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Asymptotic decay of bipolar isentropic/non-isentropic compressible Navier-Stokes-Maxwell systems

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Abstract

The initial value problems of bipolar isentropic/non-isentropic compressible Navier-Stokes-Maxwell (CNS-M) systems arising from plasmas in $\mathbb{R}^3$ are studied. The main difficulty of studying the bipolar isentropic/non-isentropic CNS-M systems lies in the appearance of the electromagnetic fields satisfying the hyperbolic Maxwell equations. The large time-decay rates of global smooth solutions with small amplitude in $L^q(\mathbb{R}^3)$ for $2 \le q \le \infty$ are established. For the bipolar non-isentropic CNS-M system, the difference of velocities of two charged carriers decay at the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}$ which is faster than the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ of the bipolar isentropic CNS-M system, meanwhile, the magnetic field decay at the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}$ for the bipolar isentropic CNS-M system. The approach adopted is the classical energy method but with some new developments, where the techniques of choosing symmetrizers and the spectrum analysis on the linearized homogeneous system play the crucial roles.

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Keywords: The bipolar isentropic/non-isentropic CNS-M system; Time-decay rates; Plasmas

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1. Introduction and main results

A plasma is a collection of fast-moving charged particles. It is believed that more than 9/10 of the matter in the universe is in the form of plasma, from sparse intergalactic plasma, to the interior of stars to neon signs. At high temperature and velocity, ions and electrons in a plasma tend to become two separate fluids due to their different physical properties (inertia, charge). One of the basic fluid models for describing plasma dynamics is the bipolar compressible Navier-Stokes-Maxwell (CNS-M) system, in which two compressible ion and electron carriers interact with their own self-consistent electromagnetic field. Generally, it takes the form of the bipolar CNS equations forced by the Electric field force $q_\nu\mathscr{E}$ and the Lorenz force $q_\nu\mathscr{U}^\nu\times\mathscr{B}$ in electromagnetic field, which is governed by the self-consistent Maxwell equations. Here the index $\nu=e,i$, stands for the electrons and the ions in plasmas, respectively. The symbols $\mathscr{E}\in\mathbb{R}^3$, $\mathscr{B}\in\mathbb{R}^3$, $\mathscr{U}^\nu\in\mathbb{R}^3$ and $\mathscr{N}^\nu>0$ represent the electric field, the magnetic field, the velocity and the density of the two charged carriers, respectively. The thermodynamic variables $\mathcal{T}^\nu>0$, $\mathscr{O}_\nu=\frac{3}{2}\mathcal{K}_B\mathcal{T}^\nu$ and $\mathscr{P}_\nu=\frac{2}{3}\mathscr{N}^\nu\mathscr{O}_\nu$ stand for the absolute temperature, the internal energy and the pressure of the charged carriers, respectively. The total energy of the charged carriers is defined by

$$\mathcal{E}_\nu=m_\nu\mathscr{N}^\nu\left(\mathscr{O}_\nu+\frac{1}{2}|\mathscr{U}^\nu|^2\right),\quad \nu=e,i.$$

In the following, we first consider the initial value problem to the scaled bipolar non-isentropic CNS-M system (see [1,4,13,16,21,25,26])

$$\begin{cases}\partial_t\mathscr{N}^\nu+\nabla\cdot\left(\mathscr{N}^\nu\mathscr{U}^\nu\right)=0,\\ \partial_t\left(m_\nu\mathscr{N}^\nu\mathscr{U}^\nu\right)+\nabla\cdot\left(m_\nu\mathscr{N}^\nu\mathscr{U}^\nu\otimes\mathscr{U}^\nu\right)+\nabla\mathscr{P}_\nu\\ \qquad=q_\nu\mathscr{N}^\nu\left(\mathscr{E}+\eta\mathscr{U}^\nu\times\mathscr{B}\right)+\mu_\nu\Delta\mathscr{U}^\nu+\mu_\nu'\nabla(\nabla\cdot\mathscr{U}^\nu),\\ \partial_t\mathcal{E}_\nu+\nabla\cdot\left(\mathcal{E}_\nu\mathscr{U}^\nu+\mathscr{P}_\nu\mathscr{U}^\nu\right)=q_\nu\mathscr{N}^\nu\mathscr{U}^\nu\mathscr{E}-\dfrac{\mathcal{E}_\nu-\mathscr{N}^\nu\mathcal{I}_*}{\tau}+\mathscr{U}^\nu\Delta\mathscr{U}^\nu,\\ \eta\lambda^2\partial_t\mathscr{E}-\nabla\times\mathscr{B}=-\eta\left(q_e\mathscr{N}^e\mathscr{U}^e+q_i\mathscr{N}^i\mathscr{U}^i\right),\quad \lambda^2\nabla\cdot\mathscr{E}=q_i\mathscr{N}^i+q_e\mathscr{N}^e,\\ \eta\partial_t\mathscr{B}+\nabla\times\mathscr{E}=0,\quad \nabla\cdot\mathscr{B}=0,\quad (t,x)\in[0,+\infty)\times\mathbb{R}^3,\end{cases}\tag{1.1}$$

where the physical parameters $\mathcal{T}_*>0$, $\mathcal{I}_*=\frac{3}{2}\mathcal{K}_B\mathcal{T}_*$, $\mathcal{K}_B>0$, $\lambda>0$, $\tau>0$, $\eta^{-1}=c=(\varepsilon_0\mu_0)^{-\frac{1}{2}}$, $\varepsilon_0>0$, $\mu_0>0$, $\mu_\nu>0$, μ_ν', $m_\nu>0$, $q_e=-1$ and $q_i=1$ are the back ground temperature, the background internal energy, the Boltzmann constant, the scaled Debye length, the energy relaxation time, the speed of light, the vacuum permittivity, permeability, the shear viscosity coefficient, the bulk viscosity coefficient, the masses, the charges of the electrons and ions, respectively. Notice that the similar terminology “Maxwell-Navier-Stokes system” was used in Masmoudi [22] but for a different system. Throughout this paper, we set

$$\mathcal{T}_*=\mathcal{K}_B=\lambda=\tau=\eta=\mu_\nu=m_\nu=1\quad\text{and}\quad\mu_\nu'=0,\quad \nu=e,i,$$

without loss of generality. This assumption is not an essential restriction in the investigation of global existence of smooth solutions for system (1.1). Otherwise, the smallness conditions on the initial data in the main results in Theorem 1.1 would depend on these parameters. Then, for smooth solutions with $\mathcal{N}^\nu, \mathcal{T}^\nu > 0$, system (1.1) is equivalent to

$$\begin{cases} \partial_t \mathcal{N}^\nu + \nabla \cdot \left(\mathcal{N}^\nu \mathcal{U}^\nu\right) = 0, \\ \partial_t \mathcal{U}^\nu + \left(\mathcal{U}^\nu \cdot \nabla\right) \mathcal{U}^\nu + \dfrac{\nabla\left(\mathcal{N}^\nu \mathcal{T}^\nu\right)}{\mathcal{N}^\nu} = q_\nu \left(\mathcal{E} + \mathcal{U}^\nu \times \mathcal{B}\right) + \dfrac{\Delta \mathcal{U}^\nu}{\mathcal{N}^\nu}, \\ \partial_t \mathcal{T}^\nu + \dfrac{2}{3}\mathcal{T}^\nu \nabla \cdot \mathcal{U}^\nu + \mathcal{U}^\nu \cdot \nabla \mathcal{T}^\nu + \dfrac{1}{3}\left|\mathcal{U}^\nu\right|^2 = -(\mathcal{T}^\nu - 1), \\ \partial_t \mathcal{E} - \nabla \times \mathcal{B} = \mathcal{N}^e \mathcal{U}^e - \mathcal{N}^i \mathcal{U}^i, \quad \nabla \cdot \mathcal{E} = \mathcal{N}^i - \mathcal{N}^e, \\ \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0, \quad \nabla \cdot \mathcal{B} = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3, \end{cases} \tag{1.2}$$

supplemented with the initial condition

$$(\mathcal{N}^\nu, \mathcal{U}^\nu, \mathcal{T}^\nu, \mathcal{E}, \mathcal{B})|_{t=0} = (\mathcal{N}^{\nu 0}, \mathcal{U}^{\nu 0}, \mathcal{T}^{\nu 0}, \mathcal{E}^0, \mathcal{B}^0), \quad \nu = e, i, \quad x \in \mathbb{R}^3, \tag{1.3}$$

which satisfies the compatibility condition

$$\nabla \cdot \mathcal{E}^0 = \mathcal{N}^{i0} - \mathcal{N}^{e0}, \quad \nabla \cdot \mathcal{B}^0 = 0, \quad x \in \mathbb{R}^3. \tag{1.4}$$

Due to the fact that $\mathcal{N}^\nu, \mathcal{T}^\nu > 0$, system (1.2) is symmetrizable hyperbolic-parabolic (see (2.25)). Then it follows from the result of Kato [17] and the pioneering work of Matsumura-Nishida [23] that the initial value problem (1.2)-(1.3) admits a unique local smooth solution as long as the initial data are smooth.

Lemma 1.1 *(Local existence to the bipolar non-isentropic CNS-M system [17,20]). Assume integer $s \geq 3$ and (1.4) holds. Suppose $(\mathcal{N}^{\nu 0} - 1, \ \mathcal{U}^{\nu 0}, \ \mathcal{T}^{\nu 0} - 1, \ \mathcal{E}^0, \ \mathcal{B}^0) \in H^s(\mathbb{R}^3)$ with $\mathcal{N}^{\nu 0}$, $\mathcal{T}^{\nu 0} \geq 2\kappa$ for some given constant $\kappa > 0$. Then there exists $T_1 > 0$ such that problem (1.2)-(1.3) has a unique smooth solution which satisfies $\mathcal{N}^\nu, \mathcal{T}^\nu \geq \kappa$ in $[0, T_1] \times \mathbb{R}^3$ and*

$$\mathcal{U}^\nu \in C^1\left(\left([0, T_1]; H^{s-2}\left(\mathbb{R}^3\right)\right) \cap C\left(\left([0, T_1]; H^s\left(\mathbb{R}^3\right)\right)\right.\right.,$$
$$(\mathcal{N}^\nu - 1, \mathcal{T}^\nu - 1, \mathcal{E}, \mathcal{B}) \in C^1\left(\left([0, T_1]; H^{s-1}\left(\mathbb{R}^3\right)\right) \cap C\left(\left([0, T_1]; H^s\left(\mathbb{R}^3\right)\right)\right.\right..$$

Before stating the main results, let us recall some previous related work. For the CNS-M systems, Fan-Li-Nakamura [7] proved the convergence of the non-isentropic CNS-M system to the incompressible magnetohydrodynamic equations in a bounded domain. Late then, Fan-Jia [6] established the local well-posedness of the non-isentropic CNS-M system with vacuum (the density may be zero). By means of the Green's function method and energy estimates, Duan [4] further proved the global existence and the large time decay rates of smooth solutions near a constant steady-state in unipolar isentropic case (in which the energy equation is not contained). By using classical energy methods and the techniques of symmetrizer, Feng-Peng-Wang [8] and Wang-Xu [32] generalized the results to the unipolar non-isentropic CNS-M systems.

On the other hand, the CNS-M systems turn into the incompressible Navier-Stokes-Maxwell (INS-M) systems when the densities of the charged carriers are invariant. For INS-M systems, by using the Fujita-Kato's method in l^1 based (for the Fourier coefficients) functional spaces, Ibrahim-Yoneda [15] proved the local existence of unique solution and loss of smoothness of the velocity and magnetic field for periodic problem. Then, Ibrahim-Keraani [14] showed the existence of global small mild solutions in three dimensions and the same results in a space 'close' to the energy space in 2 dimensions. With the help of a priori $L_t^2(L_x^\infty)$-estimates for solutions of the forced Navier-Stokes (NS) equations, Germain-Ibrahim-Masmoudi [10] proved the local existence of mild solutions for arbitrarily large data in a space similar to the scale invariant spaces classically used for NS systems and refined the results of [14]. By means of Littlewood-Paley analysis, Yue-Zhong [33] proved the global well-posedness of solutions in the Besov spaces $B_{1/22,1} \times B_{3/22,1} \times B_{3/22,1}$ provided with some sufficiently small initial data.

If we neglect the friction forces in charged fluids, the CNS-M systems become the Euler-Maxwell (E-M) systems for the ideal smooth charged carriers. For the E-M systems with velocity damping terms, Duan [3] and Duan-Liu-Zhu [5] studied the asymptotic behaviors of global solutions of the isentropic model, meanwhile, Feng-Wang-Kawashima [9] and Wang-Feng-Li [31] investigated the long time decay properties of global solutions for the non-isentropic models. We also recall the global existences results of solutions for E-M systems without damping terms in [2,11,12].

Based on the fact that the friction forces exist between any moving particles in any real fluids, the CNS-M systems for viscosity fluids are more meaningful than the E-M systems for smooth fluids. However, besides the structure of the characteristic equations of the CNS-M systems are much more complex than that of the E-M systems, the dissipative property of the CNS-M systems in $\mathbb{R}^3$ is much weaker than the one of the E-M systems, because the velocity function is no longer integrable in time-space, there are no results on the asymptotic decay of solutions to the bipolar isentropic or non-isentropic CNS-M systems up to now. A natural question is: how about the solution behaves as time goes to infinity? The goal of the present article is to consider this problem. The first main result can be stated as follows.

Theorem 1.1 *(Global existence and time decay rates for the bipolar non-isentropic CNS-M system). Let $s \geq 4$ and (1.4) hold. Then, there are $\delta_0 > 0$ and $C_0 > 0$ such that if*

$$\left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathcal{T}^{\nu 0} - 1, \mathscr{E}^0, \mathscr{B}^0)\right\|_{H^s(\mathbb{R}^3)} \leq \delta_0,$$

then problem (1.2)-(1.3) admits a unique global solution $(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathcal{T}^\nu, \mathscr{E}, \mathscr{B})$ satisfying

$$\mathscr{U}^\nu \in C^1\left([0,+\infty); H^{s-2}\left(\mathbb{R}^3\right)\right) \cap C\left([0,+\infty); H^s\left(\mathbb{R}^3\right)\right),$$
$$(\mathscr{N}^\nu - 1, \mathcal{T}^\nu - 1, \mathscr{E}, \mathscr{B}) \in C^1\left([0,+\infty); H^{s-1}\left(\mathbb{R}^3\right)\right) \cap C\left([0,+\infty); H^s\left(\mathbb{R}^3\right)\right),$$

and

$$\sup_{t\geq 0}\left\|(\mathscr{N}^\nu(t) - 1, \mathscr{U}^\nu(t), \mathcal{T}^\nu(t) - 1, \mathscr{E}(t), \mathscr{B}(t))\right\|_{H^s(\mathbb{R}^3)}$$
$$\leq C_0\left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathcal{T}^{\nu 0} - 1, \mathscr{E}^0, \mathscr{B}^0)\right\|_{H^s(\mathbb{R}^3)}.$$

Moreover, there exist $\delta_1 > 0$ *and* $C_1 > 0$ *such that if*

$$\left\|(\mathcal{N}^{\nu 0}-1, \mathcal{U}^{\nu 0}, \mathcal{T}^{\nu 0}-1, \mathcal{E}^0, \mathcal{B}^0)\right\|_{H^6(\mathbb{R}^3)} + \left\|(\mathcal{N}^{\nu 0}-1, \mathcal{U}^{\nu 0}, \mathcal{T}^{\nu 0}-1, \mathcal{E}^0, \mathcal{B}^0)\right\|_{L^1(\mathbb{R}^3)} \le \delta_1,$$

then, the solution $(\mathcal{N}^\nu, \mathcal{U}^\nu, \mathcal{T}^\nu, \mathcal{E}, \mathcal{B})$ *satisfies for any* $t \ge 0$, $\forall\, 2 \le q \le +\infty$,

$$\left\|\left(\mathcal{N}^\nu(t)-1, \mathcal{T}^\nu(t)-1\right)\right\|_{L^q(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{4}}, \tag{1.5}$$

$$\left\|\mathcal{U}^e(t)+\mathcal{U}^i(t)\right\|_{L^q(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{4}}, \tag{1.6}$$

$$\left\|\mathcal{U}^e(t)-\mathcal{U}^i(t)\right\|_{L^q(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{4}+\frac{1}{4q}}, \tag{1.7}$$

$$\|\mathcal{E}(t)\|_{L^q(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{4}}, \tag{1.8}$$

and

$$\|\mathcal{B}(t)\|_{L^q(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}. \tag{1.9}$$

Remark 1.1. It should be emphasized that the appearance of the energy relaxation term in the third equation of system (1.1) plays a key role in the proof of Theorem 1.1.

Remark 1.2. Note that in system (1.1), the masses of two charged carriers are assumed to take the same value 1. In fact, this assumption is not physical. That is, if we remove this assumption, similar results with the same time decay rate as in Theorem 1.1 can be obtained even though the disparate masses could induce a new difficulty in carrying out the linearized analysis of the coupling system for obtaining the time decay. We omit this.

Remark 1.3. It is well known that for classical CNS equations, the density, the absolute temperature and the velocity converges to its equilibrium state at the L^2 rate $(1+t)^{-\frac{3}{4}}$. However, as one can see in Theorem 1.1 that for CNS-M equations, the L^2 decay rate of the density, the absolute temperature and the total of velocity is $(1+t)^{-\frac{3}{4}}$, which is the same as the L^2 rate of the CNS equations. This shows that these estimates are optimal in comparison with the CNS equations for $q=2$. However, these decay estimates are not optimal when $q>2$. Notice that without considering the extra $(1+t)^{-\frac{1}{2}}$ decay, the time-decay rate of L^q norm has to be $(1+t)^{-\frac{3}{2}(1-\frac{1}{q})}$ for $2 \le q \le \infty$. Thus, the rate $(1+t)^{-\frac{3}{4}}$ corresponds to the case $q=2$ only.

Remark 1.4. During the discussion of the linearized CNS-M system, since the electric field $\mathcal{E}$ satisfies a time evolution equation and initial data $\mathcal{E}^0 \in L^1(\mathbb{R}^3)$, the higher time rate $(1+t)^{-\frac{3}{4}}$ of $\|\mathcal{E}\|$ is natural. Moreover, the time decay rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}$ of $\|\mathcal{U}^e(t)-\mathcal{U}^i(t)\|_{L^q}$ for the velocity component is slower than $(1+t)^{-\frac{3}{4}}$ in the case of the NS system. This point essentially results from the magnetic field effect of $\|\mathcal{B}\|_{L^q}$ which has the slowest time decay rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ among all the components of the solution to the bipolar non-isentropic CNS-M system.

Remark 1.5. The principal difference and the chief difficulty for proving the large time decay rates between the CNS-M systems and the E-M systems stem from the fact that the two characteristic equations corresponding to their linearized homogeneous systems are very different. Indeed, for the E-M systems, we only need to consider the asymptotic properties of characteristic roots when $|k|$ goes to 0 and when $|k|$ approaches $+\infty$. However, due to the much more complex structure of the CNS-M systems, we have to divide the frequency space into three parts and even more (see the proof of Lemmas 3.1 and 3.2).

Remark 1.6. The essential difficulty between the unipolar CNS-M system and the bipolar CNS-M system lies in the fact that the structure of the bipolar system is much more complex than that of the unipolar model. For removing the difficulties caused by two charged carriers, we introduce the 'total functions' and 'difference functions' for unknowns. Then the linearized homogeneous equations of the bipolar non-isentropic CNS-M system (3.1) can be written as two decoupled subsystems (3.9) and (3.12).

In comparison with the bipolar non-isentropic CNS-M system, the bipolar isentropic CNS-M systems are the other important class of equations due to their concise applications in plasmas. Besides the density $\mathscr{N}^\nu > 0$, the velocity $\mathscr{U}^\nu \in \mathbb{R}^3$, the electric field $\mathscr{E} \in \mathbb{R}^3$ and the magnetic field $\mathscr{B} \in \mathbb{R}^3$, we use $\mathscr{P}_\nu = \mathscr{P}_\nu(\mathscr{N}^\nu)$ to represent the pressure functions of two charged carriers with $\mathscr{P}'_\nu > 0$, then the bipolar isentropic CNS-M system with all the parameters equal to unity is written as

$$\begin{cases} \partial_t \mathscr{N}^\nu + \nabla \cdot \left(\mathscr{N}^\nu \mathscr{U}^\nu\right) = 0, \\ \partial_t \left(\mathscr{N}^\nu \mathscr{U}^\nu\right) + \nabla \cdot \left(\mathscr{N}^\nu \mathscr{U}^\nu \otimes \mathscr{U}^\nu\right) + \nabla \mathscr{P}_\nu \left(\mathscr{N}^\nu\right) = q_\nu \mathscr{N}^\nu \left(\mathscr{E} + \mathscr{U}^\nu \times \mathscr{B}\right) + \Delta \mathscr{U}^\nu, \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} = \mathscr{N}^e \mathscr{U}^e - \mathscr{N}^i \mathscr{U}^i, \quad \nabla \cdot \mathscr{E} = \mathscr{N}^i - \mathscr{N}^e, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3. \end{cases} \tag{1.10}$$

For smooth solutions with $\mathscr{N}^e, \mathscr{N}^i > 0$, we introduce the enthalpy functions $\mathscr{H}_\nu$ which satisfies $\mathscr{N}^\nu \mathscr{H}'_\nu(\mathscr{N}^\nu) = \mathscr{P}'_\nu(\mathscr{N}^\nu)$. Since $\mathscr{P}_\nu$ is strictly increasing on $(0, +\infty)$, so is $\mathscr{H}_\nu$. Then system (1.10) is equivalent to

$$\begin{cases} \partial_t \mathscr{N}^\nu + \nabla \cdot \left(\mathscr{N}^\nu \mathscr{U}^\nu\right) = 0, \\ \partial_t \mathscr{U}^\nu + \left(\mathscr{U}^\nu \cdot \nabla\right) \mathscr{U}^\nu + \nabla \mathscr{H}_\nu \left(\mathscr{N}^\nu\right) = q_\nu \left(\mathscr{E} + \mathscr{U}^\nu \times \mathscr{B}\right) + \dfrac{\Delta \mathscr{U}^\nu}{\mathscr{N}^\nu}, \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} = \mathscr{N}^e \mathscr{U}^e - \mathscr{N}^i \mathscr{U}^i, \quad \nabla \cdot \mathscr{E} = \mathscr{N}^i - \mathscr{N}^e, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3, \end{cases} \tag{1.11}$$

supplemented with the initial condition

$$(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})|_{t=0} = (\mathscr{N}^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0), \quad x \in \mathbb{R}^3, \quad \nu = e, i, \tag{1.12}$$

which also satisfies the compatibility condition (1.4).

It follows that system (1.11) is also symmetrizable hyperbolic-parabolic provided that $\mathscr{N}^\nu$ is positive. Then problem (1.11)-(1.12) admits a unique local smooth solution when the initial data are smooth.

Lemma 1.2 *(Local existence of solutions to the bipolar isentropic CNS-M system). Assume integer $s \geq 3$ and (1.4) holds. Suppose $(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0) \in H^s\left(\mathbb{R}^3\right)$ with $\mathscr{N}^{\nu 0} \geq 2\kappa$ for some given constant $\kappa > 0$. Then there exists $T_1 > 0$ such that problem (1.11)-(1.12) has a unique smooth solution satisfying $\mathscr{N}^\nu \geq \kappa$ in $[0, T_1] \times \mathbb{R}^3$ and*

$$\mathscr{U}^\nu \in C^1\left(\left([0, T_1]; H^{s-2}\left(\mathbb{R}^3\right)\right) \cap C\left(\left([0, T_1]; H^s\left(\mathbb{R}^3\right)\right)\right),\right.$$
$$(\mathscr{N}^\nu - 1, \mathscr{E}, \mathscr{B}) \in C^1\left(\left([0, T_1]; H^{s-1}\left(\mathbb{R}^3\right)\right) \cap C\left(\left([0, T_1]; H^s\left(\mathbb{R}^3\right)\right)\right).\right.$$

Next, let us state the second main results of this paper as follows.

Theorem 1.2 *(Global existence and time decay rates for the bipolar isentropic CNS-M system). Let $s \geq 4$ and (1.4) hold. Then, there are $\delta_0 > 0$ and $C_0 > 0$ such that if*

$$\left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)\right\|_{H^s(\mathbb{R}^3)} \leq \delta_0,$$

then problem (1.11)-(1.12) has a unique global solution $(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ satisfying

$$\mathscr{U}^\nu \in C^1\left([0, +\infty); H^{s-2}\left(\mathbb{R}^3\right)\right) \cap C\left([0, +\infty); H^s\left(\mathbb{R}^3\right)\right),$$
$$(\mathscr{N}^\nu - 1, \mathscr{E}, \mathscr{B}) \in C^1\left([0, +\infty); H^{s-1}\left(\mathbb{R}^3\right)\right) \cap C\left([0, +\infty); H^s\left(\mathbb{R}^3\right)\right),$$

and

$$\sup_{t \geq 0} \left\|(\mathscr{N}^\nu(t) - 1, \mathscr{U}^\nu(t), \mathscr{E}(t), \mathscr{B}(t))\right\|_{H^s(\mathbb{R}^3)} \leq C_0 \left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)\right\|_{H^s(\mathbb{R}^3)}.$$

Moreover, there exist $\delta_1 > 0$ and $C_1 > 0$ such that if

$$\left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)\right\|_{H^6(\mathbb{R}^3)} + \left\|(\mathscr{N}^{\nu 0} - 1, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)\right\|_{L^1(\mathbb{R}^3)} \leq \delta_1,$$

then the solution $(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ satisfies for any $t \geq 0$, $\forall\, 2 \leq q \leq +\infty$,

$$\left\|\mathscr{N}^\nu(t) - 1\right\|_{L^q(\mathbb{R}^3)} \leq C_1(1+t)^{-\frac{3}{4}}, \tag{1.13}$$

$$\left\|\mathscr{U}^e(t) + \mathscr{U}^i(t)\right\|_{L^q(\mathbb{R}^3)} \leq C_1(1+t)^{-\frac{3}{4}}, \tag{1.14}$$

$$\left\|\mathscr{U}^e(t) - \mathscr{U}^i(t)\right\|_{L^q(\mathbb{R}^3)} \leq C_1(1+t)^{-\frac{3}{4}+\frac{1}{4q}}(\ln(3+t))^{1-\frac{2}{q}}, \tag{1.15}$$

$$\|\mathscr{E}(t)\|_{L^q(\mathbb{R}^3)} \leq C_1(1+t)^{-\frac{3}{4}}, \tag{1.16}$$

and

$$\|\mathscr{B}(t)\|_{L^q(\mathbb{R}^3)} \leq C_1(1+t)^{-\frac{3}{4}+\frac{3}{4q}}. \tag{1.17}$$

Remark 1.7. The crucial difference between the isentropic CNS-M system and the non-isentropic CNS-M system results from the fact that the characteristic equation corresponding to (3.22) (the linear wave equation of 'total' functions for the non-isentropic system) which is of order three, while the characteristic equation of the density or velocity equations in the isentropic model is of order two (see (5.33)), and hence it is much more complex to obtain the time decay rate of the linearized systems.

Remark 1.8. It should be pointed out that due to the fact that temperatures are variant in the bipolar non-isentropic CNS-M systems, the difference of velocities of two carriers decay at the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}$ which is faster than that in (1.15) for the isentropic model, meanwhile, the magnetic field decay at the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ which is much slower than that in (1.17) for the isentropic system.

The main difficulty of studying the bipolar isentropic/non-isentropic CNS-M systems lies in the appearance of the electromagnetic fields satisfying the hyperbolic Maxwell equations. For overcoming this difficulty, we complete the proof of existence in Theorems 1.1 and 1.2 by using the elaborate energy estimates but with some new developments. The techniques of symmetrizer and the skew-symmetric dissipative structure of the bipolar models play the crucial roles in the proof (see [27–29]). The dissipative structure of the linearized equations around constant states were well analyzed in the previous works for unipolar CNS-M systems (see [4,32]). The current work focuses on the large time behaviors of global smooth solutions near constant equilibrium state for bipolar models. The key point here is to obtain the a priori estimates as follows,

$$\frac{d}{dt}\mathfrak{E}_s(\mathscr{W}(t)) + \mathfrak{D}_s(\mathscr{W}(t)) \leq 0,$$

where $\mathscr{W}(t)$ is the perturbation of solutions, and $\mathfrak{E}_s(\cdot)$, $\mathfrak{D}_s(\cdot)$ denote the energy functional and dissipation functional. Although along the same procedure, our construction of $\mathfrak{E}_s(\cdot)$ and $\mathfrak{D}_s(\cdot)$ is very different from that for the unipolar isentropic [4] or non-isentropic [32] cases because of the much more complex structure of bipolar systems. In particular, since the CNS-M systems are degenerate over some components of the whole solution, we have to construct some interactive functionals in order to capture the energy dissipation rate which plays a key role for the study of time-decay property of solutions to the nonlinear system.

Furthermore, for establishing the time decay rates of solutions in Theorems 1.1 and 1.2, we use the combination of the spectrum analysis on the linearized system and the refined energy estimates together with the Duhamel's principle. Here we recall the general theory of hypocoercivity [30] and [18] for the systematic study of hyperbolic-parabolic composite type systems, and also mention that the theoretical framework developed can not be directly applied to the bipolar isentropic/non-isentropic CNS-M systems.

In order to remove the difficulties caused by two charged carriers, we introduce the 'total functions' and 'difference functions' for unknowns. Then the linearized homogeneous equations of the bipolar non-isentropic CNS-M system (3.1) can be written as two decoupled subsystems (3.9) and (3.12). Simultaneously, the bipolar isentropic CNS-M systems (5.16) can be written as two decoupled subsystems (5.22) and (5.24). Thus, we first discuss the $L^p - L^q$ time decay rates of the linearized equations. For (3.9) or (5.22), the corresponding results can be obtained in the way as [32,4]. For (3.12) or (5.24), we first achieve the basic L^∞ estimates for the Fourier transform of the solution. And then we solve the dissipative linear wave system by the Fourier analysis methods. It should be emphasized that the Fourier transforms of 'total' functions $\hat{\zeta}^e + \hat{\zeta}^i$ and $\tilde{k} \cdot (\hat{\mathscr{U}}^e + \hat{\mathscr{U}}^i)$ satisfy the same dissipative linear wave equation which is of order three (see (3.22) and (3.46)), while the corresponding density and velocity equations in the isentropic case is of order two (see (5.31) and (5.35)), and hence it is much more complex to obtain the time decay rate of the linearized systems. This phenomena reflects that there is an essential difference between the non-isentropic and the isentropic cases.

For overcoming the difficulty arising from the non-isentropic case, we have to divide the whole space $\mathbb{R}^3$ into three parts as $\mathbb{R}^3 = \Omega_0 \cup \Omega_1 \cup \Omega_\infty$, and then study the properties of the roots of the characteristic equation corresponding to (3.22) on each domain, respectively. Lemma 3.1 shows the asymptotic behavior of roots on domains Ω_0 and Ω_∞. However, on the domain Ω_1, the main difficulty lies in the case that this characteristic equation might have multiple roots. To deal with this difficulty, we split Ω_1 into $\Omega_1 = \Omega_{1,0} \cup \Omega_{1,-} \cup \Omega_{1,+}$. On domain $\Omega_{1,0}$, it follows that both $|\mathfrak{r}_1 - (1-\tau)\mathfrak{r}_2 - \tau\mathfrak{r}_3|$ and $|(1-\tau)\mathfrak{r}_2 + \tau\mathfrak{r}_3|$ have a positive lower bound. Meanwhile, we find that $|\mathfrak{r}_2 - \mathfrak{r}_3|$ has a uniform lower bound over domains $\Omega_{1,-}$ and $\Omega_{1,+}$. Then we get $|\mathscr{R}_{11}| \le Ce^{-\gamma t}$, as $k \in \Omega_1$. Furthermore, the time frequency estimates on $(\hat{\zeta}_2, \hat{\mathscr{U}}_2, \hat{\mathscr{T}}_2)$ follow (see Lemma 3.2). On the other hand, with the help of techniques of decomposition of frequency space, we can get the estimates on $(\hat{\zeta}_2, \hat{\mathscr{U}}_2)$ in isentropic case (see Lemma 5.2).

Finally, by combining the preliminary works above, we are able to prove the large time behavior in Theorems 1.1 and 1.2. By means of the Duhamel principle, we apply the energy estimates above to the nonlinear problem satisfied by the error functions, whose solutions can be represented by the semigroup operator for the linearized problem. For the non-isentropic model, we obtain that the electric field, the densities, the total velocities and the temperatures of two charged carriers converge to the equilibrium states at the same rate $(1+t)^{-\frac{3}{4}}$. However, the difference of velocities of two charged carriers decay at the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}$, and the magnetic field decay at the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$. On the other hand, in the isentropic case, we find that the electric field, the densities and the total velocities of two charged carriers also converge to the steady states at the rate $(1+t)^{-\frac{3}{4}}$. Meanwhile, due to the fact that the temperatures are invariant, the difference of velocities of two charged carriers decay at the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ which is slower than that in the non-isentropic case, and the magnetic field decay at the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}$ which is faster than that in the non-isentropic case. This phenomenon on the charge transport shows the essential relation between the isentropic and the non-isentropic cases of the bipolar CNS-M systems.

We end this section by stating the arrangement of the rest of this paper. In Section 2, we give the detailed estimates for the proof of the global existence of solutions to the initial value problem of the bipolar non-isentropic CNS-M system. In Section 3, the linearized homogeneous systems corresponding to the bipolar non-isentropic CNS-M system are studied carefully to obtain the $L^p - L^q$ decay properties and the explicit representation of solutions. In Section 4, we study the large time behavior of solutions to the nonlinear problem of the bipolar non-isentropic CNS-M system and complete the proof of Theorem 1.1. In Section 5, we study the asymptotic decay of the bipolar isentropic CNS-M system and finish the proof of Theorem 1.2. In the last Section 6, we give our conclusion on the bipolar isentropic/non-isentropic CNS-M systems.

2. Global existence for the non-isentropic CNS-M system

2.1. Preliminaries

For later use, we split the whole space $\mathbb{R}^3$ into three parts as

$$\mathbb{R}^3 = \Omega_0 \cup \Omega_1 \cup \Omega_\infty,$$

where

$$\Omega_0 = \{k \in \mathbb{R}^3 | \ |k| < \epsilon\}, \quad \Omega_1 = \{k \in \mathbb{R}^3 | \ \epsilon \le |k| \le L\} \quad \text{and} \quad \Omega_\infty = \{k \in \mathbb{R}^3 | \ |k| > L\},$$

in which constant $\epsilon > 0$ is small enough and $L > 0$ is sufficiently large.

Next, we introduce some notations. For a constant $0 < \lambda < 1$, the expression $f \sim g$ means $\lambda g \le f \le \frac{1}{\lambda} g$. We denote by $\|\cdot\|_s$ the norm of the usual Sobolev space $H^s = H^s(\mathbb{R}^3)$, and by $\|\cdot\|$ the norm of $L^2 = L^2(\mathbb{R}^3)$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product over L^2. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, $\beta \le \alpha$ stands for $\beta_j \le \alpha_j$ for $j = 1, 2, 3$, and $\beta < \alpha$ stands for $\beta \le \alpha$ and $\beta \ne \alpha$.

And then, we recall the elementary properties of cubic equations, the Moser-type calculus inequalities and lemmas on time decay rates, which will be used in the proof of our main theorem.

Lemma 2.1 *([24]). For constants $a, b, c, d \in \mathbb{R}^1$, $a \ne 0$. The solutions of the cubic equation*

$$ax^3 + bx^2 + cx + d = 0, \tag{2.1}$$

can be represented as

$$x_k = -\frac{b}{3a} + e^{i\frac{2}{3}(k-1)\pi} \sqrt[3]{\mathfrak{S} + \sqrt{\mathfrak{R}}} + e^{i\frac{2}{3}(4-k)\pi} \sqrt[3]{\mathfrak{S} - \sqrt{\mathfrak{R}}}, \qquad k = 1, 2, 3, \tag{2.2}$$

where

$$\mathfrak{S} = \frac{bc}{6a^2} - \left(\frac{b}{3a}\right)^3 - \frac{d}{2a}, \quad \mathfrak{R} = \mathfrak{S}^2 + \left(\frac{c}{3a} - \left(\frac{b}{3a}\right)^2\right)^3. \tag{2.3}$$

Moreover, $\mathfrak{R}$ is called as the discriminant of (2.1). Indeed, the solutions of (2.1) will be one real root and two conjugate complex roots as $\mathfrak{R} > 0$, and three roots as $\mathfrak{R} < 0$. In case of $\mathfrak{R} = 0$, the solutions will be one real root with three multiples as $\mathfrak{S} = 0$, and two real roots, one of which with two multiples as $\mathfrak{S} \ne 0$.

Lemma 2.2 *(Moser-type calculus inequalities [19,20]). Let $s \geq 1$ be an integer. Suppose $\mathscr{U} \in H^s$, $\nabla \mathscr{U} \in L^\infty$ and $\mathscr{V} \in H^{s-1} \cap L^\infty$. Then for all multi-index α with $1 \leq |\alpha| \leq s$, it holds*

$$\partial^\alpha(\mathscr{U}\mathscr{V}) - \mathscr{U}\partial^\alpha \mathscr{V} \in L^2,$$

and

$$\|\partial^\alpha(\mathscr{U}\mathscr{V}) - \mathscr{U}\partial^\alpha \mathscr{V}\| \leq C_s\big(\|\nabla \mathscr{U}\|_{L^\infty}\|D^{|\alpha|-1}\mathscr{V}\| + \|D^{|\alpha|}\mathscr{U}\|\|\mathscr{V}\|_{L^\infty}\big),$$

where

$$\|D^{s'}\mathscr{U}\| = \sum_{|\alpha|=s'} \|\partial^\alpha \mathscr{U}\|.$$

Furthermore, if $s \geq 3$, then the embedding $H^{s-1} \hookrightarrow L^\infty$ is continuous and we have

$$\|\mathscr{U}\mathscr{V}\|_{s-1} \leq C_s\|\mathscr{U}\|_{s-1}\|\mathscr{V}\|_{s-1}, \quad \forall\, \mathscr{U}, \mathscr{V} \in H^{s-1},$$

and for all $\mathscr{U}, \mathscr{V} \in H^s$ and all smooth function $\mathscr{F}$,

$$\begin{aligned} \left\|\partial^\alpha \mathscr{F}(\mathscr{U})\right\| &\leq C_\infty(1 + \|\mathscr{U}\|_s)^{s-1}\|\mathscr{U}\|_s, \\ \|\partial^\alpha(\mathscr{U}\mathscr{V}) - \mathscr{U}\partial^\alpha \mathscr{V}\| &\leq C_s\|\mathscr{U}\|_s\|\mathscr{V}\|_{s-1}, \quad \forall\, |\alpha| \leq s. \end{aligned}$$

Lemma 2.3 *(see Lemma 2.1 in [4]). Assume that for any initial data $\mathscr{W}^0$, the linear homogeneous solution $\mathscr{W}(t) = \mathscr{A}(t)\mathscr{W}^0$ obeys the point-wise estimate*

$$\left|\hat{\mathscr{W}}(t,k)\right| \leq Ce^{-\phi(k)t}\left|\hat{\mathscr{W}}^0(k)\right|,$$

for all $t \geq 0$, $k \in \mathbb{R}^3$, where $\phi(k)$ is a strictly positive, continuous and real-valued function over $k \in \mathbb{R}^3$ and satisfies

$$\phi(k) = \begin{cases} O(1)|k|^{\sigma_+} & as\ |k| \to 0, \\ O(1)|k|^{\sigma_-} & as\ |k| \to \infty, \end{cases}$$

for two constants $\sigma_- > \sigma_+ > 0$. Let $j \geq 0$ be an integer, $1 \leq p, r \leq 2 \leq q \leq \infty$ and $l \geq 0$. Then, $\mathscr{W}(t) = \mathscr{A}(t)\mathscr{W}^0$ obeys the time-decay estimate

$$\left\|\nabla^j \mathscr{W}(t,k)\right\|_{L^q} \leq C(1+t)^{-\frac{3}{\sigma_+}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{\sigma_-}}\left\|\mathscr{W}^0(k)\right\|_{L^p} + C(1+t)^{-\frac{l}{\sigma_-}}\left\|\nabla^{\mathscr{S}(j+l,s,q)}\mathscr{W}^0(k)\right\|_{L^r},$$

for any $t \geq 0$, where and in the sequel, $\mathscr{S}(j,r,q)$ is defined by

$$\mathscr{S}(j,r,q)=\begin{cases}0, & \textit{if } j+3\left(\frac{1}{r}-\frac{1}{q}\right)<0,\\ j, & \textit{if } j\geq 0, r=q=2, \textit{ and } j \textit{ is integer},\\ \left[j+3\left(\frac{1}{r}-\frac{1}{q}\right)\right]+1, & \textit{otherwise}.\end{cases}$$

Lemma 2.4 *(see Lemmas 2.3, 2.4 and 2.6 in [32]). For any integer $j, m \geq 0$, $1 \leq p, r \leq 2 \leq q \leq +\infty$, if $|\hat{\mathscr{W}}(t,k)| \leq |k|^m e^{-O(1)t}|\hat{\mathscr{W}}_0(k)|1_{\Omega_0}$, there exists a constant $\gamma > 0$ such that*

$$\|\nabla^j \mathscr{W}\|_{L^q} \leq Ce^{-\gamma t}\|\mathscr{W}_0\|_{L^p}.$$

And if $|\hat{\mathscr{W}}(t,k)| \leq |k|^m e^{-O(1)|k|^2 t}|\hat{\mathscr{W}}_0(k)|1_{\Omega_0}$, then

$$\|\nabla^j \mathscr{W}\|_{L^q} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{j+m}{2}}\|\mathscr{W}_0\|_{L^p}.$$

Moreover, if $|\hat{\mathscr{W}}(t,k)| \leq |k|^{-m} e^{-O(1)t}|\hat{\mathscr{W}}_0(k)|1_{\Omega_\infty}$, then

$$\|\nabla^j \mathscr{W}\|_{L^q} \leq Ce^{-\gamma t}\|\nabla^{\mathscr{S}(j-m,r,q)}\mathscr{W}_0\|_{L^r}.$$

2.2. *Reformulation of problem* (1.2)-(1.3)

Now, let $(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathcal{T}^\nu, \mathscr{E}, \mathscr{B})$ be a local smooth solution to the initial value problem for system (1.2) with initial data (1.3) satisfying (1.4). Set

$$\mathscr{N}^\nu = 1+\zeta^\nu, \quad \mathcal{T}^\nu = 1+\mathscr{T}^\nu, \quad \nu = e, i. \tag{2.4}$$

Then problem (1.2)-(1.4) can be rewritten as

$$\begin{cases}\partial_t \zeta^\nu + \nabla\cdot\left(\left(1+\zeta^\nu\right)\mathscr{U}^\nu\right)=0,\\ \partial_t \mathscr{U}^\nu + \left(\mathscr{U}^\nu\cdot\nabla\right)\mathscr{U}^\nu + \frac{1}{1+\zeta^\nu}\nabla\left(\left(1+\zeta^\nu\right)\left(1+\mathscr{T}^\nu\right)\right) = q_\nu\left(\mathscr{E}+\mathscr{U}^\nu\times\mathscr{B}\right)+\frac{\Delta\mathscr{U}^\nu}{1+\zeta^\nu},\\ \partial_t\mathscr{T}^\nu + \frac{2}{3}(1+\mathscr{T}^\nu)\nabla\cdot\mathscr{U}^\nu + \mathscr{U}^\nu\cdot\nabla\mathscr{T}^\nu + \frac{1}{3}\left|\mathscr{U}^\nu\right|^2 = -\mathscr{T}^\nu,\\ \partial_t\mathscr{E} - \nabla\times\mathscr{B} - \mathscr{U}^e + \mathscr{U}^i = \zeta^e\mathscr{U}^e - \zeta^i\mathscr{U}^i, \quad \nabla\cdot\mathscr{E} = \zeta^i - \zeta^e,\\ \partial_t\mathscr{B} + \nabla\times\mathscr{E} = 0, \quad \nabla\cdot\mathscr{B} = 0, \quad (t,x)\in[0,+\infty)\times\mathbb{R}^3,\end{cases} \tag{2.5}$$

with the initial condition

$$\mathscr{W}|_{t=0} = \mathscr{W}^0 := (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0), \quad x\in\mathbb{R}^3, \quad \nu = e, i, \tag{2.6}$$

which satisfies the compatibility condition

$$\nabla\cdot\mathscr{E}^0=\zeta^{i0}-\zeta^{e0},\quad \nabla\cdot\mathscr{B}^0=0,\quad x\in\mathbb{R}^3. \tag{2.7}$$

Here, $\zeta^{\nu 0}=\mathscr{N}^{\nu 0}-1$ and $\mathscr{T}^{\nu 0}=\mathcal{T}^{\nu 0}-1$.

For proving Theorem 1.1, we always assume $s\geq 4$. Moreover, for $\mathscr{W}=(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu,\mathscr{E},\mathscr{B})$, we also use $\mathfrak{E}_s(\mathscr{W}(t))$, $\mathfrak{E}_s^h(\mathscr{W}(t))$, $\mathfrak{D}_s(\mathscr{W}(t))$ and $\mathfrak{D}_s^h(\mathscr{W}(t))$ to denote the energy functionals, the higher order energy functionals, the dissipative functionals and the higher order dissipative functionals for two charged carriers. They satisfy

$$\mathfrak{E}_s(\mathscr{W}(t))\sim\sum_{\nu=e,i}\left\|(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu)\right\|_s^2+\|(\mathscr{E},\mathscr{B})\|_s^2, \tag{2.8}$$

$$\mathfrak{E}_s^h(\mathscr{W}(t))\sim\sum_{\nu=e,i}\left\|\nabla(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu)\right\|_{s-1}^2+\|\nabla(\mathscr{E},\mathscr{B})\|_{s-1}^2, \tag{2.9}$$

$$\mathfrak{D}_s(\mathscr{W}(t))\sim\sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2+\|\nabla\mathscr{U}^\nu\|_s^2+\|\mathscr{T}^\nu\|_s^2\right)+\|\nabla\mathscr{E}\|_{s-2}^2+\left\|\nabla^2\mathscr{B}\right\|_{s-3}^2+\left\|\zeta^e-\zeta^i\right\|^2 \tag{2.10}$$

and

$$\begin{aligned}\mathfrak{D}_s^h(\mathscr{W}(t))\sim\sum_{\nu=e,i}&\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right)+\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2\\&+\left\|\nabla^3\mathscr{B}\right\|_{s-4}^2+\left\|\nabla(\zeta^e-\zeta^i)\right\|^2,\end{aligned} \tag{2.11}$$

respectively. Now, concerning the transformed initial value problem (2.5)-(2.6), we obtain the global existence result as follows.

Proposition 2.1. *Assume that $\mathscr{W}^0=(\zeta^{\nu 0},\mathscr{U}^{\nu 0},\mathscr{T}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0)$ satisfies the compatibility condition (2.7). Then, if $\mathfrak{E}_s(\mathscr{W}^0)$ is small enough, the initial value problem (2.5)-(2.6) admits a unique global solution $\mathscr{W}=(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu,\mathscr{E},\mathscr{B})$ which satisfies*

$$\begin{aligned}\mathscr{U}^\nu&\in C^1\left(\mathbb{R}^+;H^{s-2}\left(\mathbb{R}^3\right)\right)\cap C\left(\mathbb{R}^+;H^s\left(\mathbb{R}^3\right)\right),\\(\zeta^\nu,\ \mathscr{T}^\nu,\mathscr{E},\mathscr{B})&\in C^1\left(\mathbb{R}^+;H^{s-1}\left(\mathbb{R}^3\right)\right)\cap C\left(\mathbb{R}^+;H^s\left(\mathbb{R}^3\right)\right),\end{aligned} \tag{2.12}$$

and

$$\mathfrak{E}_s(\mathscr{W}(t))+\int_0^t\mathfrak{D}_s(\mathscr{W}(\tau))d\tau\leq\mathfrak{E}_s(\mathscr{W}^0),\quad\forall\, t\geq 0. \tag{2.13}$$

Based on Proposition 2.1, it is standard to prove the existence result in Theorem 1.1. Moreover, solutions of Proposition 2.1 may decay under some extra conditions on $\mathscr{W}^0=(\zeta^{\nu 0},\mathscr{U}^{\nu 0},\mathscr{T}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0)$. For that, we introduce the definition of $\omega_{s_*}(\mathscr{W}^0)$ as:

$$\omega_{s_*}(\cdot) = \|\cdot\|_{s_*} + \|\cdot\|_{L^1}, \tag{2.14}$$

for $s_* \geq 4$. Then, the time decay results we obtained are as follows.

Proposition 2.2. *Let $\mathscr{W} = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the initial value problem (2.5)-(2.6) with initial data $\mathscr{W}^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 2.1. Then, if $\omega_{s+1}(\mathscr{W}^0)$ is sufficiently small,*

$$\|\mathscr{W}(t)\|_s \leq C\omega_{s+1}(\mathscr{W}^0)(1+t)^{-\frac{3}{8}}, \quad \forall\, t \geq 0. \tag{2.15}$$

Moreover, if $\omega_{s+5}(\mathscr{W}^0)$ is sufficiently small, then, the solution also satisfies

$$\|\nabla\mathscr{W}(t)\|_{s-1} \leq C\omega_{s+5}(\mathscr{W}^0)(1+t)^{-\frac{5}{8}}, \quad \forall\, t \geq 0. \tag{2.16}$$

Thus, decay rates (1.5)-(1.9) in Theorem 1.1 follow by combining the bootstrap method and Proposition 2.2.

2.3. Global existence

In this subsection, we give the proof of Proposition 2.1 for the global existence and uniqueness of solutions to the initial value problem (2.5)-(2.6). Due to the local existence results in Lemma 1.1, the global existence of solutions satisfying (2.12) and (2.13) follows by combining the standard continuity argument and the a priori estimate as follows.

Theorem 2.1 *(The a priori estimate for the bipolar non-isentropic CNS-M system). For any $t \in (0, T)$ with $T > 0$, suppose $\mathscr{W} = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B})$ to be the solution to the initial value problem (2.5)-(2.6), in which $(\zeta^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B}) \in C^1\left([0, T]; H^{s-1}\right) \cap C\left([0, T]; H^s\right)$ and $\mathscr{U}^\nu \in C^1\left([0, T]; H^{s-2}\right) \cap C\left([0, T]; H^s\right)$. Then, if*

$$\sup_{0 \leq t \leq T} \|\mathscr{W}(t)\|_s \leq \delta_0 \tag{2.17}$$

with δ_0 sufficiently small, there exist $\mathfrak{E}_s(\cdot)$ and $\mathfrak{D}_s(\cdot)$ in the form of (2.8) and (2.10) such that for any $0 \leq t \leq T$,

$$\frac{d}{dt}\mathfrak{E}_s(\mathscr{W}(t)) + \mathfrak{D}_s(\mathscr{W}(t)) \leq C\mathfrak{E}_s(\mathscr{W}(t))^{\frac{1}{2}}\mathfrak{D}_s(\mathscr{W}(t)). \tag{2.18}$$

Proof. We will use five steps to complete the proof. Step 1 is to estimate the bipolar non-isentropic CNS part and the Maxwell part of system (1.2), respectively. And then we obtain the dissipative estimates for $\nabla\mathscr{U}^\nu$ and $\mathscr{T}^\nu$. Steps 2-4 are to establish the dissipative estimates for ζ^ν, $\mathscr{E}$ and $\mathscr{B}$ by using the skew-symmetric structure of the bipolar non-isentropic CNS-M system.

Step 1. It holds that

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{|\alpha|\le s}\sum_{\nu=e,i}\left\langle\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)\partial^\alpha\mathscr{V}^\nu,\partial^\alpha\mathscr{V}^\nu\right\rangle+\|\mathscr{E}\|_s^2+\ \|\mathscr{B}\|_s^2\right)\\&+2\sum_{\nu=e,i}\left\|\nabla\mathscr{U}^\nu\right\|_s^2+3\sum_{|\alpha|\le s}\sum_{\nu=e,i}\left\langle\frac{\mathscr{N}^\nu}{\mathcal{T}^\nu},\left|\partial^\alpha\mathscr{T}^\nu\right|^2\right\rangle\\&\le C\,\|\mathscr{W}\|_s\sum_{\nu=e,i}\left(\left\|\nabla\zeta^\nu\right\|_{s-1}^2+\left\|\nabla\mathscr{U}^\nu\right\|_s^2+\left\|\mathscr{T}^\nu\right\|_s^2\right),\end{aligned}\tag{2.19}$$

where

$$\mathscr{V}^\nu=\left(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu\right)^T,\quad \mathscr{V}=\left(\mathscr{V}^e,\mathscr{V}^i\right)^T,\quad \mathscr{W}=(\mathscr{V},\mathscr{E},\mathscr{B})^T$$

and

$$\mathscr{A}_0^\nu=\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)=\begin{pmatrix}\frac{\mathcal{T}^\nu}{\mathscr{N}^\nu}&0&0\\0&\mathscr{N}^\nu\mathrm{I}_3&0\\0&0&\frac{3}{2}\frac{\mathscr{N}^\nu}{\mathcal{T}^\nu}\end{pmatrix}.\tag{2.20}$$

Here and in the rest of this paper, I_3 is the 3×3 unit matrix and we use $(\cdot)^T$ to denote the transpose of $(\cdot)$.

In fact, the bipolar non-isentropic CNS equations in (2.5) can be rewritten as

$$\partial_t\mathscr{V}^\nu+\sum_{j=1}^3\mathscr{A}_j^\nu\left(\mathscr{N}^\nu,\mathscr{U}^\nu,\mathcal{T}^\nu\right)\partial_j\mathscr{V}^\nu=\mathscr{K}_I^\nu(\mathscr{W})+\mathscr{K}_{II}^\nu\left(\mathscr{V}^\nu\right),\tag{2.21}$$

where

$$\mathscr{A}_j^\nu\left(\mathscr{N}^\nu,\mathscr{U}^\nu,\mathcal{T}^\nu\right)=\begin{pmatrix}\mathscr{U}_j^\nu&\mathscr{N}^\nu e_j^T&0\\\frac{\mathcal{T}^\nu}{\mathscr{N}^\nu}e_j&\mathscr{U}_j^\nu\mathrm{I}_3&e_j\\0&\frac{2}{3}\mathcal{T}^\nu e_j^T&\mathscr{U}_j^\nu\end{pmatrix},\ j=1,2,3,\tag{2.22}$$

and

$$\mathscr{K}_I^\nu(\mathscr{W})=\begin{pmatrix}0\\q_\nu\left(\mathscr{E}+\mathscr{U}^\nu\times\mathscr{B}\right)\\-\frac{1}{3}\left|\mathscr{U}^\nu\right|^2\end{pmatrix},\quad \mathscr{K}_{II}^\nu\left(\mathscr{V}^\nu\right)=\begin{pmatrix}0\\\frac{\Delta\mathscr{U}^\nu}{\mathscr{N}^\nu}\\-\mathscr{T}^\nu\end{pmatrix},\tag{2.23}$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.

For $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq s$. Applying ∂^α to (2.21) and multiplying the resulting equations by the symmetrizer matrix $\mathscr{A}_0^\nu$, we have

$$\mathscr{A}_0^\nu \partial_t \partial^\alpha \mathscr{V}^\nu + \sum_{j=1}^{3} \tilde{\mathscr{A}}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) \partial_j \partial^\alpha \mathscr{V}^\nu = \mathscr{A}_0^\nu \partial^\alpha \left(\mathscr{K}_I^\nu (\mathscr{W}) + \mathscr{K}_{II}^\nu \left(\mathscr{V}^\nu\right)\right) + \mathscr{J}_\alpha^\nu, \tag{2.24}$$

where

$$\tilde{\mathscr{A}}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) = \begin{pmatrix} \frac{\mathscr{T}^\nu}{\mathscr{N}^\nu} \mathscr{U}_j^\nu & \mathscr{T}^\nu e_j^T & 0 \\ \mathscr{T}^\nu e_j & \mathscr{N}^\nu \mathscr{U}_j^\nu \mathrm{I}_3 & \mathscr{N}^\nu e_j \\ 0 & \mathscr{N}^\nu e_j^T & \frac{3}{2} \frac{\mathscr{N}^\nu}{\mathscr{T}^\nu} \mathscr{U}_j^\nu \end{pmatrix}, \quad j = 1, 2, 3, \tag{2.25}$$

and

$$\mathscr{J}_\alpha^\nu = -\sum_{j=1}^{3} \mathscr{A}_0^\nu \left(\mathscr{N}^\nu, \mathscr{T}^\nu\right) \left(\partial^\alpha \left(\mathscr{A}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) \partial_j \mathscr{V}^\nu\right) - \mathscr{A}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) \partial^\alpha \partial_j \mathscr{V}^\nu\right). \tag{2.26}$$

By Lemma 2.2 and the Cauchy-Schwarz inequality, we get

$$\left\| \mathscr{J}_\alpha^\nu \right\| \leq C \left\| \mathscr{A}_0^\nu \left(\mathscr{N}^\nu, \mathscr{T}^\nu\right) \right\|_{L^\infty} \left(\left\| \nabla \mathscr{A}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) \right\|_{s-1}^2 + \left\| \partial_j \mathscr{V}^\nu \right\|_{s-1}^2 \right) \leq C \left\| \nabla \mathscr{V}^\nu \right\|_{s-1}^2. \tag{2.27}$$

Taking the inner product of (2.24) with $2\partial^\alpha \mathscr{V}^\nu$ in L^2, we obtain

$$\begin{aligned} \frac{d}{dt} \left\langle \mathscr{A}_0^\nu \left(\mathscr{N}^\nu, \mathscr{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{V}^\nu \right\rangle =& 2\left\langle \mathscr{J}_\alpha^\nu, \partial^\alpha \mathscr{V}^\nu \right\rangle + \left\langle \operatorname{div} \mathscr{A}^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{V}^\nu \right\rangle \\ &+ 2\left\langle \mathscr{A}_0^\nu \left(\mathscr{N}^\nu, \mathscr{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \left(\mathscr{K}_I^\nu (\mathscr{W}) + \mathscr{K}_{II}^\nu \left(\mathscr{V}^\nu\right)\right) \right\rangle, \end{aligned} \tag{2.28}$$

where

$$\operatorname{div} \mathscr{A}^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right) = \partial_t \mathscr{A}_0^\nu \left(\mathscr{N}^\nu, \mathscr{T}^\nu\right) + \sum_{j=1}^{3} \partial_j \tilde{\mathscr{A}}_j^\nu \left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu\right). \tag{2.29}$$

Obviously, it follows from the density equation and the temperature equation in (2.5) that

$$\left\| \partial_t \zeta^\nu \right\| \leq C \left\| \nabla \mathscr{U}^\nu \right\|_1, \quad \left\| \partial_t \zeta^\nu \right\|_{L^\infty} \leq C \left\| \nabla \mathscr{U}^\nu \right\|_2 \tag{2.30}$$

and

$$\left\|\partial_t \mathscr{T}^\nu\right\| \leq C\left(\left\|\nabla \mathscr{U}^\nu\right\|_1+\left\|\mathscr{T}^\nu\right\|_1\right), \quad\left\|\partial_t \mathscr{T}^\nu\right\|_{L^{\infty}} \leq C\left(\left\|\nabla \mathscr{U}^\nu\right\|_2+\left\|\mathscr{T}^\nu\right\|_3\right). \tag{2.31}$$

The above inequalities together with (2.20) and (2.22) imply

$$\left\|\operatorname{div} \mathscr{A}^\nu\left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathcal{T}^\nu\right)\right\|_{L^{\infty}} \leq C\left\|\mathscr{V}^\nu\right\|_s. \tag{2.32}$$

Now, we estimate each term on the right hand side of (2.28). For the first two terms, by noticing (2.26), when $|\alpha|=0$, we get $\mathscr{J}_\alpha^\nu=0$. This together with (2.29)-(2.31), we obtain

$$\begin{aligned}
& 2\left\langle\mathscr{J}_\alpha^\nu, \partial^\alpha \mathscr{V}^\nu\right\rangle+\left\langle\operatorname{div} \mathscr{A}^\nu\left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathcal{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{V}^\nu\right\rangle \\
=& \left\langle\frac{\left|\zeta^\nu\right|^2}{\mathscr{N}^\nu}, \partial_t \mathscr{T}^\nu\right\rangle+2\left\langle\frac{\mathcal{T}^\nu}{\mathscr{N}^\nu} \nabla \cdot \mathscr{U}^\nu,\left|\zeta^\nu\right|^2\right\rangle+2\left\langle\mathscr{U}^\nu \cdot \nabla \zeta^\nu, \mathscr{T}^\nu\right\rangle+\left\langle\frac{\mathscr{U}^\nu \cdot \nabla \mathscr{T}^\nu}{\mathscr{N}^\nu},\left|\zeta^\nu\right|^2\right\rangle \\
& +2\left\langle\mathscr{U}^\nu \cdot \nabla \mathscr{T}^\nu, \zeta^\nu\right\rangle-\frac{3}{2}\left\langle\frac{\mathscr{N}^\nu}{\left|\mathcal{T}^\nu\right|^2}\left|\mathscr{T}^\nu\right|^2, \partial_t \mathscr{T}^\nu\right\rangle-\frac{3}{2}\left\langle\frac{\mathscr{N}^\nu}{\left|\mathcal{T}^\nu\right|^2}\left|\mathscr{T}^\nu\right|^2, \mathscr{U}^\nu \cdot \nabla \mathscr{T}^\nu\right\rangle \\
\leq & C\left\|\zeta^\nu\right\|_s\left(\left\|\nabla \zeta^\nu\right\|_1^2+\left\|\nabla \mathscr{U}^\nu\right\|_1^2+\left\|\mathscr{T}^\nu\right\|_1^2\right)+C\left\|\zeta^\nu\right\|_s\left(\left\|\nabla \zeta^\nu\right\|_1^2+\left\|\nabla \mathscr{U}^\nu\right\|^2\right) \\
& +C\left\|\mathscr{V}^\nu\right\|_s\left(\left\|\nabla \zeta^\nu\right\|^2+\left\|\nabla \mathscr{U}^\nu\right\|_1^2\right)+C\left\|\mathscr{V}^\nu\right\|_s\left(\left\|\nabla \zeta^\nu\right\|_1^2+\left\|\nabla \mathscr{U}^\nu\right\|_1^2+\left\|\mathscr{T}^\nu\right\|_1^2\right) \\
& +C\left\|\mathscr{V}^\nu\right\|_s\left(\left\|\nabla \mathscr{T}^\nu\right\|^2+\left\|\nabla \mathscr{U}^\nu\right\|_1^2\right)+C\left\|\mathscr{V}^\nu\right\|_s\left\|\mathscr{T}^\nu\right\|^2+C\left\|\mathscr{V}^\nu\right\|_s\left(\left\|\nabla \mathscr{U}^\nu\right\|_1^2+\left\|\mathscr{T}^\nu\right\|_2^2\right) \\
\leq & C\left\|\mathscr{V}^\nu\right\|_s\left(\left\|\nabla \zeta^\nu\right\|_1^2+\left\|\nabla \mathscr{U}^\nu\right\|_2^2+\left\|\mathscr{T}^\nu\right\|_2^2\right), \quad \text{as} \quad |\alpha|=0.
\end{aligned} \tag{2.33}$$

And when $1 \leq|\alpha| \leq s$, by (2.27) and (2.32), we have

$$2\left\langle\mathscr{J}_\alpha^\nu, \partial^\alpha \mathscr{V}^\nu\right\rangle+\left\langle\operatorname{div} \mathscr{A}^\nu\left(\mathscr{N}^\nu, \mathscr{U}^\nu, \mathcal{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{V}^\nu\right\rangle \leq C\left\|\mathscr{V}^\nu\right\|_s\left\|\nabla \mathscr{V}^\nu\right\|_{s-1}^2. \tag{2.34}$$

For the last term, by Lemma 2.2 and (2.23), we get

$$\begin{aligned}
& \left\langle\mathscr{A}_0^\nu\left(\mathscr{N}^\nu, \mathcal{T}^\nu\right) \partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{K}_I^\nu(\mathscr{W})\right\rangle \\
= & q_\nu\left\langle\partial^\alpha\left(\mathscr{N}^\nu \mathscr{U}^\nu\right), \partial^\alpha \mathscr{E}\right\rangle-\frac{1}{2}\left\langle\frac{\mathscr{N}^\nu}{\mathcal{T}^\nu} \partial^\alpha \mathscr{T}^\nu, \mathscr{U}^\nu \partial^\alpha \mathscr{U}^\nu\right\rangle \\
& +q_\nu \sum_{\beta<\alpha} C_\alpha^\beta\left(\left\langle\mathscr{N}^\nu \partial^\alpha \mathscr{U}^\nu, \partial^\beta \mathscr{U}^\nu \times \partial^{\alpha-\beta} \mathscr{B}\right\rangle-\left\langle\partial^{\alpha-\beta} \zeta^\nu \partial^\beta \mathscr{U}^\nu, \partial^\alpha \mathscr{E}\right\rangle\right) \\
& -\frac{1}{2} \sum_{\beta<\alpha} C_\alpha^\beta\left\langle\frac{\mathscr{N}^\nu}{\mathcal{T}^\nu} \partial^\alpha \mathscr{T}^\nu, \partial^{\alpha-\beta} \mathscr{U}^\nu \partial^\beta \mathscr{U}^\nu\right\rangle \\
\leq & q_\nu\left\langle\partial^\alpha\left(\mathscr{N}^\nu \mathscr{U}^\nu\right), \partial^\alpha \mathscr{E}\right\rangle+C\left\|\left(\mathscr{T}^\nu, \mathscr{E}, \mathscr{B}\right)\right\|_s\left\|\nabla \mathscr{V}^\nu\right\|_{s-1}^2
\end{aligned} \tag{2.35}$$

and

$$
\begin{aligned}
&\left\langle \mathscr{A}_0^\nu\left(\mathcal{N}^\nu, \mathcal{T}^\nu\right)\partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{K}_{II}^\nu\left(\mathscr{V}^\nu\right)\right\rangle \\
=&\left\langle \partial^\alpha \mathscr{U}^\nu, \partial^\alpha \Delta \mathscr{U}^\nu\right\rangle + \sum_{\beta<\alpha} C_\alpha^\beta \left\langle \mathcal{N}^\nu \partial^\alpha \mathscr{U}^\nu, \partial^{\alpha-\beta}\left(\frac{1}{\mathcal{N}^\nu}\right)\partial^\beta \Delta \mathscr{U}^\nu\right\rangle - \frac{3}{2}\left\langle \frac{\mathcal{N}^\nu}{\mathcal{T}^\nu}, \left|\partial^\alpha \mathscr{T}^\nu\right|^2\right\rangle \\
\leq& -\left\|\partial^\alpha \nabla \mathscr{U}^\nu\right\|^2 - \frac{3}{2}\left\langle \frac{\mathcal{N}^\nu}{\mathcal{T}^\nu}, \left|\partial^\alpha \mathscr{T}^\nu\right|^2\right\rangle + C\left\|\mathscr{W}\right\|_s \sum_{\nu=e,i}\left(\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + \left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right).
\end{aligned}
\tag{2.36}
$$

Then it follows from (2.28) and (2.33)-(2.36) that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\nu=e,i}\left\langle \mathscr{A}_0^\nu\left(\mathcal{N}^\nu, \mathcal{T}^\nu\right)\partial^\alpha \mathscr{V}^\nu, \partial^\alpha \mathscr{V}^\nu\right\rangle + 2\sum_{\nu=e,i}\left(\left\|\partial^\alpha \nabla \mathscr{U}^\nu\right\|^2 + \frac{3}{2}\left\langle \frac{\mathcal{N}^\nu}{\mathcal{T}^\nu}, \left|\partial^\alpha \mathscr{T}^\nu\right|^2\right\rangle\right) \\
\leq& 2\left\langle \partial^\alpha\left(\mathcal{N}^i \mathscr{U}^i - \mathcal{N}^e \mathscr{U}^e\right), \partial^\alpha \mathscr{E}\right\rangle + C\left\|\mathscr{W}\right\|_s \sum_{\nu=e,i}\left(\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + \left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right).
\end{aligned}
\tag{2.37}
$$

On the other hand, for $|\alpha| \leq s$, standard energy estimates on $\partial^\alpha \mathscr{E}$ and $\partial^\alpha \mathscr{B}$ from (2.5) yield

$$
\frac{d}{dt}\left(\left\|\partial^\alpha \mathscr{E}\right\|^2 + \left\|\partial^\alpha \mathscr{B}\right\|^2\right) = 2\left\langle \partial^\alpha\left(\mathcal{N}^e \mathscr{U}^e - \mathcal{N}^i \mathscr{U}^i\right), \partial^\alpha \mathscr{E}\right\rangle. \tag{2.38}
$$

Then, combining (2.37) and (2.38), summing the resulting inequalities over $|\alpha| \leq s$ yields (2.19).

Step 2. It holds that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq s-1}\sum_{\nu=e,i}\left\langle \partial^\alpha \mathscr{U}^\nu, \partial^\alpha \nabla \zeta^\nu\right\rangle + c_0 \sum_{\nu=e,i}\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + c_0\left\|\zeta^e - \zeta^i\right\|^2 \\
\leq& C\sum_{\nu=e,i}\left(\left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right) + C\left\|\mathscr{W}\right\|_s \sum_{\nu=e,i}\left(\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + \left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right),
\end{aligned}
\tag{2.39}
$$

where c_0 is a positive constant.

Step 3. It holds that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{1\leq|\alpha|\leq s-1}\left\langle \partial^\alpha\left(\mathscr{U}^e - \mathscr{U}^i\right), \partial^\alpha \mathscr{E}\right\rangle + c_0\left\|\nabla \mathscr{E}\right\|_{s-2}^2 \\
\leq& C\left\|\mathscr{W}\right\|_s\left(\sum_{\nu=e,i}\left(\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + \left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right) + \left\|\nabla \mathscr{E}\right\|_{s-2}^2\right) + \varepsilon\left\|\nabla^2 \mathscr{B}\right\|_{s-3}^2 \\
&+ C\sum_{\nu=e,i}\left(\left\|\nabla \zeta^\nu\right\|_{s-1}^2 + \left\|\nabla \mathscr{U}^\nu\right\|_s^2 + \left\|\mathscr{T}^\nu\right\|_s^2\right),
\end{aligned}
\tag{2.40}
$$

where $\varepsilon > 0$ is a small constant to be chosen later.

Step 4. It holds that

$$\begin{aligned}&\frac{d}{dt}\sum_{1\le|\alpha|\le s-2}\left\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\right\rangle+c_0\left\|\nabla^2\mathscr{B}\right\|_{s-3}^2\\&\le C\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2+C\sum_{\nu=e,i}\|\nabla\mathscr{U}^\nu\|_{s-3}^2+C\|\mathscr{B}\|_s\sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2+\|\nabla\mathscr{U}^\nu\|_{s-1}^2\right).\end{aligned}\tag{2.41}$$

Step 5. Based on the steps above, we establish (2.18). We define the energy functionals as

$$\begin{aligned}\mathfrak{E}_s(\mathscr{W}(t))=&\sum_{|\alpha|\le s}\sum_{\nu=e,i}\left\langle\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)\partial^\alpha\mathscr{V}^\nu,\partial^\alpha\mathscr{V}^\nu\right\rangle+\|\mathscr{E}\|_s^2+\|\mathscr{B}\|_s^2\\&+\mathfrak{K}_1\sum_{|\alpha|\le s-1}\sum_{\nu=e,i}\left\langle\partial^\alpha\mathscr{U}^\nu,\nabla\partial^\alpha\zeta^\nu\right\rangle+\mathfrak{K}_2\sum_{1\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle\\&+\mathfrak{K}_3\sum_{1\le|\alpha|\le s-2}\left\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\right\rangle,\end{aligned}$$

where constants $0<\mathfrak{K}_3\ll\mathfrak{K}_2\ll\mathfrak{K}_1\ll1$ are to be chosen later. It follows from the fact that $\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)$ is positive definite that

$$\mathfrak{E}_s(\mathscr{W}(t))\sim||\mathscr{W}||_s^2,$$

as long as $0<\mathfrak{K}_j\ll1$, $j=1,2,3$, are sufficiently small. Furthermore, by letting $\varepsilon>0$ and $0<\mathfrak{K}_3\ll\mathfrak{K}_2\ll\mathfrak{K}_1\ll1$ be sufficiently small with $2\varepsilon\mathfrak{K}_2\ll\mathfrak{K}_3$, summing (2.19), (2.39)$\times\mathfrak{K}_1$, (2.40)$\times\mathfrak{K}_2$ and (2.41)$\times\mathfrak{K}_3$, we get (2.18). The proof of Theorem 2.1 is finished. □

3. Linearized homogeneous equations of the non-isentropic CNS-M system

In this section, in order to obtain the time decay properties of solutions for the nonlinear systems (2.5), we study the decay properties of solutions for the linearized systems.

3.1. Linearized homogeneous systems

Now, we rewrite system (2.5) as

$$\begin{cases}\partial_t\zeta^\nu+\nabla\cdot\mathscr{U}^\nu=\mathscr{G}_{1\nu},\\\partial_t\mathscr{U}^\nu+\nabla\zeta^\nu+\nabla\mathscr{T}^\nu-q_\nu\mathscr{E}-\Delta\mathscr{U}^e=\mathscr{G}_{2\nu},\\\partial_t\mathscr{T}^\nu+\dfrac{2}{3}\nabla\cdot\mathscr{U}^\nu+\mathscr{T}^\nu=\mathscr{G}_{3\nu},\\\partial_t\mathscr{E}-\nabla\times\mathscr{B}-\mathscr{U}^e+\mathscr{U}^i=\dfrac{1}{2}(\mathscr{G}_{4e}-\mathscr{G}_{4i}),\quad\nabla\cdot\mathscr{E}=\zeta^i-\zeta^e,\\\partial_t\mathscr{B}+\nabla\times\mathscr{E}=0,\quad\nabla\cdot\mathscr{B}=0,\quad(t,x)\in[0,+\infty)\times\mathbb{R}^3,\end{cases}\tag{3.1}$$

where

$$\begin{cases} \mathscr{G}_{1\nu} = -\nabla \cdot \left(\zeta^{\nu} \mathscr{U}^{\nu}\right), \\ \mathscr{G}_{2\nu} = -\left(\mathscr{U}^{\nu} \cdot \nabla\right) \mathscr{U}^{\nu} + \dfrac{\zeta^{\nu} - \mathscr{T}^{\nu}}{1+\zeta^{\nu}} \nabla \zeta^{\nu} + q_{\nu} \left(\mathscr{U}^{\nu} \times \mathscr{B}\right) - \dfrac{\zeta^{\nu} \Delta \mathscr{U}^{\nu}}{1+\zeta^{\nu}}, \\ \mathscr{G}_{3\nu} = -\dfrac{2}{3} \mathscr{T}^{\nu} \nabla \cdot \mathscr{U}^{\nu} - \mathscr{U}^{\nu} \cdot \nabla \mathscr{T}^{\nu} - \dfrac{1}{3} \left|\mathscr{U}^{\nu}\right|^{2}, \\ \mathscr{G}_{4\nu} = 2\zeta^{\nu} \mathscr{U}^{\nu}, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^{3}. \end{cases} \tag{3.2}$$

Next, we introduce

$$\zeta_1 = \frac{\zeta^e - \zeta^i}{2}, \quad \mathscr{U}_1 = \frac{\mathscr{U}^e - \mathscr{U}^i}{2} \quad \text{and} \quad \mathscr{T}_1 = \frac{\mathscr{T}^e - \mathscr{T}^i}{2}. \tag{3.3}$$

Then, from system (3.1), $\mathscr{W}_1 = (\zeta_1,\ \mathscr{U}_1,\ \mathscr{T}_1,\ \mathscr{E},\ \mathscr{B})$ satisfies:

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot \mathscr{U}_1 = \dfrac{1}{2} \left(\mathscr{G}_{1e} - \mathscr{G}_{1i}\right), \\ \partial_t \mathscr{U}_1 + \nabla \zeta_1 + \nabla \mathscr{T}_1 + \mathscr{E} - \Delta \mathscr{U}_1 = \dfrac{1}{2} \left(\mathscr{G}_{2e} - \mathscr{G}_{2i}\right), \\ \partial_t \mathscr{T}_1 + \dfrac{2}{3} \nabla \cdot \mathscr{U}_1 + \mathscr{T}_1 = \dfrac{1}{2} \left(\mathscr{G}_{3e} - \mathscr{G}_{3i}\right), \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} - 2\mathscr{U}_1 = \dfrac{1}{2} \left(\mathscr{G}_{4e} - \mathscr{G}_{4i}\right), \quad \dfrac{1}{2} \nabla \cdot \mathscr{E} = -\zeta_1, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^{3}, \end{cases} \tag{3.4}$$

with the initial condition:

$$\mathscr{W}_1|_{t=0} = \mathscr{W}_1^0 := \left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0, \mathscr{E}^0, \mathscr{B}^0\right), \quad x \in \mathbb{R}^3,$$

which satisfies the compatibility condition:

$$\frac{1}{2} \nabla \cdot \mathscr{E}^0 = -\zeta_1^0, \quad \nabla \cdot \mathscr{B}^0 = 0, \quad x \in \mathbb{R}^3,$$

where $\left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0\right)$ is given from $\left(\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}\right)$ due to (3.3).

Moreover, we also introduce

$$\zeta_2 = \frac{\zeta^e + \zeta^i}{2}, \quad \mathscr{U}_2 = \frac{\mathscr{U}^e + \mathscr{U}^i}{2} \quad \text{and} \quad \mathscr{T}_2 = \frac{\mathscr{T}^e + \mathscr{T}^i}{2}. \tag{3.5}$$

Then $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$ satisfies:

$$\begin{cases} \partial_t \zeta_2 + \nabla \cdot \mathscr{U}_2 = \dfrac{1}{2}\left(\mathscr{G}_{1e} + \mathscr{G}_{1i}\right), \\ \partial_t \mathscr{U}_2 + \nabla \zeta_2 + \nabla \mathscr{T}_2 - \Delta \mathscr{U}_2 = \dfrac{1}{2}\left(\mathscr{G}_{2e} + \mathscr{G}_{2i}\right), \\ \partial_t \mathscr{T}_2 + \dfrac{2}{3}\nabla \cdot \mathscr{U}_2 + \mathscr{T}_2 = \dfrac{1}{2}\left(\mathscr{G}_{3e} + \mathscr{G}_{3i}\right), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{3.6}$$

with the initial condition:

$$\mathscr{W}_2|_{t=0} = \mathscr{W}_2^0 := \left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right), \quad x \in \mathbb{R}^3,$$

where $\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)$ is given from $\left(\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}\right)$ due to (3.5). Therefore, we define the solution $\mathscr{W}_1 = (\zeta_1, \mathscr{U}_1, \mathscr{T}_1, \mathscr{E}, \mathscr{B})$ and $\mathscr{W}_2 = (\zeta_2, \mathscr{U}_2, \mathscr{T}_2)$, respectively, as follows

$$\mathscr{W}_1(t) = e^{t\mathscr{L}_1}\mathscr{W}_1^0 + \frac{1}{2}\int_0^t e^{(t-\tau)\mathscr{L}_1}\left(\mathscr{G}_{1e} - \mathscr{G}_{1i}, \mathscr{G}_{2e} - \mathscr{G}_{2i}, \mathscr{G}_{3e} - \mathscr{G}_{3i}, \mathscr{G}_{4e} - \mathscr{G}_{4i}\right)(\tau)d\tau, \tag{3.7}$$

and

$$\mathscr{W}_2(t) = e^{t\mathscr{L}_2}\mathscr{W}_2^0 + \frac{1}{2}\int_0^t e^{(t-\tau)\mathscr{L}_2}\left(\mathscr{G}_{1e} + \mathscr{G}_{1i}, \mathscr{G}_{2e} + \mathscr{G}_{2i}, \mathscr{G}_{3e} + \mathscr{G}_{3i}\right)(\tau)d\tau, \tag{3.8}$$

where $e^{t\mathscr{L}_1}\mathscr{W}_1^0$ and $e^{t\mathscr{L}_2}\mathscr{W}_2^0$, respectively, denote the solutions of the homogeneous initial value problems (3.9)-(3.10) and (3.12)-(3.13), which are given as follows.

The linearized homogeneous system of (3.4) is:

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot \mathscr{U}_1 = 0 \\ \partial_t \mathscr{U}_1 + \nabla \zeta_1 + \nabla \mathscr{T}_1 + \mathscr{E} - \Delta \mathscr{U}_1 = 0, \\ \partial_t \mathscr{T}_1 + \dfrac{2}{3}\nabla \cdot \mathscr{U}_1 + \mathscr{T}_1 = 0, \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} - 2\mathscr{U}_1 = 0, \quad \dfrac{1}{2}\nabla \cdot \mathscr{E} = -\zeta_1, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{3.9}$$

with the initial condition:

$$\mathscr{W}_1|_{t=0} = \mathscr{W}_1^0 := \left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0, \mathscr{E}^0, \mathscr{B}^0\right), \quad x \in \mathbb{R}^3, \tag{3.10}$$

which satisfies the compatibility condition:

$$\frac{1}{2}\nabla \cdot \mathscr{E}^0 = -\zeta_1^0, \quad \nabla \cdot \mathscr{B}^0 = 0, \quad x \in \mathbb{R}^3. \tag{3.11}$$

And the linearized homogeneous system of (3.6) is:

$$\begin{cases} \partial_t \zeta_2 + \nabla \cdot \mathscr{U}_2 = 0, \\ \partial_t \mathscr{U}_2 + \nabla \zeta_2 + \nabla \mathscr{T}_2 - \Delta \mathscr{U}_2 = 0, \\ \partial_t \mathscr{T}_2 + \frac{2}{3} \nabla \cdot \mathscr{U}_2 + \mathscr{T}_2 = 0, \quad (t, x) \in [0, +\infty) \times \mathbb{R}^3, \end{cases} \tag{3.12}$$

with the initial condition:

$$\mathscr{W}_2|_{t=0} = \mathscr{W}_2^0 := \left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right), \quad x \in \mathbb{R}^3. \tag{3.13}$$

In the sequel, we denote $\mathscr{W}_1 = (\zeta_1, \ \mathscr{U}_1, \ \mathscr{T}_1, \ \mathscr{E}, \ \mathscr{B})$ as the solution to the initial value problem (3.9)-(3.10), and $\mathscr{W}_2 = (\zeta_2, \ \mathscr{U}_2, \ \mathscr{T}_2)$ as the one to the initial value problem (3.12)-(3.13).

For the initial value problem (3.9)-(3.10), we obtain the $L^p - L^q$ decay property as follows.

Proposition 3.1. *Let $\mathscr{W}_1(t) = e^{t\mathscr{L}_1} \mathscr{W}_1^0$ be the solution to the initial value problem (3.9)-(3.10) with initial data $\mathscr{W}_1^0 = (\zeta_1^0, \ \mathscr{U}_1^0, \ \mathscr{T}_1^0, \ \mathscr{E}^0, \ \mathscr{B}^0)$ satisfying (3.11). Then, for any $t \geq 0$, there exist constants $C > 0$ and $\gamma > 0$ such that $\mathscr{W}_1$ satisfies the following time decay property:*

$$\begin{cases} \|(\zeta_1, \mathscr{T}_1)(t)\| \leq C e^{-\gamma t} \left\|\zeta_1^0\right\|_{L^1} + C(1+t)^{-\frac{5}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}} \left\|\mathscr{T}_1^0\right\|_{L^1} \\ \qquad + C e^{-\gamma t} \left\|\left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0\right)\right\|, \\ \|\mathscr{U}_1(t)\| \leq C(1+t)^{-\frac{5}{4}} \left\|(\zeta_1^0, \mathscr{T}_1^0, \mathscr{B}^0)\right\|_{L^1} + C(1+t)^{-\frac{3}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{7}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{11}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{5}{8}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C e^{-\gamma t} \left\|\left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}} \left(\left\|\mathscr{U}_1^0\right\| + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^1}\right), \end{cases} \tag{3.14}$$

$$\begin{cases} \|\mathscr{E}(t)\| \leq C(1+t)^{-\frac{3}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}} \left\|\left(\mathscr{T}_1^0, \mathscr{B}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{11}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{15}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{9}{8}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C e^{-\gamma t} \left\|\left(\mathscr{U}_1^0 \mathscr{T}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}} \left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^1} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^3}\right), \\ \|\mathscr{B}(t)\| \leq C(1+t)^{-\frac{5}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{5}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{9}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{3}{4}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C e^{-\gamma t} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}} \left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^1} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^3}\right), \end{cases} \tag{3.15}$$

$$\begin{cases} \|(\zeta_1, \mathscr{T}_1)(t)\|_{L^\infty} \le Ce^{-\gamma t} \left\|\zeta_1^0\right\|_{L^1 \cap \dot{H}^2} + C(1+t)^{-2} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{5}{2}} \left\|\mathscr{T}_1^0\right\|_{L^1} + Ce^{-\gamma t}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^1} + \left\|\mathscr{T}_1^0\right\|_{\dot{H}^2}\right), \\ \|\mathscr{U}_1(t)\|_{L^\infty} \le C(1+t)^{-2} \left\|(\zeta_1^0, \mathscr{T}_1^0, \mathscr{B}^0)\right\|_{L^1} + C(1+t)^{-\frac{3}{2}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{5}{4}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{7}{4}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-1} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\zeta_1^0, \mathscr{T}_1^0\right)\right\|_{\dot{H}^1} + \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\|\right) \\ \qquad + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^1} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^3}\right), \end{cases} \tag{3.16}$$

$$\begin{cases} \|\mathscr{E}(t)\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-2} \left\|\left(\mathscr{T}_1^0, \mathscr{B}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{7}{4}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{9}{4}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{3}{2}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\mathscr{T}_1^0\right\|_{\dot{H}^1} + \left\|\mathscr{E}^0\right\|_{\dot{H}^2} + \left\|\left(\mathscr{U}_1^0, \mathscr{B}^0\right)\right\|\right) \\ \qquad + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^3} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^5}\right), \\ \|\mathscr{B}(t)\|_{L^\infty} \le C(1+t)^{-2} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{2}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C(1+t)^{-1} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{3}{2}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{3}{4}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^3} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^5}\right), \end{cases} \tag{3.17}$$

$$\begin{cases} \|\nabla\mathscr{U}_1(t)\| \le C(1+t)^{-\frac{7}{4}} \left\|(\zeta_1^0, \mathscr{T}_1^0, \mathscr{B}^0)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{9}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{13}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{7}{8}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{T}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\| + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^2}\right), \\ \|\nabla\mathscr{E}(t)\| \le C(1+t)^{-\frac{5}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}} \left\|\left(\mathscr{T}_1^0, \mathscr{B}^0\right)\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{13}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{17}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{11}{8}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\mathscr{E}^0\right\|_{\dot{H}^1} + \left\|\left(\mathscr{U}_1^0, \mathscr{T}_1^0, \mathscr{B}^0\right)\right\|\right) \\ \qquad + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^2} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^4}\right), \\ \|\nabla\mathscr{B}(t)\| \le C(1+t)^{-\frac{7}{4}} \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0\right)\right\|_{L^1} + C(1+t)^{-\frac{9}{4}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + C(1+t)^{-\frac{7}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1} + C(1+t)^{-\frac{11}{8}} \left\|\mathscr{E}^0\right\|_{L^1} + C(1+t)^{-\frac{5}{8}} \left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^2} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{\dot{H}^4}\right), \end{cases} \tag{3.18}$$

and

$$\begin{cases} \left\|\nabla^2\mathscr{B}(t)\right\| \le C\,(1+t)^{-\frac{9}{4}}\left\|\left(\mathscr{U}_1^0,\mathscr{E}^0\right)\right\|_{L^1} + C\,(1+t)^{-\frac{11}{4}}\left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad\qquad + C\,(1+t)^{-\frac{9}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1} + C\,(1+t)^{-\frac{13}{8}}\left\|\mathscr{E}^0\right\|_{L^1} + C\,(1+t)^{-\frac{7}{8}}\left\|\mathscr{B}^0\right\|_{L^1} \\ \qquad\qquad + Ce^{-\gamma t}\left\|\left(\mathscr{U}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\| + C\,(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{\dot{H}^3} + \left\|\left(\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{\dot{H}^5}\right), \\ \left\|\nabla^s\left(\mathscr{E}(t),\mathscr{B}(t)\right)\right\| \le C(1+t)^{-1}\left\|\left(\mathscr{U}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^2\cap\dot{H}^{s+2}}. \end{cases} \tag{3.19}$$

Proof. The proof is similar to that of Theorem 2.8 in [32], we omit it here for simplicity. □

3.2. Explicit solutions

In the following, we study the explicit Fourier transform solution $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$ to problem (3.12)-(3.13). From (3.12), we obtain

$$\partial_{ttt}\zeta_2 + \partial_{tt}(\zeta_2 - \Delta\zeta_2) - \frac{8}{3}\partial_t\Delta\zeta_2 - \Delta\zeta_2 = 0, \tag{3.20}$$

supplemented with the initial condition:

$$\begin{cases} \zeta_2\,|_{t=0} = \zeta_2^0, \\ \partial_t\zeta_2\,|_{t=0} = -\nabla\cdot\mathscr{U}_2^0, \\ \partial_{tt}\zeta_2\,|_{t=0} = \Delta\zeta_2^0 - \Delta\nabla\cdot\mathscr{U}_2^0 + \Delta\mathscr{T}_2^0. \end{cases} \tag{3.21}$$

By taking the Fourier transform on (3.20) and (3.21), we get

$$\partial_{ttt}\hat{\zeta}_2 + (1+|k|^2)\partial_{tt}\hat{\zeta}_2 + \frac{8}{3}|k|^2\,\partial_t\hat{\zeta}_2 + |k|^2\,\hat{\zeta}_2 = 0, \tag{3.22}$$

with the initial condition:

$$\begin{cases} \hat{\zeta}_2\,|_{t=0} = \hat{\zeta}_2^0, \\ \partial_t\hat{\zeta}_2\,|_{t=0} = -i|k|\tilde{k}\cdot\hat{\mathscr{U}}_2^0, \\ \partial_{tt}\hat{\zeta}_2\,|_{t=0} = -|k|^2\hat{\zeta}_2^0 + i|k|^3\tilde{k}\cdot\hat{\mathscr{U}}_2^0 - |k|^2\hat{\mathscr{T}}_2^0, \end{cases} \tag{3.23}$$

where and in the sequel $\tilde{k} = k/|k|$.

The characteristic equation corresponding to (3.22) is:

$$\mathscr{C}(\mathfrak{r}) := \mathfrak{r}^3 + (1+|k|^2)\mathfrak{r}^2 + \frac{8}{3}|k|^2\,\mathfrak{r} + |k|^2 = 0, \tag{3.24}$$

whose roots may be written as: $\mathfrak{r} = \mathfrak{r}_{1,2,3}$.

Then we write the solution of (3.22)-(3.23) as:

$$\hat{\zeta}_2 = C_1 e^{\mathfrak{x}_1 t} + C_2 e^{\mathfrak{x}_2 t} + C_3 e^{\mathfrak{x}_3 t}. \tag{3.25}$$

For the roots of the characteristic equation (3.24) and their properties, we have

Lemma 3.1. *Assume $|k| \neq 0$. Then, there exist domains Ω_0 and Ω_∞ in the form of that in section 2.1 such that $\mathscr{C}(\mathfrak{x}) = 0$, $\mathfrak{x} \in \mathbb{C}$ has one real root $\mathfrak{x}_1 \in (-1, 0)$ and two conjugate complex roots $\mathfrak{x}_{2,3}$ when $k \in \Omega_0$, or three real roots $\mathfrak{x}_{1,2,3}$ as $k \in \Omega_\infty$. Moreover, it holds*

$$\begin{cases} \mathfrak{x}_1 = -1 + \dfrac{2}{3}|k|^2 + O(1)|k|^4, \\ \mathfrak{x}_{2,3} = -\dfrac{5}{6}|k|^2 + O(1)|k|^4 \pm i|k|\sqrt{1 + O(1)|k|^2}, \end{cases} \tag{3.26}$$

whenever $|k| \to 0$ is small. And it also holds

$$\begin{cases} \mathfrak{x}_1 = -|k|^2 + \dfrac{5}{3} + \dfrac{31}{9}|k|^{-2} + O(1)|k|^{-4}, \\ \mathfrak{x}_2 = \dfrac{-4 + \sqrt{7}}{3} + \dfrac{(11 - 8\sqrt{7})(\sqrt{7} - 1)}{9\sqrt{7}}|k|^{-2} + O(1)|k|^{-4}, \\ \mathfrak{x}_3 = -\dfrac{4 + \sqrt{7}}{3} - \dfrac{(11 + 8\sqrt{7})(\sqrt{7} + 1)}{9\sqrt{7}}|k|^{-2} + O(1)|k|^{-4}, \end{cases} \tag{3.27}$$

whenever $|k| > L$ is large.

Proof. Let us denote $\rho = |k|^2$, and then (3.24) becomes

$$\mathscr{C}(\mathfrak{x}) := \mathfrak{x}^3 + (1 + \rho)\mathfrak{x}^2 + \frac{8}{3}\rho\mathfrak{x} + \rho = 0. \tag{3.28}$$

Obviously, we obtain that $\mathscr{C}(0) = \rho > 0$ and $\mathscr{C}(-1) = \frac{2}{3}\rho < 0$, which deduces that $\mathscr{C}(\mathfrak{x}) = 0$ has at least one real root in $(-1, 0)$. Denote $\phi = \phi(\rho)$ is a real root in $(-1, 0)$. Then we may rewrite

$$\mathscr{C}(\mathfrak{x}) = (\mathfrak{x} - \phi)\left(\mathfrak{x}^2 + (1 + \rho + \phi)\mathfrak{x} + \left(\frac{8}{3}\rho + (1 + \rho)\phi + \phi^2\right)\right). \tag{3.29}$$

The remaining two roots, denoted by $\varphi_\pm$, which satisfy

$$\mathfrak{x}^2 + (1 + \rho + \phi)\mathfrak{x} + \left(\frac{8}{3}\rho + (1 + \rho)\phi + \phi^2\right) = 0. \tag{3.30}$$

Then it follows that

$$\varphi_\pm = -\frac{1}{2}(1 + \rho + \phi) \pm \frac{1}{2}\sqrt{\psi}, \tag{3.31}$$

in which

$$\psi = -\left(3\phi^2 + 2\phi + 2\rho\phi + \frac{26}{3}\rho - \rho^2 - 1\right). \tag{3.32}$$

In view of Lemma 2.1, the characteristic equation (3.28) may be rewritten in the form:

$$\mathscr{C}(\mathfrak{x}) = a\mathfrak{x}^3 + b\mathfrak{x}^2 + c\mathfrak{x} + d = 0, \tag{3.33}$$

where

$$a = 1, \quad b = 1 + \rho, \quad c = \frac{8}{3}\rho, \quad d = \rho. \tag{3.34}$$

A direct computation gives

$$\mathfrak{S} = -\frac{1}{54}\left(2\rho^3 - 18\rho^2 + 9\rho + 2\right),$$
$$\mathfrak{R} = \frac{\rho}{2916}\left(-84\rho^3 + 1180\rho^2 - 435\rho + 108\right).$$

It is easy to check that $\mathfrak{R} > 0$ when $\rho > 0$ is sufficiently small, and that $\mathfrak{R} < 0$ when $\rho > 0$ is large enough. Then there exist domains Ω_0 and Ω_∞ in the form of that in section 2.1 such that $\mathfrak{R} > 0$ over Ω_0 and $\mathfrak{R} < 0$ over Ω_∞. By Lemma 2.1, we obtain that there are one real root and two conjugate complex roots over Ω_0, while there are three real roots over Ω_∞.

In case of $k \in \Omega_0$, we denote $\mathfrak{x}_1 = \phi \in (-1, 0)$ and $\mathfrak{x}_{2,3} = \varphi_\pm$. Let us expand $\phi(\rho)$ at zero point as

$$\phi(\rho) = \sum_{n=0}^{\infty} a_n \rho^n.$$

Substitute it into (3.33), we have

$$a_0 = -1, \quad a_1 = \frac{2}{3}, \quad a_2 = \frac{4}{9}.$$

Then it follows that

$$\mathfrak{x}_1 = \phi(\rho) = -1 + \frac{2}{3}\rho + O(1)\rho^2, \tag{3.35}$$

provided $\rho > 0$ is sufficiently small. In view of the expression of $\varphi_\pm$ in (3.31) and the expression of ψ in (3.32), we get the real part and the imaginary part as

$$\mathcal{Re}(\varphi_\pm) = -\frac{5}{6}\rho + O(1)\rho^2 \quad \text{and} \quad \mathcal{Im}(\varphi_\pm) = \sqrt{\rho + O(1)\rho^2},$$

respectively. Hence,

$$\mathfrak{r}_{2,3} = \varphi_{\pm} = -\frac{5}{6}\rho + O(1)\rho^2 \pm i\sqrt{\rho + O(1)\rho^2}, \tag{3.36}$$

when $\rho > 0$ is sufficiently small. Then, (3.26) follows by combining (3.35)-(3.36).

On the other hand, in case of $k \in \Omega_\infty$, we may rewrite (3.28) as

$$\left(\frac{\mathfrak{r}}{\rho}\right)^3 + (1 + \frac{1}{\rho})\left(\frac{\mathfrak{r}}{\rho}\right)^2 + \frac{8}{3\rho}\frac{\mathfrak{r}}{\rho} + \frac{1}{\rho^2} = 0. \tag{3.37}$$

Because the uniqueness of the real root in $(-1, 0)$ is unclear, we just set

$$\frac{\mathfrak{r}}{\rho} = \sum_{n=0}^{\infty} a_n \rho^{-n},$$

provided ρ is sufficiently large. Substitute it into (3.37), we have

$$a_0 = -1, \quad a_1 = \frac{5}{3}, \quad a_2 = \frac{31}{9},$$

or

$$a_0 = 0, \quad a_1 = \frac{-4 + \sqrt{7}}{3}, \quad a_2 = \frac{(11 - 8\sqrt{7})(\sqrt{7} - 1)}{9\sqrt{7}},$$

or

$$a_0 = 0, \quad a_1 = -\frac{4 + \sqrt{7}}{3}, \quad a_2 = \frac{(11 + 8\sqrt{7})(\sqrt{7} + 1)}{-9\sqrt{7}}.$$

Thus, we get

$$\begin{cases} \mathfrak{r}_1 = -\rho + \frac{5}{3} + \frac{31}{9}\rho^{-1} + O(1)\rho^{-2}, \\ \mathfrak{r}_2 = \frac{-4 + \sqrt{7}}{3} + \frac{(11 - 8\sqrt{7})(\sqrt{7} - 1)}{9\sqrt{7}}\rho^{-1} + O(1)\rho^{-2}, \\ \mathfrak{r}_3 = -\frac{4 + \sqrt{7}}{3} - \frac{(11 + 8\sqrt{7})(\sqrt{7} + 1)}{9\sqrt{7}}\rho^{-1} + O(1)\rho^{-2}, \end{cases}$$

when $|k|$ is large enough. Then (3.27) follows by noticing $\rho = |k|^2$. The proof of Lemma 3.1 is completed. □

Remark 3.1. On the domain $\Omega_1 = \{k \in \mathbb{R}^3 | \epsilon \leq |k| \leq L\}$, the main difficulty lies in the case that the characteristic equation (3.24) might have multiple roots.

Now, by the initial condition (3.23) and (3.25), we obtain

$$\begin{cases} C_1 + C_2 + C_3 = \hat{\xi}_2^0, \\ \mathfrak{r}_1 C_1 + \mathfrak{r}_2 C_2 + \mathfrak{r}_3 C_3 = -i|k|\tilde{k} \cdot \hat{\mathscr{U}}_2^0, \\ \mathfrak{r}_1^2 C_1 + \mathfrak{r}_2^2 C_2 + \mathfrak{r}_3^2 C_3 = -|k|^2 \hat{\xi}_2^0 + i|k|^3 \tilde{k} \cdot \hat{\mathscr{U}}_2^0 - |k|^2 \hat{\mathscr{T}}_2^0, \end{cases} \tag{3.38}$$

which implies

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \frac{-\mathfrak{r}_2\mathfrak{r}_3(1+\mathfrak{r}_1)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} & \frac{ik^T(1+\mathfrak{r}_1)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} & \frac{|k|^2}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} \\ \frac{-\mathfrak{r}_1\mathfrak{r}_3(1+\mathfrak{r}_2)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} & \frac{ik^T(1+\mathfrak{r}_2)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} & \frac{|k|^2}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} \\ \frac{-\mathfrak{r}_1\mathfrak{r}_2(1+\mathfrak{r}_3)}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)} & \frac{ik^T(1+\mathfrak{r}_3)}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)} & \frac{|k|^2}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)} \end{pmatrix} \begin{pmatrix} \hat{\xi}_2^0 \\ \tilde{k}\tilde{k} \cdot \hat{\mathscr{U}}_2^0 \\ \hat{\mathscr{T}}_2^0 \end{pmatrix}, \tag{3.39}$$

where we have used $\mathfrak{r}_1\mathfrak{r}_2\mathfrak{r}_3 = -|k|^2$ and $\mathfrak{r}_1 + \mathfrak{r}_2 + \mathfrak{r}_3 = -(1+|k|^2)$. Substitute (3.39) into (3.25), we get

$$\hat{\xi}_2 = (\mathscr{R}_{11}, \mathscr{R}_{12}, \mathscr{R}_{13}) \begin{pmatrix} \hat{\xi}_2^0 \\ \tilde{k}\tilde{k} \cdot \hat{\mathscr{U}}_2^0 \\ \hat{\mathscr{T}}_2^0 \end{pmatrix}, \tag{3.40}$$

where

$$\begin{cases} \mathscr{R}_{11} = -\frac{e^{\mathfrak{r}_1 t}\mathfrak{r}_2\mathfrak{r}_3(1+\mathfrak{r}_1)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} - \frac{e^{\mathfrak{r}_2 t}\mathfrak{r}_1\mathfrak{r}_3(1+\mathfrak{r}_2)}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} - \frac{e^{\mathfrak{r}_3 t}\mathfrak{r}_1\mathfrak{r}_2(1+\mathfrak{r}_3)}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)}, \\ \mathscr{R}_{12} = \frac{ik^T(1+\mathfrak{r}_1)e^{\mathfrak{r}_1 t}}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} + \frac{ik^T(1+\mathfrak{r}_2)e^{\mathfrak{r}_2 t}}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} + \frac{ik^T(1+\mathfrak{r}_3)e^{\mathfrak{r}_3 t}}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)}, \\ \mathscr{R}_{13} = \frac{|k|^2 e^{\mathfrak{r}_1 t}}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)} + \frac{|k|^2 e^{\mathfrak{r}_2 t}}{(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)} + \frac{|k|^2 e^{\mathfrak{r}_3 t}}{(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)}. \end{cases} \tag{3.41}$$

Next, we try to solve $\hat{\mathscr{T}}_2$. From (3.25), the first and the third equations in (3.12), we have

$$\partial_t \hat{\mathscr{T}}_2 + \hat{\mathscr{T}}_2 = \frac{2}{3}\partial_t \hat{\xi}_2 = \frac{2}{3}(\mathfrak{r}_1 e^{\mathfrak{r}_1 t} C_1 + \mathfrak{r}_2 e^{\mathfrak{r}_2 t} C_2 + \mathfrak{r}_3 e^{\mathfrak{r}_3 t} C_3). \tag{3.42}$$

A direct computation implies

$$\begin{aligned}\hat{\mathscr{T}}_2 =&e^{-t}\hat{\mathscr{T}}_2^0+\frac{2}{3}e^{-t}\int_0^t e^{\tau}(\mathfrak{r}_1e^{\mathfrak{r}_1\tau}C_1+\mathfrak{r}_2e^{\mathfrak{r}_2\tau}C_2+\mathfrak{r}_3e^{\mathfrak{r}_3\tau}C_3)d\tau\\ =&\frac{2}{3}(\frac{\mathfrak{r}_1e^{\mathfrak{r}_1t}C_1}{1+\mathfrak{r}_1}+\frac{\mathfrak{r}_2e^{\mathfrak{r}_2t}C_2}{1+\mathfrak{r}_2}+\frac{\mathfrak{r}_3e^{\mathfrak{r}_3t}C_3}{1+\mathfrak{r}_3}),\end{aligned} \tag{3.43}$$

where we have used the fact that

$$\hat{\mathscr{T}}_2^0=\frac{2}{3}(\frac{\mathfrak{r}_1C_1}{1+\mathfrak{r}_1}+\frac{\mathfrak{r}_2C_2}{1+\mathfrak{r}_2}+\frac{\mathfrak{r}_3C_3}{1+\mathfrak{r}_3}).$$

Then substitute (3.39) into (3.43), we have

$$\hat{\mathscr{T}}_2=(\mathscr{R}_{31},\mathscr{R}_{32},\mathscr{R}_{33})\begin{pmatrix}\hat{\zeta}_2^0\\ \tilde{k}\tilde{k}\cdot\hat{\mathscr{U}}_2^0\\ \hat{\mathscr{T}}_2^0\end{pmatrix}, \tag{3.44}$$

where

$$\begin{cases}\mathscr{R}_{31}=-\frac{2e^{\mathfrak{r}_1t}\mathfrak{r}_1\mathfrak{r}_2\mathfrak{r}_3}{3(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)}-\frac{2e^{\mathfrak{r}_2t}\mathfrak{r}_1\mathfrak{r}_2\mathfrak{r}_3}{3(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)}-\frac{2e^{\mathfrak{r}_3t}\mathfrak{r}_1\mathfrak{r}_2\mathfrak{r}_3}{3(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)},\\ \mathscr{R}_{32}=\frac{2ik^T\mathfrak{r}_1e^{\mathfrak{r}_1t}}{3(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)}+\frac{2ik^T\mathfrak{r}_2e^{\mathfrak{r}_2t}}{3(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)}+\frac{2ik^T\mathfrak{r}_3e^{\mathfrak{r}_3t}}{3(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)},\\ \mathscr{R}_{33}=\frac{2|k|^2\mathfrak{r}_1e^{\mathfrak{r}_1t}}{3(1+\mathfrak{r}_1)(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_3-\mathfrak{r}_1)}+\frac{2|k|^2\mathfrak{r}_2e^{\mathfrak{r}_2t}}{3(1+\mathfrak{r}_2)(\mathfrak{r}_1-\mathfrak{r}_2)(\mathfrak{r}_2-\mathfrak{r}_3)}+\frac{2|k|^2\mathfrak{r}_3e^{\mathfrak{r}_3t}}{3(1+\mathfrak{r}_3)(\mathfrak{r}_2-\mathfrak{r}_3)(\mathfrak{r}_3-\mathfrak{r}_1)}.\end{cases} \tag{3.45}$$

Next, again from (3.12), we obtain

$$\partial_{ttt}\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)+(1+|k|^2)\partial_{tt}\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)+\frac{8}{3}|k|^2\,\partial_t\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)+|k|^2\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)=0, \tag{3.46}$$

with the initial condition:

$$\begin{cases}\tilde{k}\cdot\hat{\mathscr{U}}_2\Big|_{t=0}=\tilde{k}\cdot\hat{\mathscr{U}}_2^0,\\ \partial_t\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)\Big|_{t=0}=-i\,|k|\,\hat{\zeta}_2^0-|k|^2\,\tilde{k}\cdot\hat{\mathscr{U}}_2-i\,|k|\,\hat{\mathscr{T}}_2^0\\ \partial_{tt}\left(\tilde{k}\cdot\hat{\mathscr{U}}_2\right)\Big|_{t=0}=i\,|k|^3\,\hat{\zeta}_2^0+(|k|^4-\frac{5}{3}|k|^2)\tilde{k}\cdot\hat{\mathscr{U}}_2+i(|k|^3+|k|)\hat{\mathscr{T}}_2^0.\end{cases} \tag{3.47}$$

It follows from (3.46)-(3.47) that

$$\tilde{k}\tilde{k}\cdot\hat{\mathscr{U}}_2=(\mathscr{R}_{21},\mathscr{R}_{22},\mathscr{R}_{23})\begin{pmatrix}\hat{\zeta}_2^0\\ \tilde{k}\tilde{k}\cdot\hat{\mathscr{U}}_2^0\\ \hat{\mathscr{T}}_2^0\end{pmatrix}, \tag{3.48}$$

where

$$\begin{cases} \mathscr{R}_{21} = \dfrac{ike^{\mathfrak{x}_1 t}(1+\mathfrak{x}_1)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)} + \dfrac{ike^{\mathfrak{x}_2 t}(1+\mathfrak{x}_2)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_2-\mathfrak{x}_3)} + \dfrac{ike^{\mathfrak{x}_3 t}(1+\mathfrak{x}_3)}{(\mathfrak{x}_2-\mathfrak{x}_3)(\mathfrak{x}_3-\mathfrak{x}_1)}, \\ \mathscr{R}_{22} = -\dfrac{\mathfrak{x}_1(1+\mathfrak{x}_1)e^{\mathfrak{x}_1 t}}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)} - \dfrac{\mathfrak{x}_2(1+\mathfrak{x}_2)e^{\mathfrak{x}_2 t}}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_2-\mathfrak{x}_3)} - \dfrac{\mathfrak{x}_3(1+\mathfrak{x}_3)e^{\mathfrak{x}_3 t}}{(\mathfrak{x}_2-\mathfrak{x}_3)(\mathfrak{x}_3-\mathfrak{x}_1)}, \\ \mathscr{R}_{23} = \dfrac{ik\mathfrak{x}_1 e^{\mathfrak{x}_1 t}}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)} + \dfrac{ik\mathfrak{x}_2 e^{\mathfrak{x}_2 t}}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_2-\mathfrak{x}_3)} + \dfrac{ik\mathfrak{x}_3 e^{\mathfrak{x}_3 t}}{(\mathfrak{x}_2-\mathfrak{x}_3)(\mathfrak{x}_3-\mathfrak{x}_1)}, \end{cases} \tag{3.49}$$

where we have used $\mathfrak{x}_1+\mathfrak{x}_2+\mathfrak{x}_3 = -(1+|k|^2)$ and $\mathfrak{x}_1\mathfrak{x}_2+\mathfrak{x}_2\mathfrak{x}_3+\mathfrak{x}_3\mathfrak{x}_1 = \frac{8}{3}|k|^2$.

Moreover, taking the curl for the second equation of (3.12) and then taking the Fourier transform on the resulting equation, we have

$$\partial_t\left(\tilde{k}\times\hat{\mathscr{U}_2}\right) + |k|^2\left(\tilde{k}\times\hat{\mathscr{U}_2}\right) = 0, \tag{3.50}$$

with the initial condition:

$$\tilde{k}\times\hat{\mathscr{U}_2}\Big|_{t=0} = \tilde{k}\times\hat{\mathscr{U}_2^0}. \tag{3.51}$$

Then from (3.50)-(3.51), we obtain

$$\tilde{k}\times\hat{\mathscr{U}_2} = e^{-|k|^2 t}\left(\tilde{k}\times\hat{\mathscr{U}_2^0}\right). \tag{3.52}$$

Moreover, by the Helmholtz decomposition $\Delta\mathscr{U} = \nabla(\nabla\cdot\mathscr{U}) - \nabla\times(\nabla\times\mathscr{U})$, we have

$$\hat{\mathscr{U}_2} = \tilde{k}\tilde{k}\cdot\hat{\mathscr{U}_2} - \tilde{k}\times\left(\tilde{k}\times\hat{\mathscr{U}_2}\right). \tag{3.53}$$

Then it follows from (3.40), (3.44), (3.48), (3.52) and (3.53) that the explicit Fourier transform solution $\hat{\mathscr{W}_2} = (\hat{\zeta}_2,\ \hat{\mathscr{U}_2},\ \hat{\mathscr{T}_2})$ as follows.

Theorem 3.1. *Let $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$ be the solution to the Initial value problem (3.12)-(3.13). For $(t,k)\in\mathbb{R}^+\times\mathbb{R}^3$ with $|k|\neq 0$,*

$$\begin{pmatrix} \hat{\zeta}_2(t,k) \\ \hat{\mathscr{U}_2}(t,k) \\ \hat{\mathscr{T}_2}(t,k) \end{pmatrix} = \begin{pmatrix} \hat{\zeta}_2(t,k) \\ \tilde{k}\tilde{k}\cdot\hat{\mathscr{U}_2}(t,k) \\ \hat{\mathscr{T}_2}(t,k) \end{pmatrix} + \begin{pmatrix} 0 \\ -\tilde{k}\times\left(\tilde{k}\times\hat{\mathscr{U}_2}\right)(t,k) \\ 0 \end{pmatrix}. \tag{3.54}$$

Then, it holds

$$\begin{pmatrix} \hat{\zeta}_2(t,k) \\ \hat{\mathscr{U}_2}(t,k) \\ \hat{\mathscr{T}_2}(t,k) \end{pmatrix} = \mathscr{R}(t,k)\begin{pmatrix} \hat{\zeta}_2^0(k) \\ \tilde{k}\tilde{k}\cdot\hat{\mathscr{U}_2^0}(k) \\ \hat{\mathscr{T}_2^0}(k) \end{pmatrix} + e^{-|k|^2 t}\begin{pmatrix} 0 \\ -\tilde{k}\times\left(\tilde{k}\times\hat{\mathscr{U}_2^0}\right)(k) \\ 0 \end{pmatrix}, \tag{3.55}$$

where the 5×5 *matrix* $\mathscr{R}(t,k)$ *is defined as:*

$$\mathscr{R}(t,k) = \begin{pmatrix} \mathscr{R}_{11}(t,k) & \mathscr{R}_{12}(t,k) & \mathscr{R}_{13}(t,k) \\ \mathscr{R}_{21}(t,k) & \mathscr{R}_{22}(t,k) & \mathscr{R}_{23}(t,k) \\ \mathscr{R}_{31}(t,k) & \mathscr{R}_{32}(t,k) & \mathscr{R}_{33}(t,k) \end{pmatrix}.$$

Here, $\mathscr{R}_{ij}$*,* $1 \le i, j \le 3$ *is explicitly determined by representations* (3.41)*,* (3.45) *and* (3.49)*.*

3.3. $L^p - L^q$ decay properties

In this subsection, we want to get the $L^p - L^q$ decay property for each component of the solution $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$. To this end, we investigate the rigorous time frequency estimates on $\hat{\mathscr{W}}_2 = (\hat{\zeta}_2,\ \hat{\mathscr{U}}_2,\ \hat{\mathscr{T}}_2)$.

Lemma 3.2. *Let* $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$ *be the solution to the initial value problem* (3.12)-(3.13)*. Then, there are constants* $\gamma > 0$*,* $C > 0$ *and domains* Ω_0*,* Ω_1 *and* Ω_∞ *in the form of that in section* 2.1 *such that for all* $(t,k) \in \mathbb{R}^+ \times \mathbb{R}^3$*,*

$$\left|\hat{\zeta}_2(t,k)\right| \le \begin{cases} C\left(|k|\, e^{-\gamma|k|^2 t} + |k|^6 e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right| + C\left(e^{-\gamma|k|^2 t} + |k|^3 e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ + C\left(|k|\, e^{-\gamma|k|^2 t} + |k|^2 e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad \text{as} \quad k \in \Omega_0, \\ Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right| + \left|\hat{\mathscr{T}}_2^0(k)\right|\right), \quad \text{as} \quad k \in \Omega_1, \\ C\left(e^{-\gamma t} + \frac{1}{|k|^2} e^{-|k|^2 t}\right)\left|\hat{\zeta}_2^0(k)\right| + \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2 t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ + C\left(e^{-\gamma t} + \frac{1}{|k|^2} e^{-|k|^2 t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad \text{as} \quad k \in \Omega_\infty, \end{cases} \tag{3.56}$$

$$\left|\hat{\mathscr{U}}_2(t,k)\right| \le \begin{cases} C\left(e^{-\gamma|k|^2 t} + |k|^3 e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right| + C\left(e^{-\gamma|k|^2 t} + |k|^2 e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ + C\left(|k|\, e^{-\gamma|k|^2 t} + |k|\, e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad \text{as} \quad k \in \Omega_0, \\ Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right| + \left|\hat{\mathscr{T}}_2^0(k)\right|\right), \quad \text{as} \quad k \in \Omega_1, \\ \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2 t}\right)\left|\hat{\zeta}_2^0(k)\right| + C\left(e^{-\gamma t} + e^{-|k|^2 t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ + \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2 t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad \text{as} \quad k \in \Omega_\infty, \end{cases} \tag{3.57}$$

and

$$\left|\hat{\mathscr{T}}_2(t,k)\right| \le \begin{cases} C\left(|k|\,e^{-\gamma|k|^2t}+|k|^2e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right|+C\left(|k|\,e^{-\gamma|k|^2t}+|k|\,e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ +C\left(e^{-\gamma|k|^2t}+e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad as \quad k\in\Omega_0, \\ Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right|+\left|\hat{\mathscr{U}}_2^0(k)\right|+\left|\hat{\mathscr{T}}_2^0(k)\right|\right), \quad as \quad k\in\Omega_1, \\ C\left(e^{-\gamma t}+\frac{1}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\zeta}_2^0(k)\right|+\frac{C}{|k|}\left(e^{-\gamma t}+e^{-|k|^2t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ +C\left(e^{-\gamma t}+\frac{1}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \quad as \quad k\in\Omega_\infty. \end{cases} \tag{3.58}$$

Proof. For establishing estimates (3.56)-(3.58), we have to get the estimates for each component in matrix $\mathscr{R}(t,k)$, namely, $\mathscr{R}_{ij}$, $(1\le i,j\le 3)$. From Lemma 3.1, we obtain that there are domains Ω_0, Ω_1 and Ω_∞ in the form of that in section 2.1 such that

$$\begin{cases} |\mathscr{R}_{11}|\le C\left(|k|\,e^{-\gamma|k|^2t}+|k|^6e^{-t}\right), \quad |\mathscr{R}_{12}|,|\mathscr{R}_{21}|\le C\left(e^{-\gamma|k|^2t}+|k|^3e^{-t}\right), \\ |\mathscr{R}_{13}|,|\mathscr{R}_{31}|\le C\left(|k|\,e^{-\gamma|k|^2t}+|k|^2e^{-t}\right), \quad |\mathscr{R}_{22}|\le C\left(e^{-\gamma|k|^2t}+|k|^2e^{-t}\right), \\ |\mathscr{R}_{23}|\le C\left(|k|\,e^{-\gamma|k|^2t}+|k|\,e^{-t}\right), \quad |\mathscr{R}_{32}|\le C\left(|k|\,e^{-\gamma|k|^2t}+|k|\,e^{-t}\right), \\ |\mathscr{R}_{33}|\le C\left(e^{-\gamma|k|^2t}+e^{-t}\right), \end{cases} \tag{3.59}$$

as $k\in\Omega_0$, and

$$\begin{cases} |\mathscr{R}_{11}|\le Ce^{-\gamma t}+\frac{C}{|k|^2}e^{-|k|^2t}, \ |\mathscr{R}_{12}|\le\frac{C}{|k|}\left(e^{-\gamma t}+e^{-|k|^2t}\right), \ |\mathscr{R}_{13}|\le Ce^{-\gamma t}+\frac{C}{|k|^2}e^{-|k|^2t}, \\ |\mathscr{R}_{21}|\le\frac{C}{|k|}\left(e^{-\gamma t}+e^{-|k|^2t}\right), \ |\mathscr{R}_{22}|\le C\left(e^{-\gamma t}+e^{-|k|^2t}\right), \ |\mathscr{R}_{23}|\le\frac{C}{|k|}\left(e^{-\gamma t}+e^{-|k|^2t}\right), \\ |\mathscr{R}_{31}|\le Ce^{-\gamma t}+\frac{C}{|k|^2}e^{-|k|^2t}, \ |\mathscr{R}_{32}|\le\frac{C}{|k|}\left(e^{-\gamma t}+e^{-|k|^2t}\right), \ |\mathscr{R}_{33}|\le Ce^{-\gamma t}+\frac{C}{|k|^2}e^{-|k|^2t}, \end{cases} \tag{3.60}$$

as $k\in\Omega_1$.

Then, (3.59)-(3.60) together with (3.55) imply that

$$\begin{aligned} \left|\hat{\zeta}_2(t,k)\right| &\le |\mathscr{R}_{11}|\left|\hat{\zeta}_2^0(k)\right|+|\mathscr{R}_{12}|\left|\hat{\mathscr{U}}_2^0(k)\right|+|\mathscr{R}_{13}|\left|\hat{\mathscr{T}}_2^0(k)\right| \\ &\le C\left(|k|\,e^{-\gamma|k|^2t}+|k|^6e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right|+C\left(e^{-\gamma|k|^2t}+|k|^3e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &\quad +C\left(|k|\,e^{-\gamma|k|^2t}+|k|^2e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad as \quad k\in\Omega_0, \end{aligned} \tag{3.61}$$

$$\begin{aligned} \left|\hat{\mathscr{U}}_2(t,k)\right| &\le |\mathscr{R}_{21}|\left|\hat{\zeta}_2^0(k)\right|+\left(|\mathscr{R}_{22}|+e^{-|k|^2t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right|+|\mathscr{R}_{23}|\left|\hat{\mathscr{T}}_2^0(k)\right| \\ &\le C\left(e^{-\gamma|k|^2t}+|k|^3e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right|+C\left(e^{-\gamma|k|^2t}+|k|^2e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &\quad +C\left(|k|\,e^{-\gamma|k|^2t}+|k|\,e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad as \quad k\in\Omega_0, \end{aligned} \tag{3.62}$$

$$\begin{aligned}\left|\hat{\mathscr{T}}_2(t,k)\right| \le& |\mathscr{R}_{31}|\left|\hat{\zeta}_2^0(k)\right| + |\mathscr{R}_{32}|\left|\hat{\mathscr{U}}_2^0(k)\right| + |\mathscr{R}_{33}|\left|\hat{\mathscr{T}}_2^0(k)\right| \\ \le& C\left(|k|\,e^{-\gamma|k|^2t} + |k|^2e^{-t}\right)\left|\hat{\zeta}_2^0(k)\right| + C\left(|k|\,e^{-\gamma|k|^2t} + |k|\,e^{-t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &+ C\left(e^{-\gamma|k|^2t} + e^{-t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad \text{as} \quad k\in\Omega_0,\end{aligned} \tag{3.63}$$

$$\begin{aligned}\left|\hat{\zeta}_2(t,k)\right| \le& \left(Ce^{-\gamma t} + \frac{C}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\zeta}_2^0(k)\right| + \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &+ \left(Ce^{-\gamma t} + \frac{C}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad \text{as} \quad k\in\Omega_\infty,\end{aligned} \tag{3.64}$$

$$\begin{aligned}\left|\hat{\mathscr{U}}_2(t,k)\right| \le& \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2t}\right)\left|\hat{\zeta}_2^0(k)\right| + C\left(e^{-\gamma t} + e^{-|k|^2t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &+ \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad \text{as} \quad k\in\Omega_\infty,\end{aligned} \tag{3.65}$$

and

$$\begin{aligned}\left|\hat{\mathscr{T}}_2(t,k)\right| \le& \left(Ce^{-\gamma t} + \frac{C}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\zeta}_2^0(k)\right| + \frac{C}{|k|}\left(e^{-\gamma t} + e^{-|k|^2t}\right)\left|\hat{\mathscr{U}}_2^0(k)\right| \\ &+ \left(Ce^{-\gamma t} + \frac{C}{|k|^2}e^{-|k|^2t}\right)\left|\hat{\mathscr{T}}_2^0(k)\right|, \qquad \text{as} \quad k\in\Omega_\infty.\end{aligned} \tag{3.66}$$

Furthermore, we begin to establish the L^∞ estimates for $\hat{\zeta}_2$, $\hat{\mathscr{U}}_2$ and $\hat{\mathscr{T}}_2$ on Ω_1. For this goal, we have to estimate $\mathscr{R}_{ij}$, $(1\le i,j\le 3)$ on Ω_1. Since all $\mathscr{R}_{ij}$ may be estimated in a similar way, we only study the estimate for $\mathscr{R}_{11}(t,k)$ for simplicity.

In view of Remark 3.1, we define

$$\mathcal{A} = \{\rho_j,\ j=1,2,\cdots,j_0\} = \{\rho \ \Big|\ \epsilon^2\le\rho\le L^2, \quad \mathfrak{R}=0\}.$$

It is easy to check that (3.24) can't have roots with three multiples. Then, according to Lemma 2.1, we may suppose ϕ to be the 1-multiple real root, and $\varphi_\pm$ the 2-multiple real roots of $\mathscr{C}(\mathfrak{x})=0$ (see (3.28)) when $|k|^2=\rho_j$. Hence,

$$\mathscr{C}(\mathfrak{x}) = (\mathfrak{x}-\phi)(\mathfrak{x}-\varphi_\pm)^2.$$

Then we have

$$\frac{d}{d\mathfrak{x}}\mathscr{C}(\mathfrak{x})\,\Big|_{(\mathfrak{x}=\phi,\rho=\rho_j)} = (\phi-\varphi_\pm)^2 > 0, \quad \psi\,\Big|_{\rho=\rho_j} = 0.$$

For some $\delta_0>0$ and $\eta>0$ small enough, there are pairwise disjoint neighborhoods $\mathfrak{B}(\rho_j,\epsilon_j)\subseteq\Omega_1$, such that

$$\min_{1\le j\le j_0}\ \inf_{\mathfrak{B}(\rho_j,\epsilon_j)}\ \frac{d}{d\mathfrak{x}}\mathscr{C}(\mathfrak{x})\,\Big|_{\mathfrak{x}=\phi} \ge \delta_0, \qquad \max_{1\le j\le j_0}\ \sup_{\mathfrak{B}(\rho_j,\epsilon_j)}\ \psi \le \eta. \tag{3.67}$$

We further split Ω_1 into

$$\Omega_1 = \Omega_{1,0} \cup \Omega_{1,-} \cup \Omega_{1,+},$$

where

$$\Omega_{1,0} = \{k \in \mathbb{R}^3 |\ \epsilon^2 \le \rho \le L^2,\ \rho \in \bigcup_{j=1}^{j_0} \mathfrak{B}(\rho_j, \epsilon_j)\},$$

$$\Omega_{1,-} = \{k \in \mathbb{R}^3 |\ \epsilon^2 \le \rho \le L^2,\ \rho \notin \bigcup_{j=1}^{j_0} \mathfrak{B}(\rho_j, \epsilon_j),\ \Re \le 0\}$$

and

$$\Omega_{1,+} = \{k \in \mathbb{R}^3 |\ \epsilon^2 \le \rho \le L^2,\ \rho \notin \bigcup_{j=1}^{j_0} \mathfrak{B}(\rho_j, \epsilon_j),\ \Re \ge 0\}.$$

Since all $\mathscr{R}_{ij}$ may be estimated in a similar way, we only study the estimate for $\mathscr{R}_{11}(t,k)$ for simplicity. When $k \in \Omega_{1,0}$, the expression of $\mathscr{R}_{11}$ may be rewritten in the form

$$\begin{aligned}\mathscr{R}_{11} = &- \mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3 \int_0^1 e^{\mathfrak{x}t}\left(\frac{1+(1+\mathfrak{x})t}{\mathfrak{x}(\mathfrak{x}_1-\mathfrak{x})} + \frac{(1+\mathfrak{x})(2\mathfrak{x}-\mathfrak{x}_1)}{\mathfrak{x}^2(\mathfrak{x}_1-\mathfrak{x})^2}\right)_{\mathfrak{x}=\tau\mathfrak{x}_3+(1-\tau)\mathfrak{x}_2} d\tau \\ &- \frac{e^{\mathfrak{x}_1 t}\mathfrak{x}_2\mathfrak{x}_3(1+\mathfrak{x}_1)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)}.\end{aligned} \tag{3.68}$$

For briefness, we still use ϕ and $\varphi_\pm$ to stand for solutions of $\mathscr{C}(\mathfrak{x}) = 0$ as usual, namely:

$$\begin{cases}\mathfrak{x}_1 = \phi, \\ \mathfrak{x}_{2,3} = \varphi_\pm = -\dfrac{1+\rho+\phi}{2} \pm \dfrac{\psi}{2}.\end{cases} \tag{3.69}$$

By (3.67), we have

$$(\mathfrak{x}_1 - \mathfrak{x}_2)(\mathfrak{x}_1 - \mathfrak{x}_3) = 3\phi^2 + 2\rho\phi + 2\phi + \frac{8}{3}\rho = \frac{d}{d\mathfrak{x}}\mathscr{C}(\mathfrak{x})\Big|_{\mathfrak{x}=\phi} \ge \delta_0. \tag{3.70}$$

In order to estimate the last term on the r.h.s. of (3.68), we have to establish the lower bound of $\mathfrak{x}_1 - (1-\tau)\mathfrak{x}_2 - \tau\mathfrak{x}_3$ and $(1-\tau)\mathfrak{x}_2 + \tau\mathfrak{x}_3$ with $\tau \in (0,1)$. By noticing (3.69), we get

$$\begin{cases}\mathfrak{x}_1 - (1-\tau)\mathfrak{x}_2 - \tau\mathfrak{x}_3 = \dfrac{1+\rho+3\phi}{2} - (1-2\tau)\dfrac{\sqrt{\psi}}{2}, \\ (1-\tau)\mathfrak{x}_2 + \tau\mathfrak{x}_3 = -\dfrac{1+\rho+\phi}{2} + (1-2\tau)\dfrac{\sqrt{\psi}}{2}, \quad \text{if } \mathfrak{x}_{2,3} \text{ are real,}\end{cases}$$

and

$$\begin{cases} \mathfrak{x}_1-(1-\tau)\mathfrak{x}_2-\tau\mathfrak{x}_3=\dfrac{1+\rho+3\phi}{2}-i(1-2\tau)\dfrac{\sqrt{-\psi}}{2}, \\ (1-\tau)\mathfrak{x}_2+\tau\mathfrak{x}_3=-\dfrac{1+\rho+\phi}{2}+i(1-2\tau)\dfrac{\sqrt{-\psi}}{2}, \quad \text{if } \mathfrak{x}_{2,3} \text{ are complex.} \end{cases}$$

Then (3.67) and the smallness of η imply that both $|\mathfrak{x}_1-(1-\tau)\mathfrak{x}_2-\tau\mathfrak{x}_3|$ and $|(1-\tau)\mathfrak{x}_2+\tau\mathfrak{x}_3|$ have a positive lower bound.

Therefore, combining the estimates above, we obtain

$$\begin{aligned} &|\mathscr{R}_{11}| \\ \le&\left|\frac{e^{\mathfrak{x}_1 t}\mathfrak{x}_2\mathfrak{x}_3(1+\mathfrak{x}_1)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)}\right|+\left|\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\int_0^1 e^{\mathfrak{x}t}\left(\frac{1+(1+\mathfrak{x})t}{\mathfrak{x}(\mathfrak{x}_1-\mathfrak{x})}+\frac{(1+\mathfrak{x})(2\mathfrak{x}-\mathfrak{x}_1)}{\mathfrak{x}^2(\mathfrak{x}_1-\mathfrak{x})^2}\right)_{\mathfrak{x}=\tau\mathfrak{x}_3+(1-\tau)\mathfrak{x}_2} d\tau\right| \\ \le& Ce^{-\gamma t}, \quad \text{as } k\in\Omega_{1,0}. \end{aligned} \tag{3.71}$$

On the other hand, when $k\in\Omega_{1,\pm}$, it is easy to see that

$$\mathfrak{x}_2-\mathfrak{x}_3=i\sqrt{-\psi}, \quad \text{as } k\in\Omega_{1,+},$$

and

$$\mathfrak{x}_2-\mathfrak{x}_3=\sqrt{\psi}, \quad \text{as } k\in\Omega_{1,-}.$$

Then $|\mathfrak{x}_2-\mathfrak{x}_3|$ has a uniform lower bound over $\Omega_{1,\pm}$. Thus, it follows that

$$\begin{aligned} |\mathscr{R}_{11}| \le& \left|\frac{e^{\mathfrak{x}_1 t}\mathfrak{x}_2\mathfrak{x}_3(1+\mathfrak{x}_1)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_3-\mathfrak{x}_1)}\right|+\left|\frac{e^{\mathfrak{x}_2 t}\mathfrak{x}_1\mathfrak{x}_3(1+\mathfrak{x}_2)}{(\mathfrak{x}_1-\mathfrak{x}_2)(\mathfrak{x}_2-\mathfrak{x}_3)}\right|+\left|\frac{e^{\mathfrak{x}_3 t}\mathfrak{x}_1\mathfrak{x}_2(1+\mathfrak{x}_3)}{(\mathfrak{x}_2-\mathfrak{x}_3)(\mathfrak{x}_3-\mathfrak{x}_1)}\right| \\ \le& Ce^{-\gamma t}, \quad \text{as } k\in\Omega_{1,\pm}. \end{aligned} \tag{3.72}$$

Hence, from (3.71)-(3.72), we obtain

$$|\mathscr{R}_{11}|\le Ce^{-\gamma t}, \quad \text{as } k\in\Omega_1. \tag{3.73}$$

Similarly, we also get

$$|\mathscr{R}_{12}|,|\mathscr{R}_{13}|,|\mathscr{R}_{21}|,|\mathscr{R}_{22}|,|\mathscr{R}_{23}|,|\mathscr{R}_{31}|,|\mathscr{R}_{32}|,|\mathscr{R}_{33}|\le Ce^{-\gamma t}, \quad \text{as } k\in\Omega_1. \tag{3.74}$$

Based on (3.73)-(3.74) and (3.55), we obtain

$$\begin{cases} \left|\hat{\zeta}_2(t,k)\right| \le |\mathscr{R}_{11}| \left|\hat{\zeta}_2^0(k)\right| + |\mathscr{R}_{12}| \left|\hat{\mathscr{U}}_2^0(k)\right| + |\mathscr{R}_{13}| \left|\hat{\mathscr{T}}_2^0(k)\right| \\ \qquad \le Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right| + \left|\hat{\mathscr{T}}_2^0(k)\right|\right), \\ \left|\hat{\mathscr{U}}_2(t,k)\right| \le |\mathscr{R}_{21}| \left|\hat{\zeta}_2^0(k)\right| + |\mathscr{R}_{22}| \left|\hat{\mathscr{U}}_2^0(k)\right| + |\mathscr{R}_{23}| \left|\hat{\mathscr{T}}_2^0(k)\right| \\ \qquad \le Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right| + \left|\hat{\mathscr{T}}_2^0(k)\right|\right), \\ \left|\hat{\mathscr{T}}_2(t,k)\right| \le |\mathscr{R}_{31}| \left|\hat{\zeta}_2^0(k)\right| + |\mathscr{R}_{32}| \left|\hat{\mathscr{U}}_2^0(k)\right| + |\mathscr{R}_{33}| \left|\hat{\mathscr{T}}_2^0(k)\right| \\ \qquad \le Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right| + \left|\hat{\mathscr{T}}_2^0(k)\right|\right), \quad \text{as } k \in \Omega_1. \end{cases} \tag{3.75}$$

Thus, (3.56)-(3.57) follow by combining (3.61)-(3.66) and (3.75). We have finished the proof of Lemma 3.2. □

Next, by applying Lemma 2.4 to the estimates in Lemma 3.2 above, it is straightforward to get the decay property for every component of the solution $\mathscr{W}_2 = (\zeta_2,\ \mathscr{U}_2,\ \mathscr{T}_2)$.

Theorem 3.2. *Let $j \ge 0$ be an integer and $1 \le p, r \le 2 \le q \le \infty$. Suppose $\mathscr{W}_2(t) = e^{t\mathscr{L}_2}\mathscr{W}_2^0$ to be the solution to the initial value problem (3.12)-(3.13). Then, for any $t \ge 0$, $\mathscr{W}_2$ satisfies the following time decay property:*

$$\begin{aligned} \left\|\nabla^j \zeta_2\right\|_{L^q} \le & Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^p} + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j+1}{2}}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^p} \\ & + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}}\left\|\mathscr{U}_2^0\right\|_{L^p} \\ & + Ce^{-\gamma t}\left(\left\|\nabla^{m(j,r,q)}\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^r} + \left\|\nabla^{m(j-1,r,q)}\mathscr{U}_2^0\right\|_{L^r}\right), \end{aligned} \tag{3.76}$$

$$\begin{aligned} \left\|\nabla^j \mathscr{U}_2\right\|_{L^q} \le & Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^p} + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^p} \\ & + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j+1}{2}}\left\|\mathscr{T}_2^0\right\|_{L^p} \\ & + Ce^{-\gamma t}\left(\left\|\nabla^{m(j-1,r,q)}\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^r} + \left\|\nabla^{m(j,r,q)}\mathscr{U}_2^0\right\|_{L^r}\right) \end{aligned} \tag{3.77}$$

and

$$\begin{aligned} \left\|\nabla^j \mathscr{T}_2\right\|_{L^q} \le & Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^p} + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j+1}{2}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^p} \\ & + C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}}\left\|\mathscr{T}_2^0\right\|_{L^p} \\ & + Ce^{-\gamma t}\left(\left\|\nabla^{m(j,r,q)}\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^r} + \left\|\nabla^{m(j-1,r,q)}\mathscr{U}_2^0\right\|_{L^r}\right). \end{aligned} \tag{3.78}$$

For later use, we list the following result which is an immediate corollary from Theorem 3.2.

Corollary 3.1. *Let $\mathscr{W}_2(t) = e^{t\mathscr{L}_2}\mathscr{W}_2^0$ be the solution to the Initial value problem (3.12)-(3.13). Then, for any $t \geq 0$, $\mathscr{W}_2$ satisfies the following time decay property:*

$$\begin{cases} \|\zeta_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{4}}\left\|\mathscr{U}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|, \\ \|\mathscr{U}_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|, \\ \|\mathscr{T}_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|, \end{cases} \tag{3.79}$$

$$\begin{cases} \|\nabla\zeta_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\mathscr{U}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\nabla\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\mathscr{U}_2^0\right\|\right), \\ \|\nabla\mathscr{U}_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\nabla\mathscr{U}_2^0\right\|\right), \\ \|\nabla\mathscr{T}_2\| \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{7}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{5}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\nabla\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\mathscr{U}_2^0\right\|\right), \end{cases} \tag{3.80}$$

and

$$\begin{cases} \|\zeta_2\|_{L^\infty} \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-2}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{2}}\left\|\mathscr{U}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\nabla^2\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\nabla\mathscr{U}_2^0\right\|\right), \\ \|\mathscr{U}_2\|_{L^\infty} \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{2}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-2}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\nabla\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\nabla^2\mathscr{U}_2^0\right\|\right), \\ \|\mathscr{T}_2\|_{L^\infty} \leq Ce^{-t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0, \mathscr{T}_2^0\right)\right\|_{L^1} + C(1+t)^{-2}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + C(1+t)^{-\frac{3}{2}}\left\|\mathscr{T}_2^0\right\|_{L^1} \\ \qquad + Ce^{-\gamma t}\left(\left\|\nabla^2\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\| + \left\|\nabla\mathscr{U}_2^0\right\|\right). \end{cases} \tag{3.81}$$

4. Time decay rates for the non-isentropic CNS-M system

4.1. Time decay rates for energy functionals

In this subsection, we prove the decay rate (2.15) in Proposition 2.2 for the energy $\|\mathscr{W}(t)\|_s^2$. We begin with the following Lemma which can be seen directly from the proof of Theorem 2.1.

Lemma 4.1. *Let $\mathscr{W} = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the initial value problem (2.5)-(2.6) with initial data $\mathscr{W}^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 2.1. Then, if $\mathfrak{E}_s(\mathscr{W}^0)$ is small enough, for any $t \geq 0$,*

$$\frac{d}{dt}\mathfrak{E}_s(\mathscr{W}(t)) + \mathfrak{D}_s(\mathscr{W}(t)) \leq 0. \tag{4.1}$$

For $p > 0$, it follows from Lemma 4.1 that

$$\begin{aligned}
&(1+t)^p\mathfrak{E}_s(\mathscr{W}(t)) + \int_0^t (1+\tau)^p\mathfrak{D}_s(\mathscr{W}(\tau))d\tau \\
\leq&\mathfrak{E}_s(\mathscr{W}^0) + Cp\int_0^t (1+\tau)^{p-1}\Big(\left\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(\tau)\right\|^2 + \|\mathscr{E}(\tau)\|^2 + \|\mathscr{B}(\tau)\|^2 + \|\nabla\mathscr{B}(\tau)\|^2 \\
&+ \left\|\left(\zeta^e + \zeta^i\right)(\tau)\right\|^2 + \left\|\left(\mathscr{U}^e + \mathscr{U}^i\right)(\tau)\right\|^2 + \mathfrak{D}_{s+1}(\mathscr{W}(\tau))\Big)d\tau,
\end{aligned}$$

where we used

$$\begin{aligned}
\mathfrak{E}_s(\mathscr{W}(t)) \leq& \left\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(t)\right\|^2 + \|\mathscr{E}(t)\|^2 + \|\mathscr{B}(t)\|^2 + \|\nabla\mathscr{B}(t)\|^2 + \left\|\left(\zeta^e + \zeta^i\right)(t)\right\|^2 \\
&+ \left\|\left(\mathscr{U}^e + \mathscr{U}^i\right)(t)\right\|^2 + \mathfrak{D}_{s+1}(\mathscr{W}(t)).
\end{aligned}$$

By using (4.1) again, we obtain

$$\begin{aligned}
&(1+t)^p\mathfrak{E}_s(\mathscr{W}(t)) + \int_0^t (1+\tau)^p\mathfrak{D}_s(\mathscr{W}(\tau))d\tau \\
\leq&C\mathfrak{E}_{s+1}(\mathscr{W}^0) + C\int_0^t (1+\tau)^{p-1}\Big(\left\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(\tau)\right\|^2 + \|\mathscr{E}(\tau)\|^2 + \|\mathscr{B}(\tau)\|^2 + \|\nabla\mathscr{B}(\tau)\|^2 \\
&+ \left\|\left(\zeta^e + \zeta^i\right)(\tau)\right\|^2 + \left\|\left(\mathscr{U}^e + \mathscr{U}^i\right)(\tau)\right\|^2\Big)d\tau, \quad \forall\, p > 0.
\end{aligned} \tag{4.2}$$

Next, we begin to establish the estimates on the integral term on the right hand side of (4.2). Applying the estimates on $\mathscr{U}_1$, $\mathscr{E}$ and $\mathscr{B}$ in (3.14)-(3.15), the estimate on $\nabla\mathscr{B}$ in (3.18) and the estimates on ζ_2 and $\mathscr{U}_2$ in (3.79) to (3.7) and (3.8), respectively, we have

$$
\begin{aligned}
&\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\| \\
\leq & C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_1^0, \mathscr{T}_1^0\right)\right\|_{L^1 \cap L^2}+C(1+t)^{-\frac{3}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1 \cap L^2}+\left\|\mathscr{E}^0\right\|_{L^1 \cap H^1}\right) \\
& +C(1+t)^{-\frac{5}{8}}\left\|\mathscr{B}^0\right\|_{L^1 \cap H^1}+C \int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(\mathscr{G}_{1e}-\mathscr{G}_{1i}, \mathscr{G}_{3e}-\mathscr{G}_{3i}\right)(\tau)\right\|_{L^1 \cap L^2} d \tau \\
& +C \int_0^t(1+t-\tau)^{-\frac{3}{4}}\left(\left\|\left(\mathscr{G}_{2e}-\mathscr{G}_{2i}\right)(\tau)\right\|_{L^1 \cap L^2}+\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1 \cap H^1}\right) d \tau,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\|\mathscr{E}(t)\| \leq & C(1+t)^{-\frac{3}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1 \cap H^1}+\left\|\mathscr{E}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^3}\right)+C(1+t)^{-\frac{5}{4}}\left\|\mathscr{T}_1^0\right\|_{L^1 \cap L^2} \\
& +C(1+t)^{-\frac{9}{8}}\left\|\mathscr{B}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^3}+C \int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(\mathscr{G}_{3e}-\mathscr{G}_{3i}\right)(\tau)\right\|_{L^1 \cap L^2} d \tau \\
& +C \int_0^t(1+t-\tau)^{-\frac{3}{4}}\left(\left\|\left(\mathscr{G}_{2e}-\mathscr{G}_{2i}\right)(\tau)\right\|_{L^1 \cap L^2}+\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1 \cap L^2 \cap \dot{H}^2}\right) d \tau,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
\|\mathscr{B}(t)\| \leq & C(1+t)^{-\frac{5}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1 \cap H^1}+C(1+t)^{-\frac{9}{8}}\left\|\mathscr{E}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^3}+C(1+t)^{-\frac{3}{4}}\left\|\mathscr{B}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^3} \\
& +C \int_0^t(1+t-\tau)^{-\frac{5}{8}}\left\|\left(\mathscr{G}_{2e}-\mathscr{G}_{2i}\right)(\tau)\right\|_{L^1 \cap L^2} d \tau \\
& +C \int_0^t(1+t-\tau)^{-\frac{9}{8}}\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1 \cap L^2 \cap \dot{H}^3} d \tau,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
&\|\nabla \mathscr{B}(t)\| \\
\leq & C(1+t)^{-\frac{7}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1 \cap L^2 \cap \dot{H}^2}+C(1+t)^{-\frac{11}{8}}\left\|\mathscr{E}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^4}+C(1+t)^{-\frac{5}{8}}\left\|\mathscr{B}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^4} \\
& +C \int_0^t(1+t-\tau)^{-\frac{7}{8}}\left\|\left(\mathscr{G}_{2e}-\mathscr{G}_{2i}\right)(\tau)\right\|_{L^1 \cap L^2 \cap \dot{H}^2} d \tau \\
& +C \int_0^t(1+t-\tau)^{-\frac{11}{8}}\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1 \cap L^2 \cap \dot{H}^4} d \tau,
\end{aligned} \tag{4.6}
$$

$$\begin{aligned}\left\|\left(\zeta^e+\zeta^i\right)(t)\right\| \leq & C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^1\cap L^2}+C(1+t)^{-\frac{3}{4}}\left\|\mathscr{U}_2^0\right\|_{L^1\cap L^2}\\ &+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{1e}+\mathscr{G}_{1i}, \mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}d\tau\\ &+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|(\mathscr{G}_{2e}+\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2}d\tau,\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}\left\|\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\| \leq & C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1\cap L^2}+C(1+t)^{-\frac{5}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1\cap L^2}\\ &+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|(\mathscr{G}_{1e}+\mathscr{G}_{1i}, \mathscr{G}_{2e}+\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2}d\tau\\ &+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}d\tau.\end{aligned}\tag{4.8}$$

It is direct to check that for any $0 \leq \tau \leq t$,

$$\|(\mathscr{G}_{1e}+\mathscr{G}_{1i}, \mathscr{G}_{2e}+\mathscr{G}_{2i}, \mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2} \leq C\mathfrak{E}_s(\mathscr{W}(\tau)) \leq C(1+\tau)^{-\frac{3}{4}}\mathfrak{E}_{s,\infty}(\mathscr{W}(t)),$$

and

$$\begin{aligned}&\|(\mathscr{G}_{1e}-\mathscr{G}_{1i}, \mathscr{G}_{3e}-\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}+\|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap H^4}\\ &\leq C(1+\tau)^{-\frac{3}{4}}\mathfrak{E}_{s,\infty}(\mathscr{W}(t)),\end{aligned}$$

where and in the rest of this paper

$$\mathfrak{E}_{s,\infty}(\mathscr{W}(t)) := \sup_{0\leqslant\tau\leqslant t}(1+\tau)^{\frac{3}{4}}\mathfrak{E}_s(\mathscr{W}(\tau)).$$

Putting the two previous inequalities into (4.3)-(4.8) implies, respectively,

$$\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\| \leq C(1+t)^{-\frac{5}{8}}\left(\left\|\left(\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap L^2}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right),\tag{4.9}$$

$$\|\mathscr{E}(t)\| \leq C(1+t)^{-\frac{3}{4}}\left(\left\|\left(\mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap \dot{H}^3}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right),\tag{4.10}$$

$$\|\mathscr{B}(t)\| \leq C(1+t)^{-\frac{5}{8}}\left(\left\|\left(\mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap H^1\cap \dot{H}^3}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right),\tag{4.11}$$

$$\|\nabla\mathscr{B}(t)\| \leq C(1+t)^{-\frac{5}{8}}\left(\left\|\left(\mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap \dot{H}^2\cap \dot{H}^4}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right),\tag{4.12}$$

$$\left\|\left(\zeta^e+\zeta^i\right)(t)\right\| \leq C(1+t)^{-\frac{3}{4}}\left(\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1\cap L^2}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right),\tag{4.13}$$

and

$$\left\|\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\| \leq C(1+t)^{-\frac{3}{4}}\left(\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1\cap L^2}+\mathfrak{E}_{s,\infty}(\mathscr{W}(t))\right). \tag{4.14}$$

Next, choosing $p=\dfrac{3}{4}+\varepsilon$ in (4.2) with $\varepsilon>0$ sufficiently small and using (4.9)-(4.14), we obtain

$$\|\mathscr{W}(t)\|_s \leq C\mathfrak{E}_s(\mathscr{W}(t))^{\frac{1}{2}} \leq C\omega_{s+1}\left(\mathscr{W}^0\right)(1+t)^{-\frac{3}{8}}, \quad \forall\, t\geq 0,$$

that is (2.15).

4.2. *Time decay rates for higher order energy functionals*

In this subsection, we investigate the decay estimate of the higher order energy $\|\nabla\mathscr{W}(t)\|_{s-1}^2$, that is (2.16) in Proposition 2.2. We start with the following Lemma.

Lemma 4.2. *Let $\mathscr{W}=(\zeta^\nu,\mathscr{U}^\nu,\mathscr{T}^\nu,\mathscr{E},\mathscr{B})$ be the solution to the initial value problem (2.5)-(2.6) with initial data $\mathscr{W}^0=(\zeta^{\nu 0},\mathscr{U}^{\nu 0},\mathscr{T}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 2.1. Then, if $\mathfrak{E}_s(\mathscr{W}^0)$ is small enough, there exist the higher order energy functionals $\mathfrak{E}_s^h(\cdot)$ and the higher order dissipative functionals $\mathfrak{D}_s^h(\cdot)$ in the form of (2.9) and (2.11) such that for any $t\geq 0$,*

$$\frac{d}{dt}\mathfrak{E}_s^h(\mathscr{W}(t))+\mathfrak{D}_s^h(\mathscr{W}(t))\leq 0. \tag{4.15}$$

Proof. The proof is similar to that in Theorem 2.1. Indeed, by choosing $|\alpha|\geq 1$, then corresponding to (2.19), (2.39), (2.40) and (2.41), it can also be checked that

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{1\leq|\alpha|\leq s}\sum_{\nu=e,i}\left\langle\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)\partial^\alpha\mathscr{V}^\nu,\partial^\alpha\mathscr{V}^\nu\right\rangle+\|\nabla\mathscr{E}\|_{s-1}^2+\|\nabla\mathscr{B}\|_{s-1}^2\right)\\
&\quad+2\sum_{\nu=e,i}\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+3\sum_{1\leq|\alpha|\leq s}\sum_{\nu=e,i}\left\langle\frac{\mathscr{N}^\nu}{\mathcal{T}^\nu},|\partial^\alpha\mathscr{T}^\nu|^2\right\rangle\\
&\leq C\|\mathscr{W}\|_s\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right),\\
&\frac{d}{dt}\sum_{1\leq|\alpha|\leq s-1}\sum_{\nu=e,i}\left\langle\partial^\alpha\mathscr{U}^\nu,\partial^\alpha\nabla\zeta^\nu\right\rangle+c_0\sum_{\nu=e,i}\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+c_0\left\|\nabla\zeta^e-\zeta^i\right\|^2\\
&\leq C\sum_{\nu=e,i}\left(\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right)\\
&\quad+C\|\mathscr{W}\|_s\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right),
\end{aligned}$$

$$\frac{d}{dt}\sum_{2\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle+c_0\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2$$
$$\le C\|\mathscr{W}\|_s\left(\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right)+\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2\right)+\varepsilon\left\|\nabla^3\mathscr{B}\right\|_{s-4}^2$$
$$+C\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2+\|\nabla\mathscr{T}^\nu\|_{s-1}^2\right),$$

and

$$\frac{d}{dt}\sum_{2\le|\alpha|\le s-2}\left\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\right\rangle+c_0\left\|\nabla^3\mathscr{B}\right\|_{s-4}^2$$
$$\le C\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2+C\sum_{\nu=e,i}\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-3}^2+C\|\mathscr{W}\|_s\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2\right).$$

Next we define the higher order energy functionals as

$$\begin{aligned}\mathfrak{E}_s^h(\mathscr{W}(t))=&\sum_{1\le|\alpha|\le s}\sum_{\nu=e,i}\left\langle\mathscr{A}_0^\nu\left(\mathscr{N}^\nu,\mathcal{T}^\nu\right)\partial^\alpha\mathscr{V}^\nu,\partial^\alpha\mathscr{V}^\nu\right\rangle+\|\nabla\mathscr{E}\|_{s-1}^2+\|\nabla\mathscr{B}\|_{s-1}^2\\&+\mathfrak{K}_1\sum_{1\le|\alpha|\le s-1}\sum_{\nu=e,i}\left\langle\partial^\alpha\mathscr{U}^\nu,\nabla\partial^\alpha\zeta^\nu\right\rangle+\mathfrak{K}_2\sum_{2\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle\\&+\mathfrak{K}_3\sum_{2\le|\alpha|\le s-2}\left\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\right\rangle.\end{aligned}\tag{4.16}$$

Similarly, we choose $0<\varepsilon\ll 1$ and $0<\mathfrak{K}_3\ll\mathfrak{K}_2\ll\mathfrak{K}_1\ll 1$ to be sufficiently small with $2\varepsilon\mathfrak{K}_2\ll\mathfrak{K}_3$, such that $\mathfrak{E}_s^h(\mathscr{W}(t))\sim\|\nabla\mathscr{W}(t)\|_{s-1}^2$, that is $\mathfrak{E}_s^h(\cdot)$ is a higher order energy functionals which satisfies (2.9), and furthermore, the summation of the four previously estimates with coefficients corresponding to (4.16) gives (4.15). We have finished the proof of Lemma 4.2. □

It follows from Lemma 4.2 that

$$\frac{d}{dt}\mathfrak{E}_s^h(\mathscr{W}(t))+\mathfrak{E}_s^h(\mathscr{W}(t))$$
$$\le C\Big(\left\|\nabla\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\|^2+\|\nabla\mathscr{E}(t)\|^2+\|\nabla\mathscr{B}(t)\|^2+\left\|\nabla^2\mathscr{B}(t)\right\|^2+\left\|\nabla^s\mathscr{E}(t)\right\|^2+\left\|\nabla^s\mathscr{B}(t)\right\|^2$$
$$+\left\|\nabla\left(\zeta^e+\zeta^i\right)(t)\right\|^2+\left\|\nabla\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\|^2\Big),$$

which gives

$$
\begin{aligned}
\mathfrak{E}_s^h(\mathscr{W}(t)) \leq & e^{-t}\mathfrak{E}_s^h(\mathscr{W}^0) + C\int_0^t e^{-(t-\tau)}\Big(\left\|\nabla\left(\mathscr{U}^e-\mathscr{U}^i\right)(\tau)\right\|^2 + \|\nabla\mathscr{E}(\tau)\|^2 + \|\nabla\mathscr{B}(\tau)\|^2 \\
& + \left\|\nabla^2\mathscr{B}(\tau)\right\|^2 + \left\|\nabla^s\mathscr{E}(\tau)\right\|^2 + \left\|\nabla^s\mathscr{B}(\tau)\right\|^2 + \left\|\nabla\left(\zeta^e+\zeta^i\right)(\tau)\right\|^2 \\
& + \left\|\nabla\left(\mathscr{U}^e+\mathscr{U}^i\right)(\tau)\right\|^2\Big)d\tau.
\end{aligned}
\tag{4.17}
$$

In what follows, we estimate the time integral term on the right hand side of (4.17).

Lemma 4.3. *Let $\mathscr{W} = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the Initial value problem (2.5)-(2.6) with initial data $\mathscr{W}^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{T}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 2.1. Then, if $\omega_{s+5}(\mathscr{W}^0)$ is small enough, for any $t \geq 0$,*

$$
\begin{aligned}
& \left\|\nabla\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\|^2 + \|\nabla(\mathscr{E},\mathscr{B})(t)\|^2 + \left\|\nabla^2\mathscr{B}(t)\right\|^2 \\
& + \left\|\nabla^s(\mathscr{E},\mathscr{B})(t)\right\|^2 + \left\|\nabla\left(\zeta^e+\zeta^i\right)(t)\right\|^2 + \left\|\nabla\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\|^2 \\
\leq & C\left(\omega_{s+5}\left(\mathscr{W}^0\right)\right)^2(1+t)^{-\frac{5}{4}}.
\end{aligned}
\tag{4.18}
$$

Proof. By applying the estimates on $\nabla\mathscr{U}_1$, $\nabla\mathscr{E}$, $\nabla\mathscr{B}$, $\nabla^2\mathscr{B}$ and $\nabla^s(\mathscr{E},\mathscr{B})$ in (3.18)-(3.19) to (3.7), respectively, and using (2.15), we have

$$
\begin{aligned}
\|\nabla\mathscr{U}_1(t)\| \leq & C(1+t)^{-\frac{7}{4}}\left\|\left(\zeta_1^0, \mathscr{T}_1^0\right)\right\|_{L^1\cap L^2} + C(1+t)^{-\frac{9}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2} \\
& + C(1+t)^{-\frac{5}{4}}\left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^2} + C(1+t)^{-\frac{7}{8}}\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^2} \\
& + C\int_0^t (1+t-\tau)^{-\frac{7}{4}}\|(\mathscr{G}_{1e}-\mathscr{G}_{1i}, \mathscr{G}_{3e}-\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}d\tau \\
& + C\int_0^t (1+t-\tau)^{-\frac{9}{8}}\|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2}d\tau \\
& + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^2}d\tau \\
\leq & C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla\mathscr{E}(t)\| \leq & C(1+t)^{-\frac{5}{4}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap \dot{H}^2} + C(1+t)^{-\frac{7}{4}}\left\|\mathscr{T}_1^0\right\|_{L^1\cap L^2} + C(1+t)^{-\frac{5}{4}}\left\|\mathscr{E}^0\right\|_{L^1\cap H^1\cap \dot{H}^4} \\
& + C(1+t)^{-\frac{11}{8}}\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^4} + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^2}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_0^t (1+t-\tau)^{-\frac{7}{4}} \|(\mathscr{G}_{3e}-\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2} d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap H^1\cap \dot{H}^4} d\tau \\
&\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla \mathscr{B}(t)\| \le& C(1+t)^{-\frac{7}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap \dot{H}^2} + C(1+t)^{-\frac{11}{8}} \left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^4} \\
&+ C(1+t)^{-\frac{5}{8}} \left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^4} \\
&+ C\int_0^t (1+t-\tau)^{-\frac{7}{8}} \|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^2} d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{11}{8}} \|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^4} d\tau \\
\le& C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{5}{8}},
\end{aligned}
$$

$$
\begin{aligned}
\left\|\nabla^2 \mathscr{B}(t)\right\| \le& C(1+t)^{-\frac{9}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap \dot{H}^3} + C(1+t)^{-\frac{13}{8}} \left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^5} \\
&+ C(1+t)^{-\frac{7}{8}} \left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^5} \\
&+ C\int_0^t (1+t-\tau)^{-\frac{9}{8}} \|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3} d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{13}{8}} \|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^5} d\tau \\
\le& C\omega_6(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\nabla^s(\mathscr{E},\mathscr{B})(t)\right\| \le& C(1+t)^{-1} \left\|\left(\mathscr{U}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^2\cap \dot{H}^{s+2}} \\
&+ C\int_0^t (1+t-\tau)^{-1} \|(\mathscr{G}_{2e}-\mathscr{G}_{2i},\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^2\cap \dot{H}^{s+2}} d\tau \\
\le& C\omega_{s+5}(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}\ln(3+t),
\end{aligned}
$$

where we have used the smallness of $\omega_{s+5}(\mathscr{W}^0)$. Furthermore, by (2.15) and applying the estimates on $\nabla\zeta_2$ and $\nabla\mathscr{U}_2$ in (3.80) to (3.8), respectively, we get

$$\begin{aligned}\left\|\nabla\left(\zeta^e+\zeta^i\right)(t)\right\| \leq & C(1+t)^{-\frac{7}{4}}\left\|\left(\zeta_2^0, \mathscr{T}_2^0\right)\right\|_{L^1\cap \dot{H}^1}+C(1+t)^{-\frac{5}{4}}\left\|\mathscr{U}_2^0\right\|_{L^1\cap L^2} \\ & +C\int_0^t(1+t-\tau)^{-\frac{7}{4}}\|(\mathscr{G}_{1e}+\mathscr{G}_{1i}, \mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap \dot{H}^1}d\tau \\ & +C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{2e}+\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2}d\tau \\ \leq & C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},\end{aligned}$$

and

$$\begin{aligned}\|\nabla\mathscr{U}_2\| \leq & C(1+t)^{-\frac{5}{4}}\left(\left\|\zeta_2^0\right\|_{L^1\cap L^2}+\left\|\mathscr{U}_2^0\right\|_{L^1\cap \dot{H}^1}\right)+C(1+t)^{-\frac{7}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1\cap L^2} \\ & +C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left(\|\mathscr{G}_{1e}+\mathscr{G}_{1i}\|_{L^1\cap L^2}+\|\mathscr{G}_{2e}+\mathscr{G}_{2i}\|_{L^1\cap \dot{H}^1}\right)(\tau)d\tau \\ & +C\int_0^t(1+t-\tau)^{-\frac{7}{4}}\|(\mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}d\tau \\ \leq & C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}.\end{aligned}$$

The proof of Lemma 4.3 is completed. □

Then, putting (4.18) into (4.17), we have

$$\mathfrak{E}_s^h(\mathscr{W}(t)) \leq e^{-t}\mathfrak{E}_s^h(\mathscr{W}^0)+C\left(\omega_{s+5}\left(\mathscr{W}^0\right)\right)^2(1+t)^{-\frac{5}{4}}.$$

Since $\mathfrak{E}_s^h(\mathscr{W}(t)) \sim \|\nabla\mathscr{W}(t)\|_{s-1}^2$ holds true for any $t \geq 0$, (2.16) follows. We have completed the proof of Proposition 2.2. □

4.3. Time decay rates in L^q

In this subsection, we investigate the decay rates of solutions $\mathscr{W}=(\zeta^\nu, \mathscr{U}^\nu, \mathscr{T}^\nu, \mathscr{E}, \mathscr{B})$ to problem (2.5)-(2.6) in L^q with $2 \leq q \leq +\infty$, and complete the proof for the second part of Theorem 1.1. In what follows, we always suppose $\omega_6(\mathscr{W}^0)$ to be sufficiently small. For $s \geq 4$, it follows from Proposition 2.2 that if $\omega_{s+1}(\mathscr{W}^0)$ is sufficiently small,

$$\|\mathscr{W}(t)\|_s \leq C\omega_{s+1}(\mathscr{W}^0)(1+t)^{-\frac{3}{8}}. \tag{4.19}$$

Next we establish the estimates on $\mathscr{B}$, $\mathscr{U}^e-\mathscr{U}^i$, $\mathscr{E}$, $\mathscr{U}^e+\mathscr{U}^i$, $\zeta^e-\zeta^i$, $\mathscr{T}^e-\mathscr{T}^i$, $\zeta^e+\zeta^i$ and $\mathscr{T}^e+\mathscr{T}^i$ in turn as follows.

Estimate on $\|\mathscr{B}\|_{L^q}$. For L^2 rate, in view of (4.19), we have

$$\|\mathscr{B}(t)\| \le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{8}}.$$

For L^∞ rate, by applying L^∞ estimate on $\mathscr{B}$ of (3.17) to (3.7), we get

$$\begin{aligned}\|\mathscr{B}(t)\|_{L^\infty} \le & C(1+t)^{-1}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap \dot{H}^3} + C(1+t)^{-\frac{3}{2}}\left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^5} \\ & + C(1+t)^{-\frac{3}{4}}\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^5} \\ & + C\int_0^t (1+t-\tau)^{-1}\|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3} d\tau \\ & + C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^5} d\tau.\end{aligned}$$

By (4.19), since

$$\begin{aligned}\|(\mathscr{G}_{2e}-\mathscr{G}_{2i})(t)\|_{L^1\cap L^2\cap \dot{H}^3} + \|(\mathscr{G}_{3e}-\mathscr{G}_{3i})(t)\|_{L^1\cap L^2\cap \dot{H}^5} & \le C\|\mathscr{W}(t)\|_5^2 \\ & \le C\left(\omega_6(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},\end{aligned}$$

we obtain

$$\|\mathscr{B}(t)\|_{L^\infty} \le C\omega_6(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}\ln(3+t).$$

Then, it follows from $L^2 - L^\infty$ interpolation that

$$\|\mathscr{B}(t)\|_{L^q} \le C\omega_6(\mathscr{W}^0)(1+t)^{-\frac{3}{4}+\frac{3}{4q}}\left(\ln(3+t)\right)^{1-\frac{2}{q}}, \quad \forall\, 2\le q\le +\infty. \tag{4.20}$$

Estimate on $\|\mathscr{U}^e-\mathscr{U}^i\|_{L^q}$. For L^2 rate, by applying the L^2 estimate on $\mathscr{U}_1$ in (3.14) to (3.7), we obtain

$$\begin{aligned}\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\| \le & C(1+t)^{-\frac{5}{8}}\left\|\left(\zeta_1^0,\mathscr{U}_1^0,\mathscr{T}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2} \\ & + C\int_0^t (1+t-\tau)^{-\frac{5}{8}}\|(\mathscr{G}_{1e}-\mathscr{G}_{1i},\mathscr{G}_{2e}-\mathscr{G}_{2i},\mathscr{G}_{3e}-\mathscr{G}_{3i},\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2} d\tau.\end{aligned}$$

By (4.19), in view of

$$\begin{aligned}\|(\mathscr{G}_{1e}-\mathscr{G}_{1i},\mathscr{G}_{2e}-\mathscr{G}_{2i},\mathscr{G}_{3e}-\mathscr{G}_{3i},\mathscr{G}_{4e}-\mathscr{G}_{4i})(t)\|_{L^1\cap L^2} & \le C\|\mathscr{W}(t)\|_4^2 \\ & \le C\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},\end{aligned}$$

we have

$$\left\|(\mathscr{U}^e - \mathscr{U}^i)(t)\right\| \le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{5}{8}}. \tag{4.21}$$

For L^∞ rate, by applying the L^∞ estimate on $\mathscr{U}_1$ in (3.16) to (3.7), we get

$$\begin{aligned}
&\left\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(t)\right\|_{L^\infty} \\
&\le C(1+t)^{-\frac{5}{4}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap H^1} + C(1+t)^{-2}\left\|\mathscr{T}_1^0\right\|_{L^1\cap \dot{H}^1} + C(1+t)^{-\frac{3}{2}}\left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^3} \\
&\quad + C(1+t)^{-1}\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^3} + C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|\mathscr{G}_{4e} - \mathscr{G}_{4i}\|_{L^1\cap L^2\cap \dot{H}^3}(\tau)d\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|\mathscr{G}_{2e} - \mathscr{G}_{2i}\|_{L^1\cap H^1}(\tau)d\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-2}\|\mathscr{G}_{3e} - \mathscr{G}_{3i}\|_{L^1\cap \dot{H}^1}(\tau)d\tau.
\end{aligned}$$

In view of

$$\|\mathscr{G}_{2e} - \mathscr{G}_{2i}\|_{L^1\cap H^1} + \|\mathscr{G}_{3e} - \mathscr{G}_{3i}\|_{L^1\cap \dot{H}^1} + \|\mathscr{G}_{4e} - \mathscr{G}_{4i}\|_{L^1\cap L^2\cap \dot{H}^3} \le \left(\omega_5(\mathscr{W}^0)\right)^2 (1+t)^{-\frac{3}{4}},$$

we have

$$\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(t)\|_{L^\infty} \le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},$$

where we have used the smallness of $\omega_5(\mathscr{W}^0)$.

Then, it follows from $L^2 - L^\infty$ interpolation that

$$\|\left(\mathscr{U}^e - \mathscr{U}^i\right)(t)\|_{L^q} \le C\omega_5(W^0)(1+t)^{-\frac{3}{4}+\frac{1}{4q}}, \quad \forall\, 2 \le q \le +\infty. \tag{4.22}$$

Estimate on $\|\mathscr{E}\|_{L^q}$. For L^2 rate, by applying the L^2 estimate on $\mathscr{E}$ in (3.15) to (3.7), we obtain

$$\begin{aligned}
&\|\mathscr{E}(t)\| \\
&\le C(1+t)^{-\frac{3}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap H^1} + \left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^3} + \left\|\mathscr{T}_1^0\right\|_{L^1\cap L^2} + \left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^3}\right) \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\left(\|(\mathscr{G}_{2e} - \mathscr{G}_{2i}, \mathscr{G}_{3e} - \mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2} + \|(\mathscr{G}_{4e} - \mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^2}\right)d\tau.
\end{aligned}$$

In view of (4.19), by

$$
\begin{aligned}
&\|(\mathscr{G}_{2e}-\mathscr{G}_{2i},\mathscr{G}_{3e}-\mathscr{G}_{3i})(t)\|_{L^1\cap L^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(t)\|_{L^1\cap L^2\cap \dot{H}^2}\\
&\le C\|\mathscr{W}(t)\|_4^2\le C\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

we have

$$
\|\mathscr{E}(t)\|\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}. \tag{4.23}
$$

For L^∞ rate, by applying the L^∞ estimates on $\mathscr{E}$ in (3.17) to (3.7), we get

$$
\begin{aligned}
&\|\mathscr{E}(t)\|_{L^\infty}\\
&\le C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap \dot{H}^3}+\left\|\mathscr{E}^0\right\|_{L^1\cap \dot{H}^2\cap \dot{H}^5}+\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap \dot{H}^5}\right)\\
&\quad+C(1+t)^{-2}\left\|\Theta_1^0\right\|_{L^1\cap \dot{H}^1}\\
&\quad+C\int_0^t(1+t-\tau)^{-\frac{3}{2}}\left(\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap L^2\cap \dot{H}^3}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1\cap \dot{H}^2\cap \dot{H}^5}\right)(\tau)d\tau\\
&\quad+C\int_0^t(1+t-\tau)^{-2}\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap \dot{H}^1}(\tau)d\tau.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap L^2\cap \dot{H}^3}+\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap \dot{H}^1}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1\cap \dot{H}^2\cap \dot{H}^5}\\
&\le C\|\mathscr{W}(t)\|_5^2\le\left(\omega_6(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

we get

$$
\|\mathscr{E}(t)\|_{L^\infty}\le C\omega_6(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},
$$

where the smallness of $\omega_6(\mathscr{W}^0)$ is used. Therefore, by L^2-L^∞ interpolation

$$
\|\mathscr{E}(t)\|_{L^q}\le C\omega_6(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},\quad \forall\, 2\le q\le+\infty. \tag{4.24}
$$

Estimate on $\|\mathscr{U}^e+\mathscr{U}^i\|_{L^q}$. For L^2 rate, by applying the L^2 estimates on $\mathscr{U}_2$ in (3.79) to (3.8), we have

$$\begin{aligned}
&\left\|\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\| \\
&\le C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0,\mathscr{U}_2^0\right)\right\|_{L^1\cap L^2}+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\|(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}+\mathscr{G}_{2i})(\tau)\|_{L^1\cap L^2}d\tau \\
&\quad+C(1+t)^{-\frac{5}{4}}\left\|\mathscr{T}_2^0\right\|_{L^1\cap L^2}+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|(\mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap L^2}d\tau.
\end{aligned}$$

By (4.19), since

$$\|(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}+\mathscr{G}_{2i},\mathscr{G}_{3e}+\mathscr{G}_{3i})(t)\|_{L^1\cap L^2}\le C\|\mathscr{W}(t)\|_4^2\le\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},$$

it follows that

$$\left\|(\mathscr{U}^e+\mathscr{U}^i)(t)\right\|\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}.$$

For L^∞ rate, we use the L^∞ estimates on $\mathscr{U}_2$ in (3.81) to (3.8) to obtain

$$\begin{aligned}
&\left\|\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\|_{L^\infty} \\
&\le C(1+t)^{-\frac{3}{2}}\left(\left\|\zeta_2^0\right\|_{L^1\cap\dot{H}^1}+\left\|\mathscr{U}_2^0\right\|_{L^1\cap\dot{H}^2}\right)+C(1+t)^{-2}\left\|\mathscr{T}_2^0\right\|_{L^1\cap\dot{H}^1} \\
&\quad+C\int_0^t(1+t-\tau)^{-\frac{3}{2}}\left(\|\mathscr{G}_{1e}+\mathscr{G}_{1i}\|_{L^1\cap\dot{H}^1}+\|\mathscr{G}_{2e}+\mathscr{G}_{2i}\|_{L^1\cap\dot{H}^2}\right)(\tau)d\tau \\
&\quad+C\int_0^t(1+t-\tau)^{-2}\|(\mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap\dot{H}^1}d\tau.
\end{aligned}$$

By (4.19), since

$$\begin{aligned}
&\|\mathscr{G}_{1e}+\mathscr{G}_{1i}\|_{L^1\cap\dot{H}^1}+\|\mathscr{G}_{2e}+\mathscr{G}_{2i}\|_{L^1\cap\dot{H}^2}+\|(\mathscr{G}_{3e}+\mathscr{G}_{3i})(\tau)\|_{L^1\cap\dot{H}^1} \\
&\le C\|\mathscr{W}(t)\|_4^2\le\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}},
\end{aligned}$$

we get

$$\|\mathscr{U}^e(t)+\mathscr{U}^i(t)\|_{L^\infty}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}.$$

Therefore, by L^2-L^∞ interpolation

$$\|\mathscr{U}^e(t)+\mathscr{U}^i(t)\|_{L^q}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}},\quad\forall\, 2\le q\le+\infty. \tag{4.25}$$

Estimate on $\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)\|_{L^q}$ *and* $\|\left(\zeta^e+\zeta^i,\mathscr{T}^e+\mathscr{T}^i\right)\|_{L^q}$. For L^2 rate, by applying the L^2 estimates on ζ_1 and $\mathscr{T}_1$ in (3.14) to (3.7), we have

$$\begin{aligned}\left\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)(t)\right\| \leq & Ce^{-\gamma t}\left\|\zeta_1^0\right\|_{L^1\cap L^2}+C(1+t)^{-\frac{5}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2}+\left\|\mathscr{E}^0\right\|_{L^1}\right)\\ &+C(1+t)^{-\frac{7}{4}}\left\|\mathscr{T}_1^0\right\|_{L^1\cap L^2}\\ &+C\int_0^t e^{-\gamma(t-\tau)}\|\mathscr{G}_{1e}-\mathscr{G}_{1i}\|_{L^1\cap L^2}(\tau)d\tau\\ &+C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\left(\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap L^2}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1}\right)(\tau)d\tau\\ &+C\int_0^t (1+t-\tau)^{-\frac{7}{4}}\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap L^2}d\tau.\end{aligned} \tag{4.26}$$

Since

$$\begin{aligned}&\|\mathscr{G}_{1e}-\mathscr{G}_{1i}\|_{L^1\cap L^2}+\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap L^2}+\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap L^2}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1}\\ &\leq C\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac{3}{4}}.\end{aligned}$$

Then (4.26) implies the decay estimate

$$\left\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)(t)\right\| \leq C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}. \tag{4.27}$$

Similarly to that for $\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)\|$, by applying the L^2 estimates on ζ_2 and $\mathscr{T}_2$ in (3.79) to (3.8), we obtain the decay estimate

$$\left\|\left(\zeta^e+\zeta^i,\mathscr{T}^e+\mathscr{T}^i\right)(t)\right\| \leq C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}. \tag{4.28}$$

Combining (4.27) and (4.28), we obtain

$$\left\|\left(\zeta^\nu,\mathscr{T}^\nu\right)(t)\right\| \leq C\omega_5(\mathscr{W}^0)(1+t)^{-\frac{3}{4}}. \tag{4.29}$$

For L^∞ rate, by applying the L^∞ estimates on ζ_1 and $\mathscr{T}_1$ in (3.16) to (3.7), we have the decay estimate

$$\left\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)(t)\right\|_{L^\infty}\le Ce^{-\gamma t}\left\|\zeta_1^0\right\|_{L^1\cap\dot H^2}+C(1+t)^{-2}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap\dot H^1}+\left\|\mathscr{E}^0\right\|_{L^1}\right)$$
$$+C(1+t)^{-\frac52}\left\|\mathscr{T}_1^0\right\|_{L^1\cap\dot H^2}+C\int_0^t e^{-\gamma(t-\tau)}\|\mathscr{G}_{1e}-\mathscr{G}_{1i}\|_{L^1\cap\dot H^2}(\tau)d\tau$$
$$+C\int_0^t(1+t-\tau)^{-2}\left(\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap\dot H^1}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1}\right)(\tau)d\tau$$
$$+C\int_0^t(1+t-\tau)^{-\frac52}\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap\dot H^2}d\tau. \tag{4.30}$$

Notice that

$$\begin{aligned}&\|\mathscr{G}_{1e}-\mathscr{G}_{1i}\|_{L^1\cap\dot H^2}+\|\mathscr{G}_{2e}-\mathscr{G}_{2i}\|_{L^1\cap\dot H^1}+\|\mathscr{G}_{3e}-\mathscr{G}_{3i}\|_{L^1\cap\dot H^2}+\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\|_{L^1}\\&\le C\left(\omega_5(\mathscr{W}^0)\right)^2(1+t)^{-\frac34}.\end{aligned} \tag{4.31}$$

Together with (4.30) yields

$$\left\|\left(\zeta^e-\zeta^i,\mathscr{T}^e-\mathscr{T}^i\right)(t)\right\|_{L^\infty}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac34}. \tag{4.32}$$

For $\|\left(\zeta^e+\zeta^i,\mathscr{T}^e+\mathscr{T}^i\right)\|_{L^\infty}$, by applying the L^∞ estimates on ζ_2 and $\mathscr{T}_2$ in (3.81) to (3.8), we have the decay estimate

$$\left\|\left(\zeta^e+\zeta^i,\mathscr{T}^e+\mathscr{T}^i\right)(t)\right\|_{L^\infty}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac34}. \tag{4.33}$$

Combining (4.32) and (4.33), we obtain

$$\left\|(\zeta^\nu,\mathscr{T}^\nu)(t)\right\|_{L^\infty}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac34}. \tag{4.34}$$

Then from (4.29) and (4.34), by L^2-L^∞ interpolation, it follows that

$$\left\|(\zeta^\nu,\mathscr{T}^\nu)(t)\right\|_{L^q}\le C\omega_5(\mathscr{W}^0)(1+t)^{-\frac34},\quad \forall\, 2\le q\le+\infty. \tag{4.35}$$

Thus, (4.35), (4.25), (4.22), (4.24) and (4.20) give (1.5), (1.6), (1.7), (1.8) and (1.9), respectively. We have finished the proof of Theorem 1.1. □

5. Global existence and asymptotic decay of the isentropic CNS-M system

5.1. *Reformulation of problem* (1.11)-(1.12)

Next, let $(\mathscr{N}^\nu,\mathscr{U}^\nu,\mathscr{E},\mathscr{B})$ be a local smooth solution to the initial value problem for system (1.11) with initial data (1.12) satisfying (1.4). Based on the transformation (2.4) about $\mathscr{N}^\nu$, we can rewrite (1.11)-(1.12) as

$$\begin{cases}\partial_t \zeta^\nu + \nabla\cdot\left(\left(1+\zeta^\nu\right)\mathscr{U}^\nu\right) = 0,\\ \partial_t \mathscr{U}^\nu + \left(\mathscr{U}^\nu\cdot\nabla\right)\mathscr{U}^\nu + \nabla\mathscr{H}_\nu\left(1+\zeta^\nu\right) = q_\nu\left(\mathscr{E}+\mathscr{U}^\nu\times\mathscr{B}\right) + \dfrac{\Delta\mathscr{U}^\nu}{1+\zeta^\nu},\\ \partial_t\mathscr{E} - \nabla\times\mathscr{B} - \mathscr{U}^e + \mathscr{U}^i = \zeta^e\mathscr{U}^e - \zeta^i\mathscr{U}^i, \quad \nabla\cdot\mathscr{E} = \zeta^i - \zeta^e,\\ \partial_t\mathscr{B} + \nabla\times\mathscr{E} = 0, \quad \nabla\cdot\mathscr{B} = 0, \quad (t,x)\in[0,+\infty)\times\mathbb{R}^3,\end{cases} \tag{5.1}$$

with the initial condition:

$$W|_{t=0} = W^0 := (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0), \quad x\in\mathbb{R}^3, \tag{5.2}$$

which also satisfies the compatibility condition (2.7).

In order to establish Theorem 1.2, for $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$, we use $\mathfrak{E}_s(W(t))$, $\mathfrak{E}_s^h(W(t))$, $\mathscr{D}_s(W(t))$ and $\mathscr{D}_s^h(W(t))$ to denote the energy functionals, the higher order energy functionals, the dissipative functionals and the higher order dissipative functionals for two charged carriers. They satisfy

$$\mathfrak{E}_s(W(t)) \sim \sum_{\nu=e,i}\left\|(\zeta^\nu,\mathscr{U}^\nu)\right\|_s^2 + \|(\mathscr{E},\mathscr{B})\|_s^2, \tag{5.3}$$

$$\mathfrak{E}_s^h(W(t)) \sim \sum_{\nu=e,i}\left\|\nabla(\zeta^\nu,u^\nu)\right\|_{s-1}^2 + \|\nabla(\mathscr{E},\mathscr{B})\|_{s-1}^2, \tag{5.4}$$

$$\mathscr{D}_s(W(t)) \sim \sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2 + \|\nabla\mathscr{U}^\nu\|_s^2\right) + \|\nabla\mathscr{E}\|_{s-2}^2 + \left\|\nabla^2\mathscr{B}\right\|_{s-3}^2 + \left\|\zeta^e - \zeta^i\right\|^2 \tag{5.5}$$

and

$$\mathscr{D}_s^h(W(t)) \sim \sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2 + \left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2\right) + \left\|\nabla^2\mathscr{E}\right\|_{s-3}^2 + \left\|\nabla^3\mathscr{B}\right\|_{s-4}^2 + \left\|\nabla(\zeta^e-\zeta^i)\right\|^2, \tag{5.6}$$

respectively. Now, for problem (5.1)-(5.2), we obtain the global existence result as follows.

Proposition 5.1. *Assume that $W^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfies the compatibility condition (2.7). Then, if $\mathfrak{E}_s(W^0)$ is small enough, the initial value problem (5.1)-(5.2) admits a unique global solution $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ satisfying*

$$\begin{aligned}&\mathscr{U}^\nu \in C^1\left(\mathbb{R}^+; H^{s-2}\left(\mathbb{R}^3\right)\right)\cap C\left(\mathbb{R}^+; H^s\left(\mathbb{R}^3\right)\right),\\ &(\zeta^\nu,\mathscr{E},\mathscr{B}) \in C^1\left(\mathbb{R}^+; H^{s-1}\left(\mathbb{R}^3\right)\right)\cap C\left(\mathbb{R}^+; H^s\left(\mathbb{R}^3\right)\right),\end{aligned} \tag{5.7}$$

and (2.13) in which $\mathfrak{E}_s(\mathscr{W}(t))$ and $\mathfrak{D}_s(\mathscr{W}(\tau))$ are replaced with $\mathfrak{E}_s(W(t))$ and $\mathscr{D}_s(W(\tau))$.

From Proposition 5.1, it is easy to get the existence result in Theorem 1.2. Furthermore, we obtain the following decay results.

Proposition 5.2. *Let $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the initial value problem (5.1)-(5.2) with initial data $W^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ which satisfies (2.7) in the sense of Proposition 5.1. Then, if $\omega_{s+1}(W^0)$ is sufficiently small,*

$$\|W(t)\|_s \leq C\omega_{s+1}(W^0)(1+t)^{-\frac{3}{8}}, \quad \forall t \geq 0. \tag{5.8}$$

Moreover, if $\omega_{s+6}(W^0)$ is sufficiently small, then, the solution also satisfies

$$\|\nabla W(t)\|_{s-1} \leq C\omega_{s+6}(W^0)(1+t)^{-\frac{5}{8}} \ln(3+t), \quad \forall t \geq 0. \tag{5.9}$$

5.2. Global existence for system (5.1)

Similarly to that in the subsection 2.3, the global existence of solutions satisfying (5.7) and (2.13) follows by combining the standard continuity argument and the a priori estimate as follows.

Theorem 5.1 *(A priori estimates for the bipolar isentropic CNS-M system). Let $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the initial value problem (5.1)-(5.2) satisfying the compatibility condition (2.7) for $t \in (0, T)$ with $T > 0$, in which $(\zeta^\nu, \mathscr{E}, \mathscr{B}) \in C^1\left([0,T]; H^{s-1}\right) \cap C([0,T]; H^s)$, and $\mathscr{U}^\nu \in C^1\left([0,T]; H^{s-2}\right) \cap C([0,T]; H^s)$. Then, if*

$$\sup_{0 \leq t \leq T} \|W(t)\|_s \leq \delta_0 \tag{5.10}$$

with δ_0 sufficiently small, there exist $\mathfrak{E}_s(\cdot)$ and $\mathscr{D}_s(\cdot)$ in the form of (5.3) and (5.5) such that for any $0 \leq t \leq T$,

$$\frac{d}{dt}\mathfrak{E}_s(W(t)) + \mathscr{D}_s(W(t)) \leq C\mathfrak{E}_s(W(t))^{\frac{1}{2}}\mathscr{D}_s(W(t)). \tag{5.11}$$

Proof. Corresponding to (2.19), (2.39), (2.40) and (2.41), we get the estimates as follows.

$$\frac{d}{dt}\|W\|_s^2 + \sum_{\nu=e,i} \|\nabla\mathscr{U}^\nu\|_s^2 \leq C\|W\|_s \sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2 + \|\nabla\mathscr{U}^\nu\|_s^2\right). \tag{5.12}$$

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\leq s-1}\sum_{\nu=e,i}\langle\partial^\alpha\mathscr{U}^\nu, \partial^\alpha\nabla\zeta^\nu\rangle + c_0\left\|\nabla\left(\zeta^e,\zeta^i\right)\right\|_{s-1}^2 + c_0\left\|\zeta^e - \zeta^i\right\|^2 \\
\leq& C\sum_{\nu=e,i}\|\nabla\mathscr{U}^\nu\|_s^2 + C\|W\|_s\sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2 + \|\nabla\mathscr{U}^\nu\|_s^2\right),
\end{aligned} \tag{5.13}$$

where c_0 is a positive constant.

$$
\begin{aligned}
&\frac{d}{dt}\sum_{1\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle+c_0\|\nabla\mathscr{E}\|_{s-2}^2\\
\le&C\|(\mathscr{E},\mathscr{B})\|_s\left(\sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2+\|\nabla\mathscr{U}^\nu\|_s^2\right)+\|\nabla\mathscr{E}\|_{s-2}^2\right)+\varepsilon\left\|\nabla^2\mathscr{B}\right\|_{s-3}^2\\
&+C\sum_{\nu=e,i}\left(\|\nabla\zeta^\nu\|_{s-1}^2+\|\nabla\mathscr{U}^\nu\|_s^2\right),
\end{aligned}
\tag{5.14}
$$

in which $\varepsilon>0$ is a small constant to be chosen later.

$$
\begin{aligned}
&\frac{d}{dt}\sum_{1\le|\alpha|\le s-2}\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\rangle+c_0\left\|\nabla^2\mathscr{B}\right\|_{s-3}^2\\
\le&C\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2+C\sum_{\nu=e,i}\|\nabla\mathscr{U}^\nu\|_{s-3}^2+C\|\mathscr{B}\|_s\sum_{\nu=e,i}\|\nabla(\zeta^\nu,\mathscr{U}^\nu)\|_{s-1}^2.
\end{aligned}
\tag{5.15}
$$

Finally, combining estimates (5.12)-(5.15), we define the energy functionals as

$$
\begin{aligned}
\mathfrak{E}_s(W(t))=&\|W\|_s^2+\mu_1\sum_{|\alpha|\le s-1}\sum_{\nu=e,i}\langle\partial^\alpha\mathscr{U}^\nu,\nabla\partial^\alpha\zeta^\nu\rangle+\mu_2\sum_{1\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle\\
&+\mu_3\sum_{1\le|\alpha|\le s-2}\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\rangle,
\end{aligned}
$$

where constants $0<\mu_3\ll\mu_2\ll\mu_1\ll1$ are to be chosen later. It follows that

$$
\mathfrak{E}_s(W(t))\sim||W||_s^2,
$$

as long as $0<\mu_j\ll1$, $j=1,2,3$, are small enough. Furthermore, by letting $\varepsilon>0$ and $0<\mu_3\ll\mu_2\ll\mu_1\ll1$ be sufficiently small with $2\varepsilon\mu_2\ll\mu_3$, summing (5.12), (5.13)$\times\mu_1$, (5.14)$\times\mu_2$ and (5.15)$\times\mu_3$, we get (5.11). The proof of Theorem 5.1 is finished. □

5.3. *Linearized homogeneous systems for system* (5.1)

Next, we rewrite system (5.1) as

$$
\begin{cases}
\partial_t\zeta^\nu+\nabla\cdot\mathscr{U}^\nu=\mathscr{G}_{1\nu},\\
\partial_t\mathscr{U}^\nu+\nabla\zeta^\nu-q_\nu\mathscr{E}-\Delta\mathscr{U}^\nu=\mathscr{G}_{2\nu}^*,\\
\partial_t\mathscr{E}-\nabla\times\mathscr{B}-\mathscr{U}^e+\mathscr{U}^i=\frac{1}{2}(\mathscr{G}_{4e}-\mathscr{G}_{4i}),\quad \nabla\cdot\mathscr{E}=\zeta^i-\zeta^e,\\
\partial_t\mathscr{B}+\nabla\times\mathscr{E}=0,\quad \nabla\cdot\mathscr{B}=0,\quad (t,x)\in[0,+\infty)\times\mathbb{R}^3,
\end{cases}
\tag{5.16}
$$

where $\mathscr{G}_{1\nu}$ and $\mathscr{G}_{4\nu}$ are defined in (3.2) and

$$
\mathscr{G}_{2\nu}^*=-\left(\mathscr{U}^\nu\cdot\nabla\right)\mathscr{U}^\nu-\nabla\mathscr{H}_\nu\left(1+\zeta^\nu\right)+\nabla\zeta^\nu+q_\nu\left(\mathscr{U}^\nu\times\mathscr{B}\right)-\frac{\zeta^\nu\Delta\mathscr{U}^\nu}{1+\zeta^\nu}.
\tag{5.17}
$$

Then it follows from system (5.16) and the transformation (3.3) on ζ_1 and $\mathscr{U}_1$ that $W_1 = (\zeta_1,\ \mathscr{U}_1,\ \mathscr{E},\ \mathscr{B})$ satisfies

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot \mathscr{U}_1 = \frac{1}{2}\left(\mathscr{G}_{1e} - \mathscr{G}_{1i}\right), \\ \partial_t \mathscr{U}_1 + \nabla \zeta_1 + \mathscr{E} - \Delta \mathscr{U}_1 = \frac{1}{2}\left(\mathscr{G}^*_{2e} - \mathscr{G}^*_{2i}\right), \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} - 2\mathscr{U}_1 = \frac{1}{2}\left(\mathscr{G}_{4e} - \mathscr{G}_{4i}\right), \quad \frac{1}{2}\nabla \cdot \mathscr{E} = -\zeta_1, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \qquad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{5.18}$$

with the initial condition

$$W_1|_{t=0} = W_1^0 := \left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right), \quad x \in \mathbb{R}^3,$$

which also satisfies (3.11). Here $\left(\zeta_1^0, \mathscr{U}_1^0\right)$ is given from $\left(\zeta^{e0}, \mathscr{U}^{e0}, \zeta^{i0}, \mathscr{U}^{i0}\right)$ due to (3.3).

Moreover, based on the transformation (3.5) about ζ_2 and $\mathscr{U}_2$, we obtain that $W_2 = (\zeta_2,\ \mathscr{U}_2)$ satisfies

$$\begin{cases} \partial_t \zeta_2 + \nabla \cdot \mathscr{U}_2 = \frac{1}{2}\left(\mathscr{G}_{1e} + \mathscr{G}_{1i}\right), \\ \partial_t \mathscr{U}_2 + \nabla \zeta_2 - \Delta \mathscr{U}_2 = \frac{1}{2}\left(\mathscr{G}^*_{2e} + \mathscr{G}^*_{2i}\right), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{5.19}$$

with the initial condition

$$W_2|_{t=0} = W_2^0 := \left(\zeta_2^0, \mathscr{U}_2^0\right), \quad x \in \mathbb{R}^3,$$

where $\left(\zeta_2^0, \mathscr{U}_2^0\right)$ is given from $\left(\zeta^{\nu 0}, \mathscr{U}^{\nu 0}\right)$ due to (3.5). Therefore, we define the solution $W_1 = (\zeta_1,\ \mathscr{U}_1,\ \mathscr{E},\ \mathscr{B})$ and $W_2 = (\zeta_2,\ \mathscr{U}_2)$, respectively, as follows

$$W_1(t) = e^{tL_1} W_1^0 + \frac{1}{2}\int_0^t e^{(t-\tau)L_1}\left(\mathscr{G}_{1e} - \mathscr{G}_{1i}, \mathscr{G}^*_{2e} - \mathscr{G}^*_{2i}, \mathscr{G}_{4e} - \mathscr{G}_{4i}\right)(\tau)d\tau, \tag{5.20}$$

and

$$W_2(t) = e^{tL_2} W_2^0 + \frac{1}{2}\int_0^t e^{(t-\tau)L_2}\left(\mathscr{G}_{1e} + \mathscr{G}_{1i}, \mathscr{G}^*_{2e} + \mathscr{G}^*_{2i}\right)(\tau)d\tau, \tag{5.21}$$

where $e^{tL_1}W_1^0$ and $e^{tL_2}W_2^0$, respectively, denote the solutions of the homogeneous initial value problems (5.22)-(5.23) and (5.24)-(5.25), which are given as follows.

The linearized homogeneous system of (5.18) is

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot \mathscr{U}_1 = 0, \\ \partial_t \mathscr{U}_1 + \nabla \zeta_1 + \mathscr{E} - \Delta \mathscr{U}_1 = 0, \\ \partial_t \mathscr{E} - \nabla \times \mathscr{B} - 2\mathscr{U}_1 = 0, \quad \dfrac{1}{2}\nabla \cdot \mathscr{E} = -\zeta_1, \\ \partial_t \mathscr{B} + \nabla \times \mathscr{E} = 0, \quad \nabla \cdot \mathscr{B} = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{5.22}$$

with the initial condition

$$W_1|_{t=0} = W_1^0 := \left(\zeta_1^0, \mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right), \quad x \in \mathbb{R}^3, \tag{5.23}$$

which satisfies the compatibility condition (3.11).

And the linearized homogeneous system of (5.19) is

$$\begin{cases} \partial_t \zeta_2 + \nabla \cdot \mathscr{U}_2 = 0, \\ \partial_t \mathscr{U}_2 + \nabla \zeta_2 - \Delta \mathscr{U}_2 = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \end{cases} \tag{5.24}$$

with the initial condition

$$W_2|_{t=0} = W_2^0 := \left(\zeta_2^0, \mathscr{U}_2^0\right), \quad x \in \mathbb{R}^3. \tag{5.25}$$

In the following, we use $W_1 = (\zeta_1,\ \mathscr{U}_1,\ \mathscr{E},\ \mathscr{B})$ to denote the solution of the initial value problem (5.22)-(5.23), and $W_2 = (\zeta_2,\ \mathscr{U}_2)$ as the one to the initial value problem (5.24)-(5.25).

For the initial value problem (5.22)-(5.23), we also get the $L^p - L^q$ decay property as follows.

Proposition 5.3. *Let $W_1(t) = e^{tL_1} W_1^0$ be the solution to the initial value problem (5.22)-(5.23) with initial data $W_1^0 = (\zeta_1^0,\ \mathscr{U}_1^0,\ \mathscr{E}^0,\ \mathscr{B}^0)$ satisfying (3.11). Then, for any $t \geq 0$, there exist constants $C > 0$ and $\gamma > 0$ such that W_1 satisfies the following time decay property*

$$\begin{cases} \|\zeta_1(t)\| \leq C(1+t)^{-\frac{5}{4}} \left\| \left(\zeta_1^0, \mathscr{U}_1^0\right) \right\|_{L^1 \cap L^2}, \\ \|\mathscr{U}_1(t)\| \leq C(1+t)^{-\frac{5}{4}} \left\|\zeta_1^0\right\|_{L^1\cap L^2} + C(1+t)^{-\frac{3}{4}} \left\| \left(\mathscr{U}_1^0, \mathscr{E}^0\right) \right\|_{L^1 \cap L^2} + C(1+t)^{-\frac{5}{8}} \left\|\mathscr{B}^0\right\|_{L^1 \cap L^2}, \\ \|\mathscr{E}(t)\| \leq C(1+t)^{-\frac{3}{4}} \left\|\mathscr{U}_1^0\right\|_{L^1 \cap L^2} + C(1+t)^{-\frac{3}{4}} \left\|\mathscr{E}^0\right\|_{L^1 \cap \dot{L^2} \cap H^2} + C(1+t)^{-\frac{9}{8}} \left\|\mathscr{B}^0\right\|_{L^1 \cap L^2 \cap \dot{H}^3}, \\ \|\mathscr{B}(t)\| \leq C(1+t)^{-\frac{5}{8}} \left\|\mathscr{U}_1^0\right\|_{L^1 \cap L^2} + C(1+t)^{-\frac{9}{8}} \left\|\mathscr{E}^0\right\|_{L^1 \cap \dot{L^2} \cap H^3} + C(1+t)^{-\frac{3}{8}} \left\|\mathscr{B}^0\right\|_{L^1 \cap L^2 \cap H^1}, \end{cases} \tag{5.26}$$

$$\begin{cases} \|\zeta_1(t)\|_{L^\infty} \le C(1+t)^{-2}\left\|\left(\mathscr{E}^0, \mathscr{U}_1^0\right)\right\|_{L^1} + Ce^{-\gamma t}\left(\left\|\zeta_1^0\right\|_{\dot{H}^2} + \left\|\mathscr{U}_1^0\right\|_{\dot{H}^1}\right), \\ \|\mathscr{U}_1(t)\|_{L^\infty} \le C(1+t)^{-1}\left(\left\|\zeta_1^0\right\|_{L^1} + \left\|\mathscr{B}^0\right\|_{L^1\cap\dot{H}^3}\right) + C(1+t)^{-\frac{5}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap\dot{H}^1} + \left\|\mathscr{E}^0\right\|_{L^1\cap\dot{H}^3}\right) \\ \qquad + Ce^{-\gamma t}\left(\left\|\zeta_1^0\right\|_{\dot{H}^1} + \left\|\mathscr{U}_1^0\right\|_{\dot{H}^2} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|\right), \\ \|\mathscr{E}(t)\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap\dot{H}^3} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap\dot{H}^5}\right) \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\mathscr{U}_1^0, \mathscr{B}^0\right)\right\|_{L^2} + \left\|\mathscr{E}^0\right\|_{\dot{H}^2}\right), \\ \|\mathscr{B}(t)\|_{L^\infty} \le C(1+t)^{-\frac{3}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap\dot{H}^3} + \left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap\dot{H}^5}\right) \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^2}\right), \end{cases} \tag{5.27}$$

and

$$\begin{cases} \|\nabla\mathscr{U}_1(t)\| \le C(1+t)^{-1}\left(\left\|\zeta_1^0\right\| + \left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap H^1}\right) \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\zeta_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\| + \left\|u_1^0\right\|_{\dot{H}^1}\right), \\ \|\nabla\mathscr{E}(t)\| \le C(1+t)^{-1}\left(\left\|u_1^0\right\|_{L^1\cap H^1} + \left\|\mathscr{E}^0\right\|_{L^1\cap H^1\cap\dot{H}^3} + \left\|B^0\right\|_{L^1\cap\dot{H}^3}\right) \\ \qquad + Ce^{-\gamma t}\left(\left\|\left(\mathscr{U}_1^0, \mathscr{B}^0\right)\right\| + \left\|\mathscr{E}^0\right\|_{\dot{H}^1}\right), \\ \|\nabla\mathscr{B}(t)\| \le C(1+t)^{-\frac{7}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2} + C(1+t)^{-\frac{5}{8}}\left\|\left(\mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap\dot{H}^3}, \\ \left\|\nabla^2\mathscr{B}(t)\right\| \le C(1+t)^{-1}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap\dot{H}^2} + \left\|\mathscr{E}^0\right\|_{L^2\cap\dot{H}^4} + \left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap\dot{H}^4}\right) \\ \qquad + Ce^{-\gamma t}\left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\|, \\ \left\|\nabla^s\left(\mathscr{E}(t), \mathscr{B}(t)\right)\right\| \le C(1+t)^{-\frac{5}{4}}\left\|\left(\mathscr{U}_1^0, \mathscr{E}^0, \mathscr{B}^0\right)\right\|_{L^2\cap\dot{H}^{s+3}}. \end{cases} \tag{5.28}$$

Proof. The proof is similar to that of Corollary 5.1 in [4], we omit it here for simplicity. □

5.4. *Explicit solutions of linearized system* (5.24)

Next, we consider the explicit Fourier transform solution $W_2 = (\zeta_2,\ \mathscr{U}_2)$ of the initial value problem (5.24)-(5.25). It follows from (5.24) that

$$\partial_{tt}\zeta_2 - \Delta\partial_t\zeta_2 - \Delta\zeta_2 = 0, \tag{5.29}$$

with the initial condition

$$\begin{cases} \zeta_2 \,|_{t=0} = \zeta_2^0, \\ \partial_t \zeta_2 \,|_{t=0} = -\nabla \cdot \mathscr{U}_2^0. \end{cases} \tag{5.30}$$

Taking the Fourier transform on (5.29) and (5.30), we have

$$\partial_{tt}\hat{\zeta}_2 + |k|^2 \partial_t \hat{\zeta}_2 + |k|^2 \hat{\zeta}_2 = 0, \tag{5.31}$$

with the initial condition

$$\begin{cases} \hat{\zeta}_2 \,|_{t=0} = \hat{\zeta}_2^0, \\ \partial_t \hat{\zeta}_2 \,|_{t=0} = -i|k|\tilde{k} \cdot \hat{\mathscr{U}}_2^0. \end{cases} \tag{5.32}$$

The characteristic equation of (5.31) is

$$\mathscr{L}(\lambda) := \lambda^2 + |k|^2\lambda + |k|^2 = 0, \tag{5.33}$$

whose roots may be written as $\lambda = \lambda_{1,2}$.

Then the solution of (5.31)-(5.32) can be written as

$$\hat{\zeta}_2 = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \hat{\zeta}_2^0 - \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} ik \cdot \hat{\mathscr{U}}_2^0. \tag{5.34}$$

For the roots of the characteristic equation (5.33) and their properties, by a direct computation, we obtain

Lemma 5.1. *Assume $|k| \neq 0$. Then, $\mathscr{L}(\lambda) = 0$, $\lambda \in \mathbb{C}$ has two real roots*

$$\lambda_{1,2} = -\frac{|k|^2}{2} \pm \frac{|k|\sqrt{|k|^2 - 4}}{2}, \quad as \quad |k| \geq 2,$$

or two conjugate complex roots

$$\lambda_{1,2} = -\frac{|k|^2}{2} \pm i\frac{|k|\sqrt{4 - |k|^2}}{2}, \quad as \quad |k| < 2.$$

Moreover, the following asymptotic behaviors hold true:

$$\left|\lambda_{1,2}\right| = O(1)\,|k|\,, \quad \lambda_1 - \lambda_2 = i|k|\sqrt{4 - |k|^2} = i\,O(1)|k|,$$

whenever $|k| \to 0$ is small.

$$\lambda_1 = -O(1), \quad \lambda_2 = -O(1)|k|^2, \quad \lambda_1 - \lambda_2 = |k|\sqrt{|k|^2 - 4} = O(1)|k|^2,$$

whenever $|k| \geq 2$ is large. Here and in the sequel $O(1)$ denotes a generic strictly positive constant.

Next, again from (5.24), we obtain

$$\partial_{tt}\left(\tilde{k}\cdot\hat{\mathscr{U}_2}\right)+|k|^2\partial_t\left(\tilde{k}\cdot\hat{\mathscr{U}_2}\right)+|k|^2\left(\tilde{k}\cdot\hat{\mathscr{U}_2}\right)=0, \tag{5.35}$$

with the initial condition

$$\begin{cases}\tilde{k}\cdot\hat{\mathscr{U}_2}\Big|_{t=0}=\tilde{k}\cdot\hat{\mathscr{U}}_2^0,\\ \partial_t\left(\tilde{k}\cdot\hat{\mathscr{U}_2}\right)\Big|_{t=0}=-|k|^2\tilde{k}\cdot\hat{\mathscr{U}_2}-i\,|k|\,\hat{\zeta}_2^0.\end{cases} \tag{5.36}$$

It follows from (5.35)-(5.36) that

$$\tilde{k}\cdot\hat{\mathscr{U}_2}=-\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}i\,|k|\,\hat{\zeta}_2^0+\frac{\lambda_1e^{\lambda_1 t}-\lambda_2e^{\lambda_2 t}}{\lambda_1-\lambda_2}k\cdot\hat{\mathscr{U}}_2^0, \tag{5.37}$$

where we have used the fact that $\lambda_1+\lambda_2=-|k|^2$.

Furthermore, taking the curl for the second equation of (5.24) and then taking the Fourier transform, it also follows that (3.50) with the initial condition (3.51). And then, we get (3.52). Then by combining (3.48), (3.52) and (3.53), we have

$$\hat{\mathscr{U}_2}=-\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}ik\hat{\zeta}_2^0+\frac{\lambda_1e^{\lambda_1 t}-\lambda_2e^{\lambda_2 t}}{\lambda_1-\lambda_2}\tilde{k}\tilde{k}\cdot\hat{\mathscr{U}}_2^0-e^{-|k|^2t}\tilde{k}\times\left(\tilde{k}\times\hat{\mathscr{U}}_2^0\right). \tag{5.38}$$

Now, it follows from (5.34) and (5.38) that the explicit Fourier transform solution $\hat{W}_2=(\hat{\zeta}_2,\ \hat{\mathscr{U}_2})$ as follows.

Theorem 5.2. *Let $W_2=(\zeta_2,\ \mathscr{U}_2)$ be the solution to the Initial value problem (5.24)-(5.25). For $(t,k)\in\mathbb{R}^+\times\mathbb{R}^3$ with $|k|\neq 0$, it holds*

$$\begin{bmatrix}\hat{\zeta}_2(t,k)\\ \hat{\mathscr{U}_2}(t,k)\end{bmatrix}=\mathscr{R}^*(t,k)\begin{bmatrix}\hat{\zeta}_2^0(k)\\ \hat{\mathscr{U}}_2^0(k)\end{bmatrix}, \tag{5.39}$$

where the 4×4 matrix $\mathscr{R}^*(t,k)$ is defined as:

$$\mathscr{R}^*(t,k)=\begin{bmatrix}\mathscr{R}^*_{11}(t,k) & \mathscr{R}^*_{12}(t,k)\\ \mathscr{R}^*_{21}(t,k) & \mathscr{R}^*_{22}(t,k)\end{bmatrix}$$
$$=\begin{bmatrix}\dfrac{\lambda_1e^{\lambda_2 t}-\lambda_2e^{\lambda_1 t}}{\lambda_1-\lambda_2} & \dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}(-ik^T)\\ \dfrac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}(-ik) & e^{-|k|^2t}\left(\mathrm{I}_3-\dfrac{k\otimes k}{|k|^2}\right)+\dfrac{\lambda_1e^{\lambda_1 t}-\lambda_2e^{\lambda_2 t}}{\lambda_1-\lambda_2}\dfrac{k\otimes k}{|k|^2}\end{bmatrix}.$$

5.5. $L^p - L^q$ decay properties for system (5.24)

In this subsection, we want to get the $L^p - L^q$ decay property for every component of the solution $W_2 = (\zeta_2, \mathscr{U}_2)$. To this end, we consider the time frequency estimates on $\hat{W}_2 = (\hat{\zeta}_2, \hat{\mathscr{U}_2})$ as follows.

Lemma 5.2. *Let $W_2 = (\zeta_2, \mathscr{U}_2)$ be the solution to the initial value problem (5.24)-(5.25). Then, there are constants $\epsilon > 0$, $L > 0$, $\gamma > 0$ and $C > 0$ such that for all $(t, k) \in \mathbb{R}^+ \times \mathbb{R}^3$,*

$$\left|\hat{\zeta}_2(t,k)\right| \le \begin{cases} Ce^{-\gamma|k|^2 t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}_2^0}(k)\right|\right), & \text{if} \quad k \le \epsilon, \\ Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}_2^0}(k)\right|\right), & \text{if} \quad \epsilon \le k \le L, \\ Ce^{-\gamma t}\left|\hat{\zeta}_2^0(k)\right| + \dfrac{C}{|k|}e^{-\gamma t}\left|\hat{\mathscr{U}_2^0}(k)\right|, & \text{if} \quad k \ge L, \end{cases} \tag{5.40}$$

and

$$\left|\hat{\mathscr{U}_2}(t,k)\right| \le \begin{cases} Ce^{-\gamma|k|^2 t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}_2^0}(k)\right|\right), & \text{if} \quad k \le \epsilon, \\ Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}_2^0}(k)\right|\right), & \text{if} \quad \epsilon \le k \le L, \\ \dfrac{C}{|k|}e^{-\gamma t}\left|\hat{\zeta}_2^0(k)\right| + \dfrac{C}{|k|^2}e^{-\gamma t}\left|\hat{\mathscr{U}_2^0}(k)\right|, & \text{if} \quad k \ge L. \end{cases} \tag{5.41}$$

Proof. In order to get estimates (5.40)-(5.41), we need to establish the estimates for every component in matrix $\mathscr{R}^*(t,k)$, namely, $\mathscr{R}^*_{ij}$, $(1 \le i, j \le 2)$. It follows from Lemma 5.1 that there exist constants $\epsilon \le 2 \le L$ satisfying $0 < \epsilon \ll 1 \ll L < \infty$ such that

$$\begin{cases} \left|\mathscr{R}^*_{11}\right| \le Ce^{-\frac{1}{2}|k|^2 t} \le Ce^{-\gamma|k|^2 t}, \\ \left|\mathscr{R}^*_{12}\right| + \left|\mathscr{R}^*_{21}\right| \le Ce^{-\frac{1}{2}|k|^2 t} \le Ce^{-\gamma|k|^2 t}, \\ \left|\mathscr{R}^*_{22}\right| \le Ce^{-|k|^2 t} + Ce^{-\frac{1}{2}|k|^2 t} \le Ce^{-\gamma|k|^2 t}, \quad \text{as} \quad |k| \le \epsilon, \end{cases} \tag{5.42}$$

and

$$\begin{cases} \left|\mathscr{R}^*_{11}\right| \le \dfrac{C}{|k|^2}e^{-\gamma|k|^2 t} + Ce^{-\gamma t}, \\ \left|\mathscr{R}^*_{12}\right| + \left|\mathscr{R}^*_{21}\right| \le \dfrac{C}{|k|}\left(e^{-\gamma|k|^2 t} + e^{-\gamma t}\right), \\ \left|\mathscr{R}^*_{22}\right| \le Ce^{-|k|^2 t} + \dfrac{C}{|k|^2}e^{-\gamma t} + Ce^{-\gamma|k|^2 t} \le \dfrac{C}{|k|^2}e^{-\gamma t} + Ce^{-\gamma|k|^2 t}, \quad \text{as} \quad |k| \ge L. \end{cases} \tag{5.43}$$

Thus, based on (5.42)-(5.43) and (5.39), we get, when $|k| \le \varepsilon$,

$$\begin{cases} \left|\hat{\zeta}_2(t,k)\right| \le \left|\mathscr{R}^*_{11}\right| \left|\hat{\zeta}^0_2(k)\right| + \left|\mathscr{R}^*_{12}\right| \left|\hat{\mathscr{U}}^0_2(k)\right| \le Ce^{-\gamma|k|^2 t}\left(\left|\hat{\zeta}^0_2(k)\right| + \left|\hat{\mathscr{U}}^0_2(k)\right|\right), \\ \left|\hat{\mathscr{U}}_2(t,k)\right| \le \left|\mathscr{R}^*_{21}\right| \left|\hat{\zeta}^0_2(k)\right| + \left|\mathscr{R}^*_{22}\right| \left|\hat{\mathscr{U}}^0_2(k)\right| \le Ce^{-\gamma|k|^2 t}\left(\left|\hat{\zeta}^0_2(k)\right| + \left|\hat{\mathscr{U}}^0_2(k)\right|\right), \end{cases} \tag{5.44}$$

and

$$\begin{cases} \left|\hat{\zeta}_2(t,k)\right| \le C\left(\frac{1}{|k|^2}e^{-\gamma|k|^2 t} + e^{-\gamma t}\right)\left|\hat{\zeta}^0_2(k)\right| + \frac{C}{|k|}\left(e^{-\gamma|k|^2 t} + e^{-\gamma t}\right)\left|\hat{\mathscr{U}}^0_2(k)\right| \\ \qquad \le Ce^{-\gamma t}\left|\hat{\zeta}^0_2(k)\right| + \frac{C}{|k|}e^{-\gamma t}\left|\hat{\mathscr{U}}^0_2(k)\right|, \\ \left|\hat{\mathscr{U}}_2(t,k)\right| \le \frac{C}{|k|}\left(e^{-\gamma|k|^2 t} + e^{-\gamma t}\right)\left|\hat{\zeta}^0_2(k)\right| + C\left(e^{-\gamma|k|^2 t} + \frac{1}{|k|^2}e^{-\gamma t}\right)\left|\hat{\mathscr{U}}^0_2(k)\right| \\ \qquad \le \frac{C}{|k|}e^{-\gamma t}\left|\hat{\zeta}^0_2(k)\right| + \frac{C}{|k|^2}e^{-\gamma t}\left|\hat{\mathscr{U}}^0_2(k)\right|, \quad \text{as } |k| \ge L. \end{cases} \tag{5.45}$$

Moreover, we establish the estimates on the domain $\{\varepsilon \le |k| \le L\}$. For briefness, we only consider the estimate for $\mathscr{R}^*_{12}(t,k)$.

By Lemma 5.1, we have

$$\lim_{|k|\to 2^+} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = te^{-2t} \le Ce^{-\gamma t},$$

and

$$\lim_{|k|\to 2^-} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = \lim_{|k|\to 2} \frac{2e^{-\frac{1}{2}|k|^2 t}\sin\left(\frac{t}{2}|k|\sqrt{4-|k|^2}\right)}{|k|\sqrt{4-|k|^2}} = te^{-2t} \le Ce^{-\gamma t}.$$

Then there is a constant $\delta > 0$ such that

$$\left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right| \le Ce^{-\gamma t}, \quad \text{as } ||k| - 2| < \delta,$$

which implies

$$\left|\mathscr{R}^*_{12}\right| = \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(-ik^T)\right| \le Ce^{-\gamma t}, \quad \text{as } ||k| - 2| < \delta. \tag{5.46}$$

On the other hand, when $\varepsilon \le |k| \le L$ and $||k| - 2| \ge \delta$, due to the fact that

$$\begin{cases} \lambda_{1,2} < 0, & \text{if } \lambda_{1,2} \text{ are two real roots,} \\ \mathcal{R}e\lambda_{1,2} = -\frac{1}{2}|k|^2, & \text{if } \lambda_{1,2} \text{ are two conjugate complex roots,} \end{cases}$$

we have

$$\left|\mathscr{R}_{12}^*\right| = \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(-ik^T)\right| \le Ce^{-\gamma t}, \quad \text{as} \ \ ||k|-2| \ge \delta \ \text{ and } \ \varepsilon \le |k| \le L. \tag{5.47}$$

Therefore, it follows from (5.46) and (5.47) that

$$\left|\mathscr{R}_{12}^*\right| = \left|\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(-ik^T)\right| \le Ce^{-\gamma t}, \quad \text{as} \ \ \varepsilon \le |k| \le L. \tag{5.48}$$

Similarly, we also obtain

$$\left|\mathscr{R}_{11}^*\right|, \left|\mathscr{R}_{21}^*\right|, \left|\mathscr{R}_{22}^*\right| \le Ce^{-\gamma t}, \quad \text{as} \ \ \varepsilon \le |k| \le L. \tag{5.49}$$

Based on (5.48), (5.49) and (5.39), we obtain, when $\varepsilon \le |k| \le L$,

$$\begin{cases} \left|\hat{\zeta}_2(t,k)\right| \le \left|\mathscr{R}_{11}^*\right| \left|\hat{\zeta}_2^0(k)\right| + \left|\mathscr{R}_{12}^*\right| \left|\hat{\mathscr{U}}_2^0(k)\right| \le Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right|\right), \\ \left|\hat{\mathscr{U}}_2(t,k)\right| \le \left|\mathscr{R}_{21}^*\right| \left|\hat{\zeta}_2^0(k)\right| + \left|\mathscr{R}_{22}^*\right| \left|\hat{\mathscr{U}}_2^0(k)\right| \le Ce^{-\gamma t}\left(\left|\hat{\zeta}_2^0(k)\right| + \left|\hat{\mathscr{U}}_2^0(k)\right|\right). \end{cases} \tag{5.50}$$

Thus, (5.40)-(5.41) follow by combining (5.44)-(5.45) and (5.50). We have finished the proof of Lemma 5.2. □

Next, by applying Lemma 2.3 to the estimates in Lemma 5.2 above, it is straightforward to get the decay property for every component of the solution $W_2 = (\zeta_2, \ \mathscr{U}_2)$.

Theorem 5.3. *Let $j \ge 0$ be an integer and $1 \le p, r \le 2 \le q \le \infty$. Suppose $W_2(t) = e^{tL_2}W_2^0$ to be the solution to the Initial value problem (5.24)-(5.25). Then, for any $t \ge 0$, W_2 satisfies the following time decay property:*

$$\begin{aligned} \left\|\nabla^j \zeta_2\right\|_{L^q} \le & C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}} \left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^p} \\ & + Ce^{-\gamma t}\left(\left\|\nabla^{m(j,r,q)}\zeta_2^0\right\|_{L^r} + \left\|\nabla^{m(j-1,r,q)}\mathscr{U}_2^0\right\|_{L^r}\right), \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} \left\|\nabla^j \mathscr{U}_2\right\|_{L^q} \le & C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2}} \left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^p} \\ & + Ce^{-\gamma t}\left(\left\|\nabla^{m(j-1,r,q)}\zeta_2^0\right\|_{L^r} + \left\|\nabla^{m(j-2,r,q)}\mathscr{U}_2^0\right\|_{L^r}\right). \end{aligned} \tag{5.52}$$

For later use, we list the following result which is an immediate corollary from Theorem 5.3.

Corollary 5.1. *Let $W_2(t) = e^{tL_2}W_2^0$ be the solution to the initial value problem (5.24)-(5.25). Then, for any $t \ge 0$, W_2 satisfies the following time decay property:*

$$\begin{cases} \|\zeta_2\| \le C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + Ce^{-\gamma t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|, \\ \|\mathscr{U}_2\| \le C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|_{L^1} + Ce^{-\gamma t}\left\|\left(\zeta_2^0, \mathscr{U}_2^0\right)\right\|, \end{cases} \tag{5.53}$$

$$\begin{cases} \|\nabla\zeta_2\| \le C(1+t)^{-\frac{5}{4}} \left\| \left(\zeta_2^0, \mathscr{U}_2^0\right) \right\|_{L^1} + Ce^{-\gamma t} \left(\left\| \zeta_2^0 \right\|_{\dot{H}^1} + \left\| \mathscr{U}_2^0 \right\| \right), \\ \|\nabla\mathscr{U}_2\| \le C(1+t)^{-\frac{5}{4}} \left\| \left(\zeta_2^0, \mathscr{U}_2^0\right) \right\|_{L^1} + Ce^{-\gamma t} \left\| \left(\zeta_2^0, \mathscr{U}_2^0\right) \right\|, \end{cases} \tag{5.54}$$

and

$$\begin{cases} \|\zeta_2\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}} \left\| \left(\zeta_2^0, \mathscr{U}_2^0\right) \right\|_{L^1} + Ce^{-\gamma t} \left(\left\| \zeta_2^0 \right\|_{\dot{H}^2} + \left\| \mathscr{U}_2^0 \right\|_{\dot{H}^1} \right), \\ \|\mathscr{U}_2\|_{L^\infty} \le C(1+t)^{-\frac{3}{2}} \left\| \left(\zeta_2^0, \mathscr{U}_2^0\right) \right\|_{L^1} + Ce^{-\gamma t} \left(\left\| \zeta_2^0 \right\|_{\dot{H}^1} + \left\| \mathscr{U}_2^0 \right\| \right). \end{cases} \tag{5.55}$$

5.6. *Time decay rates for energy functionals of system* (5.1)

In this subsection, we prove the decay rate (5.8) in Proposition 5.2 for the energy $\|W(t)\|_s^2$. We begin with the following Lemma which can be seen directly from the proof of Theorem 5.1.

Lemma 5.3. *Let $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the initial value problem* (5.1)-(5.2) *with initial data $W^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfying* (2.7) *in the sense of Proposition 5.1. Then, if $\mathfrak{E}_s(W^0)$ is small enough, for any $t \ge 0$,*

$$\frac{d}{dt}\mathfrak{E}_s(W(t)) + \mathscr{D}_s(W(t)) \le 0. \tag{5.56}$$

For $p > 0$, the same procedure as that for the non-isentropic case, from Lemma 5.3, we also have (4.2), in which $\mathfrak{E}_s(\mathscr{W}(t)$ and $\mathfrak{D}_s(\mathscr{W}(t)$ are replaced with $\mathfrak{E}_s(W(t)$ and $\mathscr{D}_s(W(t)$, respectively. Next, let us estimate the terms on the right hand side of (4.2). Applying the estimates on $\mathscr{U}_1$, $\mathscr{E}$ and $\mathscr{B}$ in (5.26), the estimate on $\nabla\mathscr{B}$ in (5.28) and the estimates on ζ_2 and $\mathscr{U}_2$ in (5.53) to (5.20) and (5.21), respectively, we have

$$\begin{aligned} &\left\| \left(\mathscr{U}^e - \mathscr{U}^i\right)(t) \right\| \\ \le& C(1+t)^{-\frac{5}{4}} \left\| \zeta_1^0 \right\|_{L^1\cap L^2} + C(1+t)^{-\frac{3}{4}} \left\| \left(\mathscr{U}_1^0, \mathscr{E}^0\right) \right\|_{L^1\cap L^2} + C(1+t)^{-\frac{5}{8}} \left\| \mathscr{B}^0 \right\|_{L^1\cap L^2} \\ &+ C\int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(\mathscr{G}_{1e} - \mathscr{G}_{1i})(\tau)\|_{L^1\cap L^2} d\tau \\ &+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}} \left\| \left(\mathscr{G}_{2e}^* - \mathscr{G}_{2i}^*, \mathscr{G}_{4e} - \mathscr{G}_{4i}\right)(\tau) \right\|_{L^1\cap L^2} d\tau, \end{aligned} \tag{5.57}$$

$$\begin{aligned} &\|\mathscr{E}(t)\| \\ \le& C(1+t)^{-\frac{3}{4}} \left\| \mathscr{U}_1^0 \right\|_{L^1\cap L^2} + C(1+t)^{-\frac{3}{4}} \left\| \mathscr{E}^0 \right\|_{L^1\cap L^2\cap \dot{H}^2} + C(1+t)^{-\frac{9}{8}} \left\| \mathscr{B}^0 \right\|_{L^1\cap L^2\cap \dot{H}^3} \\ &+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}} \left(\left\| \left(\mathscr{G}_{2e}^* - \mathscr{G}_{2i}^*\right)(\tau) \right\|_{L^1\cap L^2} + \|(\mathscr{G}_{4e} - \mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^2} \right) d\tau, \end{aligned} \tag{5.58}$$

$$
\begin{aligned}
&\|\mathscr{B}(t)\| \\
\leq & C(1+t)^{-\frac{5}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2}+C(1+t)^{-\frac{9}{8}}\left\|\mathscr{E}^0\right\|_{L^1\cap L^2\cap \dot{H}^3}+C(1+t)^{-\frac{3}{8}}\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap H^1} \\
&+C\int_0^t(1+t-\tau)^{-\frac{5}{8}}\left(\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3}\right)d\tau,
\end{aligned} \tag{5.59}
$$

$$
\begin{aligned}
&\|\nabla\mathscr{B}(t)\| \\
\leq & C(1+t)^{-\frac{7}{8}}\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2}+C\int_0^t(1+t-\tau)^{-\frac{7}{8}}\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}d\tau \\
&+C(1+t)^{-\frac{5}{8}}\left\|\left(\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap \dot{H}^3}+C\int_0^t(1+t-\tau)^{-\frac{5}{8}}\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3}d\tau,
\end{aligned} \tag{5.60}
$$

and

$$
\begin{aligned}
\left\|\left(\zeta^e+\zeta^i\right)(t)\right\| \leq & C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0,\mathscr{U}_2^0\right)\right\|_{L^1\cap L^2} \\
&+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\left\|\left(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}^*+\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}d\tau,
\end{aligned} \tag{5.61}
$$

and

$$
\begin{aligned}
\left\|\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\| \leq & C(1+t)^{-\frac{3}{4}}\left\|\left(\zeta_2^0,\mathscr{U}_2^0\right)\right\|_{L^1\cap L^2} \\
&+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\left\|\left(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}^*+\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}d\tau.
\end{aligned} \tag{5.62}
$$

It is direct to check that for any $0\leq\tau\leq t$,

$$
\left\|\left(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}^*+\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}\leq C\mathfrak{E}_s\left(W(\tau)\right)\leq C(1+\tau)^{-\frac{3}{4}}\mathfrak{E}_{s,\infty}\left(W(t)\right),
$$

and

$$
\left\|\left(\mathscr{G}_{1e}-\mathscr{G}_{1i},\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap L^2\cap \dot{H}^3}\leq C(1+\tau)^{-\frac{3}{4}}\mathfrak{E}_{s,\infty}\left(W(t)\right).
$$

Plugging the two previous inequalities into (5.57)-(5.62) implies, respectively,

$$
\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\|\leq C(1+t)^{-\frac{5}{8}}\Big(\left\|\left(\zeta^{\nu 0},\mathscr{U}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2}+\mathfrak{E}_{s,\infty}\left(W(t)\right)\Big), \tag{5.63}
$$

$$
\|\mathscr{E}(t)\|\leq C(1+t)^{-\frac{3}{4}}\Big(\left\|\left(\mathscr{U}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap \dot{H}^2}+\mathfrak{E}_{s,\infty}\left(W(t)\right)\Big), \tag{5.64}
$$

$$\|\mathscr{B}(t)\| \leq C(1+t)^{-\frac{3}{8}}\Big(\Big\|\Big(\mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\Big)\Big\|_{L^1\cap L^2\cap \dot{H}^3} + \mathfrak{E}_{s,\infty}(W(t))\Big), \tag{5.65}$$

$$\|\nabla\mathscr{B}(t)\| \leq C(1+t)^{-\frac{5}{8}}\Big(\Big\|\Big(\mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0\Big)\Big\|_{L^1\cap L^2\cap \dot{H}^3} + \mathfrak{E}_{s,\infty}(W(t))\Big), \tag{5.66}$$

$$\Big\|\Big(\zeta^e + \zeta^i\Big)(t)\Big\| \leq C(1+t)^{-\frac{3}{4}}\Big(\Big\|\Big(\zeta_2^0, \mathscr{U}_2^0\Big)\Big\|_{L^1\cap L^2} + \mathfrak{E}_{s,\infty}(W(t))\Big), \tag{5.67}$$

and

$$\Big\|\Big(\mathscr{U}^e + \mathscr{U}^i\Big)(t)\Big\| \leq C(1+t)^{-\frac{3}{4}}\Big(\Big\|\Big(\zeta_2^0, \mathscr{U}_2^0\Big)\Big\|_{L^1\cap L^2} + \mathfrak{E}_{s,\infty}(W(t))\Big). \tag{5.68}$$

Next, we also choose $p = \dfrac{3}{4} + \varepsilon$ in (4.2) with $\varepsilon > 0$ small enough and using (5.63)-(5.68), we have

$$\|W(t)\|_s \leq C\mathfrak{E}_s(W(t))^{\frac{1}{2}} \leq C\omega_{s+1}\Big(W^0\Big)(1+t)^{-\frac{3}{8}}, \quad \forall\, t \geq 0,$$

that is (5.8).

5.7. *Time decay rates for higher order energy functionals of system* (5.1)

In this subsection, we consider the decay estimate of the higher order energy $\|\nabla W(t)\|_{s-1}^2$, that is (5.9) in Proposition 5.2. We begin with the following Lemma.

Lemma 5.4. *Let $W = (\zeta^\nu, \mathscr{U}^\nu, \mathscr{E}, \mathscr{B})$ be the solution to the Initial value problem (5.1)-(5.2) with initial data $W^0 = (\zeta^{\nu 0}, \mathscr{U}^{\nu 0}, \mathscr{E}^0, \mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 5.1. Then, if $\mathfrak{E}_s(W^0)$ is small enough, there exist the higher order energy functionals $\mathfrak{E}_s^h(\cdot)$ and the higher order dissipative functionals $\mathscr{D}_s^h(\cdot)$ in the form of (5.4) and (5.6) such that for any $t \geq 0$,*

$$\frac{d}{dt}\mathfrak{E}_s^h(W(t)) + \mathscr{D}_s^h(W(t)) \leq 0. \tag{5.69}$$

Proof. By letting $|\alpha| \geq 1$, then corresponding to (5.12), (5.13), (5.14) and (5.15), we also get

$$\frac{d}{dt}\|\nabla W\|_{s-1}^2 + \sum_{\nu=e,i}\|\nabla\mathscr{U}^\nu\|_{s-1}^2 \leq C\|W\|_s\sum_{\nu=e,i}\Big(\|\nabla\zeta^\nu\|_{s-2}^2 + \|\nabla\mathscr{U}^\nu\|_{s-1}^2\Big),$$

$$\begin{aligned}&\frac{d}{dt}\sum_{1\leq|\alpha|\leq s-1}\sum_{\nu=e,i}\langle\partial^\alpha\mathscr{U}^\nu, \partial^\alpha\nabla\zeta^\nu\rangle + c_0\Big\|\nabla\Big(\zeta^e, \zeta^i\Big)\Big\|_{s-2}^2 + c_0\Big\|\nabla\Big(\zeta^e - \zeta^i\Big)\Big\|^2 \\ \leq\, & C\sum_{\nu=e,i}\Big\|\nabla^2\mathscr{U}^\nu\Big\|_{s-1}^2 + C\|W\|_s\sum_{\nu=e,i}\Big(\Big\|\nabla^2\zeta^\nu\Big\|_{s-2}^2 + \Big\|\nabla^2\mathscr{U}^\nu\Big\|_{s-1}^2\Big),\end{aligned}$$

$$
\begin{aligned}
&\frac{d}{dt}\sum_{2\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle+c_0\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2\\
\le&\varepsilon\left\|\nabla^3\mathscr{B}\right\|_{s-4}^2+C\|W\|_s\left(\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2\right)+\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2\right)\\
&+C\sum_{\nu=e,i}\left(\left\|\nabla^2\zeta^\nu\right\|_{s-2}^2+\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-1}^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\sum_{2\le|\alpha|\le s-2}\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\rangle+c_0\left\|\nabla^3\mathscr{B}\right\|_{s-4}^2\\
\le&C\left\|\nabla^2\mathscr{E}\right\|_{s-3}^2+C\sum_{\nu=e,i}\left\|\nabla^2\mathscr{U}^\nu\right\|_{s-3}^2+C\|W\|_s\sum_{\nu=e,i}\left\|\nabla^2\left(\zeta^\nu,\mathscr{U}^\nu\right)\right\|_{s-2}^2.
\end{aligned}
$$

Now, let us define the higher order energy functionals as:

$$
\begin{aligned}
\mathfrak{E}_s^h(W(t))=&\|\nabla W\|_{s-1}^2+\mu_1\sum_{1\le|\alpha|\le s-1}\sum_{\nu=e,i}\langle\partial^\alpha\mathscr{U}^\nu,\nabla\partial^\alpha\zeta^\nu\rangle\\
&+\mu_2\sum_{2\le|\alpha|\le s-1}\left\langle\partial^\alpha\left(\mathscr{U}^e-\mathscr{U}^i\right),\partial^\alpha\mathscr{E}\right\rangle+\mu_3\sum_{2\le|\alpha|\le s-2}\langle\partial^\alpha\mathscr{E},-\nabla\times\partial^\alpha\mathscr{B}\rangle.
\end{aligned}
\tag{5.70}
$$

Similarly, we choose $0<\varepsilon\ll 1$ and $0<\mu_3\ll\mu_2\ll\mu_1\ll 1$ to be sufficiently small with $2\varepsilon\mu_2\ll\mu_3$, such that $\mathfrak{E}_s^h(W(t))\sim\|\nabla W(t)\|_{s-1}^2$, that is $\mathfrak{E}_s^h(\cdot)$ is a higher order energy functionals which satisfies (5.4), and moreover, summing the four previously estimates with coefficients corresponding to (5.70) gives (5.69). This ends the proof of Lemma 5.4. □

Based on Lemma 5.4, we obtain (4.17) for the isentropic case in which $\mathfrak{E}_s^h(\mathscr{W}(t))$ and $\mathfrak{E}_s^h(\mathscr{W}^0)$ are replaced with $\mathfrak{E}_s^h(W(t))$ and $\mathfrak{E}_s^h(W^0)$. Next, we estimate the time integral term on the right hand side of (4.17) for the isentropic case.

Lemma 5.5. *Let $W=(\zeta^\nu,\mathscr{U}^\nu,\mathscr{E},\mathscr{B})$ be the solution to the initial value problem (5.1)-(5.2) with initial data $W^0=(\zeta^{\nu 0},\mathscr{U}^{\nu 0},\mathscr{E}^0,\mathscr{B}^0)$ satisfying (2.7) in the sense of Proposition 5.1. Then, if $\omega_{s+6}(W^0)$ is small enough, for any $t\ge 0$,*

$$
\begin{aligned}
&\left\|\nabla\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\|^2+\|\nabla(\mathscr{E},\mathscr{B})(t)\|^2+\left\|\nabla^s(\mathscr{E},\mathscr{B})(t)\right\|^2\\
&\quad+\left\|\nabla^2\mathscr{B}(t)\right\|^2+\left\|\nabla\left(\zeta^e+\zeta^i\right)(t)\right\|^2+\left\|\nabla\left(\mathscr{U}^e+\mathscr{U}^i\right)(t)\right\|^2\\
&\le C\left(\omega_{s+6}\left(W^0\right)\right)^2(1+t)^{-\frac{5}{4}}(\ln(3+t))^2.
\end{aligned}
\tag{5.71}
$$

Proof. Applying the estimates on $\nabla\mathscr{U}_1$, $\nabla\mathscr{E}$, $\nabla\mathscr{B}$, $\nabla^2\mathscr{B}$ and $\nabla^s(\mathscr{E},\mathscr{B})$ in (5.28) to (5.20), respectively, and using (5.8), we get

$$
\begin{aligned}
\|\nabla\mathscr{U}_1(t)\| \le & C(1+t)^{-1}\left(\left\|\zeta_1^0\right\|+\left\|\left(\mathscr{U}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap H^1}\right)\\
&+C\int_0^t(1+t-\tau)^{-1}\left(\|(\mathscr{G}_{1e}-\mathscr{G}_{1i})(\tau)\|+\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*,\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1\cap H^1}\right)d\tau\\
\le & C\omega_5(W^0)(1+t)^{-\frac{3}{4}}\ln(3+t),\\
\|\nabla\mathscr{E}(t)\| \le & C(1+t)^{-1}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap H^1}+\left\|\mathscr{E}^0\right\|_{L^1\cap H^1\cap\dot{H}^3}+\left\|\mathscr{B}^0\right\|_{L^1\cap\dot{H}^3}\right)\\
&+C\int_0^t(1+t-\tau)^{-1}\left(\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap H^1}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap H^1\cap\dot{H}^3}\right)d\tau\\
\le & C\omega_5(W^0)(1+t)^{-\frac{3}{4}}\ln(3+t),\\
\|\nabla\mathscr{B}(t)\| \le & C(1+t)^{-\frac{5}{8}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2}+\left\|\left(\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap\dot{H}^3}\right)\\
&+C\int_0^t(1+t-\tau)^{-\frac{5}{8}}\left(\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^1\cap H^1\cap\dot{H}^3}\right)d\tau\\
\le & C\omega_5(W^0)(1+t)^{-\frac{5}{8}},\\
\left\|\nabla^2\mathscr{B}(t)\right\| \le & C(1+t)^{-1}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap\dot{H}^2}+\left\|\mathscr{E}^0\right\|_{L^2\cap\dot{H}^4}+\left\|\mathscr{B}^0\right\|_{L^1\cap L^2\cap\dot{H}^4}\right)\\
&+C\int_0^t(1+t-\tau)^{-1}\left(\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2\cap\dot{H}^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(\tau)\|_{L^2\cap\dot{H}^4}\right)d\tau\\
\le & C\omega_5(W^0)(1+t)^{-\frac{3}{4}}\ln(3+t),
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\nabla^s(\mathscr{E},\mathscr{B})(t)\right\| \le & C(1+t)^{-\frac{5}{4}}\left\|\left(\mathscr{U}_1^0,\mathscr{E}^0,B^0\right)\right\|_{L^2\cap\dot{H}^{s+3}}\\
&+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*,\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^2\cap\dot{H}^{s+3}}d\tau\\
\le & C\omega_{s+6}(W^0)(1+t)^{-\frac{3}{4}}\ln(3+t),
\end{aligned}
$$

where the smallness of $\omega_{s+6}(W^0)$ is also used. Moreover, by (5.8) and applying the estimates on $\nabla\zeta_2$ and $\nabla\mathscr{U}_2$ in (5.54) to (5.21), respectively, we obtain

$$
\begin{aligned}
&\left\|\nabla\left(\zeta^{e}+\zeta^{i}\right)(t)\right\| \\
\leq& C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0,\mathscr{U}_2^0\right)\right\|_{L^1\cap\dot{H}^1}+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}^*+\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap\dot{H}^1}d\tau \\
\leq& C\omega_5(W^0)\,(1+t)^{-\frac{3}{4}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla\mathscr{U}_2\| \\
\leq& C(1+t)^{-\frac{5}{4}}\left\|\left(\zeta_2^0,\mathscr{U}_2^0\right)\right\|_{L^1\cap L^2}+C\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(\mathscr{G}_{1e}+\mathscr{G}_{1i},\mathscr{G}_{2e}^*+\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2}d\tau \\
\leq& C\omega_5(W^0)\,(1+t)^{-\frac{3}{4}}.
\end{aligned}
$$

We have finished the proof of Lemma 5.5. □

Then, plugging (4.18) into (4.17), we have

$$
\mathfrak{E}_s^h(W(t))\leq e^{-t}\mathfrak{E}_s^h(W^0)+C\left(\omega_{s+6}\left(W^0\right)\right)^2(1+t)^{-\frac{5}{4}}\left(\ln(3+t)\right)^2.
$$

Since $\mathfrak{E}_s^h(W(t))\sim\|\nabla W(t)\|_{s-1}^2$ holds true for any $t\geq 0$, (5.9) follows. The proof of Proposition 5.2 is ended. □

5.8. *Time decay rates in L^q for system (5.1)*

In this subsection, we consider the decay rates of solutions $W=(\zeta^\nu,\mathscr{U}^\nu,\mathscr{E},\mathscr{B})$ to the initial value problem (5.1)-(5.2) in L^q with $2\leq q\leq+\infty$, and prove the second part of Theorem 1.2. Throughout this subsection, we suppose $\omega_6(W^0)$ to be small enough. Firstly, for $s\geq 4$, Proposition 5.2 shows that if $\omega_{s+1}(W^0)$ is small enough,

$$
\|W(t)\|_s\leq C\omega_{s+1}(W^0)(1+t)^{-\frac{3}{8}}. \tag{5.72}
$$

Next, we also estimate $\mathscr{B}$, $\mathscr{U}^e-\mathscr{U}^i$, $\mathscr{E}$, $\mathscr{U}^e+\mathscr{U}^i$, $\zeta^e-\zeta^i$ and $\zeta^e+\zeta^i$ in turn as follows. *Estimate on $\|\mathscr{B}\|_{L^q}$.* For L^2 rate, it follows from (5.72) that

$$
\|\mathscr{B}(t)\|\leq C\omega_5(W^0)(1+t)^{-\frac{3}{8}}.
$$

For L^∞ rate, by applying L^∞ estimate on $\mathscr{B}$ of (5.27) to (5.20), we obtain

$$
\begin{aligned}
\|\mathscr{B}(t)\|_{L^\infty}\leq& C(1+t)^{-\frac{3}{4}}\left(\left\|\mathscr{U}_1^0\right\|_{L^1\cap L^2\cap\dot{H}^3}+\left\|\left(\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap\dot{H}^5}\right) \\
&+C\int_0^t(1+t-\tau)^{-\frac{3}{4}}\left(\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(\tau)\right\|_{L^1\cap L^2\cap\dot{H}^3}+\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1\cap L^2\cap\dot{H}^5}\right)d\tau.
\end{aligned}
$$

By (5.72), we also have

$$\left\|\left(\mathscr{G}_{2e}^{*}-\mathscr{G}_{2i}^{*}\right)(t)\right\|_{L^1\cap L^2\cap\dot{H}^3}+\left\|\left(\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(t)\right\|_{L^1\cap L^2\cap\dot{H}^5}\le C\left\|W(t)\right\|_5^2 \le C\left(\omega_6(W^0)\right)^2(1+t)^{-\frac{3}{4}},$$

which implies

$$\|\mathscr{B}(t)\|_{L^\infty}\le C\omega_6(W^0)(1+t)^{-\frac{3}{4}}.$$

Therefore, by L^2-L^∞ interpolation

$$\|\mathscr{B}(t)\|_{L^q}\le C\omega_6(W^0)(1+t)^{-\frac{3}{4}+\frac{3}{4q}},\quad \forall\, 2\le q\le+\infty. \tag{5.73}$$

Estimate on $\|\mathscr{U}^e-\mathscr{U}^i\|_{L^q}$. For L^2 rate, we apply the L^2 estimate on $\mathscr{U}_1$ in (5.26) to (5.20) to get

$$\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\| \le C(1+t)^{-\frac{5}{8}}\left\|\left(\zeta_1^0,\mathscr{U}_1^0,\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2} + C\int_0^t(1+t-\tau)^{-\frac{5}{8}}\left\|\left(\mathscr{G}_{1e}-\mathscr{G}_{1i},\mathscr{G}_{2e}^{*}-\mathscr{G}_{2i}^{*},\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(\tau)\right\|_{L^1\cap L^2}d\tau.$$

By (5.72), since

$$\left\|\left(\mathscr{G}_{1e}-\mathscr{G}_{1i},\mathscr{G}_{2e}^{*}-\mathscr{G}_{2i}^{*},\mathscr{G}_{4e}-\mathscr{G}_{4i}\right)(t)\right\|_{L^1\cap L^2}\le C\left\|W(t)\right\|_4^2\le C\left(\omega_5(W^0)\right)^2(1+t)^{-\frac{3}{4}},$$

we obtain

$$\left\|(\mathscr{U}^e-\mathscr{U}^i)(t)\right\|\le C\omega_5(W^0)(1+t)^{-\frac{5}{8}}. \tag{5.74}$$

For L^∞ rate, by applying the L^∞ estimate on $\mathscr{U}_1$ in (5.27) to (5.20), we have

$$\begin{aligned}&\left\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\right\|_{L^\infty}\\ \le& C(1+t)^{-1}\left(\left\|\zeta_1^0\right\|_{L^1\cap\dot{H}^1}+\left\|\mathscr{U}_1^0\right\|_{L^1\cap\dot{H}^1\cap\dot{H}^2}+\left\|\left(\mathscr{E}^0,\mathscr{B}^0\right)\right\|_{L^1\cap L^2\cap\dot{H}^3}\right)\\ &+C\int_0^t(1+t-\tau)^{-1}\\ &\times\left(\left\|\mathscr{G}_{1e}-\mathscr{G}_{1i}\right\|_{L^1\cap\dot{H}^1}+\left\|\mathscr{G}_{2e}^{*}-\mathscr{G}_{2i}^{*}\right\|_{L^1\cap\dot{H}^1\cap\dot{H}^2}+\left\|\mathscr{G}_{4e}-\mathscr{G}_{4i}\right\|_{L^1\cap L^2\cap\dot{H}^3}\right)(\tau)d\tau.\end{aligned}$$

Since

$$\|(\mathscr{G}_{1e}-\mathscr{G}_{1i})(t)\|_{L^1\cap \dot{H}^1}+\left\|\left(\mathscr{G}_{2e}^*-\mathscr{G}_{2i}^*\right)(t)\right\|_{L^1\cap \dot{H}^1\cap \dot{H}^2}+\|(\mathscr{G}_{4e}-\mathscr{G}_{4i})(t)\|_{L^1\cap L^2\cap \dot{H}^3}$$
$$\le \left(\omega_5(W^0)\right)^2(1+t)^{-\frac{3}{4}},$$

we obtain

$$\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\|_{L^\infty}\le C\omega_5(W^0)(1+t)^{-\frac{3}{4}}\ln(3+t),$$

where the smallness of $\omega_5(W^0)$ is used. Therefore, by L^2-L^∞ interpolation

$$\|\left(\mathscr{U}^e-\mathscr{U}^i\right)(t)\|_{L^q}\le C\omega_5(W^0)(1+t)^{-\frac{3}{4}+\frac{1}{4q}}\left(\ln(3+t)\right)^{1-\frac{2}{q}},\quad \forall\, 2\le q\le +\infty. \tag{5.75}$$

Estimates on $\|\mathscr{E}\|_{L^q}$, $\|\mathscr{U}^e+\mathscr{U}^i\|_{L^q}$, $\|\zeta^e-\zeta^i\|_{L^q}$ *and* $\|\zeta^e+\zeta^i\|_{L^q}$. By using the same procedure as that in subsection 4.3 for the non-isentropic case, we achieve that

$$\|\mathscr{E}(t)\|_{L^q}\le C\omega_6(W^0)(1+t)^{-\frac{3}{4}},\quad \forall\, 2\le q\le +\infty, \tag{5.76}$$
$$\|\mathscr{U}^e(t)+\mathscr{U}^i(t)\|_{L^q}\le C\omega_5(W^0)(1+t)^{-\frac{3}{4}},\quad \forall\, 2\le q\le +\infty, \tag{5.77}$$

and

$$\left\|\left(\zeta^e,\zeta^i\right)(t)\right\|_{L^q}\le C\omega_5(W^0)(1+t)^{-\frac{3}{4}},\quad \forall\, 2\le q\le +\infty. \tag{5.78}$$

Therefore, (5.78), (5.77), (5.75), (5.76) and (5.73) give (1.13), (1.14), (1.15), (1.16) and (1.17), respectively. The proof of Theorem 1.2 is completed. □

6. Conclusion

We studied the initial value problems to bipolar isentropic/non-isentropic CNS-M systems arising from plasmas in $\mathbb{R}^3$. The main difficulty of studying the bipolar CNS-M systems lies in the appearance of the electromagnetic fields satisfying the hyperbolic Maxwell equations. It should be pointed out that the dissipative property of the CNS-M systems in $\mathbb{R}^3$ is much weaker than the one of the E-M systems. After carefully considering this problem, we have three observations as follows.

(1) The principal difference and the chief difficulty for proving the large time decay rates between the CNS-M systems (for viscosity charged carriers) and the E-M systems (for ideal smooth charged carriers) stem from the fact that the two characteristic equations corresponding to their linearized homogeneous systems are very different. Indeed, for the E-M systems, we only need to consider the asymptotic properties of characteristic roots when $|k|$ goes to 0 and when $|k|$ approaches $+\infty$. However, due to the much more complex structure of the CNS-M systems, we have to divide the frequency space into three parts and even more (see the proof of Lemmas 3.1 and 3.2).

(2) The essential difficulty between the unipolar CNS-M system and the bipolar CNS-M system lies in the fact that the structure of the bipolar system is much more complex than that of

the unipolar model. For removing the difficulties caused by two charged carriers, we introduce the 'total functions' and 'difference functions' for unknowns. Then the linearized homogeneous equations of the bipolar non-isentropic CNS-M system (3.1) can be written as two decoupled subsystems (3.9) and (3.12).

(3) The important difference between the isentropic CNS-M system and the non-isentropic CNS-M system results from the fact that the characteristic equation corresponding to (3.22) (the linear wave equation of 'total' functions for the non-isentropic system) which is of order three, while the characteristic equation of the density or velocity equations in the isentropic model is of order two (see (5.33)), and hence it is much more complex to obtain the time decay rate of the linearized systems.

Based on the three previous observations, with the help of the elaborate energy method but with some new developments, the techniques of choosing symmetrizers and the spectrum analysis on the linearized homogeneous system corresponding to the CNS-M system, we established the large time-decay rates of global smooth solutions with small amplitude in $L^q(\mathbb{R}^3)$ for $2 \le q \le \infty$. For the bipolar isentropic/non-isentropic CNS-M systems, we obtain that the electric field, the densities and the total velocities of two charged carriers converge to the equilibrium states at the same rate $(1+t)^{-\frac{3}{4}}$. Simultaneously, for the bipolar non-isentropic CNS-M system, we find that the temperatures of two charged carriers also converge to the equilibrium states at the rate $(1+t)^{-\frac{3}{4}}$. However, for the bipolar non-isentropic CNS-M system, the difference of velocities of two charged carriers decays at the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}$ which is faster than the rate $(1+t)^{-\frac{3}{4}+\frac{1}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ of the bipolar isentropic CNS-M system, meanwhile, the magnetic field decay at the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}(\ln(3+t))^{1-\frac{2}{q}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}+\frac{3}{4q}}$ for the bipolar isentropic CNS-M system. This phenomenon on the charge transport is caused by the fact that temperatures are variant in the bipolar non-isentropic CNS-M systems.

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