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A decoupled, linearly implicit and high-order structure-preserving scheme for Euler–Poincaré equations ☆

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ABSTRACT

It is challenging to develop high-order structure-preserving finite difference schemes for the modified two-component Euler–Poincaré equations due to the nonlinear terms and high-order derivative terms. To overcome the difficulties, we introduce a bi-variate function and carefully choose the intermediate average variable in the temporal discretization. Then, we obtain a decoupled and linearly implicit scheme. It is shown that the fully-discrete scheme can keep both the discrete mass and energy conserved. And the fully-discrete scheme has fourth-order accuracy in the spatial direction and second-order accuracy in the temporal direction. Several numerical examples are given to confirm the theoretical results.

1. Introduction

In this paper, we aim to present an effective structure-preserving scheme for the following two-component Euler–Poincaré equation with the periodic boundary conditions:

$$\begin{cases} m_t + um_x + 2mu_x + g\rho\bar{\rho}_x = 0, \ t \in (0,T], \ x \in [0,L] \subseteq \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \ t \in (0,T], \ x \in [0,L] \subseteq \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in [0,L] \subseteq \mathbb{R}, \\ \bar{\rho}(0,x) = \bar{\rho}_0(x), \quad x \in [0,L] \subseteq \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - \alpha u_{xx}$ and $\rho = (1-\beta\partial_x^2)(\bar{\rho}-\bar{\rho}_0)$, m denotes the momentum related to the horizontal fluid velocity u, ρ is density, and $\bar{\rho}$ is locally averaged density. $\bar{\rho}_0$ is taken to be a constant, $g > 0$ is the downward constant acceleration of gravity, and two non-negative constants α and β are dispersion parameters. The Euler–Poincaré equation is widely used to describe the unidirectional propagation of shallow water waves over a flat bottom [11,20,25,26,36,37].

The theoretical results of Euler–Poincaré equations have been investigated extensively. Here we refer readers to the Refs. [10,40,44] for the global well-posedness of the solutions, the Refs. [2,19] for the global conservation of the models, and

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the Ref. [9] for the related geometric property. Besides, Eq. (1.1) has two remarkable conservation quantities, i.e.,

$$\text{Mass:} \qquad \frac{d}{dt}\int_{\mathbb{R}} \rho(t,x)dx = 0,$$

$$\text{Energy:} \qquad \frac{d}{dt}\int_{\mathbb{R}} (|u|^2 + \alpha|\nabla u|^2 + g|\gamma|^2 + g\beta|\nabla\gamma|^2)(t,x)dx = 0, \quad \gamma = \bar{\rho} - \bar{\rho}_0.$$

In the past few years, much effort has been devoted to the numerical analysis of Euler–Poincaré equations. For instance, Mclanchlan and Zhang [35] constructed a family of self-similar solutions, equipping with studying its infinite-time asymptotic behavior. For the solutions in finite time, Holden and Raynaud [18] proved the convergence of a particular finite difference scheme. For general H^1 initial data, Coclite et al. [6] suggested a finite difference scheme, and proved that this scheme converges to a dissipative weak solution strongly. Under multi-dimensions conditions, Holdahl et al. [17] studied an unconditionally stable method by combining the front-tracking method with the operator splitting method. For two-component Euler–Poincaré with $\bar{\rho}_0 = 0$, Chertock et al. [5] proposed a finite volume method and a hybrid finite-volume-particle method. More works about numerical analysis can be found in [1,4,7,8,43,47]. In these references, the numerical schemes were developed without considering the conservative mass and energy.

Since the structure-preserving numerical schemes show a great advantage in a long-term calculation, it is important for the numerical methods to keep the associated discrete invariant conserved. In early references, some researchers developed several structure-preserving methods for the following special Euler–Poincaré equation without density, i.e.,

$$m_t + um_x + 2mu_x = 0, \quad m = u - u_{xx}, \quad x \in [0, L] \subseteq \mathbb{R}, \ t \in (0, T]. \tag{1.2}$$

For example, Matsuo and his co-authors [33,34] proposed an energy-conserving Galerkin scheme and a Hamiltonian-conserving Galerkin scheme. Later, Jiang et al. [23] presented a linearly implicit energy-preserving scheme by combining the finite difference schemes and scalar variable approaches. Zhang et al. [51] applied energy method to investigate four kinds of invariants and prove their conservation. Liu and Xing [32] developed a fully-discrete scheme by using the two-stages Gauss–Legendre method and a discontinuous Galerkin method for temporal and spatial discretization, respectively. They proved that the fully-discrete method can preserve momentum and energy discretely. Gong et al. [13] developed a class of arbitrarily high-order energy-preserving algorithms by combining the quadratic auxiliary variable technique with the symplectic Runge–Kutta methods in the temporal direction and using the Fourier pseudo-spectral method in the spatial direction, respectively. The convergence results in these papers were missing.

Recently, Zhang et al. [49] proposed an invariant-preserving finite difference scheme for a special case of two-component Euler–Poincaré equations ($\alpha = 1, \ \beta = 0, \ g = 1$). They showed that the scheme is of second-order convergence accuracy in both the temporal and spatial directions. We also would like to point out there are some approaches to develop the structure-preserving scheme, i.e., invariant energy quadratization [22,24,45,46], scalar auxiliary variable approach [3,15,28,31], the relaxation approach [27,29,30,38,39] and so on [14,16,21,42]. Generally speaking, the schemes constructed by using these approaches usually preserve one invariant. Up to now, it is still challenging to investigate the high-order multiple-invariants preserving schemes and their convergence results for two-component Euler–Poincaré equations (1.1).

This paper proposes a decoupled, linearly implicit and structure-preserving difference scheme for the modified two-component Euler–Poincaré equation (1.1). The fully-discrete scheme keeps both the discrete mass and energy conserved. The fully-discrete scheme has second-order accuracy in the temporal direction and fourth-order accuracy in the spatial direction. The convergence result of the fully-discrete scheme is proved by rigorous analysis of every term in the error equations. The error estimate holds without any restrictions on the parameters of the equations.

We would like to point out that it is non-trial to develop the fully-discrete structure-preserving scheme. To the end, we introduce a bi-variate function $\varphi(u, v)$ (see Eq. (2.1)). Thanks to the bi-variate function, several fourth-order spatial discrete operators have the same conservative properties as the continuous ones. Then, the time discretization is done by using the leap-frog scheme. By carefully choosing the intermediate average variable, we get the decoupled, linearly implicit and high-order structure-preserving scheme.

The rest of the paper is structured in the following way. The fully-discrete scheme and its analytical results are shown in Section 2. The proofs of the main results are presented in Section 3. In Section 4, several numerical experiments illustrate the validity of our method and the correctness of our theoretical results. Finally, some conclusions are given in Section 5.

2. The difference scheme and main results

In this section, we will establish a finite difference scheme through some discrete numerical techniques. Before giving the finite difference scheme, some notations and lemmas are collected.

2.1. Lemmas and notations

Let $M, \ N$ be two positive integers, and $h = L/M, \ \tau = T/N$. Denote $x_i = ih, \ 0 \le i \le M, \ t_k = k\tau, \ 0 \le k \le N$, and $\Omega_{h\tau} = \{(x_i, t_k)| \ 0 \le i \le M, 0 \le k \le N\}$. For any grid function $u = \{u_i^k | i \in \mathcal{Z}, \ 0 \le k \le N\}$ defined on $\Omega_{h\tau}$, we introduce the following notations:

$$(u_i^k)_x = \frac{u_{i+1}^k - u_i^k}{h}, \quad (u_i^k)_{\bar{x}} = \frac{u_i^k - u_{i-1}^k}{h}, \quad (u_i^k)_{\hat{x}} = \frac{u_{i+1}^k - u_{i-1}^k}{2h}, \quad (u_i^k)_{\ddot{x}} = \frac{u_{i+2}^k - u_{i-2}^k}{4h},$$

$$D_x u_i^k = \frac{4}{3}(u_i^k)_{\hat{x}} - \frac{1}{3}(u_i^k)_{\ddot{x}}, \quad D_x^2 u_i^k = \frac{4}{3}\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} - \frac{1}{3}\frac{u_{i+2}^k - 2u_i^k + u_{i-2}^k}{4h^2},$$
$$\delta_t u_i^{k+\frac{1}{2}} = \frac{u_i^{k+1} - u_i^k}{\tau}, \quad \Delta_t u_i^k = \frac{u_i^{k+1} - u_i^{k-1}}{2\tau}, \quad u_i^{k+\frac{1}{2}} = \frac{1}{2}(u_i^k + u_i^{k+1}), \quad u_i^{\bar{k}} = \frac{1}{2}(u_i^{k+1} + u_i^{k-1}).$$

Define

$$\mathcal{V}_h = \{v | v = (v_i),\ v_{i+M} = v_i,\ i \in \mathcal{Z}\}.$$

Then, for any grid function $u \in \mathcal{V}_h$, denote

$$u^k = [u_1^k,\ u_2^k, \dots, u_{M-1}^k]',\ D_x u^k = (D_x u_i^k)_{i\in I_M},\ D_x^2 u^k = (D_x^2 u_i^k)_{i\in I_M},$$
$$u_{\bar{x}}^k = ((u_i^k)_{\bar{x}})_{i\in I_M},\ u_{\tilde{x}}^k = ((u_i^k)_{\tilde{x}})_{i\in I_M},\ u_{\hat{x}}^k = ((u_i^k)_{\hat{x}})_{i\in I_M},\ u_{\ddot{x}}^k = ((u_i^k)_{\ddot{x}})_{i\in I_M},$$

where $I_M = \{i\ |1 \le i \le M\}$. For any grid function $u, v \in \mathcal{V}_h$ the discrete inner product can be represented as

$$(u, v) = h \sum_{i\in I_M} u_i v_i,$$

equipping with the norms

$$\|u\| = \sqrt{(u,u)}, \quad \|u\|_1 = \sqrt{\frac{4}{3}(u_{\bar{x}}, u_{\bar{x}}) - \frac{1}{3}(u_{\hat{x}}, u_{\hat{x}})},$$
$$\|u_{\bar{x}}\| = \sqrt{h\sum_{i\in I_M} |(u_i)_{\bar{x}}|^2},\ \|u_{\tilde{x}}\| = \sqrt{h\sum_{i\in I_M} |(u_i)_{\tilde{x}}|^2},$$
$$\|u_{\hat{x}}\| = \sqrt{h\sum_{i\in I_M} |(u_i)_{\hat{x}}|^2},\ \|u_{\ddot{x}}\| = \sqrt{h\sum_{i\in I_M} |(u_i)_{\ddot{x}}|^2},$$
$$\|D_x u\| = \sqrt{h\sum_{i\in I_M} |D_x u_i|^2}.$$

Moreover, denote $(uv)_i = u_i v_i$, and define a bi-variate function for any $1 \le i \le M$,

$$\varphi(u,v)_i = \frac{4}{3}[(u_i(v_i)_{\hat{x}}) + ((uv)_i)_{\hat{x}}] - \frac{1}{3}[(u_i(v_i)_{\ddot{x}}) + ((uv)_i)_{\ddot{x}}], \tag{2.1}$$
$$\varphi(D_x^2 u, v)_i = \frac{4}{3}[(D_x^2 u_i \cdot (v_i)_{\hat{x}}) + (D_x^2 u_i \cdot v_i)_{\hat{x}}] - \frac{1}{3}[(D_x^2 u_i \cdot (v_i)_{\ddot{x}}) + (D_x^2 u_i \cdot v_i)_{\ddot{x}}]. \tag{2.2}$$

Similar definition of the bi-variate function can be found in [48,50].

We collect the following lemmas, which are helpful in the proof of our main results.

Lemma 2.1 (*Lemma 1, [12]*). *For any grid functions $u,\ v \in \mathcal{V}_h$, we have,*

$$(u_{\bar{x}}, v) = -(u, v_{\tilde{x}}), \quad (u_{\hat{x}}, v) = -(u, v_{\hat{x}}), \quad (u_{\ddot{x}}, v) = -(u, v_{\ddot{x}}),$$
$$(D_x u, v) = -(u, D_x v), \quad (u_{\hat{x}}, u) = 0, \quad (u_{\ddot{x}}, u) = 0,$$
$$(u_{\bar{x}\tilde{x}}, u) = -\|u_{\bar{x}}\|^2, \quad (u_{\hat{x}\hat{x}}, u) = -\|u_{\hat{x}}\|^2.$$

Lemma 2.2. *For any grid function $u,\ v \in V_h$, we have*

$$(\varphi(u,v), v) = 0, \quad \|u_{\ddot{x}}\| \le \|u_{\hat{x}}\| \le \|u_{\bar{x}}\| \le \|u\|_1, \quad \textit{and} \quad \|D_x u\| \le \frac{5}{3}\|u\|_1.$$

Proof. Considering (2.1), we have

$$\begin{aligned}(\varphi(u,v), v) &= \sum_{i\in I_M} \frac{4}{3}[u_i(v)_i)_{\hat{x}} v_i + ((uv)_i)_{\hat{x}} v_i] - \sum_{i\in I_M} \frac{1}{3}[u_i(v)_i)_{\ddot{x}} v_i + ((uv)_i)_{\ddot{x}} v_i] \\ &= \sum_{i\in I_M} \frac{4}{3}[(v_i)_{\hat{x}}(uv)_i - (uv)_i(v_i)_{\hat{x}}] - \sum_{i\in I_M} \frac{1}{3}[(v_i)_{\ddot{x}}(uv)_i - (uv)_i(v_i)_{\ddot{x}}] \\ &= 0.\end{aligned}$$

Noting that

$$\|u_{\bar{x}}\|^2 = h\sum_{i\in I_M} (\frac{u_{i+1} - u_i}{h})^2 = \frac{1}{h}\sum_{i\in I_M} (u_{i+1} - u_i)^2, \tag{2.3}$$

we can get the following inequalities immediately,

$$\|u_{\hat{x}}\|^2 = h\sum_{i\in I_M} (\frac{u_{i+1} - u_{i-1}}{2h})^2$$

$$
\begin{aligned}
&= \frac{1}{4h}\sum_{i\in I_M}[(u_{i+1}-u_i)+(u_i-u_{i-1})]^2\\
&\le \frac{1}{2h}\sum_{i\in I_M}(u_{i+1}-u_i)^2+\frac{1}{2h}\sum_{i\in I_M}(u_i-u_{i-1})^2\\
&\le \|u_{\bar{x}}\|^2, \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_{\ddot{x}}\|^2 &= h\sum_{i\in I_M}(\frac{u_{i+2}-u_{i-2}}{4h})^2\\
&= \frac{1}{16h}\sum_{i\in I_M}[(u_{i+2}-u_i)+(u_i-u_{i-2})]^2\\
&\le \frac{h}{2}\sum_{i\in I_M}\frac{(u_{i+2}-u_i)^2}{4h^2}+\frac{h}{2}\sum_{i\in I_M}\frac{(u_i-u_{i-2})^2}{4h^2}\\
&\le \|u_{\hat{x}}\|^2. \qquad (2.5)
\end{aligned}
$$

Besides, it holds that

$$
\begin{aligned}
\|u\|_1^2 &= (-D_x^2u,u)=\frac{4}{3}\|u_{\bar{x}}\|^2-\frac{1}{3}\|u_{\hat{x}}\|^2\\
&\ge \frac{4}{3}\|u_{\bar{x}}\|^2-\frac{1}{3}\|u_{\bar{x}}\|^2=\|u_{\bar{x}}\|^2. \qquad (2.6)
\end{aligned}
$$

From (2.3)–(2.6), we can get $\|u_{\ddot{x}}\|\le\|u_{\hat{x}}\|\le\|u_{\bar{x}}\|\le\|u\|_1$. Besides, it holds that

$$
\begin{aligned}
\|D_xu\|^2 &= (\frac{4}{3}u_{\hat{x}}-\frac{1}{3}u_{\ddot{x}},\frac{4}{3}u_{\hat{x}}-\frac{1}{3}u_{\ddot{x}})\\
&= \frac{16}{9}\|u_{\hat{x}}\|^2-\frac{8}{9}(u_{\hat{x}},u_{\ddot{x}})+\frac{1}{9}\|u_{\ddot{x}}\|^2\\
&\le \frac{16}{9}\|u_{\hat{x}}\|^2+\frac{4}{9}\|u_{\hat{x}}\|^2+\frac{4}{9}\|u_{\ddot{x}}\|^2+\frac{1}{9}\|u_{\ddot{x}}\|^2\\
&\le \frac{25}{9}\|u\|_1^2.
\end{aligned}
$$

Then, we have $\|D_xu\|\le\frac{5}{3}\|u\|_1$. □

Lemma 2.3 (*Lemma 2.3, [41]*). *For any smooth function u,v, suppose $U_i^k=u(x_i,t_k)$, $V_i^k=v(x_i,t_k)$, $i\in I_M$, $t_k=\tau k$, we have*

$$
\begin{aligned}
&\frac{4}{3}\frac{U_{i+1}^k-U_{i-1}^k}{2h}-\frac{1}{3}\frac{U_{i+2}^k-U_{i-2}^k}{4h}=\frac{dU_i^k}{dx}+\mathcal{O}(h^4),\\
&\frac{4}{3}\frac{U_{i+1}^k-2U_i^k+U_{i-1}^k}{h^2}-\frac{1}{3}\frac{U_{i+2}^k-2U_i^k+U_{i-2}^k}{4h^2}=\frac{d^2U_i^k}{dx^2}+\mathcal{O}(h^4).
\end{aligned}
$$

Lemma 2.4. *For any smooth function u,v, suppose $U_i^k=u(x_i,t_k)$, $V_i^k=v(x_i,t_k)$, $i\in I_M$, $t_k=\tau k$. Then, we have*

$$
\varphi(U^k,U^k)_i=3U_i^k\frac{dU_i^k}{dx}+\mathcal{O}(h^4), \qquad (2.7)
$$

$$
\varphi(D_x^2U^k,U^k)_i=U_i^k\frac{d^3U_i^k}{dx^3}+2\frac{dU_i^k}{dx}\frac{d^2U_i^k}{dx^2}+\mathcal{O}(h^4). \qquad (2.8)
$$

Proof. By Taylor's expansion, we have

$$
U_{i+\alpha}^k=U_i^k+\sum_{l=1}^{7}\frac{(\alpha h)^l}{l!}U_i^k+\mathcal{O}(h^8),\quad \alpha=\pm1,\ \pm2.
$$

For any function $V(x,t)$ satisfying periodic boundary conditions, we have

$$
\begin{aligned}
&\frac{4}{3}\frac{V_{i+1}^k-V_{i-1}^k}{2h}-\frac{1}{3}\frac{V_{i+2}^k-V_{i-2}^k}{4h}\\
&= \frac{4}{3}[V_i^{k,(1)}+\frac{h^2}{6}V_i^{k,(3)}+\frac{h^4}{120}V_i^{k,(5)}+\mathcal{O}(h^6)]-\frac{1}{3}(V_i^{k,(1)}+\frac{2h^2}{3}V_i^{k,(3)}+\frac{h^4}{15}V_i^{k,(5)}+\mathcal{O}(h^6))\\
&= V_i^{k,(1)}+\frac{h^4}{90}V_i^{k,(5)}+\mathcal{O}(h^6). \qquad (2.9)
\end{aligned}
$$

Then, we have

$$
V_{i+1}^kU_{i+1}^k=V_i^kU_i^k+\sum_{n=1}^{5}\frac{h^n}{n!}\sum_{l=0}^{n}V_i^{k,(l)}U_i^{k,(n-l)}+\mathcal{O}(h^6).
$$

Similarly, we can obtain

$$V_{i-1}^k U_{i-1}^k = V_i^k U_i^k + \sum_{n=1}^{5}(-1)^n \frac{h^n}{n!}\sum_{l=0}^{n} V_i^{k,(l)} U_i^{k,(n-l)} + \mathcal{O}(h^6).$$

Thus, we can get

$$\frac{V_{i+1}^k U_{i+1}^k - V_{i-1}^k U_{i-1}^k}{2h} = \sum_{l=0}^{1} V_i^{k,(l)} U^{k,(1-l)} + \frac{h^2}{3!}\sum_{l=0}^{3} V_i^{k,(l)} U_i^{k,(3-l)} + \frac{h^4}{5!}\sum_{l=0}^{5} V_i^{k,(l)} U_i^{k,(5-l)} + \mathcal{O}(h^6),$$

$$\frac{V_{i+2}^k U_{i+2}^k - V_{i-2}^k U_{i-2}^k}{4h} = \sum_{l=0}^{1} V_i^{k,(l)} U_i^{k,(1-l)} + \frac{2h^2}{3}\sum_{l=0}^{3} V_i^{k,(l)} U_i^{k,(3-l)} + \frac{8h^4}{60}\sum_{l=0}^{5} V_i^{k,(l)} U_i^{k,(5-l)} + \mathcal{O}(h^6),$$

and

$$\begin{aligned}&\frac{4}{3}\frac{V_{i+1}^k U_{i+1}^k - V_{i-1}^k U_{i-1}^k}{2h} - \frac{1}{3}\frac{V_{i+2}^k U_{i+2}^k - V_{i-2}^k U_{i-2}^k}{4h}\\ &= \sum_{l=0}^{1} V_i^{k,(l)} U_i^{k,(1-l)} - \frac{h^4}{18}\sum_{l=0}^{5} V_i^{k,(l)} U_i^{k,(5-l)} + \mathcal{O}(h^6).\end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), and letting $V_i^k = U_i^k$, we can get (2.7). Then, by Lemma 2.3, let $V_i^k = D_x^2 U_i^k = \frac{d^2 U_i^k}{dx^2} + \mathcal{O}(h^4)$, we can get (2.8). □

2.2. Finite different scheme and main results

Denote

$$U_i^k = u(x_i, t_k),\ \Pi_i^k = \rho(x_i, t_k),\ \bar{\Pi}_i^k = \bar{\rho}(x_i, t_k),\quad i \in I_M,\ 0 \le k \le N.$$

Considering (1.1) at point $(x_i, t_{\frac{1}{2}}),\ i \in I_M$, we have

$$\delta_t U_i^{\frac{1}{2}} - \alpha\delta_t D_x^2 U_i^{\frac{1}{2}} + \varphi(U^0, U^{\frac{1}{2}})_i - \alpha\varphi(D_x^2 U^0, U^{\frac{1}{2}})_i + g\Pi_i^0 D_x \bar{\Pi}_i^{\frac{1}{2}} = R_i^0, \tag{2.11}$$

$$\delta_t \bar{\Pi}_i^{\frac{1}{2}} - \beta\delta_t D_x^2 \bar{\Pi}_i^{\frac{1}{2}} + D_x(\Pi_i^0 U_i^{\frac{1}{2}}) = Q_i^0. \tag{2.12}$$

The local truncation errors satisfy

$$|R_i^0| \le c_1(\tau + h^4),\quad i \in I_M, \tag{2.13}$$

$$|Q_i^0| \le c_1(\tau + h^4),\quad i \in I_M, \tag{2.14}$$

where c_1 is a generic constance independent of M, N. Considering (1.1) at point $(x_i, t_k),\ i \in I_M,\ 1 \le k \le N$, we have

$$\Delta_t U_i^k - \alpha\Delta_t D_x^2 U_i^k + \varphi(U^k, U^{\bar{k}})_i - \alpha\varphi(D_x^2 U^k, U^{\bar{k}})_i + g\Pi_i^k D_x \bar{\Pi}_i^{\bar{k}} = R_i^k, \tag{2.15}$$

$$\Delta_t \bar{\Pi}_i^k - \beta\Delta_t D_x^2 \bar{\Pi}_i^k + D_x(\Pi_i^k U_i^{\bar{k}}) = Q_i^k. \tag{2.16}$$

The local truncation errors satisfy

$$|R_i^k| \le c_2(\tau^2 + h^4),\quad i \in I_M,\ 1 \le k \le N, \tag{2.17}$$

$$|Q_i^k| \le c_2(\tau^2 + h^4),\quad i \in I_M,\ 1 \le k \le N, \tag{2.18}$$

where c_2 is a generic constant independent of M, N. Combining with the initial and boundary condition, we have

$$\begin{aligned}&U_i^0 = u_0(x_i),\quad \bar{\Pi}_i^0 = \bar{\rho}_0(x_i),\quad i \in I_M,\\ &U_i^k = U_{i+M}^k,\ \bar{\Pi}_i^k = \bar{\Pi}_{i+M}^k,\ \Pi_i^k = \Pi_{i+M}^k,\quad i \in I_M,\ 0 \le k \le N.\end{aligned}$$

Ignoring the small terms, and replacing the solutions of the grid functions in (2.11)–(2.12) and (2.15)–(2.16) $U_i^k,\ \Pi_i^k,\ \bar{\Pi}_i^k$ by $u_i^k,\ \rho_i^k,\ \bar{\rho}_i^k$, we propose a difference scheme for (1.1),

$$\delta_t u_i^{\frac{1}{2}} - \alpha\delta_t D_x^2 u_i^{\frac{1}{2}} + \varphi(u^0, u^{\frac{1}{2}})_i - \alpha\varphi(D_x^2 u^0, u^{\frac{1}{2}})_i + g\rho_i^0 D_x \bar{\rho}_i^{\frac{1}{2}} = 0,\quad i \in I_M, \tag{2.19}$$

$$\delta_t \bar{\rho}_i^{\frac{1}{2}} - \beta\delta_t D_x^2 \bar{\rho}_i^{\frac{1}{2}} + D_x(\rho_i^0 u_i^{\frac{1}{2}}) = 0,\quad i \in I_M, \tag{2.20}$$

$$\Delta_t u_i^k - \alpha\Delta_t D_x^2 u_i^k + \varphi(u^k, u^{\bar{k}})_i - \alpha\varphi(D_x^2 u^k, u^{\bar{k}})_i + g\rho_i^k D_x \bar{\rho}_i^{\bar{k}} = 0,\quad i \in I_M,\ 1 \le k \le N, \tag{2.21}$$

$$\Delta_t \bar{\rho}_i^k - \beta\Delta_t D_x^2 \bar{\rho}_i^k + D_x(\rho_i^k u_i^{\bar{k}}) = 0,\quad i \in I_M,\ 1 \le k \le N, \tag{2.22}$$

with the initial and boundary conditions,

$$u_i^0 = u_0(x_i),\quad \bar{\rho}_i^0 = \bar{\rho}_0(x_i),\quad i \in I_M, \tag{2.23}$$

$$u_i^k = u_{i+M}^k,\ \bar{\rho}_i^k = \bar{\rho}_{i+M}^k,\ \rho_i^k = \rho_{i+M}^k, \quad i \in I_M,\ 0 \le k \le N. \tag{2.24}$$

Now, we present the theoretical results of the numerical schemes as follows.

Theorem 2.5. *Let* $\{u_i^k,\ \rho_i^k,\ \bar{\rho}_i^k \mid i \in I_M,\ 0 \le k \le N\}$ *be the numerical solutions of* (2.19)–(2.24) *and* $r_i^k = \bar{\rho}_i^k - \bar{\rho}_0$. *Then, for any* $1 \le k \le N$, *we have*

$$E^k = E^0, I^k = I^0,$$

where

$$E^0 = \|u^0\|^2 + \alpha\|u^0\|_1^2 + g(\|r^0\|^2 + \beta\|r^0\|_1^2),$$
$$E^k = \frac{1}{2}\left[\left(\|u^k\|^2 + \|u^{k-1}\|^2\right) + \alpha\left(\|u^k\|_1^2 + \|u^{k-1}\|_1^2\right) + g\left(\|r^k\|^2 + \|r^{k-1}\|^2\right) + g\beta\left(\|r^k\|_1^2 + \|r^{k-1}\|_1^2\right)\right],$$
$$I^k = h\sum_{i \in I_M} \rho_i^k.$$

Theorem 2.6. *Let* $\{U_i^k,\ \Pi_i^k,\ \bar{\Pi}_i^k \mid i \in I_M,\ 0 \le k \le N\}$ *be the solutions of the* (1.1), $\{u_i^k,\ \rho_i^k,\ \bar{\rho}_i^k \mid i \in I_M,\ 0 \le k \le N\}$ *be the solutions of* (2.19)–(2.24), *and* $e_i^k = U_i^k - u_i^k,\ \bar{f}_i^k = \bar{\Pi}_i^k - \bar{\rho}_i^k, \quad i \in I_M,\ 0 \le k \le N$. *Then there exist positive constants* h_0, τ_0 *and* λ_0 *such that when* $h \le h_0,\ \tau \le \min\{1, \tau_0,\ 1/(6c_4)\},\ \lambda \le \max\{\lambda_0,\ \lambda_0\sigma\} \le \frac{4}{5}$, *it holds that,*

$$\|e^k\|^2 \le c_6(\tau^2 + h^4)^2, \quad \|e^k\|_1^2 \le c_6(\tau^2 + h^4)^2,$$
$$\|\bar{f}^k\|^2 \le c_6(\tau^2 + h^4)^2, \quad \|\bar{f}^k\|_1^2 \le c_6(\tau^2 + h^4)^2,$$

where $0 \le k \le N$, $e^k = [e_1^k, e_2^k, \dots, e_M^k]^T,\ \bar{f} = [\bar{f}_1^k, \bar{f}_2^k, \dots, \bar{f}_M^k]^T,\ \lambda = c_0\tau/h$ *with*

$$c_0 = \sup_{0 \le x \le L,\ 0 < t \le T} \{|u(x,t)|, |u_x(x,t)|, |u_{xx}(x,t)|, |u_t(x,t)|\}$$

and

$$c_3 = \max\{\frac{5c_0}{3} + \frac{5c_0(\alpha+\beta)+3}{6}, \frac{c_0(170\alpha+60\beta+60)+45\alpha}{72\alpha(1-5\lambda/4)}, \frac{22c_0+21}{24(1-3\lambda/2)}, \frac{170c_0+45}{72(1-5\lambda/4)}\},$$
$$c_4 = \max\{c_3,\ c_2^2 L\},\ c_5 = \exp(6Tc_4)\cdot(\max\{\alpha,\beta\}(1+\frac{3}{2}\lambda)c_1^2 L + \frac{1}{2}),$$
$$c_6 = \max\{2c_5,\ \frac{2c_5}{\alpha(1-5\lambda/4)},\ \frac{2c_5}{\alpha(1-3\lambda/4)},\ \frac{2c_5}{\beta(1-5\lambda/4)}\}.$$

3. Proof of the main results

In this section, we will give detailed proofs of the theorems in the previous Section.

3.1. Proof of Theorem 2.5

Proof. Firstly, we prove energy conservation. Since $\bar{\rho}_0$ is a constant, (2.19)–(2.22) can be written as

$$\delta_t u_i^{\frac{1}{2}} - \alpha\delta_t D_x^2 u_i^{\frac{1}{2}} + \varphi(u_i^0, u_i^{\frac{1}{2}}) - \alpha\varphi(D_x^2 u_i^0, u_i^{\frac{1}{2}}) + g\rho_i^0 D_x r_i^{\frac{1}{2}} = 0, \quad i \in I_M, \tag{3.1}$$
$$\delta_t r_i^{\frac{1}{2}} - \beta\delta_t D_x^2 r_i^{\frac{1}{2}} + D_x(\rho^0 u^{\frac{1}{2}})_i = 0, \quad i \in I_M, \tag{3.2}$$
$$\Delta_t u_i^k - \alpha\Delta_t D_x^2 u_i^k + \varphi(u_i^k, u_i^{\bar{k}}) - \alpha\varphi(D_x^2 u_i^k, u_i^{\bar{k}}) + g\rho_i^k D_x r_i^k = 0, \quad i \in I_M, 1 \le k \le N, \tag{3.3}$$
$$\Delta_t r_i^k - \beta\Delta_t D_x^2 r_i^k + D_x(\rho^k u^{\bar{k}})_i = 0, \quad i \in I_M, 1 \le k \le N. \tag{3.4}$$

Taking the inner product of (3.1) with $u^{\frac{1}{2}}$, we have

$$(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) - \alpha(\delta_t D_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}) + (\varphi(u^0, u^{\frac{1}{2}}), u^{\frac{1}{2}}) - \alpha(\varphi(D_x^2 u^0, u^{\frac{1}{2}}), u^{\frac{1}{2}}) + g(\rho^0 D_x r^{\frac{1}{2}}, u^{\frac{1}{2}}) = 0.$$

By Lemma 2.1 and Lemma 2.2, we can get

$$\frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) + \frac{\alpha}{2\tau}(\|u^1\|_1^2 - \|u^0\|_1^2) = g(r^{\frac{1}{2}}, D_x(\rho^0 u^{\frac{1}{2}})). \tag{3.5}$$

Similarly, taking the inner product of (3.2) with $r^{\frac{1}{2}}$, we have

$$(\delta_t r^{\frac{1}{2}}, r^{\frac{1}{2}}) - \beta(\delta_t D_x^2 r^{\frac{1}{2}}, r^{\frac{1}{2}}) + (D_x(\rho^0 u^{\frac{1}{2}}), r^{\frac{1}{2}}) = 0.$$

Rearranging the above equation, we can get

$$\frac{1}{2\tau}(\|r^1\|^2 - \|r^0\|^2) + \frac{\beta}{2\tau}(\|r^1\|_1^2 - \|r^0\|_1^2) + (D_x(\rho^0 u^{\frac{1}{2}}), r^{\frac{1}{2}}) = 0. \tag{3.6}$$

Multiplying (3.6) with g, and summing up with (3.5), we obtain

$$\frac{1}{2\tau}[(\|u^1\|^2-\|u^0\|^2)+\alpha(\|u^1\|_1^2-\|u^0\|_1^2)]+\frac{g}{2\tau}[(\|r^1\|^2-\|r^0\|^2)+\beta(\|r^1\|_1^2-\|r^0\|_1^2)]=0,$$

which implies

$$\begin{aligned}&\frac{1}{2}[(\|u^1\|^2+\|u^0\|^2)+\alpha(\|u^1\|_1^2+\|u^0\|_1^2)+g(\|r^1\|^2+\|r^0\|^2)+g\beta(\|r^1\|_1^2+\|r^0\|_1^2)]\\&=\|u^0\|^2+\alpha\|u^0\|_1^2+g\|r^0\|^2+g\beta\|r^0\|_1^2.\end{aligned}$$

Consequently,

$$E^1=E^0. \tag{3.7}$$

Then, taking the inner product of (3.3) with $u^{\bar{k}}$, we have

$$(\Delta_t u^k,u^{\bar{k}})-\alpha(\Delta_t D_x^2 u^k,u^{\bar{k}})+(\varphi(u^k,u^{\bar{k}}),u^{\bar{k}})-\alpha(\varphi(D_x^2 u^k,u^{\bar{k}}),u^{\bar{k}})+g(\rho^k D_x r^{\bar{k}},u^{\bar{k}})=0.$$

Using Lemma 2.1 and Lemma 2.2, we obtain

$$\frac{1}{4\tau}[(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\alpha(\|u^{k+1}\|_1^2-\|u^{k-1}\|_1^2)]=g(r^{\bar{k}},D_x(\rho^k u^{\bar{k}})). \tag{3.8}$$

Taking the inner product of (3.4) with $r^{\bar{k}}$, we have

$$(\Delta_t r^k,r^{\bar{k}})-\beta(\Delta_t D_x^2 r^k,r^{\bar{k}})+(r^{\bar{k}},D_x(\rho^k u^{\bar{k}}))=0,$$

which implies

$$\frac{1}{4\tau}[(\|r^{k+1}\|^2-\|r^{k-1}\|^2)+\beta(\|r^{k+1}\|_1^2-\|r^{k-1}\|_1^2)]+(r^{\bar{k}},D_x(\rho^k u^{\bar{k}}))=0. \tag{3.9}$$

Multiplying (3.9) by g, and summing up it with (3.8), we can get

$$\begin{aligned}&\frac{1}{4\tau}[(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\alpha(\|u^{k+1}\|_1^2-\|u^{k-1}\|_1^2)]\\&+\frac{g}{4\tau}[(\|r^{k+1}\|^2-\|r^{k-1}\|^2)+\beta(\|r^{k+1}\|_1^2-\|r^{k-1}\|_1^2)]=0.\end{aligned}$$

Rearranging the above equation, we get

$$\begin{aligned}&\frac{1}{2}[(\|u^{k+1}\|^2+\|u^k\|^2)+\alpha(\|u^{k+1}\|_1^2+\|u^k\|_1^2)+g(\|r^{k+1}\|^2+\|r^{k-1}\|^2)+g\beta(\|r^{k+1}\|_1^2+\|r^k\|_1^2)]\\=&\frac{1}{2}[(\|u^k\|^2+\|u^{k-1}\|^2)+\alpha(\|u^k\|_1^2+\|u^{k-1}\|_1^2)+g(\|r^k\|^2+\|r^{k-1}\|^2)+g\beta(\|r^k\|_1^2+\|r^{k-1}\|_1^2)].\end{aligned}$$

It implies that

$$E^{k+1}=E^k,\quad 1\le k\le N. \tag{3.10}$$

Combining (3.7) with (3.10), we get the conclusion on the conservation of energy.

Next, we prove mass conservation. Taking inner product of (2.20) with 1, we can get

$$(\delta_t\bar{\rho}^{\frac{1}{2}},1)-\beta(\delta_t D_x^2\bar{\rho}^{\frac{1}{2}},1)+(D_x(\rho^0 u^{\frac{1}{2}}),1)=0.$$

Because of the periodic boundary condition, the inner product between $D_x(\rho^0 u^{\frac{1}{2}})$ and 1 is 0. Then, it follows that

$$(\bar{\rho}^1,1)-\beta(D_x^2\bar{\rho}^1,1)=(\bar{\rho}^0,1)-\beta(D_x^2\bar{\rho}^0,1),$$

which implies

$$(\rho^1,1)=(\rho^0,1). \tag{3.11}$$

After that, taking the inner product of (2.22) with 1, we have

$$(\Delta_t\bar{\rho}^k,1)-\beta(\Delta_t D_x^2\bar{\rho}^k,1)+(D_x(\rho^k u^{\bar{k}}),1)=0.$$

Similarly, we obtain

$$(\rho^{k+1},1)=(\rho^{k-1},1).$$

At last, we have

$$I^{k+1}=I^{k-1}. \tag{3.12}$$

Combining (3.11) with (3.12), we get the required result. □

3.2. Proof of Theorem 2.6

Proof. Without losing the generality, we assume $g = 1$. Subtracting (2.11)–(2.12) and (2.15)–(2.16) from (2.19)–(2.22), for $i \in I_M$, $1 \le k \le N$, the error equations can be written as

$$\delta_t e_i^{\frac{1}{2}} - \alpha\delta_t D_x^2 e_i^{\frac{1}{2}} + \varphi(u^0, e^{\frac{1}{2}})_i - \alpha\varphi(D_x^2 u^0, e^{\frac{1}{2}})_i + \rho^0 D_x \bar{f}_i^{\frac{1}{2}} = R_i^0, \tag{3.13}$$

$$\delta_t \bar{f}_i^{\frac{1}{2}} - \beta\delta_t D_x^2 \bar{f}_i^{\frac{1}{2}} + D_x(\rho^0 e^{\frac{1}{2}})_i = Q_i^0, \tag{3.14}$$

$$\begin{aligned}\Delta_t e_i^k - \alpha\Delta_t D_x^2 e_i^k + (\varphi(U^k, U^{\bar{k}})_i - \varphi(u^k, u^{\bar{k}})_i) \\ = \alpha(\varphi(D_x^2 U^k, U^{\bar{k}})_i - \varphi(D_x^2 u^k, u^{\bar{k}})_i) - (\Pi_i^k D_x \bar{\Pi}_i^{\bar{k}} - \rho_i^k D_x \bar{\rho}_i^{\bar{k}}) + R_i^k,\end{aligned} \tag{3.15}$$

$$\Delta_t \bar{f}_i^k - \beta\Delta_t D_x^2 \bar{f}_i^k + (D_x(\Pi^k U^{\bar{k}})_i - D_x(\rho^k u^{\bar{k}})_i) = Q_i^k. \tag{3.16}$$

Taking the inner product of (3.13) with $e^{\frac{1}{2}}$, we have

$$(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}) - \alpha(\delta_t D_x^2 e^{\frac{1}{2}}, e^{\frac{1}{2}}) + (\varphi(u^0, e^{\frac{1}{2}}), e^{\frac{1}{2}}) - \alpha(\varphi(D_x^2 u^0, e^{\frac{1}{2}}), e^{\frac{1}{2}}) + (\rho^0 D_x \bar{f}^{\frac{1}{2}}, e^{\frac{1}{2}}) = (R^0, e^{\frac{1}{2}}).$$

By Lemma 2.2, Lemma 2.2 and the fact that $e^0 = 0$, we can rearrange the above equation as

$$\frac{1}{2\tau}\|e^1\|^2 + \frac{\alpha}{2\tau}\|e^1\|_1^2 = \frac{1}{4}(\bar{f}^1, D_x(\rho^0 e^1)) + \frac{1}{2}(R^0, e^1). \tag{3.17}$$

Taking the inner product of (3.14) with $\bar{f}^{\frac{1}{2}}$, we have

$$(\delta_t \bar{f}^{\frac{1}{2}}, \bar{f}^{\frac{1}{2}}) - \beta(\delta_t D_x^2 \bar{f}^{\frac{1}{2}}, \bar{f}^{\frac{1}{2}}) + (\bar{f}^{\frac{1}{2}}, D_x(\rho^0 e^{\frac{1}{2}})) = (Q^0, \bar{f}^{\frac{1}{2}}). \tag{3.18}$$

Noticing (3.17), (3.18) implies

$$\frac{1}{2\tau}\|\bar{f}^1\|^2 + \frac{\beta}{2\tau}\|\bar{f}^1\|_1^2 + \frac{1}{4}(\bar{f}^1, D_x(\rho^0 e^1)) = \frac{1}{2}(Q^0, \bar{f}^1). \tag{3.19}$$

Using Lemma 2.1 and adding (3.17) and (3.19), we have

$$\begin{aligned}\frac{1}{2\tau}(\|e^1\|^2 + \alpha\|e^1\|_1^2 + \|\bar{f}^1\|^2 + \beta\|\bar{f}\|_1^2) &= \frac{1}{2}(R^0, e^1) + \frac{1}{2}(Q^0, \bar{f}^1) \\ &\le \frac{1}{2}\|R^0\|\|e^1\| + \frac{1}{2}\|Q^0\|\|\bar{f}^1\| \\ &\le \frac{1}{4\tau}\|e^1\|^2 + \frac{\tau}{4}\|R^0\|^2 + \frac{1}{4\tau}\|\bar{f}^1\|^2 + \frac{\tau}{4}\|Q^0\|^2 \\ &\le \frac{1}{4\tau}(\|e^1\|^2 + \alpha\|e^1\|_1^2 + \|\bar{f}^1\|^2 + \beta\|\bar{f}^1\|_1^2) + \frac{\tau}{2}c_1^2 L(\tau + h^4)^2,\end{aligned}$$

when $\tau \le 1$,

$$\|e^1\|^2 + \alpha\|e^1\|_1^2 + \|\bar{f}^1\|^2 + \beta\|\bar{f}^1\|_1^2 \le 2\tau^2 c_1^2 L(\tau + h^4)^2 \le 2c_1^2 L(\tau^2 + h^4)^2. \tag{3.20}$$

Taking the inner product of (3.15) with $e^{\bar{k}}$, we have

$$\begin{aligned}&(\Delta_t e^k, e^{\bar{k}}) - \alpha(\Delta_t D_x^2 e^k, e^{\bar{k}}) + (\varphi(U^k, U^{\bar{k}}) - \varphi(u^k, u^{\bar{k}}), e^{\bar{k}}) \\ &- \alpha(\varphi(D_x^2 U^k, U^{\bar{k}}) - \varphi(D_x^2 u^k, u^{\bar{k}}), e^{\bar{k}}) + (\Pi^k D_x \bar{\Pi}^{\bar{k}} - \rho^k D_x \bar{\rho}^{\bar{k}}, e^{\bar{k}}) = (R^k, e^{\bar{k}}).\end{aligned} \tag{3.21}$$

It is easy to see that

$$\begin{aligned}\varphi(U^k, U^{\bar{k}})_i - \varphi(u^k, u^{\bar{k}})_i &= \varphi(U^k, U^{\bar{k}})_i - \varphi(U^k, u^{\bar{k}})_i + \varphi(U^k, u^{\bar{k}})_i - \varphi(u^k, u^{\bar{k}})_i \\ &= \varphi(U^k, e^{\bar{k}})_i + \varphi(e^k, u^{\bar{k}})_i - \varphi(e^k, U^{\bar{k}})_i - \varphi(e^k, U^{\bar{k}})_i \\ &= \varphi(U^k, e^{\bar{k}})_i + \varphi(e^k, U^{\bar{k}})_i - \varphi(e^k, e^{\bar{k}})_i.\end{aligned} \tag{3.22}$$

Similarly, we have

$$\varphi(D_x^2 u^k, u^{\bar{k}})_i - \varphi(D_x^2 u^k, u^{\bar{k}})_i = \varphi(D_x^2 U^k, e^{\bar{k}})_i + \varphi(D_x^2 e^k, U^{\bar{k}})_i - \varphi(D_x^2 e^k, e^{\bar{k}})_i, \tag{3.23}$$

$$\Pi_i^k D_x \bar{\Pi}_i^{\bar{k}} - \rho_i^k D_x \bar{\rho}_i^{\bar{k}} = \Pi_i^k D_x \bar{f}_i^{\bar{k}} + f_i^k D_x \bar{\Pi}_i^{\bar{k}} - f_i^k D_x \bar{f}_i^{\bar{k}}. \tag{3.24}$$

Combining (3.22), (3.23) with Lemma 2.1, we can get

$$\begin{aligned}(\varphi(U^k, U^{\bar{k}}) - \varphi(u^k, u^{\bar{k}}), e^{\bar{k}}) &= (\varphi(e^k, U^{\bar{k}}), e^{\bar{k}}) \\ &= \frac{4}{3}(e^k U_{\hat{x}}^{\bar{k}}, e^{\bar{k}}) - \frac{4}{3}(e^k U^{\bar{k}}, e_{\hat{x}}^{\bar{k}}) - \frac{1}{3}(e^k U_{\ddot{x}}^{\bar{k}}, e^{\bar{k}}) + \frac{1}{3}(e^k U^{\bar{k}}, e_{\ddot{x}}^{k}),\end{aligned} \tag{3.25}$$

and

$$(\varphi(D_x^2U^k,U^{\bar{k}})-\varphi(D_x^2u^k,u^{\bar{k}}),e^{\bar{k}})=(\varphi(D_x^2e^k,U^{\bar{k}}),e^{\bar{k}})$$
$$=\frac{4}{3}(D_x^2e^k\cdot U_{\hat{x}}^{\bar{k}},e^{\bar{k}})-\frac{4}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\hat{x}}^{\bar{k}})-\frac{1}{3}(D_x^2e^k\cdot U_{\ddot{x}}^{\bar{k}},e^{\bar{k}})+\frac{1}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\ddot{x}}^{\bar{k}}). \tag{3.26}$$

Taking the inner product of (3.24) with $e^{\bar{k}}$, we have

$$(\Pi^kD_x\bar{\Pi}^{\bar{k}}-\rho^kD_x\bar{\rho}^{\bar{k}},e^{\bar{k}})=-(\bar{f}^{\bar{k}},D_x(\Pi^ke^{\bar{k}}))+(f^kD_x\bar{\Pi}^{\bar{k}},e^{\bar{k}})+(\bar{f}^{\bar{k}},D_x(f^ke^{\bar{k}})). \tag{3.27}$$

Combining (3.25)–(3.27) with (3.21), we have

$$\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\frac{\alpha}{4\tau}(\|e^{k+1}\|_1^2-\|e^{k-1}\|_1^2)$$
$$=-\frac{4}{3}(e^kU_{\hat{x}}^{\bar{k}},e^{\bar{k}})+\frac{4}{3}(e^kU^{\bar{k}},e_{\hat{x}}^{\bar{k}})+\frac{1}{3}(e^kU_{\ddot{x}}^{\bar{k}},e^{\bar{k}})-\frac{1}{3}(e^kU^{\bar{k}},e_{\ddot{x}}^{\bar{k}})+\frac{4\alpha}{3}(D_x^2e^k\cdot U_{\hat{x}}^{\bar{k}},e^{\bar{k}})$$
$$-\frac{4\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\hat{x}}^{\bar{k}})-\frac{\alpha}{3}(D_x^2e^k\cdot U_{\ddot{x}}^{\bar{k}},e^{\bar{k}})+\frac{\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\ddot{x}}^{\bar{k}})+(\bar{f}^{\bar{k}},D_x(\Pi^ke^{\bar{k}}))$$
$$-(f^kD_x\bar{\Pi}^{\bar{k}},e^{\bar{k}})-(\bar{f}^{\bar{k}},D_x(f^ke^{\bar{k}})). \tag{3.28}$$

Taking the inner product of (3.16) with $\bar{f}^{\bar{k}}$, we have

$$(\Delta_t\bar{f}^k,\bar{f}^{\bar{k}})-\beta(\Delta_tD_x^2\bar{f}^k,\bar{f}^{\bar{k}})+(D_x(\Pi^ku^{\bar{k}}),\bar{f}^{\bar{k}})-(D_x(\rho^ku^{\bar{k}}),\bar{f}^{\bar{k}})=(Q^k,\bar{f}^{\bar{k}}), \tag{3.29}$$

where

$$D_x(\Pi^kU^{\bar{k}})_i-D_x(\rho^ku^{\bar{k}})_i=D_x(\Pi^ke^{\bar{k}})_i+D_x(f^kU^{\bar{k}})_i-D_x(f^ke^{\bar{k}})_i. \tag{3.30}$$

Substituting (3.29)–(3.30) into (3.28), we can calculate

$$\frac{1}{4\tau}[(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\alpha(\|e^{k+1}\|_1^2-\|e^{k-1}\|_1^2)+(\|f^{k+1}\|^2-\|f^{k-1}\|^2)+\beta(\|f^{k+1}\|_1^2-\|f^{k-1}\|_1^2)]$$
$$=\left[-\frac{4}{3}(e^kU_{\hat{x}}^{\bar{k}},e^{\bar{k}})+\frac{1}{3}(e^kU_{\ddot{x}}^{\bar{k}},e^{\bar{k}})\right]+\left[\frac{4}{3}(e^kU^{\bar{k}},e_{\hat{x}}^{\bar{k}})-\frac{1}{3}(e^kU^{\bar{k}},e_{\ddot{x}}^{k})\right]+\left[\frac{4\alpha}{3}(D_x^2e^k\cdot U_{\hat{x}}^{\bar{k}},e^{\bar{k}})\right.$$
$$\left.-\frac{\alpha}{3}(D_x^2e^k\cdot U_{\ddot{x}}^{\bar{k}},e^{\bar{k}})\right]+\left[-\frac{4\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\hat{x}}^{\bar{k}})+\frac{\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\ddot{x}}^{\bar{k}})\right]-(D_x(f^kU^{\bar{k}}),\bar{f}^{\bar{k}})$$
$$-(f^kD_x\bar{\Pi}^{\bar{k}},e^{\bar{k}})+(R^k,e^{\bar{k}})+(Q^k,\bar{f}^{\bar{k}})$$
$$:=\sum_{i=1}^{8}P_i. \tag{3.31}$$

Next, we estimate each term of (3.31). With the help of Lemma 2.2, we have

$$P_1=-\frac{4}{3}(e^kU_{\hat{x}}^{\bar{k}},e^{\bar{k}})+\frac{1}{3}(e^kU_{\ddot{x}}^{\bar{k}},e^{\bar{k}})$$
$$\le\frac{4}{3}\|e^k\|\cdot\|U_{\hat{x}}^{\bar{k}}\|_\infty\cdot\|e^{\bar{k}}\|+\frac{1}{3}\|e^k\|\cdot\|U_{\ddot{x}}^{\bar{k}}\|_\infty\cdot\|e^{\bar{k}}\|\le\frac{5}{3}c_0\|e^k\|\|e^{\bar{k}}\|,$$
$$P_2=\frac{4}{3}(e^kU^{\bar{k}},e_{\hat{x}}^{\bar{k}})-\frac{1}{3}(e^kU^{\bar{k}},e_{\ddot{x}}^{k})=(e^kU^{\bar{k}},\frac{4}{3}e_{\hat{x}}^{\bar{k}}-\frac{1}{3}e_{\ddot{x}}^{\bar{k}})=(e^kU^{\bar{k}},D_xe^{\bar{k}})$$
$$\le\|e^k\|\|U^{\bar{k}}\|_\infty\|D_xe^{\bar{k}}\|\le\frac{5}{3}c_0\|e^k\|\|e^{\bar{k}}\|_1,$$
$$P_7=(R^k,e^{\bar{k}})\le\|R^k\|\cdot\|e^{\bar{k}}\|,$$
$$P_8=(Q^k,f^{\bar{k}})\le\|Q^k\|\cdot\|\bar{f}^{\bar{k}}\|.$$

Then we estimate P_3 as

$$P_3=\frac{4\alpha}{3}(D_x^2e^k\cdot U_{\hat{x}}^{\bar{k}},e^{\bar{k}})-\frac{\alpha}{3}(D_x^2e^k\cdot U_{\ddot{x}}^{\bar{k}},e^{\bar{k}})$$
$$=\alpha(D_x^2e^k(\frac{4}{3}U_{\hat{x}}^{\bar{k}}-\frac{1}{3}U_{\ddot{x}}^{\bar{k}}),e^{\bar{k}})=\alpha(D_x^2e^k\cdot D_xU^{\bar{k}},e^{\bar{k}})$$
$$=\frac{4\alpha}{3}(e_{\bar{x}x}^k,D_xU^{\bar{k}}\cdot e^{\bar{k}})-\frac{\alpha}{3}(e_{\hat{x}\hat{x}}^k,D_xU^{\bar{k}}\cdot e^{\bar{k}})$$
$$=-\frac{4\alpha}{3}(e_{\bar{x}}^k,(e^{\bar{k}}D_xU^{\bar{k}})_{\bar{x}})+\frac{\alpha}{3}(e_{\hat{x}}^k,(e^{\bar{k}}D_xU^{\bar{k}})_{\hat{x}})$$
$$=-\frac{4\alpha}{3}\sum_{i\in I_M}(e_i^k)_{\bar{x}}(D_xU_{i+1}^{\bar{k}}\cdot e_{i+1}^{\bar{k}}-D_xU_i^{\bar{k}}\cdot e_i^{\bar{k}})+\frac{\alpha}{6}\sum_{i\in I_M}(e_i^k)_{\hat{x}}(D_xU_{i+1}^{\bar{k}}\cdot e_{i+1}^{\bar{k}}-D_xU_{i-1}^{\bar{k}}\cdot e_{i-1}^{\bar{k}})$$
$$=-\frac{4\alpha}{3}h\sum_{i\in I_M}(e_i^k)_{\bar{x}}[D_xU_{i+1}^{\bar{k}}(e_i^{\bar{k}})_{\bar{x}}+(D_xU_{i+1}^{\bar{k}})_{\bar{x}}e_i^{\bar{k}}]$$
$$+\frac{\alpha}{3}h\sum_{i\in I_M}(e_i^k)_{\hat{x}}[D_xU_{i-1}^{\bar{k}}(e_i^{\bar{k}})_{\hat{x}}+(D_xU_i^{\bar{k}})_{\hat{x}}e_{i+1}^{\bar{k}}]$$

$$
\begin{aligned}
&\le \frac{4\alpha}{3}(\|e^k_{\bar{x}}\|\|D_xU^{\bar{k}}\|_\infty\|e^{\bar{k}}_{\bar{x}}\| + \|e^k_{\bar{x}}\|\|D_xU^{\bar{k}}_{\bar{x}}\|_\infty\|e^{\bar{k}}\|)\\
&\quad+\frac{\alpha}{3}(\|e^k_{\hat{x}}\|\|D_xU^{\bar{k}}\|_\infty\|e^{\bar{k}}_{\hat{x}}\| + \|e^k_{\hat{x}}\|\|D_xU^{\bar{k}}_{\hat{x}}\|_\infty\|e^{\bar{k}}\|)\\
&\le \frac{5\alpha}{3}c_0\|e^k\|_1\cdot\|e^{\bar{k}}\|_1 + \frac{5\alpha}{3}c_0\|e^{\bar{k}}\|\|e^k\|_1.
\end{aligned}
$$

Then, we estimate P_4 as,

$$
\begin{aligned}
P_4 &= -\frac{4\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},(e^{\bar{k}})_{\hat{x}}) + \frac{\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},(e^{\bar{k}})_{\ddot{x}})\\
&= -\frac{4\alpha}{3}(\frac{4}{3}(e^k)_{\bar{x}\bar{x}}\cdot U^{\bar{k}} - \frac{1}{3}(e^k)_{\hat{x}\hat{x}}\cdot U^{\bar{k}}, e^{\bar{k}}_{\hat{x}}) + \frac{\alpha}{3}(\frac{4}{3}(e^k)_{\bar{x}\bar{x}}\cdot U^{\bar{k}} - \frac{1}{3}(e^k)_{\hat{x}\hat{x}}\cdot U^{\bar{k}}, e^{\bar{k}}_{\ddot{x}})\\
&= -\frac{16\alpha}{9}(U^{\bar{k}}(e^k)_{\bar{x}\bar{x}}, e^{\bar{k}}_{\hat{x}}) + \frac{4\alpha}{9}(U^{\bar{k}}(e^k)_{\hat{x}\hat{x}}, e^{\bar{k}}_{\hat{x}}) + \frac{4\alpha}{9}(U^{\bar{k}}(e^k)_{\bar{x}\bar{x}}, e^{\bar{k}}_{\ddot{x}}) - \frac{\alpha}{9}(U^{\bar{k}}(e^k)_{\hat{x}\hat{x}}, e^{\bar{k}}_{\ddot{x}})\\
&= -\frac{16\alpha}{9}h\sum_{i\in I_M}U_i^{\bar{k}}\frac{(e_i^k)_{\bar{x}}-(e_{i-1}^k)_{\bar{x}}}{h}\cdot\frac{e_{i+1}^{\bar{k}}-e_{i-1}^{\bar{k}}}{2h} + \frac{4\alpha}{9}h\sum_{i\in I_M}U_i^{\bar{k}}\frac{(e_{i+1}^k)_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}}{2h}\cdot\frac{e_{i+1}^{\bar{k}}-e_{i-1}^{\bar{k}}}{2h}\\
&\quad+\frac{4\alpha}{9}h\sum_{i\in I_M}U_i^{\bar{k}}\frac{(e_i^k)_{\bar{x}}-(e_{i-1}^k)_{\bar{x}}}{h}\cdot\frac{e_{i+2}^{\bar{k}}-e_{i-2}^{\bar{k}}}{4h} - \frac{\alpha}{9}h\sum_{i\in I_M}U_i^{\bar{k}}\frac{(e_{i+1}^k)_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}}{2h}\cdot\frac{e_{i+2}^{\bar{k}}-e_{i-2}^{\bar{k}}}{4h}\\
&= -\frac{8\alpha}{9}\sum_{i\in I_M}U_i^{\bar{k}}((e_i^k)_{\bar{x}} - (e_{i-1}^k)_{\bar{x}})((e_i^{\bar{k}})_{\bar{x}} + (e_{i-1}^{\bar{k}})_{\bar{x}})\\
&\quad+\frac{\alpha}{9}\sum_{i\in I_M}U_i^{\bar{k}}((e_{i+1}^k)_{\hat{x}} - (e_{i-1}^{\bar{k}})_{\hat{x}})((e_i^{\bar{k}})_{\bar{x}} + (e_{i-1}^{\bar{k}})_{\bar{x}})\\
&\quad+\frac{2\alpha}{9}\sum_{i\in I_M}U_i^{\bar{k}}((e_i^k)_{\bar{x}} - (e_{i-1}^k)_{\bar{x}})((e_{i+1}^{\bar{k}})_{\hat{x}} + (e_{i-1}^{\bar{k}})_{\hat{x}})\\
&\quad-\frac{\alpha}{36}\sum_{i\in I_M}U_i^{\bar{k}}((e_{i+1}^k)_{\hat{x}} - (e_{i-1}^k)_{\hat{x}})((e_{i+1}^{\bar{k}})_{\hat{x}} + (e_{i-1}^{\bar{k}})_{\hat{x}})\\
&= \alpha\sum_{i=1}^4 J_i.
\end{aligned}
\tag{3.32}
$$

Then each term in the (3.32) is estimated step by step as follows

$$
\begin{aligned}
J_1 &= \frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}((e_i^k)_{\bar{x}} - (e_{i-1}^k)_{\bar{x}})((e_i^{\bar{k}})_{\bar{x}} + (e_{i-1}^{\bar{k}})_{\bar{x}})\\
&= -\frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}} + (e_i^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}}]\\
&= -\frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}}] - \frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_i^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}}]\\
&:= A_1^k + A_2^k,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^k &= -\frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}} + \frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_{i-1}^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}}\\
&= -\frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}} + \frac{8}{9}\sum_{i\in I_M}U_{i+1}^{\bar{k}}(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}}\\
&= \frac{8}{9}h\sum_{i\in I_M}(U_i^{\bar{k}})_{\bar{x}}(e_i^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}}\\
&\le \frac{8}{9}c_0\|e^k_{\bar{x}}\|\|e^{\bar{k}}_{\bar{x}}\| \le \frac{8}{9}c_0\|e^k\|_1\|e^{\bar{k}}\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
A_2^k &= -\frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_i^k)_{\bar{x}}(e_{i-1}^{\bar{k}})_{\bar{x}} + \frac{8}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_{i-1}^k)_{\bar{x}}(e_i^{\bar{k}})_{\bar{x}}\\
&= \frac{4}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_{i-1}^k)_{\bar{x}}[(e_i^{k+1})_{\bar{x}} + (e_i^{k-1})_{\bar{x}}] - \frac{4}{9}\sum_{i\in I_M}U_i^{\bar{k}}(e_i^k)_{\bar{x}}[(e_{i-1}^{k+1})_{\bar{x}} + (e_{i-1}^{k-1})_{\bar{x}}]\\
&= \frac{4}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i-1}^k)_{\bar{x}}(e_i^{k+1})_{\bar{x}} - (e_i^k)_{\bar{x}}(e_{i-1}^{k+1})_{\bar{x}}] - \frac{4}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_i^k)_{\bar{x}}(e_{i-1}^{k-1})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_i^{k-1})_{\bar{x}}]\\
&= \frac{4}{9}(\sum_{i\in I_M}U_i^{k+\frac{1}{2}}[(e_{i-1}^k)_{\bar{x}}(e_i^{k+1})_{\bar{x}} - (e_i^k)_{\bar{x}}(e_{i-1}^{k+1})_{\bar{x}}] - \sum_{i\in I_M}U_i^{k-\frac{1}{2}}[(e_i^k)_{\bar{x}}(e_{i-1}^{k-1})_{\bar{x}} - (e_{i-1}^k)_{\bar{x}}(e_i^{k-1})_{\bar{x}}])
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{9}\sum_{i\in I_M}(U_i^{\bar k}-U_i^{k+\frac12})[(e_{i-1}^k)_{\tilde x}(e_i^{k+1})_{\tilde x}-(e_i^k)_{\tilde x}(e_{i-1}^{k+1})_{\tilde x}]\\
&-\frac{4}{9}\sum_{i\in I_M}(U_i^{\bar k}-U_i^{k-\frac12})[(e_i^k)_{\tilde x}(e_{i-1}^{k-1})_{\tilde x}-(e_{i-1}^k)_{\tilde x}(e_i^{k-1})_{\tilde x}]\\
=\;&\frac{4}{9}(F_1^{k+1}-F_1^k)-\frac{2\tau}{9}\sum_{i\in I_M}\delta_t U_i^{k-1}[(e_{i-1}^k)_{\tilde x}(e_i^{k+1})_{\tilde x}-(e_i^k)_{\tilde x}(e_{i-1}^{k+1})_{\tilde x}]\\
&-\frac{2\tau}{9}\sum_{i\in I_M}\delta_t U_i^k[(e_{i-1}^k)_{\tilde x}(e_i^{k-1})_{\tilde x}-(e_i^k)_{\tilde x}(e_{i-1}^{k-1})_{\tilde x}]\\
\le\;&\frac{4}{9}(F_1^{k+1}-F_1^k)+\frac{4\tau}{9h}\|\delta_t U^{k-1}\|_\infty\|e_{\tilde x}^k\|\|e_{\tilde x}^{k+1}\|+\frac{4\tau}{9h}\|\delta_t U^k\|_\infty\|e_{\tilde x}^k\|\|e_{\tilde x}^{k-1}\|\\
\le\;&\frac{4}{9}(F_1^{k+1}-F_1^k)+\frac{4\tau}{9h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{4\tau}{9h}c_0\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}
$$

where $F_1^k=\sum_{i\in I_M}U_i^{k-\frac12}[(e_{i-1}^k)_{\tilde x}(e_i^{k-1})_{\tilde x}-(e_i^k)_{\tilde x}(e_{i-1}^{k-1})_{\tilde x}]$. So that

$$
J_1\le\frac{8}{9}c_0\|e^k\|_1\|e^{\bar k}\|_1+\frac{4}{9}(F_1^{k+1}-F_1^k)+\frac{4\tau}{9h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{4\tau}{9h}c_0\|e^k\|_1\|e^{k-1}\|_1.
$$

Next, we consider the second term

$$
\begin{aligned}
J_2\;=\;&\frac19\sum_{i\in I_M}U_i^{\bar k}[(e_{i+1}^k)_{\hat x}-(e_{i-1}^k)_{\hat x}][(e_i^{\bar k})_{\tilde x}+(e_{i-1}^{\bar k})_{\tilde x}]\\
=\;&\frac29\sum_{i\in I_M}U_i^{\bar k}[(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}]\\
=\;&\frac29\sum_{i\in I_M}U_i^{\bar k}[(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}-(e_i^k)_{\hat x}(e_{i-1}^{\bar k})_{\hat x}]+\frac29\sum_{i\in I_M}U_i^{\bar k}[(e_i^k)_{\hat x}(e_{i-1}^{\bar k})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}]\\
:=\;&A_3^k+A_4^k,
\end{aligned}
$$

where

$$
\begin{aligned}
A_3^k\;=\;&\frac29\sum_{i\in I_M}U_i^{\bar k}[(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}-(e_i^k)_{\hat x}(e_{i-1}^{\bar k})_{\hat x}]\\
=\;&\frac29\sum_{i\in I_M}U_i^{\bar k}(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}-\frac29\sum_{i\in I_M}U_{i+1}^{\bar k}(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}\\
=\;&-\frac29 h\sum_{i\in I_M}(U_i^{\bar k})_{\tilde x}(e_{i+1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}\\
\le\;&\frac29 c_0\|e_{\hat x}^k\|\|e_{\hat x}^{\bar k}\|\le\frac29 c_0\|e^k\|_1\|e^{\bar k}\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
A_4^k\;=\;&\frac29\sum_{i\in I_M}U_i^{\bar k}[(e_i^k)_{\hat x}(e_{i-1}^{\bar k})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{\bar k})_{\hat x}]\\
=\;&\frac19\sum_{i\in I_M}U_i^{\bar k}[(e_i^k)_{\hat x}(e_{i-1}^{k+1})_{\hat x}+(e_i^k)_{\hat x}(e_{i-1}^{k-1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k+1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k-1})_{\hat x}]\\
=\;&\frac19\sum_{i\in I_M}U_i^{\bar k}[(e_i^k)_{\hat x}(e_{i-1}^{k+1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k+1})_{\hat x}]-\frac19\sum_{i\in I_M}U_i^{\bar k}[(e_{i-1}^k)_{\hat x}(e_i^{k-1})_{\hat x}-(e_i^k)_{\hat x}(e_{i-1}^{k-1})_{\hat x}]\\
=\;&\frac19\Big(\sum_{i\in I_M}U_i^{k+\frac12}[(e_i^k)_{\hat x}(e_{i-1}^{k+1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k+1})_{\hat x}]-\sum_{i\in I_M}U_i^{k-\frac12}[(e_i^{k-1})_{\hat x}(e_{i-1}^k)_{\hat x}-(e_{i-1}^{k-1})_{\hat x}(e_i^k)_{\hat x}]\Big)\\
&-\frac{\tau}{18}\sum_{i\in I_M}\delta_t U_i^{k-1}[(e_i^k)_{\hat x}(e_{i-1}^{k+1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k+1})_{\hat x}]\\
&-\frac{\tau}{18}\sum_{i\in I_M}\delta_t U_i^k[(e_{i-1}^k)_{\hat x}(e_i^{k-1})_{\hat x}-(e_i^k)_{\hat x}(e_{i-1}^{k-1})_{\hat x}]\\
=\;&\frac19(F_2^k-F_2^{k-1})-\frac{\tau}{18}\sum_{i\in I_M}\delta_t U_i^k[(e_{i-1}^k)_{\hat x}(e_i^{k-1})_{\hat x}-(e_i^k)_{\hat x}(e_{i-1}^{k-1})_{\hat x}]\\
&-\frac{\tau}{18}\sum_{i\in I_M}\delta_t U_i^{k-1}[(e_i^k)_{\hat x}(e_{i-1}^{k+1})_{\hat x}-(e_{i-1}^k)_{\hat x}(e_i^{k+1})_{\hat x}]\\
\le\;&\frac19(F_2^k-F_2^{k-1})+\frac{\tau}{9h}c_0\|e_{\hat x}^k\|\|e_{\hat x}^{k+1}\|+\frac{\tau}{9h}c_0\|e_{\hat x}^k\|\|e_{\hat x}^{k-1}\|\\
\le\;&\frac19(F_2^k-F_2^{k-1})+\frac{\tau}{9h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{\tau}{9h}c_0\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}
$$

where $F_2^k = \sum_{i\in I_M} U_i^{k-\frac{1}{2}}[(e_i^{k-1})_{\hat{x}}(e_{i-1}^k)_{\hat{x}} - (e_{i-1}^{k-1})_{\hat{x}}(e_i^k)_{\hat{x}}]$. Therefore,

$$J_2 \le \frac{2}{9}c_0\|e^k\|_1\|e^{\bar{k}}\|_1 + \frac{1}{9}(F_2^k - F_2^{k-1}) + \frac{\tau}{9h}c_0\|e^k\|_1\|e^{k+1}\|_1 + \frac{\tau}{9h}c_0\|e^k\|_1\|e^{k-1}\|_1.$$

Next, for J_3, we give the following estimation,

$$\begin{aligned}
J_3 &= \frac{2}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}][(e_{i+1}^{\bar{k}})_{\hat{x}} + (e_{i-1}^{\bar{k}})_{\hat{x}}] \\
&= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}][(e_{i+1}^{\bar{k}})_{\tilde{x}} + (e_i^{\bar{k}})_{\tilde{x}} + (e_{i-1}^{\bar{k}})_{\tilde{x}} + (e_{i-2}^{\bar{k}})_{\tilde{x}}] \\
&= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}][(e_i^{\bar{k}})_{\tilde{x}} + (e_{i-1}^{\bar{k}})_{\tilde{x}}] + \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}][(e_{i+1}^{\bar{k}})_{\tilde{x}} + (e_{i-2}^{\bar{k}})_{\tilde{x}}] \\
&:= A_5^k + A_6^k,
\end{aligned}$$

where $A_5^k = -\frac{1}{8}J_1$, and

$$\begin{aligned}
A_6^k &= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}][(e_{i+1}^{\bar{k}})_{\tilde{x}} + (e_{i-2}^{\bar{k}})_{\tilde{x}}] \\
&= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} + (e_i^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}}] \\
&= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} - (e_{i-2}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}}] \\
&\quad + \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_i^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} - (e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}}] \\
&\quad + \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_{i-2}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}}] \\
&\quad + \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}}] \\
&:= \sum_{j=1}^{4} D_j.
\end{aligned}$$

To calculate the result, we need to estimate each term of J_3.

$$\begin{aligned}
D_1 &= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}(e_i^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} - \frac{1}{9}\sum_{i\in I_M} U_{i+2}^{\bar{k}}(e_i^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} \\
&= -\frac{2}{9}h\sum_{i\in I_M}(U_{i+1}^{\bar{k}})_{\hat{x}}(e_i^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}} \\
&\le \frac{2}{9}\|U_{\hat{x}}^{\bar{k}}\|_\infty\|e_{\tilde{x}}^k\|\|e_x^{\bar{k}}\| \\
&\le \frac{2}{9}c_0\|e^k\|_1\|e^{\bar{k}}\|_1.
\end{aligned}$$

Then, we have

$$\begin{aligned}
D_2 = &= \frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}(e_i^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} - \frac{1}{9}\sum_{i\in I_M} U_{i-1}^{\bar{k}}(e_i^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} \\
&= \frac{1}{9}h\sum_{i\in I_M}(U_{i-1}^{\bar{k}})_{\tilde{x}}(e_i^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} \\
&\le \frac{1}{9}\|U_{\tilde{x}}^{\bar{k}}\|_\infty\|e_{\tilde{x}}^k\|\|e_{\tilde{x}}^{\bar{k}}\| \\
&\le \frac{c_0}{9}\|e^k\|_1\|e^{\bar{k}}\|_1.
\end{aligned}$$

By a similar argument as A_2^k, we can estimate D_3 as

$$\begin{aligned}
D_3 &= -\frac{1}{9}\sum_{i\in I_M} U_i^{\bar{k}}[(e_{i-1}^k)_{\tilde{x}}(e_{i-2}^{\bar{k}})_{\tilde{x}} - (e_{i-2}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}}] \\
&= -\frac{1}{9}\sum_{i\in I_M} U_{i+1}^{\bar{k}}[(e_i^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}} - (e_{i-1}^k)_{\tilde{x}}(e_i^{\bar{k}})_{\tilde{x}}] \\
&\le \frac{1}{18}(F_3^{k+1} - F_3^k) + \frac{\tau}{18h}c_0\|e^k\|_1\|e^{k+1}\|_1 + \frac{\tau}{18h}c_0\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}$$

where $F_3^k=\sum_{i\in I_M}U_{i+1}^{k-\frac{1}{2}}[(e_i^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_i^{k-1})_{\tilde{x}}]$. Next, we estimate D_4 as

$$\begin{aligned}
D_4 &= \frac{1}{9}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{\bar{k}})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{\bar{k}})_{\tilde{x}}]\\
&= \frac{1}{18}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k+1})_{\tilde{x}}+(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k+1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k-1})_{\tilde{x}}]\\
&= \frac{1}{18}\Big(\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k+1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k+1})_{\tilde{x}}]-\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k-1})_{\tilde{x}}-(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}]\Big)\\
&= \frac{1}{18}\sum_{i\in I_M}U_i^{k+\frac{1}{2}}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k+1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k+1})_{\tilde{x}}]\\
&\quad -\frac{1}{18}\sum_{i\in I_M}U_i^{k-\frac{1}{2}}[(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k-1})_{\tilde{x}}-(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}]\\
&\quad -\frac{\tau}{36}\sum_{i\in I_M}\delta_tU_i^{k-1}[(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k+1})_{\tilde{x}}-(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k+1})_{\tilde{x}}]\\
&\quad -\frac{\tau}{36}\sum_{i\in I_M}\delta_tU_i^{k}[(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k-1})_{\tilde{x}}-(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}]\\
&\le \frac{1}{18}(F_4^{k+1}-F_4^k)+\frac{\tau}{18h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{\tau}{18h}c_0\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}$$

where $F_4^k=\sum_{i\in I_M}U_i^{k-\frac{1}{2}}[(e_{i-1}^k)_{\tilde{x}}(e_{i+1}^{k-1})_{\tilde{x}}-(e_{i+1}^k)_{\tilde{x}}(e_{i-1}^{k-1})_{\tilde{x}}]$. Thus, the final estimation of A_6^k is

$$A_6^k\le\frac{c_0}{3}\|e^k\|_1\|e^{\bar{k}}\|_1+\frac{1}{18}[(F_3^{k+1}-F_3^k)+(F_4^{k+1}-F_4^k)]+\frac{\tau}{9h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{\tau}{9h}c_0\|e^k\|_1\|e^{k-1}\|_1.$$

Then, we estimate J_4 as

$$\begin{aligned}
J_4 &= -\frac{1}{36}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}][(e_{i+1}^{\bar{k}})_{\hat{x}}+(e_{i-1}^{\bar{k}})_{\hat{x}}]\\
&= -\frac{1}{36}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}+(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}]\\
&= -\frac{1}{36}\sum_{i\in i_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}]\\
&\quad -\frac{1}{36}\sum_{i\in i_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}]\\
&:= A_7^k+A_8^k,
\end{aligned}$$

where

$$\begin{aligned}
A_7^k &= -\frac{1}{36}\sum_{i\in i_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}]\\
&= -\frac{1}{36}\sum_{i\in I_M}U_{i-1}^k(e_i^k)_{\hat{x}}(e_i^{\bar{k}})_{\hat{x}}+\frac{1}{36}\sum_{i\in I_M}U_{i+1}^k(e_i^k)_{\hat{x}}(e_i^{\bar{k}})_{\hat{x}}\\
&= \frac{1}{18}h\sum_{i\in I_M}(U_i^{\bar{k}})_{\hat{x}}(e_i^k)_{\hat{x}}(e_i^{\bar{k}})_{\hat{x}}\\
&\le \frac{1}{18}\|U_{\hat{x}}^{\bar{k}}\|_\infty\|e_{\hat{x}}^k\|\|e_{\hat{x}}^{\bar{k}}\|\\
&\le \frac{c_0}{18}\|e^k\|_1\|e^{\bar{k}}\|_1,
\end{aligned}$$

and by a similar proof as D_4, we have

$$\begin{aligned}
A_8^k &= -\frac{1}{36}\sum_{i\in i_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{\bar{k}})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{\bar{k}})_{\hat{x}}]\\
&= -\frac{1}{72}\sum_{i\in I_M}U_i^{\bar{k}}[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k+1})_{\hat{x}}+(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k-1})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k+1})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k-1})_{\hat{x}}]\\
&= \frac{1}{72}\sum_{i\in I_M}U_i^{k+\frac{1}{2}}[(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k+1})_{\hat{x}}-(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k+1})_{\hat{x}}]\\
&\quad -\frac{1}{72}\sum_{i\in I_M}U_i^{k-\frac{1}{2}}[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k-1})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k-1})_{\hat{x}}]\\
&\quad +\frac{\tau}{144}\sum_{i\in I_M}\delta_tU_i^{k-1}[(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k+1})_{\hat{x}}-(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k+1})_{\hat{x}}]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\tau}{144}\sum_{i\in I_M}\delta_t U_i^k[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k-1})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k-1})_{\hat{x}}]\\
\le\ &\frac{1}{72}(F_5^{k+1}-F_5^k)+\frac{\tau}{72h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{\tau}{72h}c_0\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}
$$

where $F_5^k=\sum_{i\in I_M}U_i^{k-\frac{1}{2}}[(e_{i+1}^k)_{\hat{x}}(e_{i-1}^{k-1})_{\hat{x}}-(e_{i-1}^k)_{\hat{x}}(e_{i+1}^{k-1})_{\hat{x}}]$. So that, the estimation of J_4 is

$$
J_4\le\frac{c_0}{18}\|e^k\|_1\|e^{\bar{k}}\|_1+\frac{1}{72}(F_5^{k+1}-F_5^k)+\frac{\tau}{72h}c_0\|e^k\|_1\|e^{k+1}\|_1+\frac{\tau}{72h}c_0\|e^k\|_1\|e^{k-1}\|_1.
$$

Consequently, since $c_0\tau/h<1$ and $\alpha>0$, we have

$$
\begin{aligned}
P_4\ =\ &\frac{4\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\hat{x}}^{\bar{k}})+\frac{\alpha}{3}(D_x^2e^k\cdot U^{\bar{k}},e_{\ddot{x}}^{\bar{k}})\\
=\ &\alpha\sum_{i=1}^{4}J_i=\alpha(J_1+J_2-\frac{1}{8}J_1+A_6^k+J_4)\\
\le\ &\frac{\alpha}{2}(F^{k+1}-F^k)+\frac{25\alpha}{18}c_0\|e^k\|_1\|e^{\bar{k}}\|_1+\frac{45\alpha}{72}\|e^k\|_1\|e^{k+1}\|_1+\frac{45\alpha}{72}\|e^k\|_1\|e^{k-1}\|_1,
\end{aligned}
$$

where

$$
F^k=\frac{7}{9}F_1^k+\frac{2}{9}F_2^k+\frac{1}{9}F_3^k+\frac{1}{9}F_4^k+\frac{1}{36}F_5^k.
$$

Now, we consider the rest terms of (3.31),

$$
\begin{aligned}
P_5\ =\ &-(D_x(f^kU^{\bar{k}}),\bar{f}^{\bar{k}})=(f^kU^{\bar{k}},D_x\bar{f}^{\bar{k}})\\
=\ &(f^kU^{\bar{k}},\frac{4}{3}\bar{f}_{\hat{x}}^{\bar{k}}-\frac{1}{3}\bar{f}_{\ddot{x}}^{\bar{k}})\\
=\ &(\bar{f}^kU^{\bar{k}}-\beta D_x^2\bar{f}^kU^{\bar{k}},\frac{4}{3}\bar{f}_{\hat{x}}^{\bar{k}}-\frac{1}{3}\bar{f}_{\ddot{x}}^{\bar{k}})\\
=\ &\frac{4}{3}(\bar{f}^kU^{\bar{k}},\bar{f}_{\hat{x}}^{\bar{k}})-\frac{1}{3}(\bar{f}^kU^{\bar{k}},\bar{f}_{\ddot{x}}^{\bar{k}})-\frac{4}{3}\beta(D_x^2\bar{f}^kU^{\bar{k}},\bar{f}_{\hat{x}}^{\bar{k}})+\frac{1}{3}\beta(D_x^2\bar{f}^kU^{\bar{k}},\bar{f}_{\ddot{x}}^{\bar{k}})\\
:=\ &\sum_{j=1}^{4}B_j.
\end{aligned}
$$

Next, we estimate each term of P_5.

$$
\begin{aligned}
B_1\ =\ &\frac{4}{3}(\bar{f}^kU^{\bar{k}},\bar{f}_{\hat{x}}^{\bar{k}})=\frac{2}{3}\sum_{i\in I_M}U_i^{\bar{k}}\bar{f}_i^k(\bar{f}_{i+1}^{\bar{k}}-\bar{f}_{i-1}^{\bar{k}})\\
=\ &\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^k\bar{f}_{i+1}^{k+1}+\bar{f}_i^k\bar{f}_{i+1}^{k-1}-\bar{f}_i^k\bar{f}_{i-1}^{k+1}-\bar{f}_i^k\bar{f}_{i-1}^{k-1})\\
=\ &\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^k\bar{f}_{i+1}^{k+1}-\bar{f}_i^k\bar{f}_{i-1}^{k+1})-\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^{k-1}\bar{f}_{i+1}^k-\bar{f}_i^{k-1}\bar{f}_{i-1}^k)\\
&+\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^k\bar{f}_{i+1}^{k-1}+\bar{f}_i^{k-1}\bar{f}_{i+1}^k)-\frac{1}{3}\sum_{i\in I_M}U_{i+1}^{\bar{k}}(\bar{f}_{i+1}^k\bar{f}_i^{k-1}+\bar{f}_{i+1}^{k-1}\bar{f}_i^k)\\
=\ &\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^k\bar{f}_{i+1}^{k+1}-\bar{f}_i^k\bar{f}_{i-1}^{k+1})-\frac{1}{3}\sum_{i\in I_M}U_i^{\bar{k}}(\bar{f}_i^{k-1}\bar{f}_{i+1}^k-\bar{f}_i^{k-1}\bar{f}_{i-1}^k)\\
&-\frac{h}{3}\sum_{i\in I_M}(U_i^{\bar{k}})_x(\bar{f}_{i+1}^k\bar{f}_i^{k-1}+\bar{f}_{i+1}^{k-1}\bar{f}_i^k)\\
=\ &\frac{1}{3}\sum_{i\in I_M}U_i^{k+\frac{1}{2}}(\bar{f}_i^k\bar{f}_{i+1}^{k+1}-\bar{f}_i^k\bar{f}_{i-1}^{k+1})-\frac{1}{3}\sum_{i\in I_M}U_i^{k-\frac{1}{2}}(\bar{f}_i^{k-1}\bar{f}_{i+1}^k-\bar{f}_i^{k-1}\bar{f}_{i-1}^k)\\
&-\frac{\tau}{6}\sum_{i\in I_M}\delta_tU_i^{k-1}(\bar{f}_i^k\bar{f}_{i+1}^{k+1}-\bar{f}_i^k\bar{f}_{i-1}^{k+1})-\frac{\tau}{6}\sum_{i\in I_M}\delta_tU_i^k(\bar{f}_i^{k-1}\bar{f}_{i+1}^k-\bar{f}_i^{k-1}\bar{f}_{i-1}^k)\\
&-\frac{h}{3}\sum_{i\in I_M}(U_i^{\bar{k}})_x(\bar{f}_{i+1}^k\bar{f}_i^{k-1}+\bar{f}_{i+1}^{k-1}\bar{f}_i^k)\\
\le\ &\frac{1}{3}(G_1^{k+1}-G_1^k)+\frac{\tau c_0}{3h}\|\bar{f}^k\|\|\bar{f}^{k+1}\|+\frac{\tau c_0}{3h}\|\bar{f}^k\|\|\bar{f}^{k-1}\|+\frac{2c_0}{3}\|\bar{f}^k\|\|\bar{f}^{k-1}\|\\
\le\ &\frac{1}{3}(G_1^{k+1}-G_1^k)+\frac{1}{3}\|\bar{f}^k\|\|\bar{f}^{k+1}\|+\frac{2c_0+1}{3}\|\bar{f}^k\|\|\bar{f}^{k-1}\|,
\end{aligned}
$$

where $G_1^k=\sum_{i\in I_M}U_i^{k-\frac{1}{2}}(\bar{f}_i^{k-1}\bar{f}_{i+1}^k-\bar{f}_i^{k-1}\bar{f}_{i-1}^k)$. Then we get the estimation of B_2 by following a similar procedure as B_1.

$$
B_2\ =\ -\frac{1}{3}(\bar{f}^kU^{\bar{k}},\bar{f}_{\ddot{x}}^{\bar{k}})=-\frac{1}{12}\sum_{i\in I_M}U_i^{\bar{k}}\bar{f}_i^k(\bar{f}_{i+2}^{\bar{k}}-\bar{f}_{i-2}^{\bar{k}})
$$

$$
\begin{aligned}
&= -\frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^k \bar f_{i+2}^{k+1} + \bar f_i^k \bar f_{i+2}^{k-1} - \bar f_i^k \bar f_{i-2}^{k+1} - \bar f_i^k \bar f_{i-2}^{k-1})\\
&= \frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^k \bar f_{i-2}^{k+1} - \bar f_i^k \bar f_{i+2}^{k+1}) - \frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^{k-1} \bar f_{i-2}^{k} - \bar f_i^{k-1} \bar f_{i+2}^{k})\\
&\quad + \frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^k \bar f_{i-2}^{k-1} + \bar f_i^{k-1} \bar f_{i-2}^{k}) - \frac{1}{24}\sum_{i\in I_M} U_{i-2}^{\bar k}(\bar f_{i-2}^k \bar f_{i}^{k-1} + \bar f_{i-2}^{k-1} \bar f_{i}^{k})\\
&= \frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^k \bar f_{i-2}^{k+1} - \bar f_i^k \bar f_{i+2}^{k+1}) - \frac{1}{24}\sum_{i\in I_M} U_i^{\bar k}(\bar f_i^{k-1} \bar f_{i-2}^{k} - \bar f_i^{k-1} \bar f_{i+2}^{k})\\
&\quad + \frac{h}{12}\sum_{i\in I_M} (U_{i-1}^{\bar k})_{\hat x}(\bar f_i^k \bar f_{i-2}^{k-1} + \bar f_i^{k-1} \bar f_{i-2}^{k})\\
&= \frac{1}{24}\sum_{i\in I_M} U_i^{k+\frac12}(\bar f_i^k \bar f_{i-2}^{k+1} - \bar f_i^k \bar f_{i+2}^{k+1}) - \frac{1}{24}\sum_{i\in I_M} U_i^{k-\frac12}(\bar f_i^{k-1} \bar f_{i-2}^{k} - \bar f_i^{k-1} \bar f_{i+2}^{k})\\
&\quad - \frac{\tau}{48}\sum_{i\in I_M} \delta_t U_i^{k-1}(\bar f_i^k \bar f_{i-2}^{k+1} - \bar f_i^k \bar f_{i+2}^{k+1}) - \frac{\tau}{48}\sum_{i\in I_M} \delta_t U_i^{k}(\bar f_i^{k-1} \bar f_{i-2}^{k} - \bar f_i^{k-1} \bar f_{i+2}^{k})\\
&\quad + \frac{h}{12}\sum_{i\in I_M} (U_{i-1}^{\bar k})_{\hat x}(\bar f_i^k \bar f_{i-2}^{k-1} + \bar f_i^{k-1} \bar f_{i-2}^{k})\\
&\le \frac{1}{24}(G_2^{k+1} - G_2^k) + \frac{\tau c_0}{24h}\|\bar f^k\|\|\bar f^{k+1}\| + \frac{\tau c_0}{24h}\|\bar f^k\|\|\bar f^{k-1}\| + \frac{c_0}{6}\|\bar f^k\|\|\bar f^{k-1}\|\\
&\le \frac{1}{24}(G_2^{k+1} - G_2^k) + \frac{1}{24}\|\bar f^k\|\|\bar f^{k+1}\| + \frac{1+4c_0}{24}\|\bar f^k\|\|\bar f^{k-1}\|,
\end{aligned}
$$

where $G_2^k = \sum_{i\in I_M} U_i^{k-\frac12}(\bar f_i^{k-1}\bar f_{i-2}^k - \bar f_i^{k-1}\bar f_{i+2}^k)$. We can find that the argument of $B_3 + B_4$ is similar to P_4. Therefore, it is easy to estimate $B_3 + B_4$ as

$$
B_3 + B_4 \le \frac{25}{18}\beta c_0\|\bar f^k\|_1\|\bar f^{\bar k}\|_1 + \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k+1}\|_1 + \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k-1}\|_1 + \frac{\beta}{2}(G_3^{k+1} - G_3^k).
$$

The definition of G_3^k is similar to the definition of F^k, where we only need to change e_i^k into $\bar f_i^k$. Moreover, we define $G^k = \frac{2}{3}G_1^k + \frac{1}{12}G_2^k + \beta G_3^k$. Finally, we get

$$
\begin{aligned}
P_5 &= -(D_x(f^k U^{\bar k}), \bar f^{\bar k})\\
&\le \frac12(G^{k+1} - G^k) + \frac38\|\bar f^k\|\|\bar f^{k+1}\| + \frac{20c_0+9}{24}\|\bar f^k\|\|\bar f^{k-1}\|\\
&\quad + \frac{25c_0}{18}\beta\|\bar f^k\|_1\|\bar f^{\bar k}\|_1 + \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k+1}\|_1 + \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k-1}\|_1.
\end{aligned}
$$

At last, we estimate P_6 as

$$
\begin{aligned}
P_6 &= -(f^k D_x\bar\Pi^{\bar k}, e^{\bar k}) = (\bar f^k D_x\bar\Pi^{\bar k}, e^{\bar k}) - \beta(D_x^2\bar f^k D_x\bar\Pi^{\bar k}, e^{\bar k})\\
&\le \|\bar f^k\|\|D_x\bar\Pi^{\bar k}\|_\infty\|e^{\bar k}\| + \frac{5\beta}{3}c_0\|\bar f^k\|_1\|e^{\bar k}\|_1 + \frac{5\beta}{3}c_0\|e^{\bar k}\|\|\bar f^k\|_1\\
&\le c_0\|\bar f^k\|\|e^{\bar k}\| + \frac{5\beta}{3}c_0\|\bar f^k\|_1\|e^{\bar k}\|_1 + \frac{5\beta}{3}c_0\|e^{\bar k}\|\|\bar f^k\|_1,
\end{aligned}
$$

where we use the estimate of P_3. Substituting P_i ($1 \le i \le 8$) into (3.31), and combining with Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) + \frac{\alpha}{4\tau}(\|e^{k+1}\|_1^2 - \|e^{k-1}\|_1^2) + \frac{1}{4\tau}(\|\bar f^{k+1}\|^2 - \|\bar f^{k-1}\|^2) + \frac{\beta}{4\tau}(\|\bar f^{k+1}\|_1^2 - \|\bar f^{k-1}\|_1^2)\\
\le& \frac{5c_0}{3}\|e^k\|\|e^{\bar k}\| + \frac{5c_0}{3}\|e^k\||e^{\bar k}| + \frac{5\alpha c_0}{3}|e^k\|e^{\bar k}| + \frac{5\alpha c_0}{3}\|e^{\bar k}\|\|e^k\|_1\\
&+ \frac{25\alpha c_0}{18}\|e^k\|_1\|e^{\bar k}\|_1 + \frac{45\alpha}{72}\|e^k\|_1\|e^{k+1}\|_1 + \frac{45\alpha}{72}\|e^k\|_1\|e^{k-1}\|_1 + \frac{\alpha}{2}(F^{k+1} - F^k)\\
&+ \frac12(G^{k+1} - G^k) + \frac38\|\bar f^k\|\|\bar f^{k+1}\| + \frac{20c_0+9}{24}\|\bar f^k\|\|\bar f^{k-1}\| + \frac{25\beta c_0}{18}\|\bar f^k\|_1\|\bar f^{\bar k}\|_1 + \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k+1}\|_1\\
&+ \frac{45\beta}{72}\|\bar f^k\|_1\|\bar f^{k-1}\|_1 + c_0\|\bar f^k\|\|e^{\bar k}\| + \frac{5\beta}{3}c_0\|\bar f^k\|_1\|e^{\bar k}\|_1 + \frac{5\beta}{3}c_0\|e^{\bar k}\|\|\bar f^k\|_1 + \|R^k\|\|e^{\bar k}\| + \|Q^k\|\|\bar f^{\bar k}\|\\
\le& \frac{5c_0}{6}\|e^k\|^2 + \frac{5c_0}{6}\|e^{\bar k}\|^2 + \frac{5c_0}{6}\|e^k\|^2 + \frac{5c_0}{6}\|e^{\bar k}\|_1^2 + \frac{5\alpha c_0}{6}\|e^k\|_1^2 + \frac{5\alpha c_0}{6}\|e^{\bar k}\|_1^2\\
&+ \frac{5\alpha c_0}{6}\|e^{\bar k}\|^2 + \frac{5\alpha c_0}{6}\|e^k\|_1^2 + \frac{25\alpha c_0}{36}\|e^k\|_1^2 + \frac{25\alpha c_0}{36}\|e^{\bar k}\|_1^2 + \frac{45\alpha}{144}\|e^k\|_1^2 + \frac{45\alpha}{144}\|e^{k+1}\|_1^2\\
&+ \frac{45\alpha}{144}\|e^k\|_1^2 + \frac{45\alpha}{144}\|e^{k-1}\|_1^2 + \frac{3}{16}\|\bar f^k\|^2 + \frac{3}{16}\|\bar f^{k+1}\|^2 + \frac{20c_0+9}{48}\|\bar f^k\|^2 + \frac{20c_0+9}{48}\|\bar f^{k-1}\|^2\\
&+ \frac{25\beta c_0}{36}\|\bar f^k\|_1^2 + \frac{25\beta c_0}{36}\|\bar f^{\bar k}\|_1^2 + \frac{45\beta}{144}\|\bar f^k\|_1^2 + \frac{45\beta}{144}\|\bar f^{k+1}\|_1^2 + \frac{45\beta}{144}\|\bar f^k\|_1^2 + \frac{45\beta}{144}\|\bar f^{k-1}\|_1^2
\end{aligned}
$$

$$
\begin{aligned}
&+\ \frac{c_0}{2}\|\bar{f}^k\|^2+\frac{c_0}{2}\|e^{\bar{k}}\|^2+\frac{5\beta c_0}{6}\|\bar{f}^k\|_1^2+\frac{5\beta c_0}{6}\|e^{\bar{k}}\|_1^2+\frac{5\beta c_0}{6}\|\bar{f}^k\|_1^2+\frac{5\beta c_0}{6}\|e^{\bar{k}}\|^2\\
&+\ \frac{1}{2}\|R^k\|^2+\frac{1}{2}\|e^{\bar{k}}\|^2+\frac{1}{2}\|Q^k\|^2+\frac{1}{2}\|\bar{f}^{\bar{k}}\|^2\\
\le&\Big(\frac{2c_0}{3}+\frac{5c_0(\alpha+\beta)+3}{12}\Big)\|e^{k+1}\|^2+\frac{5c_0}{3}\|e^k\|^2+\Big(\frac{2c_0}{3}+\frac{5c_0(\alpha+\beta)+3}{12}\Big)\|e^{k-1}\|^2\\
&+\ \frac{2c_0(55\alpha+30\beta+30)+45\alpha}{144}\|e^{k+1}\|_1^2+\frac{\alpha(45+170c_0)}{72}\|e^k\|_1^2\\
&+\ \frac{2c_0(55\alpha+30\beta+30)+45\alpha}{144}\|e^{k-1}\|_1^2+\frac{7}{16}\|\bar{f}^{k+1}\|^2+\frac{22c_0+9}{24}\|\bar{f}^k\|^2+\frac{20c_0+21}{48}\|\bar{f}^{k-1}\|^2\\
&+\ \frac{\beta(50c_0+45)}{144}\|\bar{f}^{k+1}\|_1^2+\frac{\beta(170c_0+45)}{72}\|\bar{f}\|_1^2+\frac{\beta(50c_0+45)}{144}\|\bar{f}^{k-1}\|_1^2\\
&+\ \frac{\alpha}{2}(F^{k+1}-F_k)+\frac{1}{2}(G^{k+1}-G^k)+\frac{1}{2}\|R^k\|^2+\frac{1}{2}\|Q^k\|^2.
\end{aligned}
\tag{3.33}
$$

From the definition of F_i^k, we know that for $1\le i\le 5$, we can get

$$
\tau|F_i^k|\le 2\frac{c_0\tau}{h}\|e^k\|_1\|e^{k-1}\|_1=2\lambda\|e^k\|_1\|e^{k-1}\|_1.
$$

Therefore,

$$
\tau|F^k|\le\frac{5}{4}\lambda\|e^k\|_1\|e^{k-1}\|_1\le\frac{5}{8}\lambda\|e^k\|_1^2+\frac{5}{8}\lambda\|e^{k-1}\|_1^2. \tag{3.34}
$$

Similarly, we can estimate G_i^k as

$$
\begin{aligned}
&\tau|G_1^k|\le 2\frac{\tau c_0}{h}\|\bar{f}^k\|\|\bar{f}^{k-1}\|=2\lambda\|\bar{f}^k\|\|\bar{f}^{k-1}\|,\\
&\tau|G_2^k|\le 2\frac{\tau c_0}{h}\|\bar{f}^k\|\|\bar{f}^{k-1}\|=2\lambda\|\bar{f}^k\|\|\bar{f}^{k-1}\|,\\
&\tau\beta|G_3^k|\le\frac{5\beta}{4}\lambda\|\bar{f}^k\|_1\|\bar{f}^{k-1}\|_1.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tau|G^k|\ &\le\ \frac{2\tau}{3}|G_1^k|+\frac{\tau}{12}|G_2^k|+\tau\beta|G_3^k|\\
&\le\ \frac{3\lambda}{2}\|\bar{f}^k\|\|\bar{f}^{k-1}\|+\frac{5\beta\lambda}{4}\|\bar{f}^k\|_1\|\bar{f}^{k-1}\|_1\\
&\le\ \frac{3\lambda}{4}(\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2)+\frac{5\beta\lambda}{8}(\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2).
\end{aligned}
\tag{3.35}
$$

Denote

$$
\begin{aligned}
H^k\ =\ &\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}+\alpha\frac{\|e^k\|_1^2+\|e^{k-1}\|_1^2}{2}+\frac{\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2}{2}+\beta\frac{\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2}{2}\\
&-\tau\alpha F^k-\tau G^k.
\end{aligned}
\tag{3.36}
$$

It is obvious that

$$
\begin{aligned}
&\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}+\alpha(1-\frac{5}{4}\lambda)\frac{\|e^k\|_1^2+\|e^{k-1}\|_1^2}{2}+(1-\frac{3}{2}\lambda)\frac{\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2}{2}\\
&+\beta(1-\frac{5}{4}\lambda)\frac{\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2}{2}\le H^k\le\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}+\alpha(1+\frac{5}{4}\lambda)\frac{\|e^k\|_1^2+\|e^{k-1}\|_1^2}{2}\\
&+(1+\frac{3}{2}\lambda)\frac{\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2}{2}+\beta(1+\frac{5}{4}\lambda)\frac{\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2}{2}.
\end{aligned}
\tag{3.37}
$$

Submitting (3.34)–(3.37) into (3.33), we have

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+\alpha\frac{\|e^{k+1}\|_1^2+\|e^k\|_1^2}{2}+\frac{\|\bar{f}^{k+1}\|^2+\|\bar{f}^k\|^2}{2}+\beta\frac{\|\bar{f}^{k+1}\|_1^2+\|\bar{f}^k\|_1^2}{2}-\tau\alpha F^{k+1}-\tau G^{k+1}\right)\\
&-\ \frac{1}{2\tau}\left(\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}+\alpha\frac{\|e^k\|_1^2+\|e^{k-1}\|_1^2}{2}+\frac{\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2}{2}+\beta\frac{\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2}{2}-\tau\alpha F^k-\tau G^k\right)\\
\le&c_3\left[\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}+\alpha(1-\frac{5}{4}\lambda)\frac{\|e^k\|_1^2+\|e^{k-1}\|_1^2}{2}+(1-\frac{3}{2}\lambda)\frac{\|\bar{f}^k\|^2+\|\bar{f}^{k-1}\|^2}{2}\right.\\
&\left.+\beta(1-\frac{5}{4}\lambda)\frac{\|\bar{f}^k\|_1^2+\|\bar{f}^{k-1}\|_1^2}{2}\right]+c_3\left[\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+\alpha(1-\frac{5}{4}\lambda)\frac{\|e^{k+1}\|_1^2+\|e^k\|_1^2}{2}\right.\\
&\left.+(1-\frac{3}{2}\lambda)\frac{\|\bar{f}^{k+1}\|^2+\|\bar{f}^k\|^2}{2}+\beta(1-\frac{5}{4}\lambda)\frac{\|\bar{f}^{k+1}\|_1^2+\|\bar{f}^k\|_1^2}{2}\right]+c_2^2L(\tau^2+h^4)^2
\end{aligned}
$$

$$\le c_3(H^{k+1}+H^k)+c_2^2L(\tau^2+h^4)^2.$$

Therefore, we can calculate

$$\frac{1}{2\tau}(H^{k+1}-H^k)\le c_4(H^{k+1}+H^k)+c_4(\tau^2+h^4)^2.$$

When $2c_4\tau\le 1/3$, by using Grönwall inequality, we have

$$H^{k+1}\le \exp(6Tc_4)(H^1+\frac{1}{2}(\tau^2+h^4)^2). \tag{3.38}$$

When $k=1$, we have the following estimation,

$$\begin{aligned} H^1 &\le \frac{1}{2}\|e^1\|^2+\frac{\alpha}{2}(1+\frac{5\lambda}{4})\|e^1\|_1^2+\frac{1}{2}(1+\frac{3\lambda}{2})\|\bar f^1\|^2+\frac{\beta}{2}(1+\frac{5\lambda}{4})\|\bar f^1\|_1^2\\ &\le \max\{\alpha,\beta\}\cdot(1+\frac{3}{2}\lambda)c_1^2L(\tau^2+h^4)^2. \end{aligned} \tag{3.39}$$

Substituting (3.39) into (3.38), we get

$$H^{k+1}\le \exp(6Tc_4)\cdot[\max\{\alpha,\beta\}(1+\frac{3}{2}\lambda)c_1^2L+\frac{1}{2}](\tau^2+h^4)^2=c_5(\tau^2+h^4)^2.$$

From (3.37), we know that

$$\frac{1}{2}[\|e^{k+1}\|^2+\alpha(1-\frac{5\lambda}{4})\|e^{k+1}\|_1^2+(1-\frac{3\lambda}{2})\|\bar f^{k+1}\|^2+\beta(1-\frac{5\lambda}{4})\|\bar f^{k+1}\|_1^2]\le c_5(\tau^2+h^4)^2.$$

Consequently, we find that for any $1\le k\le N$,

$$\|e^{k+1}\|^2\le 2c_5(\tau^2+h^4)^2\ ,\ \|e^{k+1}\|_1^2\le\frac{2c_5}{\alpha(1-5\lambda/4)}(\tau^2+h^4)^2,$$

$$\|\bar f^{k+1}\|^2\le\frac{2c_5}{\alpha(1-3\lambda/2)}(\tau^2+h^4)^2\ ,\ \|\bar f^{k+1}\|_1^2\le\frac{2c_5}{\beta(1-5\lambda/4)}(\tau^2+h^4)^2.$$

Combining with (3.20), the proof is complete. □

4. Numerical experiment

This section shows several numerical examples to verify the conservation and error estimate. As in [49], we show the errors for two variations by using L^2−norm and semi-norm in temporal direction and spatial direction, respectively. We denote

$$\begin{aligned} \|Eu_{space}(h,\tau)\|_1 &= \max_{0\le k\le N}\sqrt{\frac{4}{3}h\sum_{i\in I_M}\left(u_i^k(h,\tau)-u_{2i}^k(h/2,\tau)\right)_{\tilde x}^2-\frac{1}{3}h\sum_{i\in I_M}\left(u_i^k(h,\tau)-u_{2i}^k(h/2,\tau)\right)_{\hat x}^2},\\ \|Eu_{space}(h,\tau)\| &= \max_{0\le k\le N}\sqrt{h\sum_{i\in I_M}\left|u_i^k(h,\tau)-u_{2i}^k(h/2,\tau)\right|^2},\\ \|Eu_{time}(h,\tau)\|_1 &= \max_{0\le k\le N}\sqrt{\frac{4}{3}h\sum_{i\in I_M}\left(u_i^k(h,\tau)-u_i^{2k}(h,\tau/2)\right)_{\tilde x}^2-\frac{1}{3}h\sum_{i\in I_M}\left(u_i^k(h,\tau)-u_i^{2k}(h,\tau/2)\right)_{\hat x}^2},\\ \|Eu_{time}(h,\tau)\| &= \max_{0\le k\le N}\sqrt{h\sum_{i\in I_M}\left|u_i^k(h,\tau)-u_i^{2k}(h,\tau/2)\right|^2}. \end{aligned} \tag{4.1}$$

and $\|E\bar\rho_{time}(h,\tau)\|_1$, $\|E\bar\rho_{time}(h,\tau)\|$, $\|E\bar\rho_{space}(h,\tau)\|_1$, $\|E\bar\rho_{space}(h,\tau)\|$ have the similar definitions.

The spatial convergence orders of velocity u are defined for sufficiently small h and τ as

$$rate_h^1=\log_2\frac{\|Eu_{space}(2h,\tau)\|_1}{\|Eu_{space}(h,\tau)\|_1},\quad rate_h^2=\log_2\frac{\|Eu_{space}(2h,\tau)\|}{\|Eu_{space}(h,\tau)\|}, \tag{4.2}$$

and the temporal convergence orders are

$$rate_\tau^1=\log_2\frac{\|Eu_{time}(h,2\tau)\|_1}{\|Eu_{time}(h,\tau)\|_1},\quad rate_\tau^2=\log_2\frac{\|Eu_{time}(h,2\tau)\|}{\|Eu_{time}(h,\tau)\|}. \tag{4.3}$$

The convergence orders of magnitude $\bar\rho$ have the similar definitions. Next, the energy discrepancies D^n and the mass discrepancies M^n are defined as followed:

$$D^n=\frac{|E^n-E^0|}{|E^0|},\quad M^n=\frac{|I^n-I^0|}{|I^0|}.$$

Example 4.1 (*Dark Break Problem [49]*). We consider problems with periodic boundary conditions on the domain $[-L,L]$. The dam-break initial conditions are given as follows,

$$u_0(x)=0,\quad \bar\rho_0(x)=1+\tanh(x+a)-\tanh(x-a),$$

where a represents the dam-breaking parameter. Numerical experiments are shown by using two sets of parameters.

Table 1
Spatial convergence rates with $\tau = 1/1000$ of Case I.

h	$\|Eu_{space}(h,\tau)\|_1$	$rate_h^1$	$\|Eu_{space}(h,\tau)\|$	$rate_h^2$	$\|E\bar{\rho}_{space}(h,\tau)\|_1$	$rate_h^1$	$\|E\bar{\rho}_{space}(h,\tau)\|$	$rate_h^2$
1/5	1.9165e−03	*	4.8896e−04	*	7.8347e−03	*	1.5088e−03	*
1/10	1.3456e−04	3.8321	3.2546e−05	3.9092	6.0786e−04	3.6881	1.0701e−04	3.8176
1/20	8.6373e−06	3.9616	2.0662e−06	3.9774	3.9807e−05	3.9326	6.8866e−06	3.9578
1/40	5.4325e−07	3.9909	1.2964e−07	3.9944	2.5140e−06	3.9850	4.3341e−07	3.9900

Table 2
Temporal convergence rates with $h = 3\pi/125$ of Case I.

τ	$\|Eu_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|Eu_{time}(h,\tau)\|$	$rate_\tau^2$	$\|E\bar{\rho}_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|E\bar{\rho}_{time}(h,\tau)\|$	$rate_\tau^2$
1/8	5.2909e−03	*	5.3405e−03	*	7.8042e−03	*	4.5872e−03	*
1/16	1.3574e−03	1.9627	1.3708–03	1.9619	1.9843e−03	1.9756	1.1800e−03	1.9589
1/32	3.4232e−04	1.9874	3.4565e−04	1.9877	5.0109e−04	1.9855	2.9832e−04	1.9838
1/64	8.5861e−05	1.9953	8.6680e−06	1.9955	1.2593e−04	1.9925	7.4940e−05	1.9931

Table 3
Spatial convergence rates with $\tau = 1/1000$ of Case II.

h	$\|Eu_{space}(h,\tau)\|_1$	$rate_h^1$	$\|Eu_{space}(h,\tau)\|$	$rate_h^2$	$\|E\bar{\rho}_{space}(h,\tau)\|_1$	$rate_h^1$	$\|E\bar{\rho}_{space}(h,\tau)\|$	$rate_h^2$
1/5	9.7735e−04	*	2.2077e−04	*	5.1291e−04	*	1.3277e−04	*
1/10	7.0632e−05	3.7905	1.5169e−05	3.8633	3.5324e−05	3.8600	8.9217e−06	3.8955
1/20	4.5827e−06	3.9461	9.7246e−07	3.9633	2.2661e−06	3.9624	5.6867e−07	3.9717
1/40	2.8911e−07	3.9865	6.1145e−08	3.9913	1.4264e−07	3.9898	3.5782e−08	3.9903

Table 4
Temporal convergence rates with $h = 1/100$ of Case II.

τ	$\|Eu_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|Eu_{time}(h,\tau)\|$	$rate_\tau^2$	$\|E\bar{\rho}_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|E\bar{\rho}_{time}(h,\tau)\|$	$rate_\tau^2$
1/8	9.9558e−05	*	4.7284e−05	*	1.3954e−05	*	5.0343e−06	*
1/16	2.4999e−05	1.9937	1.1876e−05	1.9933	3.4992e−06	1.9955	1.2635e−06	1.9944
1/32	6.2609e−06	1.9974	2.9742e−06	1.9975	8.7583e−07	1.9983	3.1631e−07	1.9980
1/64	1.5665e−06	1.9988	7.4409e−07	1.9989	2.1911e−07	1.9990	7.9137e−08	1.9989

Case 1. $\alpha = 1,\ \beta = 0,\ g = 1,\ \bar{\rho}_0 = 0,\ a = 4$ on the periodic domain $(x, t) \in [-12\pi, 12\pi] \times (0, 1]$.

Case 2. $\alpha = 0.3,\ \beta = 1,\ g = 1,\ \bar{\rho}_0 = 1,\ a = 0.2$ on the periodic domain $(x, t) \in [-8, 8] \times (0, 1]$.

Convergence tests. We solve the problem with different parameters. When we choose the parameters in Case I, the errors of two variables u and $\bar{\rho}$ are shown in Table 1 by refining spacial step size with a fixed temporal step-size $\tau = 1/1000$. The errors presented in Table 2 are calculated by refining temporal step size with a fixed spacial step-size $h = 3\pi/125$. We can find that the present scheme is second-order in time and fourth-order in space. In Table 3, the errors are calculated by refining spacial step size with a fixed temporal step-size $\tau = 1/1000$, when we choose the parameters in Case II. Table 4 presents the errors by refining temporal step size with a fixed spatial time-size $h = 0.01$. Table 3 and Table 4 show the same convergence rates in space and time.

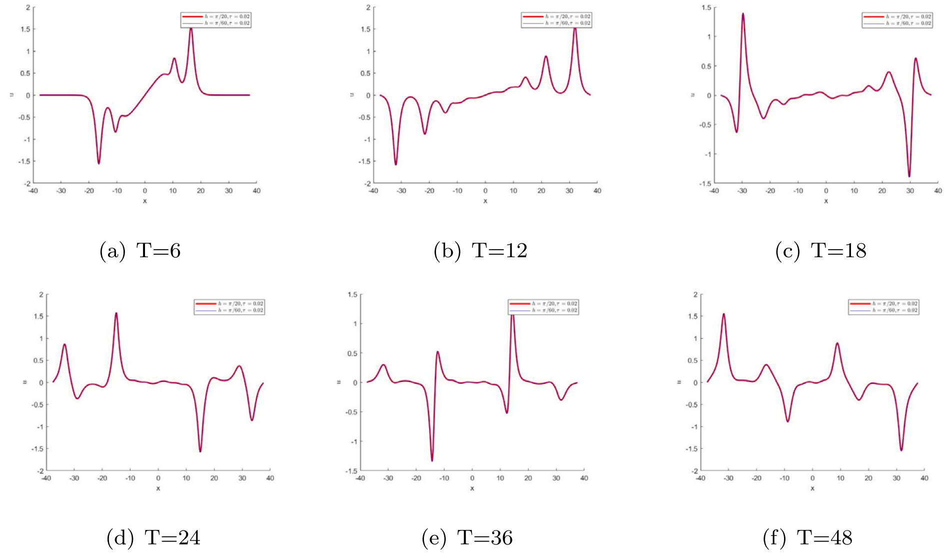
Long time evolution. We observe the process of velocity $u(x, t)$ and magnitude $\rho(x, t)$ in a long-time period for both Case I and Case II. For Case I, we set the solutions of step size $h = \pi/20, \tau = 0.02$ as the numerical solutions and the solutions of step size $h = \pi/60, \tau = 0.02$ as the reference exact solutions. Then we display the profiles at six different points in time $T = \{6, 12, 18, 24, 36, 48\}$ in Fig. 1 and Fig. 2, where u and ρ present good agreements. For Case II, we set the solutions of step size $h = 1/10, \tau = 0.01$ as the numerical solutions and the solutions of step size $h = 1/40, \tau = 0.01$ as the reference exact solutions, the numerical results at time points $T = \{1, 10, 20, 30, 40, 50\}$ in Fig. 3 and Fig. 4 are all consistent with the reference solutions. The numerical solutions reach the same phase and peak values as the reference solution, and show symmetry, even after a long period of time. The scheme can achieve high resolutions and produce accurate results, even for large values of h.

Furthermore, Fig. 5 simulates the process of long-term two-component Euler–Poincaré equations in Case I. The result displays a steepening behavior. The amplitude of the solution develops into zero right at the moment of overlap, as well as the slope becomes vertical. Then, the peaks develop and depart again as time increases. These performances are consistent with the conclusion in [49]. For Case II, the same consequence can be observed in Fig. 6.

Conservations. We show the mass- and energy-conservation properties of the fully discrete schemes (2.19)–(2.24). In order to better express the conservation of our present scheme, we also compare our scheme with other schemes. By using implicit Euler method in time discretization and extrapolate method to deal with the nonlinear terms, a linearized scheme is given as following:

Linearized Euler method:

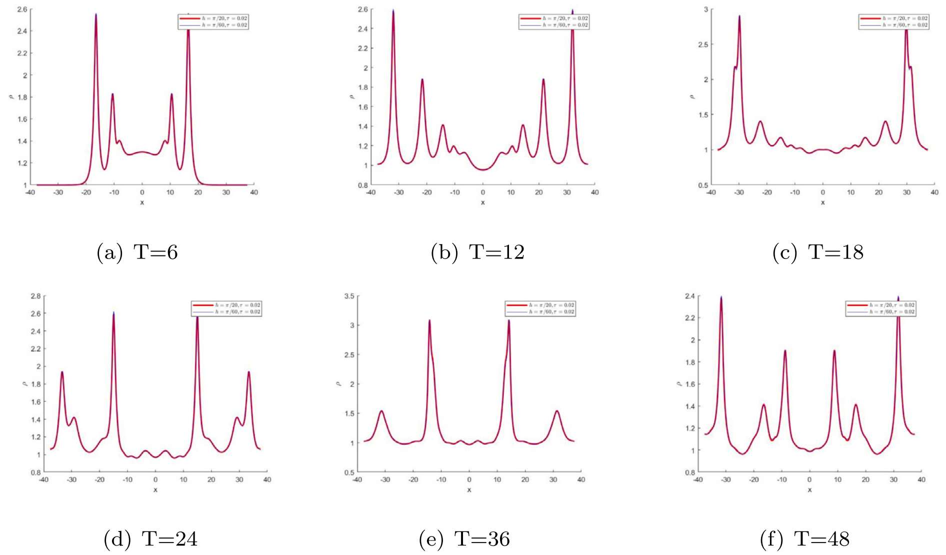
$$\frac{u_i^{n+1} - u_i^n}{\tau} - \alpha \frac{D_x^2 u_i^{n+1} - D_x^2 u_i^n}{\tau} + \varphi(u^n, u^{n+1}) - \alpha\varphi(D_x^2 u^n, u^{n+1}) + g\rho_i^n \delta_x \bar{\rho}_i^{n+1} = 0,$$



(a) T=6 (b) T=12 (c) T=18

(d) T=24 (e) T=36 (f) T=48

Fig. 1. The numerical solutions profiles of the velocity $u(x,t)$ at different points in time in Case I for the step size $h = \pi/20, \tau = 0.02$ as the numerical solutions and $h = \pi/60, \tau = 0.02$ as the reference exact solutions.



(a) T=6 (b) T=12 (c) T=18

(d) T=24 (e) T=36 (f) T=48

Fig. 2. The numerical solutions profiles of the magnitude $\rho(x,t)$ at different points in time in Case I for the step size $h = \pi/20, \tau = 0.02$ as the numerical solutions and $h = \pi/60, \tau = 0.02$ as the reference exact solutions.

$$
\begin{aligned}
&\frac{\bar{\rho}_i^{n+1} - \bar{\rho}_i^n}{\tau} - \beta \frac{D_x^2 \bar{\rho}_i^{n+1} - D_x^2 \bar{\rho}_i^n}{\tau} + D_x(\rho^n u^{n+1})_i = 0,\\
&\rho_i^{n+1} = (\mathcal{I} - \beta D_x^2)\bar{\rho}_i^{n+1}.
\end{aligned} \tag{4.4}
$$

Then, another fully implicit numerical scheme is established by using Crank–Nicolson method. The fully-discrete method is written as:

Crank–Nicolson method

$$
\begin{aligned}
&\frac{u_i^{n+1} - u_i^n}{\tau} - \alpha \frac{D_x^2 u_i^{n+1} - D_x^2 u_i^n}{\tau} + \varphi(u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) - \alpha \varphi(D_x^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) + g\rho_i^{n+\frac{1}{2}} D_x \bar{\rho}_i^{n+\frac{1}{2}} = 0,\\
&\frac{\bar{\rho}_i^{n+1} - \bar{\rho}_i^n}{\tau} - \beta \frac{D_x^2 \bar{\rho}_i^{n+1} - D_x^2 \bar{\rho}_i^n}{\tau} + D_x(\rho^{n+\frac{1}{2}} u^{n+\frac{1}{2}})_i = 0,\\
&\rho_i^{n+1} = (\mathcal{I} - \beta D_x^2)\bar{\rho}_i^{n+1}.
\end{aligned} \tag{4.5}
$$

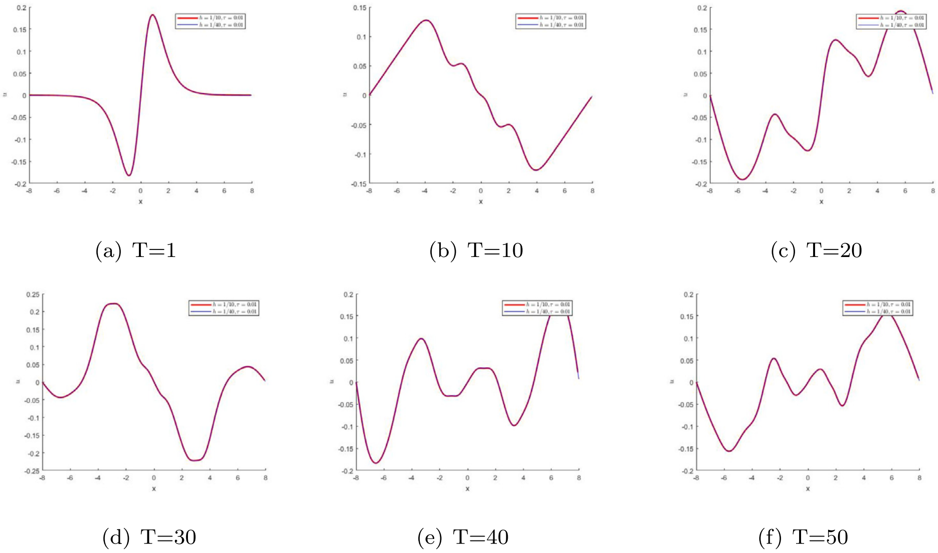


Fig. 3. The numerical solutions profiles of the velocity $u(x,t)$ at different points in time in Case II for the step size $h = 1/10, \tau = 0.01$ as the numerical solutions and $h = 1/40, \tau = 0.01$ as the reference exact solutions.

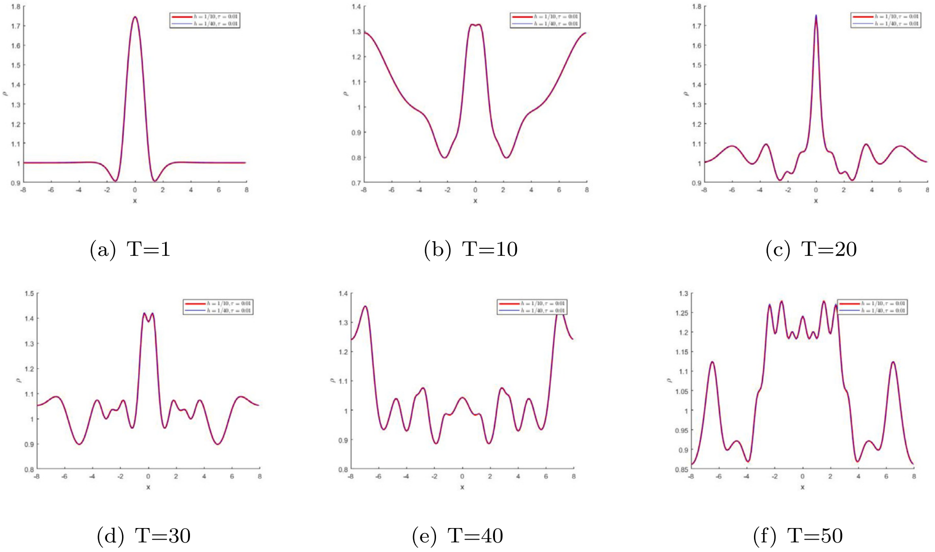


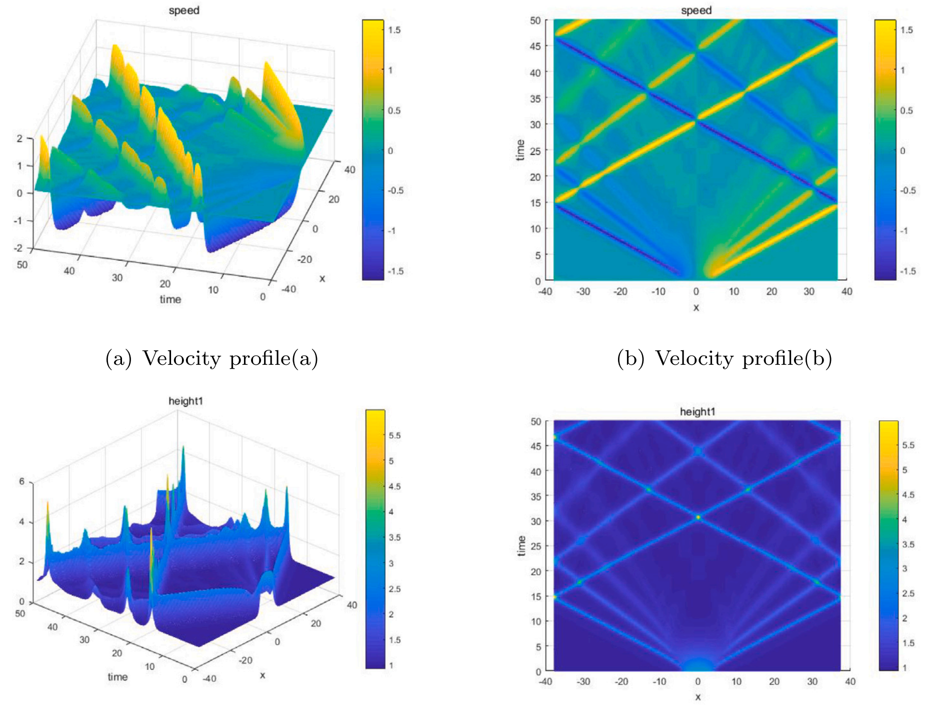
Fig. 4. The numerical solutions profiles of the magnitude $\rho(x,t)$ at different points in time in Case II for the step size $h = 1/10, \tau = 0.01$ as the numerical solutions and $h = 1/40, \tau = 0.01$ as the reference exact solutions.

Furthermore, we compare the evolution of two conservation of different methods. They are denoted as ''Linearized Euler", ''Crank–Nicolson", and our method (2.19)–(2.24) is denoted as ''the present scheme" in Fig. 7 of Case I and Fig. 8 of Case II. By the analysis of Theorem 2.5, it is easy to find that all methods mentioned above can conserve mass over a long time interval. However, the linearized Euler method cannot preserve energy. Even though the fully implicit method exhibits good energy conservation, it cost more time and storage space than our present schemes.

Example 4.2 (*Solitary Wave Propagation [5]*)**.** We consider a solitary wave with the following initial data:

$$\bar{\rho}_0(x) = 1 + \frac{1}{10}\,\mathrm{sech}^2\left(\sqrt{\frac{3}{40}}(x-70)\right), \quad u_0(x) = \frac{\sqrt{g}}{10}\,\mathrm{sech}^2\left(\sqrt{\frac{3}{40}}(x-70)\right),$$

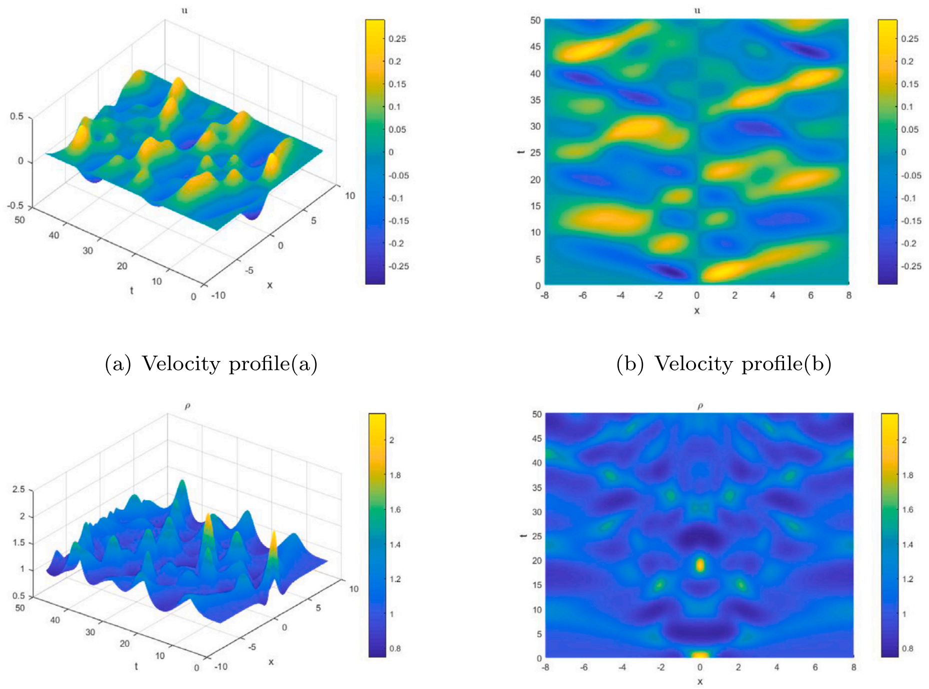
where we take $x \in [0, 200]$, $g = 9.81$, $\bar{\rho}_0 = 0$, $\alpha = 0.6$, and $\beta = 0.2$.



(a) Velocity profile(a) (b) Velocity profile(b)

(c) Magnitude profile(a) (d) Magnitude profile(b)

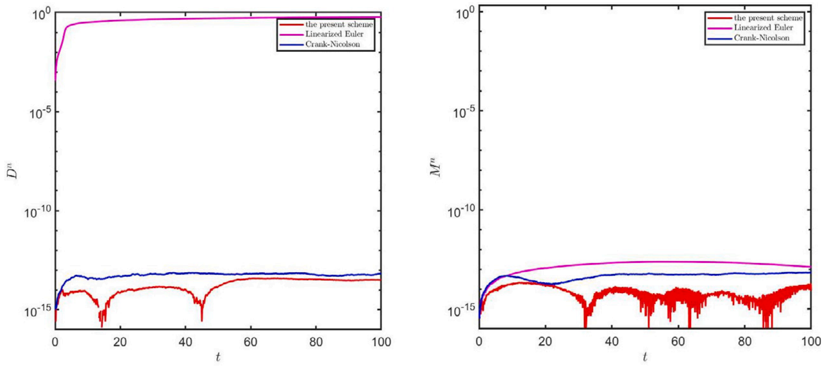
Fig. 5. The simulated profiles of the velocity $u(x,t)$ and the magnitude $\rho(x,t)$ at two view angles in Case I.



(a) Velocity profile(a) (b) Velocity profile(b)

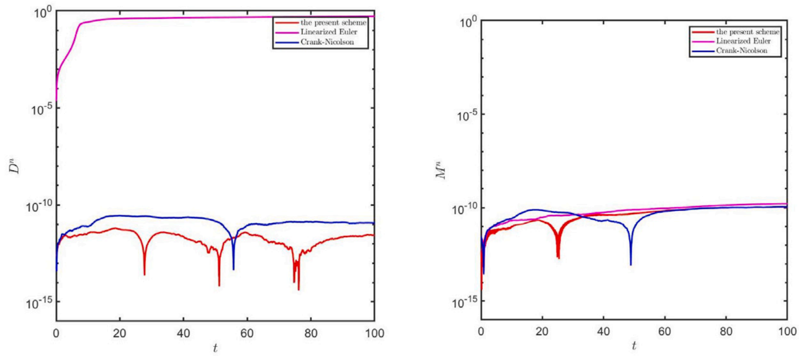
(c) Magnitude profile(a) (d) Magnitude profile(b)

Fig. 6. The simulated profiles of the velocity $u(x,t)$ and the magnitude $\rho(x,t)$ at two view angles in Case II.



(a) Evolution of D^n. (b) Evolution of M^n.

Fig. 7. Evolution of D^n (left) and M^n (right) for the different discrete schemes of Case I for $\tau = 1/10,\ h = \pi/40$.



(a) Evolution of D^n (b) Evolution of M^n

Fig. 8. Evolutions of D^n (left) and M^n (right) for the different discrete schemes of Case II for $\tau = 1/20,\ h = 1/20$.

Table 5
Temporal convergence rates with $h = 0.1$ of Example 4.2.

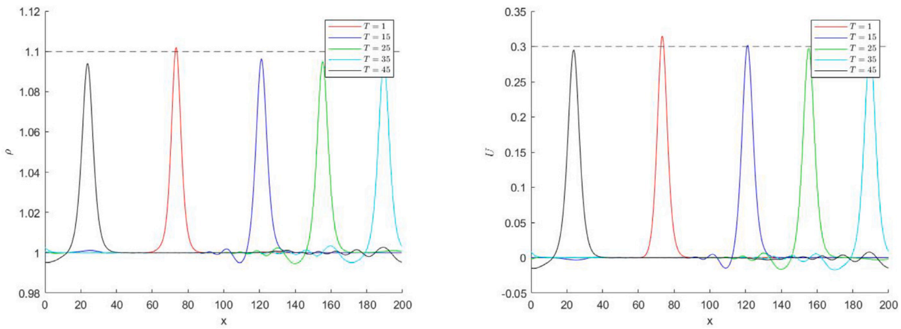
τ	$\|Eu_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|Eu_{time}(h,\tau)\|$	$rate_\tau^2$	$\|E\bar{\rho}_{time}(h,\tau)\|_1$	$rate_\tau^1$	$\|E\bar{\rho}_{time}(h,\tau)\|$	$rate_\tau^2$
1/8	4.0008e−03	*	5.7142e−03	*	1.5392e−03	*	2.0398e−03	*
1/16	1.0553e−03	1.9226	1.4864e−03	1.9427	4.0938e−04	1.9107	5.3393e−04	1.9337
1/32	2.7084e−04	1.9622	3.7560e−04	1.9846	1.0784e−04	1.9246	1.3533e−04	1.9801
1/64	6.9863e−05	1.9549	9.4600e−05	1.9893	2.7770e−05	1.9573	3.4919e−05	1.9544

Table 6
Special convergence rates with $\tau = 1/1000$ of Example 4.2.

h	$\|Eu_{space}(h,\tau)\|_1$	$rate_h^1$	$\|Eu_{space}(h,\tau)\|$	$rate_h^2$	$\|E\bar{\rho}_{space}(h,\tau)\|_1$	$rate_h^1$	$\|E\bar{\rho}_{space}(h,\tau)\|$	$rate_h^2$
1/2	2.1518e−04	*	2.0572e−04	*	1.0618e−04	*	9.0810e−05	*
1/4	1.4026e−05	3.9463	1.3232e−05	3.9585	4.9855e−06	3.9260	5.8878e−06	3.9470
1/8	8.8493e−07	3.9864	8.3295e−07	3.9897	4.4183e−07	3.9828	3.7128e−07	3.9872
1/16	5.5439e−08	3.9966	5.2155e−08	3.9974	3.3565e−08	3.7184	2.3270e−08	3.9960

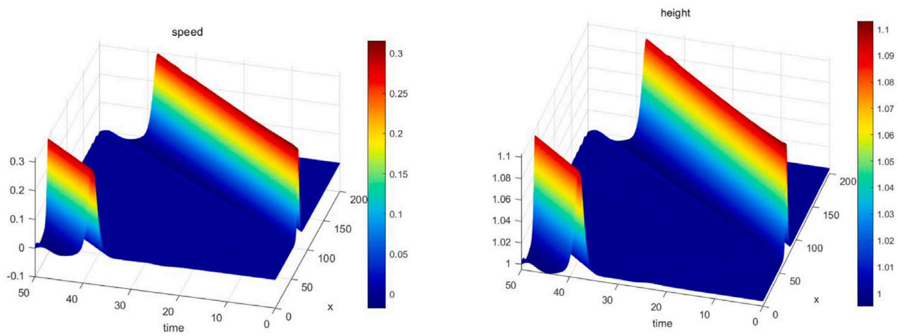
Convergence tests. We calculate the errors and the convergence rates of the numerical solutions by (4.1)–(4.3). Since the parameters are already given, errors of two variables u and $\bar{\rho}$ are shown in Table 5 by refining spacial step size with a fixed temporal step-size $\tau = 1/1000$. The errors presented in Table 6 are calculated by refining temporal step size with a fixed spatial time step size $h = 0.1$. We can see the numerical scheme has the convergence order of 2 in time and 4 in space.

Long time evolution. Ignoring bottom, lateral boundary shear, and dissipation, the shape of a solitary wave cannot be changed when traveling over a horizontal bottom. This immutability is depicted well in Fig. 9, where the changes in the aspect of velocity and magnitude of the wave in the spatial region at $t = \{1, 15, 25, 35, 45\}$ are both shown. Since $\alpha \neq 0$, the solitary wave is evolving towards



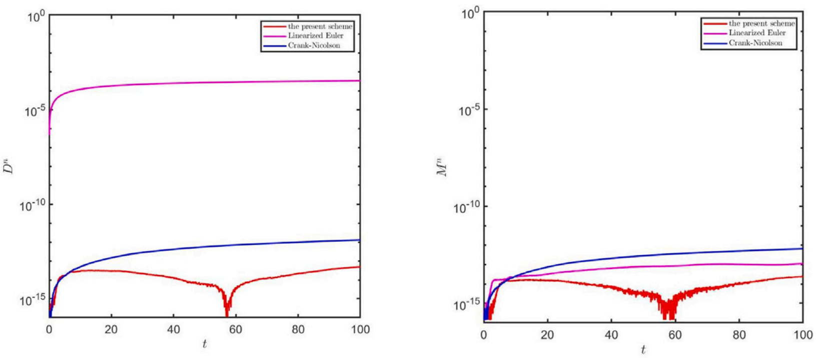
(a) Evolution of the magnitude (b) Evolution of the velocity

Fig. 9. Time evolution of the magnitude ρ (left) and velocity u (right) of Example 4.2 for $\tau = 1/20,\ h = 1/2$.



(a) Velocity profile (b) Magnitude profile

Fig. 10. The simulated profiles of the velocity $u(x,t)$ (left) and the magnitude $\rho(x,t)$ (right) in Example 4.2.



(a) Evolution of D^n (b) Evolution of M^n

Fig. 11. Evolutions of D^n (left) and M^n (right) for the different discrete schemes of Example 4.2 for $\tau = 1/20,\ h = 1/2$.

a shock wave, both the wave magnitude decays and the solitary wave profile is getting destroyed. The evolution is consistence with the conclusion in [5]. Furthermore, we simulate the long-term solitary wave propagation process and present the profiles in Fig. 10.

Conservations. In order to examine whether our present scheme can preserve invariants, we compare our methods with other discrete schemes (4.4)–(4.5). We plot the results in Fig. 11. These results further verify our theoretical results and show that our numerical scheme has great advantages.

5. Conclusion

In this paper, we have proposed a decoupled, linearly implicit scheme for the modified two-component Euler–Poincaré equation. The scheme is proved to keep both the discrete mass and energy conserved. And it has second-order accuracy in the temporal direction and fourth-order accuracy in the spatial direction. Furthermore, the error estimate of the fully-discrete scheme has been given by rigorous analysis. Several numerical experiments are given to show that the numerical results are consistent with the theoretical ones.

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