta_2}$ in L, M, S and XS scale

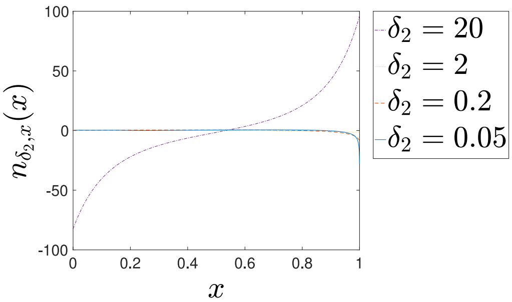
Fig. 5. Difference quotients $\frac{(n_{\delta_1}-n_{\delta_2})(x)}{\delta_1-\delta_2}$, $\frac{(E_{\delta_1}-E_{\delta_2})(x)}{\delta_1-\delta_2}$, $\frac{(n_{\delta_1}-n_{\delta_2})_x(x)}{\delta_1-\delta_2}$ and $\frac{(E_{\delta_1}-E_{\delta_2})_x(x)}{\delta_1-\delta_2}$ induced by the small change $\delta_1-\delta_2$ of boundary data.



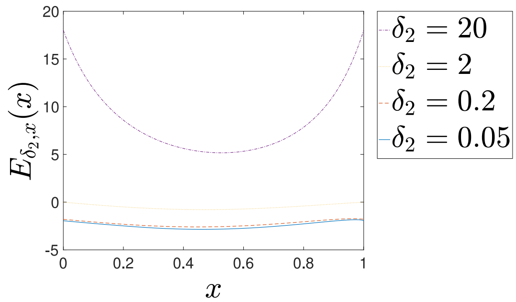
(a) Convergence of $n_{\delta_2}(x)$ to $n(x)$



(b) Highly possible convergence of $E_{\delta_2}(x)$ to $E(x)$



(c) Highly possible convergence of $n_{\delta_2,x}(x)$ to $n_x(x)$



(d) Convergence of $E_{\delta_2,x}(x)$ to $E_x(x)$

Fig. 6. A tendency for $(n_{\delta_2}, E_{\delta_2})(x)$ to be getting closer to a fixed profile $(n, E)(x)$ as δ_2 becomes smaller.

a surprise for us. What enlightens us is the convergence trend presented in Fig. 6(b) and Fig. 6(c), which makes us believe that the convergence of $\{(n_{\delta_2,x}, E_{\delta_2})\}_{\delta_2>0}$ as $\delta_2 \to 0^+$ should be true in a certain topology. For more details, see Remark 1.2.

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