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Global convergence rates in zero-relaxation limits for non-isentropic Euler-Maxwell equations

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Abstract

We consider non-isentropic Euler-Maxwell equations with relaxation times (small physical parameters) arising in the models of magnetized plasma and semiconductors. For smooth periodic initial data sufficiently close to constant steady-states, we prove the uniformly global existence of smooth solutions with respect to the parameter, and the solutions converge global-in-time to the solutions of the energy-transport equations in a slow time scaling as the relaxation time goes to zero. We also establish error estimates between the smooth periodic solutions of the non-isentropic Euler-Maxwell equations and those of energy-transport equations. The proof is based on stream function techniques and the classical energy method but with some new developments.

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Keywords: Convergence rates; Non-isentropic Euler-Maxwell equations; Energy-transport equations; Stream function

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1. Introduction

The three-dimensional non-isentropic Euler-Maxwell equations describe the motions of particles in plasmas, which can be written as (see [1,11,20])

$$\begin{cases} \partial_{t'} n + \operatorname{div}(nu) = 0, \\ \partial_{t'}(nu) + \operatorname{div}(nu \otimes u) + \nabla(n\theta) = -n(E + u \times B) - \frac{nu}{\tau_p}, \\ \partial_{t'}\theta + u \cdot \nabla\theta + \frac{2}{3}\theta \operatorname{div} u = -\frac{2|u|^2}{3}\left(\frac{1}{2\tau_w} - \frac{1}{\tau_p}\right) - \frac{1}{\tau_\omega}(\theta - 1), \\ \partial_{t'} E - \nabla \times B = nu, \quad \operatorname{div} E = 1 - n, \\ \partial_{t'} B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \quad (t', x) \in (0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{1.1}$$

where $\mathbb{T}^3 = \left(\frac{\mathbb{R}}{2\pi}\right)^3$ is a torus, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable and $t' > 0$ is the usual time. Here $n > 0$ is the scaled density, u is the velocity field, θ is the electron temperature, E is the electric field and B is the magnetic field, which are all functions of (t', x). The physical parameters τ_p and τ_ω are the momentum relaxation time and the energy relaxation time, respectively.

Obviously, (1.1) admits a steady-state

$$(n, u, \theta, E, B) = U_e \triangleq (1, 0, 1, 0, B_e), \tag{1.2}$$

where $B_e \in \mathbb{R}^3$ is an arbitrary constant vector.

Next, we introduce

$$t = \varepsilon t', \quad \tau_p = \varepsilon \in (0, 1], \quad \tau_w = \frac{1}{\varepsilon},$$

and

$$n^\varepsilon(t, x) = n\left(t', x\right), \quad u^\varepsilon(t, x) = \frac{1}{\varepsilon} u\left(t', x\right), \quad \theta^\varepsilon(t, x) = \theta\left(t', x\right),$$
$$E^\varepsilon(t, x) = E\left(t', x\right), \quad B^\varepsilon(t, x) = B\left(t', x\right).$$

Then (1.1) turns into

$$\begin{cases} \partial_t n^\varepsilon + \operatorname{div}\left(n^\varepsilon u^\varepsilon\right) = 0, \\ \partial_t\left(n^\varepsilon u^\varepsilon\right) + \operatorname{div}\left(n^\varepsilon u^\varepsilon \otimes u^\varepsilon\right) + \dfrac{1}{\varepsilon^2}\nabla\left(n^\varepsilon \theta^\varepsilon\right) = \dfrac{1}{\varepsilon^2}\left(-n^\varepsilon\left(E^\varepsilon + \varepsilon u^\varepsilon \times B^\varepsilon\right) - n^\varepsilon u^\varepsilon\right), \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon + \dfrac{2}{3}\theta^\varepsilon \operatorname{div} u^\varepsilon = \dfrac{|u^\varepsilon|^2}{3}\left(2 - \varepsilon^2\right) - \left(\theta^\varepsilon - 1\right), \\ \varepsilon\partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \operatorname{div} E^\varepsilon = 1 - n^\varepsilon, \\ \varepsilon\partial_t E^\varepsilon - \nabla \times B^\varepsilon = \varepsilon n^\varepsilon u^\varepsilon, \quad \operatorname{div} B^\varepsilon = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{T}^3, \end{cases} \tag{1.3}$$

with initial conditions

$$(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)_{t=0} = U_0^\varepsilon \triangleq \left(n_0^\varepsilon, \frac{u_0^\varepsilon}{\varepsilon}, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon\right), \tag{1.4}$$

which satisfy the compatibility conditions

$$\operatorname{div} E_0^\varepsilon = 1 - n_0^\varepsilon, \quad \operatorname{div} B_0^\varepsilon = 0, \quad x \in \mathbb{T}^3. \tag{1.5}$$

Formally, as $\varepsilon \to 0$, if denoting the limits of $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ as $(\bar{n}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B})$, we get that the formal limiting equations of (1.3) are

$$\begin{cases} \partial_t \bar{n} + \operatorname{div}(\bar{n}\bar{u}) = 0, \\ \nabla(\bar{n}\bar{\theta}) = -\bar{n}\bar{E} - \bar{n}\bar{u}, \\ \partial_t \bar{\theta} + \bar{u} \cdot \nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta} \operatorname{div} \bar{u} = \dfrac{2|\bar{u}|^2}{3} - (\bar{\theta} - 1), \\ \nabla \times \bar{E} = 0, \quad \operatorname{div} \bar{E} = 1 - \bar{n}, \\ \nabla \times \bar{B} = 0, \quad \operatorname{div} \bar{B} = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{T}^3. \end{cases} \tag{1.6}$$

Apparently, it follows from the fifth equation in (1.6) that $\bar{B}$ is a constant vector. And since $\nabla \times \bar{E} = 0$, there is a potential function $\bar{\phi}$ satisfying $\bar{E} = \nabla\bar{\phi}$. Hence, (1.6) implies the energy-transport model

$$\begin{cases} \partial_t \bar{n} - \Delta(\bar{n}\bar{\theta}) - \operatorname{div}(\bar{n}\nabla\bar{\phi}) = 0, \\ \partial_t \bar{\theta} + \bar{u} \cdot \nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta} \operatorname{div} \bar{u} = \dfrac{2|\bar{u}|^2}{3} - (\bar{\theta} - 1), \\ \Delta\bar{\phi} = 1 - \bar{n}, \end{cases} \tag{1.7}$$

and

$$\bar{E} = \nabla\bar{\phi}, \quad \bar{u} = -\nabla(\bar{\phi} + \bar{\theta}) - \bar{\theta}\nabla \ln \bar{n}. \tag{1.8}$$

For the uniqueness of $\bar{\phi}$, the following is needed,

$$m_{\bar{\phi}}(t) \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \bar{\phi}(t, x)dx = 0, \quad \forall t \geq 0. \tag{1.9}$$

Asymptotic limit problems with small parameters for important physical models are well-known research highlights. Especially, the relaxation time usually plays a crucial role in many physical models for the fluid dynamics [9,13,16]. And the zero relaxation time limit is a well-known problem in asymptotic analysis and singular perturbation theories [10,12,24,27–29]. In [9], Hajjej-Peng investigated the local-in-time initial layer problem to the isentropic Euler-Maxwell system. Next, Wasiolek [26] proved the global existence of smooth solutions and the global-in-time convergence of the isentropic Euler-Maxwell system with respects to the parameters by uniform energy estimates. Later, Li-Peng-Zhao [17] showed us the global-in-time convergence rates for isentropic Euler-Maxwell equations.

For the non-isentropic Euler-Maxwell system (1.3)-(1.4), when all parameters are taken to be unity, Feng-Wang-Kawashima [6] proved the L^q time decay rate for the global smooth solution by carrying out uniform energy estimates, which showed us that the density and temperature of electron converge to the steady states with the same rate. We also mention the important developments about the wellposedness of solutions to Euler-Maxwell equations in [2–4,7,8,18, 22,21,23,25,26]. However, these results did not take into account the dependence with respect to the physical parameters.

Recently, Feng-Li-Mei-Wang [5] studied the local-in-time initial layer problem to the non-isentropic Euler-Maxwell system. However, the global convergence and convergence rate for non-isentropic Euler-Maxwell system (1.3)-(1.4) are still open, of course, it is more interesting to investigate the global convergence as well as the convergence rates. To solve these questions is the main purpose of the present paper. Our main results are stated as follows.

Theorem 1.1 *(Global existence). Let $k \geq 3$ be an integer, there exist constants $\varepsilon_0 > 0, \varpi_0 > 0$ and $C > 0$, independent of ε and any time, such that, for all $\varepsilon \in (0, \varepsilon_0]$, if*

$$\left\|n_0^\varepsilon - 1\right\|_k + \left\|u_0^\varepsilon\right\|_k + \left\|\theta_0^\varepsilon - 1\right\|_k + \left\|E_0^\varepsilon\right\|_k + \left\|B_0^\varepsilon - B_e\right\|_k \leq \varpi_0,$$

then the system (1.3)-(1.4) admits a unique global solution $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$, satisfying

$$n^\varepsilon - 1, \varepsilon u^\varepsilon, \theta^\varepsilon - 1, E^\varepsilon, B^\varepsilon - B_e \in C\left(\mathbb{R}^+; H^k\right) \cap C^1\left(\mathbb{R}^+; H^{k-1}\right).$$

Moreover, it holds,

$$\begin{aligned} &\left\|n^\varepsilon(t) - 1\right\|_k^2 + \left\|\varepsilon u^\varepsilon(t)\right\|_k^2 + \left\|\theta^\varepsilon(t) - 1\right\|_k^2 + \left\|E^\varepsilon(t)\right\|_k^2 + \left\|B^\varepsilon(t) - B_e\right\|_k^2 \\ &+ \int_0^t \left(\left\|n^\varepsilon(\tau) - 1\right\|_k^2 + \left\|u^\varepsilon(\tau)\right\|_k^2 + \left\|\theta^\varepsilon(\tau) - 1\right\|_k^2\right) d\tau \\ \leq& C\left(\left\|n_0^\varepsilon - 1\right\|_k^2 + \left\|u_0^\varepsilon\right\|_k^2 + \left\|\theta_0^\varepsilon - 1\right\|_k^2 + \left\|E_0^\varepsilon\right\|_k^2 + \left\|B_0^\varepsilon - B_e\right\|_k^2\right), \quad \forall t \geq 0. \end{aligned} \tag{1.10}$$

Theorem 1.2 *(Zero-relaxation limit). Let $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ be the global solution given by Theorem 1.1. Assume that there exist constants $n_0 > 0$ and $\theta_0 > 0$, which are independent of ε, satisfying as $\varepsilon \to 0$,*

$$n_0^\varepsilon \rightharpoonup n_0, \text{ weakly in } H^k, \tag{1.11}$$

and

$$\theta_0^\varepsilon \rightharpoonup \theta_0, \text{ weakly in } H^k. \tag{1.12}$$

Then there exist functions $\bar{n}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B}$, with $\bar{n} - 1, \bar{\theta} - 1, \bar{E}, \bar{B} - B_e \in L^\infty\left(\mathbb{R}^+; H^k\right)$ and $\bar{u} \in L^2\left(\mathbb{R}^+; H^k\right)$, such that, as $\varepsilon \to 0$, it holds

$$\left(n^\varepsilon - 1, \theta^\varepsilon - 1, E^\varepsilon, B^\varepsilon - B_e\right) \rightharpoonup \left(\bar{n} - 1, \bar{\theta} - 1, \bar{E}, \bar{B} - B_e\right), \text{ weakly-* in } \left(L^\infty\left(\mathbb{R}^+; H^k\right)\right)^3, \tag{1.13}$$

and

$$u^{\varepsilon} \rightharpoonup \bar{u},\ \text{weakly in } L^2\left(\mathbb{R}^+; H^k\right). \tag{1.14}$$

Moreover, for any $T > 0$ and any $k_1 \in [0, k)$, it holds, as $\varepsilon \to 0$,

$$n^{\varepsilon} \to \bar{n}, \quad \theta^{\varepsilon} \to \bar{\theta},\ \text{strongly in } C\left([0, T]; H^{k_1}\right), \tag{1.15}$$

and $(\bar{n}, \bar{\phi}, \bar{\theta})$ is the unique global smooth solution of the following energy-transport model

$$\begin{cases} \partial_t \bar{n} - \Delta(\bar{n}\bar{\theta}) - \operatorname{div}(\bar{n}\nabla\bar{\phi}) = 0, \\ \partial_t \bar{\theta} - \left(\nabla(\bar{\phi} + \bar{\theta}) + \bar{\theta}\nabla \ln \bar{n}\right) \cdot \nabla\bar{\theta} - \dfrac{2}{3}\bar{\theta} \operatorname{div}\left(\nabla(\bar{\phi} + \bar{\theta}) + \bar{\theta}\nabla \ln \bar{n}\right) = \dfrac{2|\bar{u}|^2}{3} - (\bar{\theta} - 1), \\ \Delta\bar{\phi} = 1 - \bar{n}, \end{cases} \tag{1.16}$$

with the initial condition

$$(\bar{n}, \bar{\theta})|_{t=0} = (n_0, \theta_0). \tag{1.17}$$

Note that $\bar{\phi}$ is unique up to addition by a constant. Additionally, it holds

$$\bar{B} = B_e, \quad \bar{E} = \nabla\bar{\phi}, \quad \bar{u} = -\nabla(\bar{\phi} + \bar{\theta}) - \bar{\theta}\nabla \ln \bar{n}, \tag{1.18}$$

where B_e is a constant vector.

Theorem 1.3 *(Convergence rates). Let $k \geq 3$ be an integer, $(n^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon}, E^{\varepsilon}, B^{\varepsilon})$ and $(\bar{n}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B})$ respectively be the unique smooth solutions to (1.3)-(1.4) and (1.7)-(1.8). Denote $\bar{E}_0 = \bar{E}(0, \cdot)$. There exists a constant $\delta > 0$, which is independent of ε, such that if*

$$\left\|n_0^{\varepsilon} - 1\right\|_k + \left\|u_0^{\varepsilon}\right\|_k + \left\|\theta_0^{\varepsilon} - 1\right\|_k + \left\|E_0^{\varepsilon}\right\|_k + \left\|B_0^{\varepsilon} - B_e\right\|_k \leq \delta, \tag{1.19}$$

and for any given positive constants p and C_1 independent of ε satisfying

$$\left\|u_0^{\varepsilon}\right\|_{k-2} + \left\|E_0^{\varepsilon} - \bar{E}_0\right\|_{k-1} + \left\|B_0^{\varepsilon} - B_e\right\|_{k-1} \leq C_1 \varepsilon^p, \tag{1.20}$$

then for $p_1 = \min\{p, 1\}$ and $\varepsilon \in (0, 1]$, there exists a constant C_2 independent of ε, such that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\Big(\left\|n^\varepsilon(t)-\bar{n}(t)\right\|_{k-2}^2+\varepsilon^2\left\|u^\varepsilon(t)-\bar{u}(t)\right\|_{k-2}^2+\left\|\theta^\varepsilon(t)-\bar{\theta}(t)\right\|_{k-2}^2+\left\|E^\varepsilon(t)-\bar{E}(t)\right\|_{k-1}^2\\
&\qquad+\left\|B^\varepsilon(t)-B_e\right\|_{k-1}^2\Big)\\
&+\int_0^{+\infty}\Big(\left\|n^\varepsilon(\tau)-\bar{n}(\tau)\right\|_{k-1}^2+\left\|u^\varepsilon(\tau)-\bar{u}(\tau)\right\|_{k-2}^2+\left\|\theta^\varepsilon(\tau)-\bar{\theta}(\tau)\right\|_{k-1}^2+\left\|E^\varepsilon(\tau)-\bar{E}(\tau)\right\|_{k-1}^2\\
&\qquad+\left\|\nabla B^\varepsilon(\tau)\right\|_{k-2}^2\Big)\,d\tau\\
&\le C_2\varepsilon^{2p_1}.
\end{aligned}
$$

Now let us describe key steps of proof for Theorems 1.1-1.3 and explain the vital difference in proof between the local-in-time convergence rates (see Theorem 4.1 in [5]) and the global-in-time convergence rates in Theorem 1.3 for the non-isentropic Euler-Maxwell system (1.3)-(1.4).

In fact, in the local-in-time convergence result, the convergence rate was clearly shown and it depends on the local existence time. In [5], Feng-Li-Mei-Wang used the asymptotic expansion and energy estimates to prove the local-in-time convergence result. However, when we establish the global-in-time convergence rates, since the classical energy estimates for the symmetrizable hyperbolic system are not sufficient, the error estimate for $n^\varepsilon-\bar{n}\in L^2(\mathbb{R}^+;L^2(\mathbb{T}^3))$ is not trivial (see (4.21)). Then by means of classical energy estimates, we cannot derive it. Therefore, we have to employ the stream function techniques to get this error estimate.

More precisely, we first prove the uniformly global existence and global-in-time convergence of smooth solutions for equations (1.3)-(1.4) by using energy estimates but with some new developments, see the proofs of Theorems 1.1-1.2 in Section 3. Next, we should point out that, note that the presence of the temperature equation, it is much more complicated and difficult to obtain the uniform energy estimates. On one hand, the proof of Theorems 1.1-1.3 for the non-isentropic Euler-Maxwell system includes every difficulty appeared in the isentropic Euler-Maxwell system. On the other hand, we should deal with the additional troubles caused by the absolute temperature θ (see the proof of Lemma 4.6, for instantance). In addition to this, we find that the zero-relaxation limits presented here are also the transport equations but have different forms comparing with the limit of the isentropic Euler-Maxwell system in [17]. The original system is hyperbolic. The limiting system is parabolic. The error system is neither hyperbolic nor parabolic, which leads to loss of a strictly convex entropy and the structure of symmetrizable hyperbolicity, and furthermore loss of the L^2 error estimate and the estimate of the highest order derivatives. Therefore, it is difficult to obtain the convergence rates of the system with the help of the classical energy methods. Here, we employ stream function techniques to establish error estimates between smooth periodic solutions of the non-isentropic Euler-Maxwell equations and those of the limit equations in Theorem 1.3, which creatively expand the research results on the convergence rates of solutions for the non-isentropic Euler-Maxwell equations.

We conclude this section by stating the arrangement of the rest of this paper. In Section 2, we give the important preliminaries such as the local existence theory and Moser-type calculus inequalities. In Section 3, we study the global existence and convergence of solutions for non-isentropic Euler-Maxwell system (1.3)-(1.4) and prove Theorems 1.1-1.2. In Section 4, we investigate the global convergence rate for the non-isentropic Euler-Maxwell system (1.3)-(1.4) and finish the proof of Theorem 1.3.

2. Preliminaries

For later use, let us introduce some notations. For any integer k, we denote the usual spaces $H^k\left(\mathbb{T}^3\right), L^2\left(\mathbb{T}^3\right)$ and $L^\infty\left(\mathbb{T}^3\right)$, by H^k, L^2 and L^∞, respectively. Furthermore, we denote by $\|.\|_k$ the usual norm of H^k, and by $\|.\|$ and $\|.\|_\infty$ the norms of L^2 and L^∞, respectively. For a multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$, we denote

$$\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} \quad \text{with} \quad |\alpha|=\alpha_1+\alpha_2+\alpha_3.$$

Now let us recall some results on Moser-type calculus inequalities in Sobolev spaces, and the local existence of smooth solutions for symmetrizable hyperbolic systems.

Lemma 2.1 *(Moser-type inequality [15,19]). Let $k\geq 1$ be an integer. Suppose $u\in H^k$, $\nabla u\in L^\infty$, and $v\in H^{k-1}\cap L^\infty$. Then for every $\alpha\in\mathbb{N}^3$ with $|\alpha|\leq k$, it holds $\partial^\alpha(uv)-u\partial^\alpha v\in L^2$, and*

$$\left\|\partial^\alpha(uv)-u\partial^\alpha v\right\|\leq C_k\left(\|\nabla u\|_\infty\left\|D^{k-1}v\right\|+\left\|D^k u\right\|\|v\|_\infty\right),$$

where C_k denotes a constant only depending on k, and

$$\left\|D^k u\right\|=\sum_{|\alpha|=k}\left\|\partial^\alpha u\right\|.$$

In particular, when $k\geq 3$, the Sobolev inequality yields

$$\left\|\partial^\alpha(uv)-u\partial^\alpha v\right\|\leq C_k\|\nabla u\|_{k-1}\|v\|_{k-1}. \tag{2.1}$$

Lemma 2.2 *(Local existence of smooth solutions; [14,19]). Let $k\geq 3$ and $(n_0^\varepsilon-1, u_0^\varepsilon, \theta_0^\varepsilon-1, E_0^\varepsilon, B_0^\varepsilon-B_e)\in H^k$ with $n_0^\varepsilon\geq 1/2$, $\theta_0^\varepsilon\geq 1/2$. Then there exists $T_\varepsilon>0$ such that the problem (1.3)-(1.4) has a unique smooth solution $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ satisfying*

$$(n^\varepsilon-1, \varepsilon u^\varepsilon, \theta^\varepsilon-1, E^\varepsilon, B^\varepsilon-B_e)\in C\left([0,T_\varepsilon);H^k\right)\cap C^1\left([0,T_\varepsilon);H^{k-1}\right).$$

Throughout this paper, a basic assumption on the initial data is

$$\left(n_0^\varepsilon-1, u_0^\varepsilon, \theta_0^\varepsilon-1, E_0^\varepsilon, B_0^\varepsilon-B_e\right)\in H^k, \quad \text{with} \quad n_0^\varepsilon\geq\frac{1}{2}, \quad \theta_0^\varepsilon\geq\frac{1}{2}, \quad \text{for } \varepsilon\in(0,1].$$

3. Global existence and convergence of solutions for non-isentropic Euler-Maxwell system (1.3)-(1.4)

3.1. Global existence of solutions

In this subsection, we establish the global existence result for the non-isentropic Euler-Maxwell equations (1.3)-(1.4) uniformly with respect to ε. For simplicity, we drop the superscript ε in $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$, and set

$$N = n - 1, \quad \Theta = \theta - 1, \quad G = B - B_e,$$
$$W^I = (N, u, \Theta)^T, \quad W = (N, u, \Theta, E, G)^T, \quad W^\varepsilon = (N, \varepsilon u, \Theta, E, G)^T.$$

Since system (1.3) is symmetrizable and hyperbolic, Lemma 2.2 can be applied directly. Let $T > 0$ and W be a smooth solution of (1.3), defined on time interval $[0, T]$, and $k \geq 3$ be an integer. From now, we denote $C > 0$ as a generic constant independent of any time and the parameters ε. For convenience, let us introduce

$$\mathcal{N}_T = \sup_{0 \leq t \leq T} \left(\|N(t)\|_k^2 + \|\varepsilon u(t)\|_k^2 + \|\Theta(t)\|_k^2 + \|E(t)\|_k^2 + \|G(t)\|_k^2 \right)^{1/2},$$

and

$$\mathcal{K}\left(W^I\right) = \|N\|_k^2 + \|u\|_k^2 + \|\Theta\|_k^2.$$

We first prove two lemmas as follows, then it holds

Lemma 3.1. *For any $\varepsilon \in (0, 1]$,*

$$\frac{d}{dt}\left(\sum_{0 \leq |\alpha| \leq k} \left\langle A_0(N)\partial^\alpha W^I, \partial^\alpha W^I \right\rangle + \|E\|_k^2 + \|G\|_k^2 \right) + \|u\|_k^2 + \|\Theta\|_k^2 \leq C\mathcal{N}_T \mathcal{K}\left(W^I\right). \quad (3.1)$$

Lemma 3.2. *It holds*

$$\frac{d}{dt}\left\langle A_0(N)\partial^\alpha W^I, \partial^\alpha W^I \right\rangle \leq -2\left\langle n\partial^\alpha E, \partial^\alpha u \right\rangle - \left\|\partial^\alpha u\right\|^2 - \left\|\partial^\alpha \Theta\right\|^2 + C\mathcal{N}_T \mathcal{K}\left(W^I\right). \quad (3.2)$$

Proof. The first three equations in (1.3) can be written in the following form

$$\partial_t W^I + \sum_{j=1}^3 A_j\left(W^I\right)\partial_{x_j} W^I = \begin{pmatrix} 0 \\ -\frac{1}{\varepsilon^2}\left(E + \varepsilon u \times (G + B_e)\right) - \frac{u}{\varepsilon^2} \\ \frac{2-\varepsilon^2}{3}|u|^2 - \Theta \end{pmatrix}, \quad (3.3)$$

with

$$A_j\left(W^I\right) = \begin{pmatrix} u_j & (1+N)e_j^T & 0 \\ \frac{1}{\varepsilon^2}\frac{1+\Theta}{1+N}e_j & u_j I_3 & \frac{1}{\varepsilon^2}e_j \\ 0 & \frac{2}{3}\theta e_j^T & u_j \end{pmatrix}.$$

Here I_3 is the 3×3 unit matrix, (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, and u_j denotes the j-th component of $u \in \mathbb{R}^3$. Since $n \geq 1/2, \theta \geq 1/2$, it is easy to know that the matrix

$$A_0(N)=\begin{pmatrix} \frac{1+\Theta}{1+N} & 0 & 0 \\ 0 & \varepsilon^2(1+N)I_3 & 0 \\ 0 & 0 & \frac{3(1+N)}{2(1+\Theta)} \end{pmatrix}$$

is symmetrically and positively definite, at the same time, the matrix

$$\tilde{A}_j\left(W^I\right)=A_0(N)A_j\left(W^I\right)=\begin{pmatrix} \frac{1+\Theta}{1+N}u_j & (1+\Theta)e_j^T & 0 \\ (1+\Theta)e_j & \varepsilon^2(1+N)u_jI_3 & (1+N)e_j \\ 0 & (1+N)e_j^T & \frac{3(1+N)}{2(1+\Theta)}u_j \end{pmatrix}$$

is symmetric for all $1\le j\le 3$.

Let $\alpha\in\mathbb{N}^3$ with $0\le|\alpha|\le k$. By applying ∂^α to (3.3), and taking the inner product with $2A_0(N)\partial^\alpha W^I$ in $L^2\left(\mathbb{T}^3\right)$, we have

$$\begin{aligned} &\frac{d}{dt}\left\langle A_0(N)\partial^\alpha W^I,\partial^\alpha W^I\right\rangle \\ &=\left\langle\partial_t\left(A_0(N)\right)\partial^\alpha W^I,\partial^\alpha W^I\right\rangle+\sum_{j=1}^{3}\left\langle\partial_{x_j}\left(\tilde{A}_j\left(W^I\right)\right)\partial^\alpha W^I,\partial^\alpha W^I\right\rangle-2\left\langle n\partial^\alpha E,\partial^\alpha u\right\rangle \\ &\quad-2\varepsilon\left\langle(1+N)\partial^\alpha u,\partial^\alpha(u\times G)\right\rangle-2\left\langle n\partial^\alpha u,\partial^\alpha u\right\rangle+\left(2-\varepsilon^2\right)\left\langle\partial^\alpha\left(|u|^2\right),\frac{n}{\theta}\partial^\alpha\Theta\right\rangle \\ &\quad-3\left\langle\partial^\alpha\Theta,\frac{n}{\theta}\partial^\alpha\Theta\right\rangle+2\left\langle J^\alpha,\partial^\alpha W^I\right\rangle \end{aligned}\tag{3.4}$$

with $J^0=0$, and, for any α satisfying $1\le|\alpha|\le k$,

$$J^\alpha=A_0(N)\sum_{j=1}^{3}\left(A_j\left(W^I\right)\partial^\alpha\left(\partial_{x_j}W^I\right)-\partial^\alpha\left(A_j\left(W^I\right)\partial_{x_j}W^I\right)\right)=\sum_{j=1}^{3}J_j^\alpha,$$

with the corresponding $J_1^\alpha, J_2^\alpha, J_3^\alpha$.

Let us rewrite equality (3.4) as

$$\frac{d}{dt}\left\langle A_0(N)\partial^\alpha W^I,\partial^\alpha W^I\right\rangle=I_1+\sum_{j=1}^{3}I_{2,j}+I_3+I_4+I_5+I_6+I_7+\sum_{j=1}^{3}I_{8,j},$$

with the natural correspondence for $I_1,\dots,I_{8,3}$. Then we analyze each of these terms.

By developing I_1, $I_{2,j}$ and $I_{8,j}$, using repeatedly relaxation of spacial embedding in Sobolev space and inequalities (2.1), we have

$$I_1\le C\mathcal{N}_T\mathcal{K}\left(W^I\right),\quad I_{2,j}\le C\mathcal{N}_T\mathcal{K}\left(W^I\right),\quad I_{8,j}\le C\mathcal{N}_T\mathcal{K}\left(W^I\right).\tag{3.5}$$

Noticing $n \in [\frac{1}{2}, \frac{3}{2}]$, we obtain

$$I_5 = -2\left\langle (1+N)\partial^\alpha u, \partial^\alpha u\right\rangle \le -\left\|\partial^\alpha u\right\|^2. \tag{3.6}$$

We note that I_4 can be rewritten as

$$I_4 = -2\varepsilon\left\langle (1+N)\partial^\alpha (u\times G), \partial^\alpha u\right\rangle = -2\varepsilon\left\langle (1+N)\left(\partial^\alpha (u\times G) - \partial^\alpha u \times G\right), \partial^\alpha u\right\rangle, \tag{3.7}$$

then inequality (2.1) yields

$$I_4 \le C\|\nabla G\|_{k-1}\|u\|_{k-1}\left\|\partial^\alpha u\right\| \le C\|G\|_k\|u\|_k^2 \le C\mathcal{N}_T\mathcal{K}\left(W^I\right). \tag{3.8}$$

Using (2.1), the Young's inequality and Cauchy-Schwarz inequality, we obtain

$$I_6 = \left(2-\varepsilon^2\right)\left\langle \partial^\alpha\left(|u|^2\right), \frac{n}{\theta}\partial^\alpha\Theta\right\rangle \leqslant C\mathcal{N}_T\mathcal{K}\left(W^I\right), \tag{3.9}$$

$$I_7 = -3\left\langle \partial^\alpha\Theta, \frac{n}{\theta}\partial^\alpha\Theta\right\rangle \leqslant -\left\|\partial^\alpha\Theta\right\|^2. \tag{3.10}$$

Adding (3.5)-(3.6), (3.8)-(3.10), and using (3.4), we have proved Lemma 3.2. □

Lemma 3.3. *It holds*

$$\frac{d}{dt}\left(\left\|\partial^\alpha E\right\|^2 + \left\|\partial^\alpha G\right\|^2\right) \le 2\left\langle n\partial^\alpha E, \partial^\alpha u\right\rangle + C\mathcal{N}_T\mathcal{K}\left(W^I\right). \tag{3.11}$$

Proof. We start with the fifth equation in (1.3)

$$\partial_t E - \frac{1}{\varepsilon}\operatorname{curl}(G) = nu.$$

Applying ∂^α and taking the inner product with $\partial^\alpha E$, we obtain

$$\frac{d}{dt}\left\|\partial^\alpha E\right\|^2 - \frac{2}{\varepsilon}\left\langle \operatorname{curl}\left(\partial^\alpha G\right), \partial^\alpha E\right\rangle = 2\left\langle \partial^\alpha(nu), \partial^\alpha E\right\rangle, \tag{3.12}$$

then we take the fourth equation in (1.3)

$$\partial_t G + \frac{1}{\varepsilon}\operatorname{curl}(E) = 0.$$

Applying ∂^α and taking the inner product with $\partial^\alpha G$, we have

$$\frac{d}{dt}\left\|\partial^\alpha G\right\|^2 + \frac{2}{\varepsilon}\left\langle \operatorname{curl}\left(\partial^\alpha E\right), \partial^\alpha G\right\rangle = 0. \tag{3.13}$$

Adding (3.12) and (3.13), and using the vector formula, we have

$$\begin{aligned}\frac{d}{dt}\left(\left\|\partial^\alpha E\right\|^2+\left\|\partial^\alpha G\right\|^2\right) =& 2\left\langle\partial^\alpha(nu),\partial^\alpha E\right\rangle \\ =& 2\left\langle n\partial^\alpha u,\partial^\alpha E\right\rangle+2\left\langle\partial^\alpha(Nu)-N\partial^\alpha u,\partial^\alpha E\right\rangle.\end{aligned}\tag{3.14}$$

For the last term, due to inequality (1.11), satisfies

$$\left\langle\partial^\alpha(Nu)-N\partial^\alpha u,\partial^\alpha E\right\rangle\le C\mathcal{N}_T\mathcal{K}\left(W^I\right),$$

together with (3.14), it implies Lemma 3.3. □

Proof of Lemma 3.1. Adding (3.2) and (3.11) leads to

$$\frac{d}{dt}\left(\left\langle A_0(N)\partial^\alpha W^I,\partial^\alpha W^I\right\rangle+\left\|\partial^\alpha E\right\|^2+\left\|\partial^\alpha G\right\|^2\right)+\left\|\partial^\alpha u\right\|^2+\left\|\partial^\alpha\Theta\right\|^2\le C\mathcal{N}_T\mathcal{K}\left(W^I\right),$$

summing up this result over $0\le|\alpha|\le k$, we arrive at Lemma 3.1. □

Next, we begin to prove Theorem 1.1. In fact, we need the following estimate :

$$\|W^\varepsilon(t)\|_k^2+\int_0^t\left(\|N(\tau)\|_k^2+\|u(\tau)\|_k^2+\|\Theta(\tau)\|_k^2\right)d\tau\le C\|W^\varepsilon(0)\|_k^2,\quad\forall t\in[0,T].\tag{3.15}$$

In fact, this estimate is similar to the proof obtained with Euler-Poisson system (see [21]), which is mostly due to the fact E^ε and n^ε are linked with $\operatorname{div}E^\varepsilon=1-n^\varepsilon$ in (1.3). The following identity is frequently used in the proof, which is satisfied for any vectors $f,g\in\mathbb{R}^3$:

$$g\cdot\operatorname{curl}(f)-f\cdot\operatorname{curl}(g)=\operatorname{div}(f\times g),$$

and we repeatedly use the continuous embedding $H^{k-1}\hookrightarrow L^\infty$ and the following inequality

$$\|f\|_\infty\le C\|f\|_{k-1},\quad\forall f\in H^k.$$

Lemma 3.4. *There exists a positive constant $c_0>0$, which is independent of ε and any time, satisfying*

$$\begin{aligned}\|N\|_k^2\le& c_0\frac{d}{dt}\sum_{0\le|\beta|\le k-1}\left(\left\langle\frac{1}{1+N}\partial^\beta N,\partial^\beta N\right\rangle+\left\langle\partial^\beta(\operatorname{div}u),\partial^\beta N\right\rangle\right)\\&+C\mathcal{N}_T\mathcal{K}\left(W^I\right)+c_0\|\nabla\Theta\|_{k-1}^2+c_0\|u\|_k^2.\end{aligned}$$

Proof. We begin with the second equation in (1.3):

$$\partial_t u+(u\cdot\nabla)u+\frac{1}{\varepsilon^2}\frac{\nabla(n\theta)}{n}=-\frac{1}{\varepsilon^2}(E+\varepsilon u\times B)-\frac{u}{\varepsilon^2}.$$

By applying ∂^β to (1.3), with $0 \le |\beta| \le k-1$, and taking the inner product in $L^2\left(\mathbb{T}^3\right)$ with $\varepsilon^2 \partial^\beta(\nabla N)$, we obtain

$$
\begin{aligned}
&\left\langle \frac{\theta}{n} \partial^\beta(\nabla N), \partial^\beta(\nabla N)\right\rangle + \left\langle \partial^\beta(\nabla \theta), \partial^\beta(\nabla N)\right\rangle + \left\langle \partial^\beta E, \partial^\beta(\nabla N)\right\rangle \\
=& \left\langle \frac{\theta}{n} \partial^\beta(\nabla N) - \partial^\beta\left(\frac{\theta}{n}\nabla N\right), \partial^\beta(\nabla N)\right\rangle - \varepsilon^2\left\langle \partial^\beta\left(\partial_t u\right), \partial^\beta(\nabla N)\right\rangle \\
&- \varepsilon^2\left\langle \partial^\beta[(u\cdot\nabla)u], \partial^\beta(\nabla N)\right\rangle - \varepsilon\left\langle \partial^\beta(u\times G), \partial^\beta(\nabla N)\right\rangle \\
&- \varepsilon\left\langle (\partial^\beta u)\times B_e, \partial^\beta(\nabla N)\right\rangle - \left\langle \partial^\beta u, \partial^\beta(\nabla N)\right\rangle.
\end{aligned}
\tag{3.16}
$$

Next we analyze each of them in the following lemma.

Lemma 3.5. *They hold,*

$$\left\langle \frac{\theta}{n}\partial^\beta(\nabla N), \partial^\beta(\nabla N)\right\rangle \geqslant \bar{\kappa}\left\|\partial^\beta(\nabla N)\right\|^2, \tag{3.17}$$

$$\left\langle \partial^\beta(\nabla\Theta), \partial^\beta(\nabla N)\right\rangle \le \varepsilon'\left\|\partial^\beta(\nabla N)\right\|^2 + C_{\varepsilon'}\left\|\partial^\beta(\nabla\Theta)\right\|^2, \tag{3.18}$$

$$\left\langle \partial^\beta E, \partial^\beta(\nabla N)\right\rangle = \left\|\partial^\beta N\right\|^2, \tag{3.19}$$

$$\left\langle \frac{\theta}{n}\partial^\beta(\nabla N) - \partial^\beta\left(\frac{\theta}{n}\nabla N\right), \partial^\beta(\nabla N)\right\rangle \le C\mathcal{N}_T\|N\|_k^2, \tag{3.20}$$

$$\varepsilon^2\left\langle \partial_t\left(\partial^\beta u\right), \partial^\beta(\nabla N)\right\rangle \le \varepsilon^2\frac{d}{dt}\left\langle \partial^\beta(\operatorname{div}(u)), \partial^\beta N\right\rangle + C\|u\|_k^2, \tag{3.21}$$

$$\varepsilon^2\left\langle \partial^\beta((u.\nabla)u), \partial^\beta(\nabla N)\right\rangle \le C\|u\|_k^2, \tag{3.22}$$

$$-\varepsilon\left\langle \partial^\beta(u\times G), \partial^\beta(\nabla N)\right\rangle \le C\|u\|_k^2 + C\mathcal{N}_T\|N\|_k^2, \tag{3.23}$$

$$\varepsilon\left\langle \left(\partial^\beta u\right)\times B_e, \partial^\beta(\nabla N)\right\rangle \le C\|u\|_k\|\nabla N\|_{k-1} \le \delta\|\nabla N\|_{k-1}^2 + C_\delta\|u\|_k^2, \quad \forall\delta>0, \tag{3.24}$$

and

$$-\left\langle \partial^\beta u, \partial^\beta(\nabla N)\right\rangle \le C\|u\|_k\|\nabla N\|_{k-1} \le \delta\|\nabla N\|_{k-1}^2 + C_\delta\|u\|_k^2, \quad \forall\delta>0, \tag{3.25}$$

where $C_\delta > 0$ is a positive constant only depending on δ.

Proof. Since $(n_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)$ stays in a small neighborhood of $(1, 0, 1, 0, B_e)$, independent of the parameter ε, this allows us to suppose that n, θ are in a bounded interval. Thus we can obtain (3.17).

By the Cauchy-Schwarz inequality and the Young's inequality, we get (3.18), where $C_{\varepsilon'}$ and ε' are usual constants with $C_{\varepsilon'} > \varepsilon'$.

Using the first compatibility equation in (1.3), we obtain (3.19):

$$\left\langle \partial^\beta E, \partial^\beta(\nabla N)\right\rangle = -\left\langle \partial^\beta(\nabla E), \partial^\beta N\right\rangle = \left\langle \partial^\beta N, \partial^\beta N\right\rangle = \left\|\partial^\beta N\right\|^2.$$

By using the Cauchy-Schwarz inequality and the inequality (2.1), we have (3.20):

$$\begin{aligned}\left\langle \frac{\theta}{n}\partial^\beta(\nabla N)-\partial^\beta\left(\frac{\theta}{n}\nabla N\right),\partial^\beta(\nabla N)\right\rangle &\leqslant \left\|\frac{\theta}{n}\partial^\beta(\nabla N)-\partial^\beta\left(\frac{\theta}{n}\nabla N\right)\right\|\cdot\left\|\partial^\beta(\nabla N)\right\| \\ &\leqslant C_k\left\|\nabla\left(\frac{\theta}{n}\right)\right\|_{k-1}\|\nabla N\|_{k-1} \\ &\leqslant C_k\mathcal{N}_T\|N\|_k^2.\end{aligned}$$

Integrating by parts yield (3.21). We directly arrive at (3.22) due to (3.21). We prove (3.23) due to

$$\begin{aligned}-\varepsilon\left\langle\partial^\beta(u\times G),\partial^\beta(\nabla N)\right\rangle &\le C\|\nabla G\|_\infty\left\|\partial^\beta u\right\|\|N\|_k+C\|\nabla u\|_\infty\left\|\partial^\beta G\right\|\|N\|_k \\ &\le C\|G\|_k\|u\|_k\|N\|_k \\ &\le C\|u\|_k^2+C\mathcal{N}_T\|N\|_k^2.\end{aligned}$$

A direct bounding implies that

$$\varepsilon\left\langle\left(\partial^\beta u\right)\times B_e,\partial^\beta(\nabla N)\right\rangle\le C\|u\|_k\|\nabla N\|_{k-1}\le\delta\|\nabla N\|_{k-1}^2+C_\delta\|u\|_k^2,$$

for any $\delta>0$. Then we obtain (3.24).

The proof of (3.25) is more complicated and it needs some calculus. Firstly, we note that

$$-\left\langle\partial^\beta u,\partial^\beta(\nabla N)\right\rangle=\left\langle\partial^\beta(\operatorname{div}(u)),\partial^\beta N\right\rangle.$$

Then using the first equation in (1.3) yields

$$\begin{aligned}-\left\langle\partial^\beta u,\partial^\beta(\nabla N)\right\rangle=&-\left\langle\frac{\partial^\beta(\partial_t N)}{1+N},\partial^\beta N\right\rangle-\left\langle\frac{u\cdot\partial^\beta(\nabla N)}{1+N},\partial^\beta N\right\rangle \\ &+\left\langle\frac{\partial^\beta(\partial_t N)}{1+N}-\partial^\beta\left(\frac{\partial_t N}{1+N}\right),\partial^\beta N\right\rangle \\ &+\left\langle\frac{u\cdot\partial^\beta(\nabla N)}{1+N}-\partial^\beta\left(\frac{u\cdot\nabla N}{1+N}\right),\partial^\beta N\right\rangle.\end{aligned}\tag{3.26}$$

Now we analyze each term on the right-hand side of (3.26). The first one satisfies

$$\begin{aligned}-\left\langle\frac{\partial^\beta(\partial_t N)}{1+N},\partial^\beta N\right\rangle+\frac{1}{2}\frac{d}{dt}\left\langle\frac{1}{1+N}\partial^\beta N,\partial^\beta N\right\rangle &=-\frac{1}{2}\left\langle\frac{\partial_t N}{(1+N)^2}\partial^\beta N,\partial^\beta N\right\rangle \\ &\le C\|\partial_t N\|_\infty\left\|\partial^\beta N\right\|^2 \\ &\le C\|\operatorname{div}(nu)\|_\infty\|N\|_k^2 \\ &\le C\mathcal{N}_T\mathcal{K}\left(W^I\right).\end{aligned}\tag{3.27}$$

The second one on the right-hand side of (3.26) satisfies

$$\left\langle \frac{u \cdot \partial^\beta (\nabla N)}{1+N}, \partial^\beta N \right\rangle \le C\mathcal{N}_T \mathcal{K}\left(W^I\right). \tag{3.28}$$

By noticing inequality (2.1), we find that the third term satisfies

$$\left\langle \frac{\partial^\beta (\partial_t N)}{1+N} - \partial^\beta \left(\frac{\partial_t N}{1+N}\right), \partial^\beta N \right\rangle \le C\mathcal{N}_T \mathcal{K}\left(W^I\right), \tag{3.29}$$

and the last term satisfies

$$\left\langle \frac{u \cdot \partial^\beta (\nabla N)}{1+N} - \partial^\beta \left(\frac{u \cdot \nabla N}{1+N}\right), \partial^\beta N \right\rangle \le C\mathcal{N}_T \mathcal{K}\left(W^I\right), \tag{3.30}$$

combining (3.26)-(3.30), we arrive at (3.25). The proof of Lemma 3.5 is finished. □

End of the proof of Lemma 3.4.
We obtain the following inequality by summing up (3.17)-(3.25), for any $\delta > 0$,

$$\begin{aligned}&(\bar{\kappa} - \varepsilon') \left\|\partial^\beta (\nabla N)\right\|^2 - c_{\varepsilon'} \left\|\partial^\beta (\nabla \Theta)\right\|^2 + \left\|\partial^\beta N\right\|^2 \\ &\le \varepsilon^2 \frac{d}{dt} \left\langle \partial^\beta (\mathrm{div}(u)), \partial^\beta N \right\rangle + C_\delta \|u\|_k^2 + \delta \|\nabla N\|_{k-1}^2 + C\mathcal{N}_T \mathcal{K}\left(W^I\right).\end{aligned}$$

Then summing up β, $0 \le |\beta| \le k-1$, taking δ sufficiently small, we have

$$\begin{aligned}&\|\nabla N\|_{k-1}^2 - \|\nabla \Theta\|_{k-1}^2 + \|N\|_{k-1}^2 \\ &\le C\varepsilon^2 \frac{d}{dt} \sum_{0 \le |\beta| \le k-1} \left\langle \partial^\beta (\mathrm{div}\, u), \partial^\beta N \right\rangle + C\|u\|_k^2 + C\mathcal{N}_T \mathcal{K}\left(W^I\right).\end{aligned}$$

Finally, it is obvious to remark that there exists a constant $C_1 > 0$ satisfying

$$\|\nabla N\|_{k-1}^2 + \|N\|_{k-1}^2 \ge C_1 \|N\|_k^2,$$

combining with Lemma 3.1 implies Lemma 3.4. □

Proof of Theorem 1.1. As we have seen, the proofs of Lemmas 3.1-3.4 are pretty technical. Multiplying the result of Lemma 3.4 by any positive constant $\kappa > 0$ to be chosen, and adding it to result of Lemma 3.1, we then integrate the final result over $[0, t]$ to obtain

$$
\begin{aligned}
&\sum_{0\le|\alpha|\le k}\Big\langle A_0(N(t))\partial^\alpha W^I(t),\partial^\alpha W^I(t)\Big\rangle+\|E(t)\|_k^2+\|G(t)\|_k^2+\kappa\|N(t)\|_{k-1}^2\\
&-c_0\kappa\varepsilon^2\sum_{0\le|\beta|\le k-1}\left\langle\partial^\beta(\operatorname{div}(u(t))),\partial^\beta(N(t))\right\rangle\\
&+\int_0^t\Big[(1-c_0\kappa)\,\|u(\tau)\|_k^2+\kappa\|\nabla N(\tau)\|_{k-1}^2+(1-\kappa)\|\Theta(\tau)\|_k^2\Big]d\tau\\
&\le C\|W^\varepsilon(0)\|_k^2+C\mathcal{N}_T\int_0^t\mathcal{K}\Big(W^I(\tau)\Big)\,d\tau.
\end{aligned}
$$

Now, considering the fact that $A_0(N)$ is symmetric positive definite, and taking κ small enough, and combining with Lemma 3.4, we obtain

$$
\|W^\varepsilon(t)\|_k^2+\int_0^t\mathcal{K}\Big(W^I(\tau)\Big)\,d\tau\le C\left(\|W^\varepsilon(0)\|_k^2+\mathcal{N}_T\int_0^t\mathcal{K}\Big(W^I(\tau)\Big)\,d\tau\right). \tag{3.31}
$$

Taking $\mathcal{N}_T$ sufficiently small, and noticing that the last term on the right-hand side of (3.31) can be controlled by the left-hand side of (3.31), thus we obtain (3.15). So far, by the series of energy estimates before, recalling the zero-relaxation limit, we find that (1.10) is a direct consequence of estimate (3.15). It indicates the global existence of the solutions for the non-isentropic Euler-Maxwell system (1.3)-(1.4). The proof of Theorem 1.1 is completed. □

3.2. Convergence of the solution as $\varepsilon\to 0$

In this subsection, we study the global-in-time convergence from (1.3)-(1.4) to (1.16)-(1.17).

Proof of Theorem 1.2. We recall the non-isentropic Euler-Maxwell system (1.3)

$$
\begin{cases}
\partial_t n^\varepsilon+\operatorname{div}\left(n^\varepsilon u^\varepsilon\right)=0,\\
\partial_t\left(n^\varepsilon u^\varepsilon\right)+\operatorname{div}\left(n^\varepsilon u^\varepsilon\otimes u^\varepsilon\right)+\dfrac{1}{\varepsilon^2}\nabla\left(n^\varepsilon\theta^\varepsilon\right)=\dfrac{1}{\varepsilon^2}\left(-n^\varepsilon\left(E^\varepsilon+\varepsilon u^\varepsilon\times B^\varepsilon\right)-n^\varepsilon u^\varepsilon\right),\\
\partial_t\theta^\varepsilon+u^\varepsilon\cdot\nabla\theta^\varepsilon+\dfrac{2}{3}\theta^\varepsilon\operatorname{div}u^\varepsilon=\dfrac{|u^\varepsilon|^2}{3}\left(2-\varepsilon^2\right)-\left(\theta^\varepsilon-1\right),\\
\varepsilon\partial_t B^\varepsilon+\nabla\times E^\varepsilon=0,\quad \operatorname{div}E^\varepsilon=1-n^\varepsilon,\\
\varepsilon\partial_t E^\varepsilon-\nabla\times B^\varepsilon=\varepsilon n^\varepsilon u^\varepsilon,\quad \operatorname{div}B^\varepsilon=0,
\end{cases}
$$

and particularly the estimate (1.10) gives that

$$
\sup_{t\ge 0}\left(\left\|n^\varepsilon(t)-1\right\|_k^2+\left\|\theta^\varepsilon(t)-1\right\|_k^2+\left\|E^\varepsilon(t)\right\|_k^2+\left\|B^\varepsilon(t)-B_e\right\|_k^2\right)+\int_0^{+\infty}\left\|u^\varepsilon(\tau)\right\|_k^2d\tau\le c\varpi_0^2.
$$

Consequently, the sequences $(n^\varepsilon - 1)_{\varepsilon>0}, (\theta^\varepsilon - 1)_{\varepsilon>0}, (E^\varepsilon)_{\varepsilon>0}$ and $(B^\varepsilon - B_e)_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}_+; H^k)$, and $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R}_+; H^k)$.

Moreover, in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{K}^3)$, we have

$$\begin{aligned}
&\varepsilon^2\left(\partial_t u^\varepsilon + \left(u^\varepsilon \cdot \nabla\right) u^\varepsilon\right) + \varepsilon\left(u^\varepsilon \times B^\varepsilon\right) \to 0, && \text{as} \quad \varepsilon \to 0,\\
&\varepsilon\left(\partial_t E^\varepsilon - n^\varepsilon u^\varepsilon\right) \to 0, && \text{as} \quad \varepsilon \to 0,\\
&\varepsilon \partial_t B^\varepsilon \to 0, && \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

and that there exist functions $\bar{n} - 1, \bar{\theta} - 1, \bar{E}, \bar{B} - B_e \in L^\infty(\mathbb{R}_+; H^k)$ and $\bar{u} \in L^2(\mathbb{R}_+; H^k)$, such that, the convergence (1.13)-(1.14) holds up to subsequences. Furthermore, we deduce that $(\partial_t n^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0, T; H^{k-1})$ for any $T > 0$ by using the first equation of (1.16).

Let $T > 0$. The sequence $(n^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0, T; H^k)$. Then, by classical compactness theorems, for any $k_1 \in [0, k)$, the sequence $(n^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C([0, T]; H^{k_1})$. Moreover, up to a subsequence, we obtain the strong convergence (1.15) by the uniqueness of the limit.

Therefore, we can pass the limit in each of the nonlinear terms of (1.3) to arrive at

$$\begin{cases}
\partial_t \bar{n} - \Delta(\bar{n}\bar{\theta}) - \operatorname{div}(\bar{n}\nabla\bar{\phi}) = 0,\\
\partial_t \bar{\theta} + \bar{u} \cdot \nabla\bar{\theta} + \dfrac{2}{3}\bar{\theta} \operatorname{div} \bar{u} = \dfrac{2|\bar{u}|^2}{3} - (\bar{\theta} - 1),\\
\Delta\bar{\phi} = 1 - \bar{n}.
\end{cases}$$

This directly implies the existence of $\bar{\phi}$ satisfying $\bar{E} = \nabla\bar{\phi}$. Consequently, we obtain the above energy-transportation system (1.7).

Now we consider the initial condition of $\bar{n}$ and $\bar{\theta}$. Since the strong convergence (1.15) is uniform with respect to $t \in [0, T]$, then,

$$n^\varepsilon(0, .) \to \bar{n}(0, .), \quad \theta^\varepsilon(0, .) \to \bar{\theta}(0, .) \quad \text{in } H^{k_1}, \quad \text{as} \quad \varepsilon \to 0.$$

Noticing $n^\varepsilon(0, .) = n_0^\varepsilon$ and $\theta^\varepsilon(0, .) = \theta_0^\varepsilon$, the equality (1.17) comes from (1.11)-(1.12) and the uniqueness of the limit. Finally, as is known to all, the solution of the energy-transport system (1.16) with smooth initial data (1.17) is unique. Thus the convergence of the whole sequence $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ can be obtained. The proof of Theorem 1.2 is completed. □

4. Global convergence rate for system (1.3)-(1.4)

The study of the error estimates is based on the results on the uniform global existence and the global-in-time convergence from (1.3)-(1.4) to (1.16)-(1.17) established in the section above. The result on the convergence rate for (1.3)-(1.4) is stated as Theorem 1.3. First of all, let us recall the concept of stream function. For a conservative equation

$$\partial_t u + \operatorname{div} v = 0,$$

we call φ a stream function to this equation if φ satisfies

$$\partial_t \varphi = v, \quad \operatorname{div} \varphi = -u.$$

Next, we devote to finding an appropriate stream function for the non-isentropic Euler-Maxwell system (1.3)-(1.4). We obtain the following conservative equation by subtracting the first equation of system (1.6) from that of system (1.3),

$$\partial_t \left(n^\varepsilon - \bar{n}\right) + \operatorname{div}\left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}\right) = 0.$$

Then the stream function φ satisfies

$$\begin{cases} \partial_t \varphi = n^\varepsilon u^\varepsilon - \bar{n}\bar{u}, \\ \operatorname{div} \varphi = -\left(n^\varepsilon - \bar{n}\right). \end{cases}$$

The error of the electric field $\varphi = E^\varepsilon - \bar{E}$ is a natural candidate of the stream function, since

$$\operatorname{div}\left(E^\varepsilon - \bar{E}\right) = -\left(n^\varepsilon - \bar{n}\right).$$

However, we can only get the information of $\operatorname{div}\left(\partial_t \bar{E}\right)$ from the limiting system since we lose the information of $\partial_t \bar{E}$ after taking the limit $\varepsilon \to 0$. As a result, we notice that $\partial_t \varphi$ is not equal to $n^\varepsilon u^\varepsilon - \bar{n}\bar{u}$, but takes the following form with the additional divergence free term K:

$$\partial_t \varphi = n^\varepsilon u^\varepsilon - \bar{n}\bar{u} + K.$$

Therefore, $E^\varepsilon - \bar{E}$ can be regarded as the stream function of the modified conservative equation below

$$\partial_t \left(n^\varepsilon - \bar{n}\right) + \operatorname{div}\left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u} + K\right) = 0. \tag{4.1}$$

Now, we begin to study the error estimates. Let $(n^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ be the unique smooth solution to (1.3)-(1.4), and $(\bar{n}, \bar{u}, \bar{\theta}, \bar{E})$ be the unique solution to the energy-transport model (1.7)-(1.8). Then, we denote

$$\left(N^\varepsilon, \Xi^\varepsilon, \Theta^\varepsilon, F^\varepsilon, G^\varepsilon\right) = \left(n^\varepsilon - \bar{n}, u^\varepsilon - \bar{u}, \theta^\varepsilon - \bar{\theta}, E^\varepsilon - \bar{E}, B^\varepsilon - B_e\right), \tag{4.2}$$

$$\left(n_\alpha, u_\alpha, \theta_\alpha, E_\alpha, B_\alpha\right) = \left(\partial^\alpha n^\varepsilon, \partial^\alpha u^\varepsilon, \partial^\alpha \theta^\varepsilon, \partial^\alpha E^\varepsilon, \partial^\alpha B^\varepsilon\right), \tag{4.3}$$

$$\left(\bar{n}_\alpha, \bar{u}_\alpha, \bar{\theta}_\alpha, \bar{E}_\alpha, \bar{B}_\alpha\right) = \left(\partial^\alpha \bar{n}, \partial^\alpha \bar{u}, \partial^\alpha \bar{\theta}, \partial^\alpha \bar{E}, \partial^\alpha \bar{B}\right), \tag{4.4}$$

and

$$\left(N_\alpha, \Xi_\alpha, \Theta_\alpha, F_\alpha, G_\alpha\right) = \left(\partial^\alpha N^\varepsilon, \partial^\alpha \Xi^\varepsilon, \partial^\alpha \Theta^\varepsilon, \partial^\alpha F^\varepsilon, \partial^\alpha G^\varepsilon\right). \tag{4.5}$$

We prove the following result firstly.

Lemma 4.1. *Assume* $\|\bar{n}_0 - 1\|_k$, $\left\|\bar{\theta}_0 - 1\right\|_k$ *are sufficiently small, then the solution* $(\bar{n}, \bar{u}, \bar{\theta}, \bar{E})$ *to the system* (1.7)-(1.8) *satisfies*

$$\|\bar{n}(t) - 1\|_k^2 + \int_0^t \|\bar{n}(\tau) - 1\|_{k+1}^2 d\tau \le C \|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0, \tag{4.6}$$

$$\|\bar{u}(t)\|_{k-1}^2 + \|\partial_t \bar{u}(t)\|_{k-3}^2 + \int_0^t \left(\|\bar{u}(\tau)\|_k^2 + \|\partial_t \bar{u}(\tau)\|_{k-2}^2\right) d\tau \le C \|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0, \tag{4.7}$$

$$\|\bar{\theta}(t) - 1\|_k^2 + \int_0^t \|\bar{\theta}(\tau) - 1\|_k^2 d\tau \le C \left\|\bar{\theta}_0 - 1\right\|_k^2, \quad \forall t > 0, \tag{4.8}$$

and

$$\|\bar{E}(t)\|_k^2 + \left\|\partial_t \bar{E}(t)\right\|_{k-1}^2 + \int_0^t \left(\|\bar{E}(\tau)\|_{k+1}^2 + \left\|\partial_t \bar{E}(\tau)\right\|_k^2\right) d\tau \le C \|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0. \tag{4.9}$$

Proof. Let $\bar{\rho} = \bar{n} - 1$ and $\alpha \in \mathbb{N}^3$ be a multi-index with $|\alpha| \le k$. According to (1.7), for any $t > 0$, we have

$$\begin{cases} \partial_t \bar{\rho} - \operatorname{div}(\nabla(\bar{n}\bar{\theta})) - \operatorname{div}(\bar{n}\nabla\bar{\phi}) = 0, \\ \Delta\bar{\phi} = -\bar{\rho}, \quad m_{\bar{\phi}}(t) = 0, \end{cases} \tag{4.10}$$

then applying ∂^α to the first equation in (4.10), multiplying the resulting equation with $\partial^\alpha \bar{\rho}$, and integrating it with respect to x, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\partial^\alpha \bar{\rho}\right\|^2 + \left\langle \bar{\theta}\partial^\alpha(\nabla\bar{\rho}), \partial^\alpha(\nabla\bar{\rho})\right\rangle + \left\langle \bar{n}\partial^\alpha(\nabla\bar{\theta}), \partial^\alpha(\nabla\bar{\rho})\right\rangle + \left\langle \partial^\alpha(\nabla\bar{\phi}), \nabla\left(\partial^\alpha\bar{\rho}\right)\right\rangle \\ =&\left\langle \bar{\theta}\partial^\alpha(\nabla\bar{\rho}) - \partial^\alpha(\bar{\theta}\nabla\bar{\rho}), \partial^\alpha(\nabla\bar{\rho})\right\rangle + \left\langle \bar{n}\partial^\alpha(\nabla\bar{\theta}) - \partial^\alpha(\bar{n}\nabla\bar{\theta}), \partial^\alpha(\nabla\bar{\rho})\right\rangle \\ &- \left\langle \partial^\alpha(\bar{\rho}\nabla\bar{\phi}), \nabla\left(\partial^\alpha\bar{\rho}\right)\right\rangle, \end{aligned} \tag{4.11}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in L^2. In view of the Poisson equation in (4.10), we have

$$\left\langle \partial^\alpha(\nabla\bar{\phi}), \nabla\left(\partial^\alpha\bar{\rho}\right)\right\rangle = \left\|\partial^\alpha\bar{\rho}\right\|^2.$$

Noticing that $\bar{n}$, $\bar{\theta}$ are close to 1 and they are uniformly bounded, using the Cauchy-Schwarz inequality and the Young's inequality, there exist positive constants $P_1, \varepsilon', C_{\varepsilon'}$, such that

$$\left\langle \bar{\theta}\partial^\alpha(\nabla\bar{\rho}), \partial^\alpha(\nabla\bar{\rho})\right\rangle \geqslant P_1 \left\|\partial^\alpha(\nabla\bar{\rho})\right\|^2,$$

and

$$\left\langle \bar{n}\partial^\alpha(\nabla\bar{\theta}), \partial^\alpha(\nabla\bar{\rho})\right\rangle \leqslant \varepsilon' \left\|\partial^\alpha(\nabla\bar{\rho})\right\|^2 + C_{\varepsilon'}\left\|\partial^\alpha(\nabla\bar{\theta})\right\|^2.$$

Applying the Poincaré inequality to the Poisson equation in (4.10) yields

$$\left\|\nabla\partial^\alpha\bar{\phi}\right\| \le C\left\|\partial^\alpha\bar{\rho}\right\|. \tag{4.12}$$

With the help of Lemma 2.1, we have

$$\left\langle \bar{\theta}\partial^\alpha(\nabla\bar{\rho}) - \partial^\alpha(\bar{\theta}\nabla\bar{\rho}), \partial^\alpha(\nabla\bar{\rho})\right\rangle - \left\langle \partial^\alpha(\bar{\rho}\nabla\bar{\phi}), \nabla\left(\partial^\alpha\bar{\rho}\right)\right\rangle \leqslant C\|\bar{\rho}\|_k\|\bar{\rho}\|_{k+1}^2,$$

and

$$\left\langle \bar{n}\partial^\alpha(\nabla\bar{\theta}) - \partial^\alpha(\bar{n}\nabla\bar{\theta}), \partial^\alpha(\nabla\bar{\rho})\right\rangle \leqslant C\|\bar{\theta}\|_k\|\bar{\rho}\|_{k+1}^2.$$

Then, the sum of (4.11) for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le k$ implies

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}\|_k^2 + \|\bar{\rho}\|_k^2 + P_1\|\nabla\bar{\rho}\|_k^2 \leqslant C\left(\|\bar{\rho}\|_k + \|\bar{\theta}\|_k\right)\left(\|\bar{\rho}\|_{k+1}^2 + \|\bar{\theta}\|_{k+1}^2\right).$$

By noting that $\|\bar{\rho}\|_k$, $\|\bar{\theta}\|_k$ are small enough, integrating the above inequality over $[0,t]$, (4.6) follows.

Moreover, it follows from (4.6) and (4.12) that

$$\|\bar{E}(t)\|_k^2 + \int_0^t \|\bar{E}(\tau)\|_{k+1}^2 d\tau \le C\|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0. \tag{4.13}$$

Meanwhile, for a multi-index β with $|\beta| \le k-1$, from (1.8), it is easy to get

$$\left\|\partial^\beta\bar{u}\right\| \le C\|\bar{\rho}\|_{|\beta|+1}. \tag{4.14}$$

Applying $\partial_t\partial^\beta$ to both sides of the Poisson equation in (4.10), we have

$$\Delta\partial_t\partial^\beta\bar{\phi} = \operatorname{div}\partial^\beta(\bar{n}\bar{u}).$$

Then, it follows from (1.8) and (4.14) that

$$\left\|\partial_t\partial^\beta\bar{E}\right\| \le C\left\|\partial^\beta(\bar{n}\bar{u})\right\| \le C\|\bar{\rho}\|_{|\beta|+1}. \tag{4.15}$$

By (4.6), it implies that

$$\left\|\partial_t\bar{E}(t)\right\|_{k-1}^2 + \int_0^t \left\|\partial_t\bar{E}(\tau)\right\|_k^2 d\tau \le C\|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0.$$

This, together with (4.13), implies (4.9).

In the same way, by combining (4.6) and (4.14), we obtain

$$\|\bar{u}(t)\|_{k-1}^2 + \int_0^t \|\bar{u}(\tau)\|_k^2 d\tau \le C\,\|\bar{n}_0 - 1\|_k^2, \quad \forall t > 0. \tag{4.16}$$

For a multi-index γ with $|\gamma| \le k-3$, applying $\partial_t \partial^\gamma$ to the equation for $\bar{u}$ in (1.8) implies

$$\partial_t \partial^\gamma \bar{u} = -\partial^\gamma \left(\frac{\nabla \bar{n}}{\bar{n}} \partial_t \bar{\theta} - \frac{\bar{\theta}}{(\bar{n})^2} \nabla \bar{n} \partial_t \bar{n} + \frac{\bar{\theta}}{\bar{n}} \nabla \partial_t \bar{n} + \nabla \partial_t \bar{\theta} + \partial_t \bar{E} \right).$$

In view of $\partial_t \bar{n} = -\operatorname{div}(\bar{n}\bar{u})$, by (4.6) and (4.14), we have

$$\left\| \partial_t \partial^\gamma \bar{u} \right\| \le C \|\bar{u}\|_{|\gamma|+2} + \left\| \partial_t \bar{E} \right\|_{|\gamma|}.$$

This, together with (4.16), yields (4.7).

From the second equation in (1.7), we have

$$\bar{\theta} - 1 = \frac{2|\bar{u}|^2}{3} - \partial_t \bar{\theta} - \bar{u} \cdot \nabla \bar{\theta} - \frac{2}{3} \bar{\theta} \operatorname{div} \bar{u}.$$

Let $\bar{\Theta} = \bar{\theta} - 1$, then applying ∂^α to the second equation in (1.7) and taking the inner product with $\partial^\alpha \bar{\Theta}$ in L^2 yields

$$\langle \partial^\alpha \bar{\Theta}, \partial^\alpha \bar{\Theta} \rangle = \left\langle \partial^\alpha \left(\frac{2|\bar{u}|^2}{3} \right), \partial^\alpha \bar{\Theta} \right\rangle - \langle \partial_t \partial^\alpha \bar{\Theta}, \partial^\alpha \bar{\Theta} \rangle - \langle \partial^\alpha (\bar{u} \cdot \nabla \bar{\theta}), \partial^\alpha \bar{\Theta} \rangle - \left\langle \frac{2}{3} \partial^\alpha (\bar{\theta} \operatorname{div} \bar{u}), \partial^\alpha \bar{\Theta} \right\rangle.$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \left\| \partial^\alpha \bar{\Theta} \right\|^2 + \left\| \partial^\alpha \bar{\Theta} \right\|^2 + \langle \bar{u} \cdot \partial^\alpha (\nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \rangle + \left\langle \frac{2}{3} \partial^\alpha \bar{\Theta} \operatorname{div} \bar{u}, \partial^\alpha \bar{\Theta} \right\rangle \\
&\leqslant \left\| \partial^\alpha \left(\frac{2|\bar{u}|^2}{3} \right) \right\| \cdot \left\| \partial^\alpha \bar{\Theta} \right\| + \langle \bar{u} \cdot \partial^\alpha (\nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \rangle - \langle \partial^\alpha (\bar{u} \cdot \nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \rangle \\
&+ \left\langle \frac{2}{3} \partial^\alpha \bar{\Theta} \operatorname{div} \bar{u}, \partial^\alpha \bar{\Theta} \right\rangle - \left\langle \frac{2}{3} \partial^\alpha (\bar{\theta} \operatorname{div} \bar{u}), \partial^\alpha \bar{\Theta} \right\rangle.
\end{aligned} \tag{4.17}$$

Noticing $\bar{u}$ is close to 0 and uniformly bounded, there exists a constant C^*, such that

$$\langle \bar{u} \cdot \partial^\alpha (\nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \rangle \leqslant C^* \left\| \bar{\Theta} \right\|_k \left\| \bar{\Theta} \right\|_{k+1}.$$

By the Cauchy-Schwarz inequality, the Young's inequality and (4.7), we obtain

$$\left| \left\langle \frac{2}{3} \partial^\alpha \bar{\Theta} \operatorname{div} \bar{u}, \partial^\alpha \bar{\Theta} \right\rangle \right| \leqslant \left\| \frac{2}{3} \partial^\alpha \bar{\Theta} \right\| \|\operatorname{div} \bar{u}\| \left\| \partial^\alpha \bar{\Theta} \right\| \leqslant C \left\| \bar{\Theta} \right\|_k^2,$$

$$\left\| \partial^\alpha \left(\frac{2|\bar{u}|^2}{3} \right) \right\| \left\| \partial^\alpha \bar{\Theta} \right\| \leqslant \frac{4}{3} \|\bar{u}\| \left\| \partial^\alpha \bar{u} \right\| \left\| \partial^\alpha \bar{\Theta} \right\| \leqslant C_{\varepsilon'} \left\| \bar{\Theta} \right\|_k^2 + \varepsilon' \|\bar{u}\|_k^2,$$

where $C_{\varepsilon'}$ and ε' are usual constants with $C_{\varepsilon'} > \varepsilon'$.

By Lemma 2.1, we have

$$\left\langle \bar{u} \cdot \partial^\alpha (\nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \right\rangle - \left\langle \partial^\alpha (\bar{u} \cdot \nabla \bar{\Theta}), \partial^\alpha \bar{\Theta} \right\rangle \leqslant C \|\bar{\Theta}\|_k \|\bar{\Theta}\|_{k+1}^2,$$

and

$$\left| \left\langle \frac{2}{3} \partial^\alpha \bar{\Theta} \operatorname{div} \bar{u} - \frac{2}{3} \partial^\alpha (\bar{\theta} \operatorname{div} \bar{u}), \partial^\alpha \bar{\Theta} \right\rangle \right| \leqslant C \|\bar{\Theta}\|_k \|\bar{\Theta}\|_{k+1}^2.$$

Then, the sum of (4.17) for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le k$ yields

$$\frac{1}{2} \frac{d}{dt} \|\bar{\Theta}\|_k^2 + \|\bar{\Theta}\|_k^2 \leqslant C \|\bar{\Theta}\|_k \|\bar{\Theta}\|_{k+1}.$$

Since $\|\bar{\Theta}\|_k$ is sufficiently small, integrating the above inequality over $[0, t]$ gives (4.8). □

From (1.7)-(1.8), we obtain

$$\operatorname{div} \partial_t \nabla \bar{\phi} = -\partial_t \bar{n} = \operatorname{div}(\bar{n} \bar{u}),$$

which indicates that there exists a function Ψ such that

$$\begin{cases} \partial_t \nabla \bar{\phi} - \bar{n} \bar{u} = \nabla \times \Psi, \\ \operatorname{div} \Psi = 0. \end{cases} \tag{4.18}$$

We add the following restriction condition to make Ψ uniquely determined,

$$m_\Psi(t) = \int_{\mathbb{T}^3} \Psi(t, x) dx = 0, \quad \forall t \ge 0. \tag{4.19}$$

The estimate for the function Ψ is as follows.

Lemma 4.2. *The solution Ψ to (4.18)-(4.19) satisfies*

$$\Psi \in L^\infty \left(\mathbb{R}^+; H^k \right) \quad \textit{and} \quad \partial_t \Psi \in L^2 \left(\mathbb{R}^+; H^{k-1} \right).$$

Proof. By the similar way as that in Lemma 2.2 of [17], we can prove the Lemma 4.2. We omit it for the sake of simplicity. □

Next, by modifying slightly the proof of Lemma 2.3 in [17], we can also obtain the following uniform estimate with respect to ε.

Lemma 4.3. *It holds that*

$$\int_0^t \left(\left\| E^\varepsilon(\tau) \right\|_{k-1}^2 + \left\| \nabla G^\varepsilon(\tau) \right\|_{k-2}^2 \right) d\tau \le C \left\| U_0^\varepsilon - U_e \right\|_k^2, \quad \forall t \ge 0, \tag{4.20}$$

where G^ε is defined by (4.2).

Next, by taking the difference of the first equations in (1.6) and (1.3), we have

$$\partial_t N^\varepsilon + \operatorname{div}\left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}\right) = 0.$$

It follows from (1.3), (1.7)-(1.8) and (4.18) that

$$\operatorname{div} F^\varepsilon = -N^\varepsilon, \tag{4.21}$$

and

$$\partial_t F^\varepsilon = \left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}\right) + \frac{1}{\varepsilon}\nabla \times G^\varepsilon - \nabla \times \Psi. \tag{4.22}$$

According to the concept of stream function at the beginning of this section, F^ε is a stream function of (4.1) with $K = \varepsilon^{-1}\nabla \times G^\varepsilon - \nabla \times \Psi$.

In the following, we prove the estimates for the error function $(N^\varepsilon, \Xi^\varepsilon, \Theta^\varepsilon, F^\varepsilon, G^\varepsilon)$.

Lemma 4.4. *It holds*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \left(\left\| N^\varepsilon(t) \right\|_{k-2}^2 + \left\| \Theta^\varepsilon(t) \right\|_{k-2}^2 + \left\| F^\varepsilon(t) \right\|_{k-1}^2 + \left\| G^\varepsilon(t) \right\|_{k-1}^2 \right) \\
&+ \int_0^{+\infty} \left(\left\| N^\varepsilon(\tau) \right\|_{k-1}^2 + \left\| \Theta^\varepsilon(\tau) \right\|_{k-1}^2 + \left\| F^\varepsilon(\tau) \right\|_{k-1}^2 + \left\| \nabla G^\varepsilon(\tau) \right\|_{k-2}^2 \right) d\tau \le C\varepsilon^{2p_1},
\end{aligned}$$

in which p_1 is defined in Theorem 1.3.

In details, we use several lemmas to complete the proof of Lemma 4.4. For $T > 0$, we first take the difference of the second equations in (1.8) and (1.3) to get

$$\begin{aligned}
&\varepsilon^2 \left(\partial_t \left(n^\varepsilon u^\varepsilon\right) + \operatorname{div}\left(n^\varepsilon u^\varepsilon \otimes u^\varepsilon\right)\right) + \nabla\left(n^\varepsilon \theta^\varepsilon\right) - \nabla\left(\bar{n}\bar{\theta}\right) \\
&= -\left(n^\varepsilon E^\varepsilon - \bar{n}\bar{E}\right) - \varepsilon n^\varepsilon \left(u^\varepsilon \times B^\varepsilon\right) - \left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}\right).
\end{aligned}$$

For $\alpha \in \mathbb{N}^3$ with $|\alpha| \le k-1$, by applying ∂^α to the above equation, multiplying the resulting equation with F_α, integrating it with respect to x and t, we have

$$
\begin{aligned}
0 = &\int_0^T \varepsilon^2 \left\langle F_\alpha, \partial^\alpha \partial_t \left(n^\varepsilon u^\varepsilon\right)\right\rangle dt + \int_0^T \left\langle F_\alpha, \partial^\alpha \left(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}\right)\right\rangle dt \\
&+ \int_0^T \left\langle F_\alpha, \partial^\alpha \left(\nabla \left(n^\varepsilon \theta^\varepsilon\right) - \nabla \left(\bar{n}\bar{\theta}\right)\right)\right\rangle dt \\
&+ \int_0^T \left\langle F_\alpha, \partial^\alpha \left(n^\varepsilon F^\varepsilon\right)\right\rangle dt + \int_0^T \left\langle F_\alpha, \partial^\alpha \left(N^\varepsilon \bar{E}\right)\right\rangle dt \\
&+ \int_0^T \left\langle F_\alpha, \varepsilon^2 \partial^\alpha \left(\operatorname{div}\left(n^\varepsilon u^\varepsilon \otimes u^\varepsilon\right)\right) + \varepsilon \partial^\alpha \left(n^\varepsilon u^\varepsilon \times B^\varepsilon\right)\right\rangle dt \triangleq \sum_{j=1}^{6} \mathcal{R}_j,
\end{aligned}
\tag{4.23}
$$

with the corresponding $\mathcal{R}_j$ where j is from 1 to 6. The parameters p_1 and δ in Lemma 4.7-4.8 are defined in Theorem 1.3, and $\mu > 0$ is an adequately small constant.

Lemma 4.5 *(Estimate of $\mathcal{R}_2$). For all $|\alpha| \le k-1$, it holds*

$$
\mathcal{R}_2 \ge \frac{1}{2}\left\|F_\alpha(T)\right\|^2 + \frac{1}{4}\left\|G_\alpha(T)\right\|^2 - C\varepsilon^{2p_1} - C\delta \sup_{0\le t\le T}\left\|G^\varepsilon(t)\right\|_{k-1}^2 - C\mu \int_0^T \left\|\nabla G^\varepsilon(\tau)\right\|_{k-2}^2 d\tau.
\tag{4.24}
$$

Proof. By (4.18), (4.22), (1.3), (1.10), (1.20), Lemma 4.2 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\mathcal{R}_2 =& \frac{1}{2}\left\|F_\alpha(T)\right\|^2 + \frac{1}{2}\left\|G_\alpha(T)\right\|^2 - \frac{1}{2}\left\|F_\alpha(0)\right\|^2 - \frac{1}{2}\left\|G_\alpha(0)\right\|^2 - \int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \Psi\right\rangle dt \\
&+ \int_0^T \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \Psi\right\rangle dt \\
\ge& \frac{1}{2}\left\|F_\alpha(T)\right\|^2 + \frac{1}{2}\left\|G_\alpha(T)\right\|^2 - C\varepsilon^{2p_1} - \int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \Psi\right\rangle dt + \int_0^T \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \Psi\right\rangle dt.
\end{aligned}
$$

Then it follows from $\bar{n} \in L^\infty\left(\mathbb{R}^+; H^k\right)$ and $\bar{u} \in L^\infty\left(\mathbb{R}^+; H^{k-1}\right)$ that Ψ is continuous at $t = 0$. Hence, by the Young's inequality

$$
\begin{aligned}
\left|\int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \Psi\right\rangle dt\right| &\le \varepsilon \left|\left\langle G_\alpha(T), \partial^\alpha \Psi(T)\right\rangle\right| + \varepsilon \left|\left\langle G_\alpha(0), \partial^\alpha \Psi(0)\right\rangle\right| \\
&\le \frac{1}{4}\left\|G_\alpha(T)\right\|^2 + C\varepsilon^{2p_1}.
\end{aligned}
$$

It is sufficient to prove the following inequality

$$\left|\int_0^T \langle \varepsilon G_\alpha, \partial_t \partial^\alpha \Psi\rangle dt\right| \le C\varepsilon^2 + C\delta \sup_{0\le t\le T} \left\|G^\varepsilon(t)\right\|_{k-1}^2 + \mu \int_0^T \left\|\nabla G^\varepsilon(\tau)\right\|_{k-2}^2 d\tau. \tag{4.25}$$

In fact, for $1 \le |\alpha| \le k-1$, (4.25) obviously holds by using (4.20) and Lemma 4.2. It needs more calculations when $\alpha = 0$. Noticing $\operatorname{div} G^\varepsilon = 0$, there is a function η^ε such that

$$\begin{cases} \nabla \times \eta^\varepsilon = G^\varepsilon, \\ \operatorname{div} \eta^\varepsilon = 0. \end{cases}$$

Let us introduce a restriction condition to make η^ε to be uniquely determined,

$$\int_{\mathbb{T}^3} \eta^\varepsilon(t,x)dx = 0, \quad \forall t \ge 0.$$

Since $\operatorname{div} \eta^\varepsilon = 0$, it follows from the Poincaré inequality and (1.10) that

$$\left\|\eta^\varepsilon(t)\right\|_{k-1} \le C\left\|\nabla\eta^\varepsilon(t)\right\|_{k-1} = C\left\|G^\varepsilon(t)\right\|_{k-1}, \quad \forall t > 0. \tag{4.26}$$

Then by the expression for Ψ in (4.18), (1.7)-(1.8) and taking integration by parts, we get

$$\langle \varepsilon G^\varepsilon, \partial_t \Psi\rangle = \langle \varepsilon\eta^\varepsilon, \operatorname{div}(\bar n \bar u)\bar u\rangle + \langle \varepsilon\eta^\varepsilon, \bar n \partial_t \nabla\bar\phi\rangle + \left\langle \varepsilon\eta^\varepsilon, \bar n \partial_t\left(\frac{\nabla(\bar n\bar\theta)}{\bar n}\right)\right\rangle. \tag{4.27}$$

For the last term in (4.27), we obtain

$$\left\langle \varepsilon\eta^\varepsilon, \bar n \partial_t\left(\frac{\nabla(\bar n\bar\theta)}{\bar n}\right)\right\rangle = -\langle \varepsilon\eta^\varepsilon, \bar\theta\nabla\ln\bar n\partial_t\bar n\rangle - \langle\varepsilon\eta^\varepsilon, \nabla\bar\theta\partial_t\bar n\rangle.$$

Furthermore, substituting it into (4.27), we have

$$\langle \varepsilon G^\varepsilon, \partial_t \Psi\rangle = \langle \varepsilon\eta^\varepsilon, \operatorname{div}(\bar n \bar u)\bar u\rangle - \langle \varepsilon\eta^\varepsilon, \partial_t\bar\phi\nabla\bar n\rangle - \langle \varepsilon\eta^\varepsilon, \bar\theta\nabla\ln\bar n\partial_t\bar n\rangle - \langle\varepsilon\eta^\varepsilon, \nabla\bar\theta\partial_t\bar n\rangle. \tag{4.28}$$

Now let us estimate the four terms on the right-hand side of (4.28). First,

$$\left|\varepsilon\langle\eta^\varepsilon, \operatorname{div}(\bar n\bar u)\bar u\rangle\right| \le C\varepsilon\left\|G^\varepsilon\right\|_{k-1}\|\bar u\|_{k-1}^2.$$

Next, by the Poisson equation in (4.10) and noticing the estimate (4.15), we have

$$\left|\varepsilon\langle\eta^\varepsilon, \partial_t\bar\phi\nabla\bar n\rangle\right| \le C\varepsilon\left\|\eta^\varepsilon\right\|_{k-1}\|\nabla\bar n\|\left\|\partial_t\nabla\bar\phi\right\| \le C\varepsilon\left\|G^\varepsilon\right\|_{k-1}\|\bar n - 1\|_{k-1}^2.$$

Additionally, from (1.8), it is easy to get

$$-\left(\bar\theta\nabla\ln\bar n + \nabla\bar\theta\right) = \bar u + \nabla\bar\phi = \bar u + \bar E.$$

Thus, for the last two terms on the right-hand side of (4.28), we have

$$
\begin{aligned}
&\left|\left\langle \varepsilon\eta^{\varepsilon}, \bar{\theta}\nabla \ln \bar{n}\partial_t \bar{n}\right\rangle + \left\langle \varepsilon\eta^{\varepsilon}, \nabla\bar{\theta}\partial_t \bar{n}\right\rangle\right| \\
&= \left|\left\langle \varepsilon\eta^{\varepsilon}, (\bar{u}+\bar{E})\partial_t \bar{n}\right\rangle\right| \\
&= \left| \left\langle \varepsilon\eta^{\varepsilon}, (\bar{u}+\nabla\bar{\phi})\partial_t \bar{n}\right| \right. \\
&\leqslant \left|\left\langle \varepsilon\eta^{\varepsilon}, \bar{u}\partial_t \bar{n}\right\rangle\right| + \left|\left\langle \varepsilon\eta^{\varepsilon}, \nabla\bar{\phi}\partial_t \bar{n}\right\rangle\right| \\
&\leqslant C\varepsilon \left\|G^{\varepsilon}\right\|_{k-1} \|\bar{u}\|_{k-1}^2 + C\varepsilon \left\|G^{\varepsilon}\right\|_{k-1} \|\bar{n}-1\|_{k-1}^2 \\
&\leqslant C\varepsilon \left\|G^{\varepsilon}\right\|_{k-1} \left(\|\bar{n}-1\|_{k-1}^2 + \|\bar{u}\|_{k-1}^2\right).
\end{aligned}
$$

Combining all the estimates above, we obtain

$$
\left|\left\langle \varepsilon G^{\varepsilon}, \partial_t \Psi\right\rangle\right| \leqslant C\varepsilon \left\|G^{\varepsilon}\right\|_{k-1} \left(\|\bar{n}-1\|_{k-1}^2 + \|\bar{u}\|_{k-1}^2\right).
$$

Integrating it over $[0, T]$, using the estimates (4.6), (4.7) and (4.26) and noticing the Young's inequality, we get

$$
\begin{aligned}
\left|\int_0^T \left\langle \varepsilon G^{\varepsilon}, \partial_t \Psi\right\rangle dt\right| &\leq C\int_0^T \left\|G^{\varepsilon}\right\|_{k-1}^2 \left(\|\bar{n}-1\|_{k-1}^2 + \|\bar{u}\|_{k-1}^2\right) dt \\
&\quad + C\varepsilon^2 \int_0^T \left(\|\bar{n}-1\|_{k-1}^2 + \|\bar{u}\|_{k-1}^2\right) dt \\
&\leq C\delta \sup_{0\leq t\leq T} \left\|G^{\varepsilon}(t)\right\|_{k-1}^2 + C\varepsilon^2,
\end{aligned}
$$

which indicates (4.25). The proof of Lemma 4.5 is finished. □

Lemma 4.6 *(Estimate of $\mathcal{R}_3$). For all $|\alpha| \leq k-1$, there exists a constant $c_1 > 0$, such that*

$$
\mathcal{R}_3 \geq \int_0^T \left(c_1 \|N_\alpha(\tau)\|^2 - \frac{c_1}{4}\|\Theta_\alpha(\tau)\|^2\right) d\tau - C\delta \int_0^T \left(\left\|N^{\varepsilon}(\tau)\right\|_{k-1}^2 + \left\|\Theta^{\varepsilon}(\tau)\right\|_{k-1}^2\right) d\tau, \tag{4.29}
$$

and

$$
\begin{aligned}
\mathcal{R}_3 + C\varepsilon^{2p_1} &\geq \|\Theta_\alpha(T)\|^2 + c_1 \int_0^T \left(\|N_\alpha(\tau)\|^2 + \|\Theta_\alpha(\tau)\|^2\right) d\tau \\
&\quad - C\delta \int_0^T (\|N^{\varepsilon}(\tau)\|_{k-1}^2 + \left\|\Theta^{\varepsilon}(\tau)\right\|_{k-1}^2) d\tau.
\end{aligned} \tag{4.30}
$$

Proof. By recalling and using (4.21), we have

$$\mathcal{R}_3 = -\int_0^T \left\langle \operatorname{div} F_\alpha, \partial^\alpha \left(n^\varepsilon \theta^\varepsilon - \bar{n}\bar{\theta}\right)\right\rangle dt = \int_0^T \left\langle N_\alpha, \partial^\alpha \left(n^\varepsilon \theta^\varepsilon - \bar{n}\bar{\theta}\right)\right\rangle dt.$$

Let us introduce

$$P(n) = \bar{\theta} n, \quad P'(n) = \bar{\theta} > 0, \quad Q(\theta) = n^\varepsilon \theta, \quad Q'(\theta) = n^\varepsilon > 0.$$

Then we obtain

$$n^\varepsilon \theta^\varepsilon - \bar{n}\bar{\theta} = \int_0^1 (P'\left(\tilde{n}^\varepsilon(s)\right) N^\varepsilon + Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta^\varepsilon) ds,$$

where

$$\tilde{n}^\varepsilon(s) = \bar{n} + sN^\varepsilon, \tilde{\theta}^\varepsilon(s) = \bar{\theta} + s\Theta^\varepsilon.$$

According to (1.10), (4.6), and (4.8), it is easy to obtain that $\tilde{n}^\varepsilon(s) - 1$, $\tilde{\theta}^\varepsilon(s) - 1$ are sufficiently small in $L^\infty\left(\mathbb{R}^+ \times \mathbb{T}^3\right)$, uniformly with respect to $\varepsilon \in (0, 1]$ and $s \in [0, 1]$. Additionally, we have

$$\begin{aligned}
\partial^\alpha \left(n^\varepsilon \theta^\varepsilon - \bar{n}\bar{\theta}\right) = & \int_0^1 P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha ds + \int_0^1 \left(\partial^\alpha \left(P'\left(\tilde{n}^\varepsilon(s)\right) N^\varepsilon\right) - P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha\right) ds \\
& + \int_0^1 Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha ds + \int_0^1 \left(\partial^\alpha \left(Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta^\varepsilon\right) - Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha\right) ds.
\end{aligned}$$

First, using (1.10), (1.19), (1.20), (4.6) and (4.8), it is easy to find that $\tilde{n}^\varepsilon$, $\tilde{\theta}^\varepsilon$ are uniformly bounded from below for all $\varepsilon \in (0, 1]$ and $s \in [0, 1]$ when δ is sufficiently small. Consequently, the monotonicity and continuity of the pressure function $P(\rho)$ and the temperature function $Q(\theta)$ indicate that there exists a positive constant $c_1^\star > 0$, such that

$$\int_0^1 \left\langle N_\alpha, P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha\right\rangle ds \geq 4c_1^\star \left\|N_\alpha\right\|^2.$$

Noticing that

$$\int_0^1 \left\langle N_\alpha, Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha\right\rangle ds = \int_0^1 Q'\left(\tilde{\theta}^\varepsilon(s)\right) \left\langle N_\alpha, \Theta_\alpha\right\rangle ds,$$

by using the Cauchy-Schwarz inequality and the Young's inequality, there exists a positive constant $\tilde{c_1}' > 0$, such that

$$\langle N_\alpha, \Theta_\alpha \rangle \geq -\tilde{c_1}' \|N_\alpha\|^2 - \frac{\tilde{c_1}'}{4} \|\Theta_\alpha\|^2 .$$

Denoting $c_1' = \tilde{c_1}' \max_{0\leq s\leq 1} Q'\left(\tilde{\theta}^\varepsilon(s)\right)$, then $\frac{c_1'}{4} = \frac{\tilde{c_1}'}{4} \max_{0\leq s\leq 1} Q'\left(\tilde{\theta}^\varepsilon(s)\right)$, additionally, we note $c_1' \leq c_1 \leq c_1^\star$, then we have

$$\int_0^1 \left\langle N_\alpha, P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha \right\rangle ds \geq 4c_1 \|N_\alpha\|^2 , \tag{4.31}$$

$$\int_0^1 \left\langle N_\alpha, Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha \right\rangle ds \geq -c_1 \|N_\alpha\|^2 - \frac{c_1}{4} \|\Theta_\alpha\|^2 . \tag{4.32}$$

Letting δ be sufficiently small, the weak convergence from $\left(n_0^\varepsilon, E_0^\varepsilon\right)$ to $\left(\bar{n}_0, \bar{E}_0\right)$ yields that

$$\|\bar{n}_0 - 1\|_k + \left\|\bar{E}_0\right\|_k \leq \delta.$$

Thus, by using the Moser-type inequalities, we obtain

$$\begin{aligned} \left\|\partial^\alpha \left(P'\left(\tilde{n}^\varepsilon(s)\right) N^\varepsilon\right) - P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha\right\| &\leq C\delta \left\|N^\varepsilon\right\|_{k-1} , \\ \left\|\partial^\alpha \left(Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta^\varepsilon\right) - Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha\right\| &\leq C\delta \left\|\Theta^\varepsilon\right\|_{k-1} , \end{aligned}$$

which, along with the Young's inequality and the Cauchy-Schwarz inequality, implies

$$\int_0^1 \left\langle N_\alpha, \partial^\alpha \left(P'\left(\tilde{n}^\varepsilon(s)\right) N^\varepsilon\right) - P'\left(\tilde{n}^\varepsilon(s)\right) N_\alpha \right\rangle ds \geq -c_1 \|N_\alpha\|^2 - C\delta \left\|N^\varepsilon\right\|_{k-1}^2 , \tag{4.33}$$

and

$$\int_0^1 \left\langle N_\alpha, \partial^\alpha \left(Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta^\varepsilon\right) - Q'\left(\tilde{\theta}^\varepsilon(s)\right) \Theta_\alpha \right\rangle ds \geq -c_1 \|N_\alpha\|^2 - C\delta \left\|\Theta^\varepsilon\right\|_{k-1}^2 . \tag{4.34}$$

Hence, by combining (4.31), (4.32), (4.33) and (4.34), it yields (4.29).

Subtracting the second equation of the system (1.7) from the third equation of the system (1.3), we have

$$\partial_t \Theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon - \bar{u} \cdot \nabla\bar{\theta} + \frac{2}{3}\theta^\varepsilon \operatorname{div} u^\varepsilon - \frac{2}{3}\bar{\theta} \operatorname{div} \bar{u} = \frac{2-\varepsilon^2}{3} \left|u^\varepsilon\right|^2 - \frac{2|\bar{u}|^2}{3} - \Theta^\varepsilon.$$

Applying ∂^α to the above equation yields

$$\begin{aligned}&\partial_t\partial^\alpha\Theta^\varepsilon+\partial^\alpha\left(u^\varepsilon\cdot\nabla\theta^\varepsilon\right)-\partial^\alpha(\bar u\cdot\nabla\bar\theta)+\frac{2}{3}\partial^\alpha\left(\theta^\varepsilon\operatorname{div}u^\varepsilon\right)-\frac{2}{3}\partial^\alpha(\bar\theta\operatorname{div}\bar u)\\&=\frac{2-\varepsilon^2}{3}\partial^\alpha\left(\left|u^\varepsilon\right|^2\right)-\frac{2}{3}\partial^\alpha\left(|\bar u|^2\right)-\partial^\alpha\Theta^\varepsilon,\end{aligned}$$

noticing that

$$\partial^\alpha\left(u^\varepsilon\cdot\nabla\theta^\varepsilon\right)-\partial^\alpha(\bar u\cdot\nabla\bar\theta)=\partial^\alpha\left(u^\varepsilon\cdot\nabla\Theta^\varepsilon\right)+\partial^\alpha\left(u^\varepsilon\cdot\nabla\bar\theta\right),$$

and

$$\frac{2}{3}\partial^\alpha\left(\theta^\varepsilon\operatorname{div}u^\varepsilon\right)-\frac{2}{3}\partial^\alpha(\bar\theta\operatorname{div}\bar u)=\frac{2}{3}\partial^\alpha\left(\Theta^\varepsilon\operatorname{div}u^\varepsilon\right)+\frac{2}{3}\partial^\alpha\left(\bar\theta\operatorname{div}u^\varepsilon\right),$$

then we have

$$\begin{aligned}&\partial_t\partial^\alpha\Theta^\varepsilon+\partial^\alpha\left(u^\varepsilon\cdot\nabla\Theta^\varepsilon\right)+\partial^\alpha\left(u^\varepsilon\cdot\nabla\bar\theta\right)+\frac{2}{3}\partial^\alpha\left(\Theta^\varepsilon\operatorname{div}u^\varepsilon\right)+\frac{2}{3}\partial^\alpha\left(\bar\theta\operatorname{div}u^\varepsilon\right)\\&=\frac{2-\varepsilon^2}{3}\partial^\alpha\left(\left|u^\varepsilon\right|^2\right)-\frac{2}{3}\partial^\alpha\left(|\bar u|^2\right)-\partial^\alpha\Theta^\varepsilon.\end{aligned}$$

Taking the inner product with $\partial^\alpha\Theta^\varepsilon$ in L^2 yields

$$\begin{aligned}&\langle\partial_t\Theta_\alpha,\Theta_\alpha\rangle+\langle\Theta_\alpha,\Theta_\alpha\rangle+\left\langle\partial^\alpha\left(u^\varepsilon\cdot\nabla\Theta^\varepsilon\right),\Theta_\alpha\right\rangle+\left\langle\partial^\alpha\left(u^\varepsilon\cdot\nabla\bar\theta\right),\Theta_\alpha\right\rangle\\&+\left\langle\frac{2}{3}\partial^\alpha\left(\Theta^\varepsilon\operatorname{div}u^\varepsilon\right),\Theta_\alpha\right\rangle+\left\langle\frac{2}{3}\partial^\alpha\left(\bar\theta\operatorname{div}u^\varepsilon\right),\Theta_\alpha\right\rangle\\&=\left\langle\frac{2-\varepsilon^2}{3}\partial^\alpha\left(\left|u^\varepsilon\right|^2\right),\Theta_\alpha\right\rangle-\left\langle\frac{2}{3}\partial^\alpha\left(|\bar u|^2\right),\Theta_\alpha\right\rangle.\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Theta_\alpha\|^2+\|\Theta_\alpha\|^2+\left\langle u^\varepsilon\cdot\partial^\alpha\left(\nabla\Theta^\varepsilon\right),\partial^\alpha\Theta^\varepsilon\right\rangle+\left\langle u^\varepsilon\cdot\partial^\alpha(\nabla\bar\theta),\partial^\alpha\Theta^\varepsilon\right\rangle\\&+\left\langle\frac{2}{3}\partial^\alpha\Theta^\varepsilon\operatorname{div}u^\varepsilon,\partial^\alpha\Theta^\varepsilon\right\rangle+\left\langle\frac{2}{3}\partial^\alpha\bar\theta\operatorname{div}u^\varepsilon,\partial^\alpha\Theta^\varepsilon\right\rangle\\&\leqslant\left\|\partial^\alpha\left(\frac{2-\varepsilon^2}{3}\left|u^\varepsilon\right|^2\right)\right\|\left\|\partial^\alpha\Theta^\varepsilon\right\|+\left\|\partial^\alpha\left(\frac{2}{3}|\bar u|^2\right)\right\|\left\|\partial^\alpha\Theta^\varepsilon\right\|+\left\langle u^\varepsilon\cdot\partial^\alpha\left(\nabla\Theta^\varepsilon\right),\partial^\alpha\Theta^\varepsilon\right\rangle\\&-\left\langle\partial^\alpha\left(u^\varepsilon\cdot\nabla\Theta^\varepsilon\right),\partial^\alpha\Theta^\varepsilon\right\rangle+\left\langle U^\varepsilon\cdot\partial^\alpha(\nabla\bar\theta),\partial^\alpha\Theta^\varepsilon\right\rangle-\left\langle\partial^\alpha\left(u^\varepsilon\cdot\nabla\bar\theta\right),\partial^\alpha\Theta^\varepsilon\right\rangle\\&+\left\langle\frac{2}{3}\partial^\alpha\Theta^\varepsilon\operatorname{div}u^\varepsilon,\partial^\alpha\Theta^\varepsilon\right\rangle-\left\langle\frac{2}{3}\partial^\alpha\left(\Theta^\varepsilon\operatorname{div}u^\varepsilon\right),\partial^\alpha\Theta^\varepsilon\right\rangle\\&+\left\langle\frac{2}{3}\partial^\alpha\bar\theta\operatorname{div}u^\varepsilon,\partial^\alpha\Theta^\varepsilon\right\rangle-\left\langle\frac{2}{3}\partial^\alpha\left(\bar\theta\operatorname{div}u^\varepsilon\right),\partial^\alpha\Theta^\varepsilon\right\rangle.\end{aligned}\tag{4.35}$$

Noticing (1.10) and u^ε is uniformly bounded, there exists a constant $C > 0$, such that

$$\begin{aligned}\left\langle u^\varepsilon \cdot \partial^\alpha (\nabla \Theta^\varepsilon), \partial^\alpha \Theta^\varepsilon \right\rangle &\le C \left\|\Theta^\varepsilon\right\|_{k-1} \left\|\Theta^\varepsilon\right\|_k ,\\ \left\langle u^\varepsilon \cdot \partial^\alpha (\nabla \bar{\theta}), \partial^\alpha \Theta^\varepsilon \right\rangle &\le \left\|u^\varepsilon\right\| \left\|\partial^\alpha (\nabla \bar{\theta})\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le C \left\|\Theta^\varepsilon\right\|_{k-1} \left\|\Theta^\varepsilon\right\|_k ,\\ \left\langle \frac{2}{3}\partial^\alpha \Theta^\varepsilon \operatorname{div} u^\varepsilon, \partial^\alpha \Theta^\varepsilon \right\rangle &\le \left\|\frac{2}{3}\partial^\alpha \Theta^\varepsilon\right\| \left\|\operatorname{div} u^\varepsilon\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 ,\end{aligned}$$

and

$$\left\langle \frac{2}{3}\partial^\alpha \bar{\theta} \operatorname{div} u^\varepsilon, \partial^\alpha \Theta^\varepsilon \right\rangle \le \left\|\frac{2}{3}\partial^\alpha \bar{\theta}\right\| \left\|\operatorname{div} u^\varepsilon\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 .$$

By the Cauchy-Schwarz inequality, the Young's inequality, (1.10) and (4.7), we obtain

$$\left\|\partial^\alpha (\frac{2-\varepsilon^2}{3} \left|u^\varepsilon\right|^2)\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le \frac{4-2\varepsilon^2}{3} \left\|u^\varepsilon\right\| \left\|\partial^\alpha u^\varepsilon\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le C_{\varepsilon'} \left\|\Theta^\varepsilon\right\|_{k-1}^2 + \varepsilon' \left\|u^\varepsilon\right\|_{k-1}^2 ,$$

and

$$\left\|\partial^\alpha (\frac{2}{3} |\bar{u}|^2)\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le \frac{4}{3} \|\bar{u}\| \left\|\partial^\alpha \bar{u}\right\| \left\|\partial^\alpha \Theta^\varepsilon\right\| \le C_{\varepsilon'} \left\|\Theta^\varepsilon\right\|_{k-1}^2 + \varepsilon' \|\bar{u}\|_{k-1}^2 ,$$

where $C_{\varepsilon'}$ and ε' are usual constants with $C_{\varepsilon'} > \varepsilon'$.

By the Moser-type inequality [15,19], we have

$$\begin{aligned}\left\langle u^\varepsilon \cdot \partial^\alpha (\nabla \Theta^\varepsilon), \partial^\alpha \Theta^\varepsilon \right\rangle - \left\langle \partial^\alpha \left(u^\varepsilon \cdot \nabla \Theta^\varepsilon\right), \partial^\alpha \Theta^\varepsilon \right\rangle &\le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 ,\\ \left\langle u^\varepsilon \cdot \partial^\alpha (\nabla \bar{\theta}), \partial^\alpha \Theta^\varepsilon \right\rangle - \left\langle \partial^\alpha (u^\varepsilon \cdot \nabla \bar{\theta}), \partial^\alpha \Theta^\varepsilon \right\rangle &\le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 ,\\ \left\langle \frac{2}{3}\partial^\alpha \Theta^\varepsilon \operatorname{div} u^\varepsilon, \partial^\alpha \Theta^\varepsilon \right\rangle - \left\langle \frac{2}{3}\partial^\alpha \left(\Theta^\varepsilon \operatorname{div} u^\varepsilon\right), \partial^\alpha \Theta^\varepsilon \right\rangle &\le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 ,\end{aligned}$$

and

$$\left\langle \frac{2}{3}\partial^\alpha \bar{\theta} \operatorname{div} u^\varepsilon, \partial^\alpha \Theta^\varepsilon \right\rangle - \left\langle \frac{2}{3}\partial^\alpha \left(\bar{\theta} \operatorname{div} u^\varepsilon\right), \partial^\alpha \Theta^\varepsilon \right\rangle \le C \left\|\Theta^\varepsilon\right\|_{k-1}^2 .$$

Consequently, according to (4.35) and adding the above inequalities for all $|\alpha| \le k-1$, noticing (1.10), (1.19), (1.20), (4.6) and (4.8), it is easy to find that when δ, $\left\|\bar{\theta}_0 - 1\right\|_k$ are sufficiently small, $\tilde{n}^\varepsilon(s) - 1$, $\tilde{\theta}^\varepsilon(s) - 1$ are sufficiently small in $L^\infty\left(\mathbb{R}^+ \times \mathbb{T}^3\right)$, uniformly with respect to $\varepsilon \in (0, 1]$ and $s \in [0, 1]$, then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \left\|\Theta_\alpha\right\|^2 + \left\|\Theta_\alpha\right\|^2 &\le C \left\|\Theta^\varepsilon\right\|_{k-1} \left\|\Theta^\varepsilon\right\|_k \\ &\le C(\left\|\theta^\varepsilon - 1\right\|_{k-1} + \left\|\bar{\theta} - 1\right\|_{k-1})(\left\|\theta^\varepsilon - 1\right\|_k + \left\|\bar{\theta} - 1\right\|_k) \\ &\le C\varepsilon^{2p_1}.\end{aligned} \tag{4.36}$$

Multiplying the above inequality (4.36) by $\frac{5c_1}{4}$, then integrating over $[0, T]$, and adding it to (4.29) yields

$$\|\Theta_\alpha(T)\|^2 + c_1 \int_0^T \left(\|N_\alpha(\tau)\|^2 + \|\Theta_\alpha(\tau)\|^2\right) d\tau - C\delta \int_0^T (\|N^\varepsilon(\tau)\|_{k-1}^2 + \left\|\Theta^\varepsilon(\tau)\right\|_{k-1}^2) d\tau$$
$$\leqslant \mathcal{R}_3 + C\varepsilon^{2p_1},$$

which proves (4.30). □

Next, we borrow two lemmas from [17] without proof as follows.

Lemma 4.7 *(Estimate of $\mathcal{R}_1$, see Lemma 2.5 in [17]). For all $|\alpha| \le k-1$, it holds*

$$|\mathcal{R}_1| \le C\varepsilon^{2p_1} + \frac{1}{4}\|F_\alpha(T)\|^2 + \mu \int_0^T \left\|\nabla G^\varepsilon(\tau)\right\|_{k-3}^2 d\tau. \tag{4.37}$$

Lemma 4.8 *(Estimates of $\mathcal{R}_4, \mathcal{R}_5$ and $\mathcal{R}_6$, see Lemma 2.8 in [17]). For all $|\alpha| \le k-1$, there exists a constant $c_2 > 0$, such that*

$$\mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6 \ge c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \left\|N^\varepsilon(\tau)\right\|_{k-1}^2 d\tau - C\varepsilon^2. \tag{4.38}$$

Since $\mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6 = -\mathcal{R}_1$, combining Lemmas 4.7-4.8, (4.21) and (4.23), and summing up all $|\alpha| \le k-2$, we have

$$\left\|N^\varepsilon(T)\right\|_{k-2}^2 + \left\|\Theta^\varepsilon(T)\right\|_{k-2}^2 + \left\|F^\varepsilon(T)\right\|_{k-1}^2 + \left\|G^\varepsilon(T)\right\|_{k-1}^2$$
$$+ \int_0^T \left(\left\|N^\varepsilon(\tau)\right\|_{k-1}^2 + \left\|\Theta^\varepsilon(\tau)\right\|_{k-1}^2 + \left\|F^\varepsilon(\tau)\right\|_{k-1}^2\right) d\tau$$
$$\le C\varepsilon^{2p_1} + C\delta \sup_{0\le t\le T} \left\|G^\varepsilon(t)\right\|_{k-1}^2 + C\mu \int_0^T \left\|\nabla G^\varepsilon(\tau)\right\|_{k-2}^2 d\tau,$$

provided that δ is small enough. By taking the superior limit with respect to T, we obtain

$$\sup_{t\in\mathbb{R}^+} \left(\left\|N^\varepsilon(t)\right\|_{k-2}^2 + \left\|\Theta^\varepsilon(t)\right\|_{k-2}^2 + \left\|F^\varepsilon(t)\right\|_{k-1}^2 + \left\|G^\varepsilon(t)\right\|_{k-1}^2\right)$$
$$+ \int_0^{+\infty} \left(\left\|N^\varepsilon(\tau)\right\|_{k-1}^2 + \left\|\Theta^\varepsilon(\tau)\right\|_{k-1}^2 + \left\|F^\varepsilon(\tau)\right\|_{k-1}^2\right) d\tau \tag{4.39}$$

$$\leq C\varepsilon^{2p_1} + C\mu \int_0^{+\infty} \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 \, d\tau.$$

The proof of Lemma 4.4 is ended after borrowing Lemma 2.9 in [17] as follows.

Lemma 4.9. *It holds*

$$\int_0^{+\infty} \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 \, d\tau \leq C\varepsilon^{2p_1}. \tag{4.40}$$

Proof. We omit it for the sake of simplicity. □

The proof of Theorem 1.3 follows from the estimate below.

Lemma 4.10. *It holds*

$$\sup_{t\in\mathbb{R}^+} \varepsilon^2 \|\Xi^\varepsilon(t)\|_{k-2}^2 + \int_0^{+\infty} \|\Xi^\varepsilon(t)\|_{k-2}^2 \, dt \leq C\varepsilon^{2p_1}. \tag{4.41}$$

Proof. It is easy to find that the second equation in (1.3) is equivalent to

$$\varepsilon^2 \partial_t u^\varepsilon + \varepsilon^2 \left(u^\varepsilon \cdot \nabla\right) u^\varepsilon + \frac{\nabla\left(n^\varepsilon \theta^\varepsilon\right)}{n^\varepsilon} = -E^\varepsilon - \varepsilon u^\varepsilon \times B^\varepsilon - u^\varepsilon.$$

We obtain the following equation by subtracting the equation for $\bar{u}$ in (1.8) from the above equation

$$\varepsilon^2 \partial_t \Xi^\varepsilon + \Xi^\varepsilon = \mathcal{H}_1^\varepsilon, \tag{4.42}$$

where

$$\mathcal{H}_1^\varepsilon = -\varepsilon^2 \left(u^\varepsilon \cdot \nabla\right) u^\varepsilon - \left(\frac{\nabla\left(n^\varepsilon \theta^\varepsilon\right)}{n^\varepsilon} - \frac{\nabla(\bar{n}\bar{\theta})}{\bar{n}}\right) - F^\varepsilon - \varepsilon u^\varepsilon \times B^\varepsilon - \varepsilon^2 \partial_t \bar{u}.$$

Let $T > 0$. For all $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq k-2$, by applying ∂^α to (4.42), taking the inner product with w_α in L^2 and integrating over $[0, T]$, we have

$$\|\varepsilon \Xi_\alpha(T)\|^2 + \int_0^T \|\Xi_\alpha(\tau)\|^2 \, d\tau \leq C \int_0^T \left\|\partial^\alpha \mathcal{H}_1^\varepsilon(\tau)\right\|^2 d\tau + \|\varepsilon \Xi_\alpha(0)\|^2. \tag{4.43}$$

Because $k \geq 3$, we use the Moser-type inequality for the quadratic term

$$\left\|\partial^\alpha (u^\varepsilon v^\varepsilon)\right\| \leq C\|u^\varepsilon\|_{k-1}\|v^\varepsilon\|_{k-1}, \quad \forall u^\varepsilon, v^\varepsilon \in H^k, \quad |\alpha| \leq k-1.$$

Thus, for $|\alpha| \le k-2$, we obtain

$$\left\|\partial^\alpha\left(u^\varepsilon \times G^\varepsilon\right)\right\| \le C\left\|u^\varepsilon\right\|_k\left\|G^\varepsilon\right\|_k, \quad \left\|\partial^\alpha\left(u^\varepsilon \cdot \nabla\right) u^\varepsilon\right\| \le C\left\|u^\varepsilon\right\|_k^2.$$

Then by (1.10), (4.6), (4.8) and Lemma 4.4, we have

$$\left\|\partial^\alpha\left(\frac{\nabla\left(n^\varepsilon \theta^\varepsilon\right)}{n^\varepsilon}-\frac{\nabla(\bar{n}\bar{\theta})}{\bar{n}}\right)\right\|=\left\|\partial^\alpha\left(\theta^\varepsilon \nabla \ln n^\varepsilon+\nabla \theta^\varepsilon-\bar{\theta} \nabla \ln \bar{n}-\nabla \bar{\theta}\right)\right\| \leqslant C \varepsilon^{2 p_1}.$$

Thus, noticing (1.10) and (4.7) together with Lemma 4.4, we have

$$\int_0^T\left\|\partial^\alpha \mathcal{H}_1^\varepsilon(\tau)\right\|^2 d \tau \le C \varepsilon^{2 p_1}.$$

Additionally, (1.20) implies

$$\left\|\varepsilon \Xi_\alpha(0)\right\|^2 \le C \varepsilon^{2 p_1}.$$

Substituting these estimates into (4.43) and summing up all $|\alpha| \le k-2$ leads to

$$\left\|\varepsilon \Xi^\varepsilon(T)\right\|_{k-2}^2+\int_0^T\left\|\Xi^\varepsilon(\tau)\right\|_{k-2}^2 d \tau \le C \varepsilon^{2 p_1},$$

which proves (4.41). □

Proof of Theorem 1.3. Theorem 1.3 follows by combining Lemma 4.4 and Lemma 4.10. □

Data availability

No data was used for the research described in the article.

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