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Convergence to Steady-States of Compressible Navier–Stokes–Maxwell Equations

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Abstract

In this paper, we consider the compressible Navier–Stokes–Maxwell equations with a non-constant background density in $\mathbb{R}^3$. We first show the existence and uniqueness of the non-trivial equilibrium (steady-state) of the system when the background density is a small variation of certain constant state, then we prove the asymptotic stability of the steady-state once the initial perturbation around the steady-state is small. Furthermore, by establishing the time-decay estimates for the corresponding linearized homogeneous equations, we artfully derive the time-algebraic convergence rates. The proof is based on the time-weighted energy method but with some new developments on the weight settings.

Keywords Compressible Navier–Stokes–Maxwell equations · Convergence to steady-states · Time decay rates

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1 Introduction and main results

1.1 Introduction

This paper is concerned with the compressible Navier–Stokes–Maxwell equations, where the background density n_b is a function of spatial variable and the electrons flow is isentropic (see Duan (2012) where $n_b = const.$). This system of dynamical equations represents the model of plasma dynamics for the electric contact flows and ionized gases Krall and Trivelpiece (1986); Nicholson (1983), and it usually presents in the form of

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + u \cdot \nabla u + \dfrac{1}{n} \nabla p = -(E + u \times B) + \dfrac{1}{n} \Delta u, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \\ \operatorname{div} E = n_b - n, \\ \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Here, $n = n(t, x) > 0$ is the electron density, $u = u(t, x) \in \mathbb{R}^3$ is the electron velocity, $E = E(t, x) \in \mathbb{R}^3$, $B = B(t, x) \in \mathbb{R}^3$, for $t > 0$, $x \in \mathbb{R}^3$, denote electronic and magnetic fields, respectively. The pressure function $p(\cdot)$ of the flow depends only on the density and satisfies the power law $p(n) = n^\gamma$ and $\gamma > 1$. $n_b(x)$ stands for the equilibrium background ion density satisfying

$$n_b(x) \to 1, \quad \text{as} \quad |x| \to \infty.$$

System (1.1) is supplemented with the initial condition

$$(n, u, E, B)|_{t=0} = (n_0, u_0, E_0, B_0), \quad x \in \mathbb{R}^3, \tag{1.2}$$

which satisfies the compatibility conditions

$$\operatorname{div} E_0 = n_b(x) - n_0, \quad \operatorname{div} B_0 = 0, \quad x \in \mathbb{R}^3. \tag{1.3}$$

Different from the previous study Duan (2012) with the constant background density n_b, our background density is variable in x, so the expected steady states of system (1.1) should be non-trivial rather than the constant states $(1, 0, 0, 0)$. Here, the equilibrium equations to the initial value problem (1.1)-(1.2) targeted in this paper are $(n_*, u_*, E_*, B_*)(x)$ with $u_* = 0$, $B_* = 0$, but $n_*(x) \not\equiv constant$, $E_*(x) \not\equiv constant$, satisfying

$$\begin{cases} \nabla p\left(n_*\right)=-n_* E_*, \\ \nabla \times E_*=0, \\ \operatorname{div} E_*=n_b(x)-n_*, \quad x \in \mathbb{R}^3 . \end{cases} \tag{1.4}$$

The existence of the equilibrium solutions to problem (1.1)–(1.2) was addressed in Liu and Zhu (2013).

Lemma 1.1 (see Theorem 1 in Liu and Zhu (2013)) *For integers $m \geq 2$ and $k \geq 0$, suppose that $\|n_b-1\|_{W_k^{m,2}}$ is small enough. Then, the equilibrium problem (1.4) admits a unique solution $(n_*-1, E_*) \in W_k^{m,2} \times W_k^{m-1,2}$ which satisfies*

$$\|n_*-1\|_{W_k^{m,2}} \leq C\|n_b-1\|_{W_k^{m,2}}, \quad \|E_*\|_{W_k^{m-1,2}} \leq C\|n_b-1\|_{W_k^{m,2}}, \tag{1.5}$$

for some constant C, where the weighted norm $\|\cdot\|_{W_k^{m,2}}$ is defined by

$$\|g\|_{W_k^{m,2}}=\left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^3}(1+|x|)^k\left|\partial_x^\alpha g(x)\right|^2 d x\right)^{\frac{1}{2}} .$$

Navier–Stokes–Maxwell equations have been one of the interesting topics and extensively studied. For incompressible Navier–Stokes–Maxwell equations, by means of the Fujita-Kato's method in l^1-based functional spaces, Ibrahim-Yoneda Ibrahim and Yoneda (2012) showed the local existence of unique solution and loss of smoothness of the velocity and magnetic field for periodic problem. Later, Ibrahim-Keraani Ibrahim and Keraani (2011) proved the existence of global small mild solutions in 3 dimensions and the same results in a space 'close' to the energy space in two dimensions. By using the *a priori* $L_t^2(L_x^\infty)$-estimates for solutions of the forced Navier–Stokes equations, Germain-Ibrahim-Masmoudi Germain et al. (2014) showed the local existence of mild solutions for arbitrarily large data in a space similar to the scale invariant spaces classically used for Navier–Stokes equations and refined the results in Ibrahim and Keraani (2011). With the help of Littlewood–Paley analysis, Yue-Zhong Yue and Zhong (2016) established the global well-posedness of solutions in the Besov spaces $B_{1/22,1} \times B_{3/22,1} \times B_{3/22,1}$ provided that the initial data are sufficiently small.

For the compressible Navier–Stokes–Maxwell equations, Fan-Li-Nakamura Fan et al. (2016) showed the convergence of the non-isentropic equations to the incompressible MHD equations in a bounded domain. By using the Green's function method and energy estimates, Duan Duan (2012) and Chen-Li-Zhang Chen et al. (2016) further proved the large time decay rates of global smooth solutions near a constant steady-state for one-fluid isentropic model. By use of the weighted energy methods and the techniques of symmetrizer, Wang-Xu Wang and Xu (2015) generalized their results to the non-isentropic models. Recently, Feng-Li-Mei-Wang Feng et al. (2021) studied the large time decay rates of global smooth solutions near a constant steady-state for the two-fluid models.

When we neglect the friction forces in charged fluids, the Navier–Stokes–Maxwell equations become the Euler–Maxwell equations. By using the fractional Godunov scheme as well as the compensated compactness argument, Chen-Jerome-Wang Chen et al. (2000) proved global existence of weak solutions to the initial-boundary value problem in one space dimension for arbitrarily large initial data in L^1. Jerome Jerome (2003) provided a local smooth solution theory for the Cauchy problem over $\mathbb{R}^3$ by adapting the classical semigroup-resolvent approach of Kato Kato (1975). Peng-Wang Peng and Wang (2008) established convergence of the compressible Euler–Maxwell system to the incompressible Euler system for well-prepared smooth initial data. Much more studies have been made for the Euler–Maxwell equations in all kinds of cases; see Hajjej and Peng (2012); Wasiolek (2016); Yang and Wang (2011); Yang and Hu (2019); Zhao (2021); Deng et al. (2017); Guo et al. (2016); Germain and Masmoudi (2014); Duan (2011); Feng et al. (2014); Peng et al. (2011); Ueda et al. (2012); Xu (2011); Liu and Zhu (2013); Peng (2015); Dumas et al. (2021) and references therein for discussion and analysis of the different issues such as the asymptotic limits on small physical parameters Hajjej and Peng (2012); Wasiolek (2016); Yang and Wang (2011); Yang and Hu (2019); Zhao (2021), the existence of global smooth irrotational flow Deng et al. (2017); Guo et al. (2016); Germain and Masmoudi (2014), longtime behavior of global solutions near a constant equilibrium state Duan (2011); Feng et al. (2014); Peng et al. (2011); Ueda et al. (2012); Xu (2011), large time-decay rates of small non-constant steady-state solutions Liu and Zhu (2013), stability of large non-constant equilibrium solutions Peng (2015) and the instability of WKB solution Dumas et al. (2021).

Note that, in the previous studies mentioned above, they focus on the case of $n_b = const$. However, the physical case for the dynamic system is with non-constant equilibrium background $n_b(x)$. The natural but also challenging question is how the solutions to problem (1.1)–(1.2) behave in large time, provided that n_b depends on x? This will be the main target of the present paper.

Before stating the main results, let us introduce some notations.

Notations. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, $\beta \le \alpha$ stands for $\beta_j \le \alpha_j$ for $j = 1, 2, 3$, and $\beta < \alpha$ stands for $\beta \le \alpha$ and $\beta \ne \alpha$.

C denotes some positive (generally large) constant and λ denotes some positive (generally small) constant, where both C and λ may take different values in different places. For two quantities a and b, $a \sim b$ means $\lambda a \le b \le \frac{1}{\lambda} a$ for a generic constant $0 \le \lambda \le 1$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space $L^2(\mathbb{R}^3)$, i.e.,

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(x) g(x) dx, \quad \forall f = f(x), \ g = g(x) \in L^2(\mathbb{R}^3).$$

For all integer $s \in \mathbb{N}$, we denote by H^s, L^2 and L^∞ the usual Sobolev spaces $H^s(\mathbb{R}^3)$, $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, and by $\|\cdot\|_s$, $\|\cdot\|$ and $\|\cdot\|_{L^\infty}$ the corresponding norms, respectively.

Now we are stating the main result.

Theorem 1.1 *Let integer $s \geq 4$. Assume that $\|n_b - 1\|_{W_0^{s+1,2}}$ is sufficiently small and (1.3) holds. There exist two constants $\delta_0 > 0$ and $C_0 > 0$ such that, if*

$$\|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_s \leq \delta_0,$$

then problem (1.1)-(1.2) admits a unique global solution $(n, u, E, B)(t, x)$ which satisfies

$$\begin{aligned}
&(n - n_*, u, E - E_*, B) \in C([0, \infty); H^s(\mathbb{R}^3)),\\
&n - n_* \in L^2((0, \infty); H^s(\mathbb{R}^3)), \quad \nabla u \in L^2((0, \infty); H^s(\mathbb{R}^3)),\\
&\nabla(E - E_*) \in L^2((0, \infty); H^{s-2}(\mathbb{R}^3)), \quad \nabla^2 B \in L^2((0, \infty); H^{s-3}(\mathbb{R}^3))
\end{aligned}$$

and

$$\sup_{t \geq 0} \|(n - n_*, u, E - E_*, B)\|_s \leq C_0 \|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_s.$$

Furthermore, there exist $\delta_1 > 0$ and $C_1 > 0$ such that, if

$$\|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_{s+2} + \|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_{L^1} \leq \delta_1$$

and $\|n_b - 1\|_{W_0^{s+3,2}}$ is small enough, then the solution $(n, u, E, B)(t, x)$ satisfies that, for any $t \geq 0$,

$$\|(n - n_*, u, E - E_*, B)\|_s \leq C_1 (1 + t)^{-\frac{3}{8}}, \tag{1.6}$$

$$\|\nabla(n - n_*, u, E - E_*, B)\|_{s-1} \leq C_1 (1 + t)^{-\frac{5}{8}}. \tag{1.7}$$

More precisely, if

$$\|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_6 + \|(n_0 - n_*, u_0, E_0 - E_*, B_0)\|_{L^1} \leq \delta_1$$

and $\|n_b - 1\|_{W_0^{7,2}}$ is small enough, then we have

$$\|(n - n_*, u)\| \leq C_1 (1 + t)^{-\frac{5}{8}}. \tag{1.8}$$

Remark 1.1 In the proof of the global existence of Theorem 1.1, we modify the energy estimates by choosing a weight function $1 + \sigma_* + \Phi(\sigma_*)$ which plays a vital role in closing the energy estimates.

Remark 1.2 We only capture the same time decay properties of u and B as that in Duan (2012) except both $n - n_*$ and $E - E_*$ decay as $(1 + t)^{-\frac{5}{8}}$ in a slower way, because the nonhomogeneous terms containing ρ_* decay at most the same as $\left(\mathfrak{E}_s^h(\cdot)\right)^{\frac{1}{2}}$.

Remark 1.3 In the process of establishing the time decay rates of the perturbed solution, the main difficulty is to deal with these linear nonhomogeneous terms containing ρ_*, which cannot bring good enough time decay rates as expected, whereas in Duan (2012), the nonhomogeneous terms are at least quadratically nonlinear. In order to remove this obstacle, we make iteration for inequalities (4.8) and (4.9) together.

The proof of Theorem 1.1 is based on the classical energy method but with some new developments. We proceed first by using the time-weighted energy estimates, and then by combining them with the dissipation time-decay bounds for the linearized equation. In particular, we choose a new time-weight function $1 + \sigma_* + \Phi(\sigma_*)$ to establish the energy estimates and introduce two new functions $\mathfrak{Y}(t) = \sup\limits_{0\le\tau\le t} (1+\tau)^{\frac{5}{4}} \mathfrak{E}_s^h(\overline{V}(\tau))$ and $\mathfrak{F}_0(t) = \sup\limits_{0\le\tau\le t} (1+\tau)^{\frac{5}{4}} \|(\overline{\rho}, \overline{u})\|^2$ to obtain the time decay rates for each component of the solution to problem (1.1)–(1.2).

Regarding the other models involving a non-constant background density, for example, the Vlasov–Poisson–Boltzmann equations and the non-isentropic compressible Navier–Stokes–Maxwell equations, there are also some significant contributions, see Duan and Yang (2009); Duan and Strain (2011); Duan et al. (2007a, b); Feng et al. (2021) and the references therein. In Duan and Yang (2009), by using the combination of Fourier analysis and energy estimates, Duan and Yang considered the stability of the equilibrium states which were given by an elliptic equation. In Duan and Strain (2011), Duan and Strain obtained the optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$. The optimal convergence rates for the compressible Navier–Stokes equations with potential forces were obtained by Duan-Ukai-Yang-Zhao Duan et al. (2007a) and Duan-Liu-Ukai-Yang Duan et al. (2007b), respectively. Recently, for the non-isentropic compressible Navier–Stokes–Maxwell equations, Feng-Li-Wang Feng et al. (2021) proved the global existence of smooth solutions once the size of the non-constant equilibrium states is small enough, and the initial perturbations around the equilibrium states are also small enough.

In what follows, let us state the main idea for the proof of Theorem 1.1 and the vital difference of the study on the time decay rates between the Euler–Maxwell equations and the Navier–Stokes–Maxwell equations. The key point is to establish the a priori estimates

$$\mathfrak{E}_s(\overline{V}(t)) + \lambda \int_0^t \mathfrak{D}_s(\overline{V}(s))ds \le \mathfrak{E}_s(\overline{V}_0),$$

where $\overline{V}(t)$ is the perturbation of solutions, and $\mathfrak{E}_s(\cdot)$, $\mathfrak{D}_s(\cdot)$ denote the energy functional and energy dissipation rate functional. Here, if we make the energy estimates like what Duan did in Duan (2012), it is difficult to control the highest-order derivative of $\overline{E}$ because of the regularity-loss type in the sense that $(\overline{E}, \overline{B})$ is time-space integrable up to $s - 1$ order only. We overcome this difficulty by looking at the evolutions of several different quantities (i.e., quadratic forms of the unknown functions) which have different gains and losses, and construct the energy functional as a suitable linear combination of them. Specifically, we introduce the weight function $1 + \sigma_* + \Phi(\sigma_*)$

which plays a key role in closing the energy estimates (see (3.1) and (3.2)). Furthermore, for the time decay rates of perturbed solutions in Theorem 1.1, we cannot analyze the corresponding linearized homogeneous system of (1.1) around the steady state $(n_*, 0, E_*, 0)$ directly. In this case, the Fourier transform doesn't work due to the fact that the coefficients are no longer invariant. Here, the main idea follows from Liu and Zhu (2013) by combining energy estimates with the linearized results in Duan (2012). However, the techniques used here are different from Liu and Zhu (2013). For instance, in order to estimate $\|B\|$, Liu-Zhu Liu and Zhu (2013) introduce two functions

$$\mathcal{E}_{s,\infty}(\bar{V}(t)) = \sup_{0\le s\le t} \{(1+s)^{\frac{3}{2}}\mathcal{E}_s(\bar{V}(s)) + (1+s)^{\frac{5}{2}}\mathcal{E}_s^h(\bar{V}(s))\}$$

and

$$L_0(t) = \sup_{0\le s\le t} (1+s)^{\frac{5}{2}} \left(\|\bar{\rho}\|^2_{L^2(\mathbb{R}^3)} + \|\bar{u}\|^2_{L^2(\mathbb{R}^3)}\right).$$

Since the function $\mathcal{E}_{s,\infty}(\bar{V}(t))$ contains both $\mathcal{E}_s$ and $\mathcal{E}_s^h$, they have to consider two cases, namely, $1 < l < 2$ and $2 < l < 3$. Hence, the regularity of the initial data are needed to be $s+4$. However, for the Navier–Stokes–Maxwell equations, we introduce

$$\mathfrak{Y}(t) = \sup_{0\le s\le t} (1+s)^{\frac{5}{4}}\mathfrak{E}_s^h(\overline{V}(s)), \quad \mathfrak{F}_0(t) = \sup_{0\le s\le t} (1+s)^{\frac{5}{4}}\|(\overline{\rho}, \overline{u})\|^2$$

to estimate $\|B\|$. The regularity of the initial data is lower than that in Liu and Zhu (2013), $s+2$ is enough. This is based on the fact that the function $\mathfrak{Y}(t)$ contains only $\mathfrak{E}_s^h$. Indeed, after carefully carrying out the estimates, we obtain that both $\mathfrak{Y}(t)$ and $\mathfrak{F}_0(t)$ can be controlled by $C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}$. Therefore, it follows that $\|(\overline{\rho}, \overline{u})\| \le C(1+t)^{-\frac{5}{8}}\|\overline{V}_0\|_{L^1\cap H^{s+2}}$, the details can be seen in Sect. 4.

We end this section by stating the arrangement of the rest of this paper. In the next section, we give a useful lemma and the transformation of the initial value problem. In Sect. 3, we show the detailed estimates for the proof of the global existence of solutions. In the last section, we study the time decay rates of global smooth solutions by combining the $L^p - L^q$ time decay property of the linearized homogeneous system with time-weighted estimate and complete the proof of Theorem 1.1.

2 Preliminaries

First of all, let us recall the following Moser-type calculus inequalities, which will be often used later for energy estimates.

Lemma 2.1 *(Moser-type calculus inequalities, see Klainerman and Majda (1981); Majda (1984).) Let $s \ge 3$ be an integer. Suppose $u \in H^s$, $\nabla u \in L^\infty$, $v \in H^{s-1} \cap L^\infty$ and f is a smooth function. Then, for all multi-index α with $1 \le |\alpha| \le s$, one has*

$\partial^\alpha(uv) - u\partial^\alpha v \in L^2$ *and*

$$
\begin{aligned}
&\|\partial^\alpha(uv) - u\partial^\alpha v\| \le C\big(\|\nabla u\|_{L^\infty}\|D^{|\alpha|-1}v\| + \|D^{|\alpha|}u\|\|v\|_{L^\infty}\big),\\
&\left\|\partial^\alpha f(u)\right\| \le C(1+\|u\|_s)^{s-1}\|u\|_s,
\end{aligned}
$$

where the constant C may depend on $\|u\|_{L^\infty}$ and s, and

$$
\|D^{s'}u\| = \sum_{|\alpha|=s'} \|\partial^\alpha u\|.
$$

Next we are going to reduce system (1.1)-(1.2) by introducing a suitable transformation. Suppose (n, u, E, B) to be a smooth solution to system (1.1) with the initial condition (1.2) which satisfies (1.3). We introduce the transformation

$$
\begin{cases}
\sigma(t,x) = 2(\gamma-1)^{-1}\left(\left(n\left(\gamma^{-\frac{1}{2}}t, x\right)\right)^{\frac{\gamma-1}{2}} - 1\right), \quad v = \gamma^{-1}u\left(\gamma^{-\frac{1}{2}}t, x\right),\\
\widetilde{E} = \gamma^{-\frac{1}{2}}E\left(\gamma^{-\frac{1}{2}}t, x\right), \quad \widetilde{B} = \gamma^{-\frac{1}{2}}B\left(\gamma^{-\frac{1}{2}}t, x\right).
\end{cases}
\tag{2.1}
$$

Then, Eq. (1.1) becomes

$$
\begin{cases}
\partial_t\sigma + \left(\dfrac{\gamma-1}{2}\sigma + 1\right)\mathrm{div}v + v\cdot\sigma = 0,\\
\partial_t v + v\cdot\nabla v + \left(\dfrac{\gamma-1}{2}\sigma+1\right)\nabla\sigma\\
= -\left(\gamma^{-\frac{1}{2}}\widetilde{E} + v\times\widetilde{B}\right) + \gamma^{-\frac{1}{2}}\dfrac{\Delta v}{\Phi(\sigma)+\sigma+1},\\
\partial_t\widetilde{E} - \gamma^{-\frac{1}{2}}\nabla\times\widetilde{B} = \gamma^{-\frac{1}{2}}v + \gamma^{-\frac{1}{2}}(\Phi(\sigma)+\sigma)v,\\
\partial_t\widetilde{B} + \gamma^{-\frac{1}{2}}\nabla\times\widetilde{E} = 0,\\
\mathrm{div}\widetilde{E} = -\gamma^{-\frac{1}{2}}(\Phi(\sigma)+\sigma) + \gamma^{-\frac{1}{2}}(n_b(x)-1), \quad \mathrm{div}\widetilde{B} = 0, \quad t>0, \quad x\in\mathbb{R}^3,
\end{cases}
\tag{2.2}
$$

supplemented with the initial condition

$$
V|_{t=0} = (\sigma, v, \widetilde{E}, \widetilde{B})|_{t=0} = V_0 := (\sigma_0, v_0, \widetilde{E}_0, \widetilde{B}_0), \quad x\in\mathbb{R}^3,
\tag{2.3}
$$

which satisfies the compatibility condition

$$
\mathrm{div}\widetilde{E}_0 = -\gamma^{-\frac{1}{2}}(\Phi(\sigma_0)+\sigma_0) + \gamma^{-\frac{1}{2}}(n_b(x)-1), \quad \mathrm{div}\widetilde{B}_0 = 0, \quad x\in\mathbb{R}^3.
\tag{2.4}
$$

Here $\Phi(\cdot)$ is defined by

$$
\Phi(\sigma) = (\frac{\gamma-1}{2}\sigma+1)^{\frac{2}{\gamma-1}} - \sigma - 1,
\tag{2.5}
$$

and $V_0 = (\sigma_0, v_0, \widetilde{E}_0, \widetilde{B}_0)$ is given from (n_0, u_0, E_0, B_0) according to transformation (2.1).

Moreover, we also introduce the transformation

$$\sigma_* = \frac{2}{\gamma - 1}\left(n_*(x)^{\frac{\gamma-1}{2}} - 1\right), \quad \widetilde{E}_* = \gamma^{-\frac{1}{2}} E_*(x), \tag{2.6}$$

then the equilibrium Eqs. (1.4) turn into

$$\begin{cases} \left(\frac{\gamma-1}{2}\sigma_* + 1\right)\nabla\sigma_* = -\gamma^{-\frac{1}{2}}\widetilde{E}_*, \\ \nabla\times\widetilde{E}_* = 0, \\ \mathrm{div}\widetilde{E}_* = \gamma^{-\frac{1}{2}}(n_b(x) - 1) - \gamma^{-\frac{1}{2}}(\Phi(\sigma_*) + \sigma_*). \end{cases} \tag{2.7}$$

Based on the existence result in Lemma 1.1, we want to investigate the stability of the equilibrium state $(\sigma_*, 0, \widetilde{E}_*, 0)$. Let us introduce the perturbations $(\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})$ by

$$\overline{\sigma} = \sigma - \sigma_*, \quad \overline{v} = v, \quad \overline{E} = \widetilde{E} - \widetilde{E}_*, \quad \overline{B} = \widetilde{B}.$$

It follows from (2.2) and (2.7) that

$$\begin{cases} \partial_t\overline{\sigma} + \left(\frac{\gamma-1}{2}\overline{\sigma} + 1\right)\nabla\cdot\overline{v} + \overline{v}\cdot\overline{\sigma} + \overline{v}\cdot\sigma_* + \frac{\gamma-1}{2}\sigma_*\mathrm{div}\overline{v} = 0, \\ \partial_t\overline{v} + \overline{v}\cdot\nabla\overline{v} + \left(\frac{\gamma-1}{2}\overline{\sigma} + 1\right)\nabla\overline{\sigma} + \frac{\gamma-1}{2}\overline{\sigma}\nabla\sigma_* + \frac{\gamma-1}{2}\sigma_*\nabla\overline{\sigma} \\ \qquad = -\left(\gamma^{-\frac{1}{2}}\overline{E} + \overline{v}\times\overline{B}\right) + \gamma^{-\frac{1}{2}}\frac{\Delta\overline{v}}{\Phi(\overline{\sigma}+\sigma_*) + \overline{\sigma} + \sigma_* + 1}, \\ \partial_t\overline{E} - \gamma^{-\frac{1}{2}}\nabla\times\overline{B} = \gamma^{-\frac{1}{2}}\overline{v} + \gamma^{-\frac{1}{2}}(\Phi(\overline{\sigma}+\sigma_*) + \overline{\sigma} + \sigma_*)\overline{v}, \\ \partial_t\overline{B} + \gamma^{-\frac{1}{2}}\nabla\times\overline{E} = 0, \\ \mathrm{div}\overline{E} = -\gamma^{-\frac{1}{2}}(\Phi(\overline{\sigma}+\sigma_*) - \Phi(\sigma_*)) - \overline{\sigma}, \quad \mathrm{div}\overline{B} = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{2.8}$$

supplemented with the initial condition

$$\overline{V}|_{t=0} = (\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})|_{t=0} = \overline{V}_0 := (\sigma_0 - \sigma_*, v_0, \widetilde{E}_0 - \widetilde{E}_*, \widetilde{B}_0), \tag{2.9}$$

which satisfies the compatibility condition

$$\mathrm{div}\overline{E}_0 = -\gamma^{-\frac{1}{2}}(\Phi(\overline{\sigma}_0 + \sigma_*) - \Phi(\sigma_*)) - \overline{\sigma_0}, \quad \mathrm{div}\overline{B}_0 = 0, \quad x \in \mathbb{R}^3. \tag{2.10}$$

Notice that system (2.8) is quasi-linear symmetric hyperbolic–parabolic, then the local existence of smooth solutions to the initial value problem (2.8)–(2.9) can be obtained from the classical results of Kato Kato (1975) and the pioneering work of Matsumura-Nishida Matsumura and Nishida (1979, 1980).

3 Stability of equilibrium solution

In the following, we assume the integer $s \geq 4$. Meanwhile, for $\overline{V} = (\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})$, let us define the full instant energy functional $\mathfrak{E}_s(\overline{V}(t))$ and the high-order instant energy functional $\mathfrak{E}_s^h(\overline{V}(t))$ by

$$\begin{aligned}\mathfrak{E}_s(\overline{V}(t)) = &\sum_{|\alpha|\leq s} \langle 1+\sigma_* + \Phi(\sigma_*), |\partial^\alpha \overline{\sigma}|^2 + |\partial^\alpha \overline{v}|^2\rangle + \|(\overline{E}, \overline{B})\|_s^2 \\ &+ \mathfrak{K}_1 \sum_{|\alpha|\leq s-1} \left\langle \partial^\alpha \overline{v}, \nabla \partial^\alpha \overline{\sigma}\right\rangle + \mathfrak{K}_2 \sum_{|\alpha|\leq s-2} \left\langle \partial^\alpha \nabla \times \overline{E}, \partial^\alpha \nabla \times \overline{v}\right\rangle \\ &+ \mathfrak{K}_3 \sum_{1\leq|\alpha|\leq s-2} \left\langle \partial^\alpha(-\nabla \times \overline{B}), \partial^\alpha \overline{E}\right\rangle,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathfrak{E}_s^h(\overline{V}(t)) = &\sum_{1\leq|\alpha|\leq s} \langle 1+\sigma_* + \Phi(\sigma_*), |\partial^\alpha \overline{\sigma}|^2 + |\partial^\alpha \overline{v}|^2\rangle + \|\nabla(\overline{E}, \overline{B})\|_{s-1}^2 \\ &+ \mathfrak{K}_1 \sum_{1\leq|\alpha|\leq s-1} \left\langle \partial^\alpha \overline{v}, \partial^\alpha \nabla \overline{\sigma}\right\rangle + \mathfrak{K}_2 \sum_{1\leq|\alpha|\leq s-2} \left\langle \partial^\alpha \nabla \times \overline{E}, \partial^\alpha \nabla \times \overline{v}\right\rangle \\ &+ \mathfrak{K}_3 \sum_{2\leq|\alpha|\leq s-2} \left\langle \partial^\alpha(-\nabla \times \overline{B}), \partial^\alpha \overline{E}\right\rangle,\end{aligned} \tag{3.2}$$

respectively, where $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$ are small positive constants to be chosen later. Notice that since all constants $\mathfrak{K}_i\,(i = 1, 2, 3)$ are small enough, we have

$$\mathfrak{E}_s(\overline{V}(t)) \sim \|(\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})\|_s^2, \quad \mathfrak{E}_s^h(\overline{V}(t)) \sim \|\nabla(\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})\|_{s-1}^2.$$

We further define the dissipation rates $\mathfrak{D}_s(\overline{V}(t))$, $\mathfrak{D}_s^h(\overline{V}(t))$ by

$$\mathfrak{D}_s(\overline{V}(t)) \sim \|\overline{\sigma}\|_s^2 + \|\nabla \overline{v}\|_s^2 + \|\nabla \overline{E}\|_{s-2}^2 + \|\nabla^2 \overline{B}\|_{s-3}^2 \tag{3.3}$$

and

$$\mathfrak{D}_s^h(\overline{V}(t)) \sim \|\nabla \overline{\sigma}\|_{s-1}^2 + \|\nabla^2 \overline{v}\|_{s-1}^2 + \|\nabla^2 \overline{E}\|_{s-3}^2 + \|\nabla^3 \overline{B}\|_{s-4}^2. \tag{3.4}$$

Then, concerning the transformed initial value problem (2.8)-(2.9), we get the following global existence result.

Proposition 3.1 *Assume that* $\|n_b - 1\|_{W_0^{s+1,2}}$ *is sufficiently small and* (2.10) *holds. There exist* $\mathfrak{E}_s(\cdot)$ *and* $\mathfrak{D}_s(\cdot)$ *in the form of* (3.1) *and* (3.3) *such that, if* $\mathfrak{E}_s(\overline{V}_0) > 0$ *is sufficiently small, then problem* (2.8)-(2.9) *admits a unique global solution* $\overline{V} =$

$(\sigma-\sigma_*, v, \widetilde{E}-\widetilde{E}_*, \widetilde{B})$ which satisfies, for any $t\geq 0$,

$$\begin{gathered}\overline{V}\in C([0,\infty);H^s(\mathbb{R}^3)),\\ \sigma-\sigma_*\in L^2((0,\infty);H^s(\mathbb{R}^3)),\quad \nabla v\in L^2((0,\infty);H^s(\mathbb{R}^3)),\\ \nabla\widetilde{E}-\nabla\widetilde{E}_*\in L^2((0,\infty);H^{s-2}(\mathbb{R}^3)),\quad \nabla^2\widetilde{B}\in L^2((0,\infty);H^{s-3}(\mathbb{R}^3)),\end{gathered}\tag{3.5}$$

and

$$\mathfrak{E}_s(\overline{V}(t))+\lambda\int_0^t\mathfrak{D}_s(\overline{V}(s))ds\leq\mathfrak{E}_s(\overline{V}_0).\tag{3.6}$$

3.1 The a priori estimates

In the following, we prove that the equilibrium solution obtained in Lemma 1.1 is asymptotic stable under small initial perturbation. We start to use the classical energy method but with some new developments to establish the a priori estimates for smooth solutions to problem (2.8)-(2.9). For this purpose, we introduce

$$\delta=\|\overline{\sigma}_*\|_{W_0^{s+1,2}}=\sqrt{\sum_{|\alpha|\leq s+1}\int_{\mathbb{R}^3}|\partial_x^\alpha\sigma_*|^2dx}\tag{3.7}$$

for convenience. By (2.6), direct computation gives

$$\sigma_*=\frac{2}{\gamma}\frac{Q_*}{(\frac{\gamma-1}{\gamma}Q_*+1)^{\frac12}+1}\sim Q_*,$$

which implies that $\delta\leq C\|Q_*\|_{W_0^{s+1,2}}\leq C\|n_b-1\|_{W_0^{s+1,2}}$ is small enough. Here, Q_* is defined by

$$(\gamma-1)Q_*=\gamma(n_*^{\gamma-1}-1).$$

The main task of this subsection is to prove

Theorem 3.1 *(The a priori estimates). Suppose $0<T\leq\infty$ to be given. Assume that $\overline{V}=(\overline{\sigma},\overline{v},\overline{E},\overline{B})\in C([0,T);H^s(\mathbb{R}^3))$ is smooth for $T>0$ with*

$$\sup_{0\leq t<T}\|\overline{V}(t)\|_s\leq 1\tag{3.8}$$

and suppose that $\overline{V}$ solves system (2.8) for $t\in(0,T)$. Then, there exist $\mathfrak{E}_s(\cdot)$ and $\mathfrak{D}_s(\cdot)$ in the form of (3.1) and (3.3) such that, for any $0\leq t<T$,

$$\frac{d}{dt}\mathfrak{E}_s(\overline{V}(t))+\lambda\mathfrak{D}_s(\overline{V}(t))\leq C\left(\mathfrak{E}_s(\overline{V}(t))^{\frac12}+\delta\right)\mathfrak{D}_s(\overline{V}(t)).\tag{3.9}$$

Proof We use five steps to complete the proof.

Step 1. It holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\sum_{|\alpha|\le s}\langle 1+\sigma_*+\Phi(\sigma_*), |\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\rangle+\|(\overline{E},\overline{B})\|_s^2\right)\\&\quad+\gamma^{-\frac{1}{2}}\sum_{|\alpha|\le s}\|\nabla\partial^\alpha\overline{v}\|^2\\&\quad\le C\left(\|\overline{V}\|_s+\delta\|\overline{\sigma},\overline{v}]\|_s+\delta)(\|\nabla\overline{v}\|_s^2+\|\overline{\sigma}\|_s^2+\|\nabla\overline{E}\|_{s-2}^2\right).\end{aligned}\tag{3.10}$$

In fact, for $\alpha\in\mathbb{N}^3$ with $|\alpha|\le s$, applying ∂^α to the first two equations of (2.8) and taking the inner product of them with $(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}$ and $(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}$ in $L^2(\mathbb{R}^3)$, respectively, and then using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle 1+\sigma_*+\Phi(\sigma_*), |\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\rangle+\gamma^{-\frac{1}{2}}\left\langle\partial^\alpha\overline{E}, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}\right\rangle\\&\quad-\gamma^{-\frac{1}{2}}\left\langle(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}, \frac{\Delta\partial^\alpha\overline{v}}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\right\rangle\\&=I_1(t)+\sum_{\beta<\alpha}C_\alpha^\beta\Big\langle(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}, \partial^{\alpha-\beta}\\&\quad\left(\frac{1}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\right)\Delta\partial^\beta\overline{v}\Big\rangle\\&\quad-\sum_{\beta<\alpha}C_\alpha^\beta I_{\alpha,\beta}^{(\sigma)}(t)-\sum_{\beta<\alpha}C_\alpha^\beta I_{\alpha,\beta}^{(v)}(t),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}I_1(t)=&\frac{1}{2}\left\langle\nabla\cdot\overline{v}, (1+\sigma_*+\Phi(\sigma_*))(|\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2)\right\rangle\\&+\frac{\gamma-1}{2}\left\langle\nabla\overline{\sigma}\cdot\partial^\alpha\overline{v}, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}\right\rangle\\&-\left\langle\overline{v}\times\partial^\alpha\overline{B}, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}\right\rangle\\&+\frac{\gamma-1}{2}\left\langle\nabla\sigma_*\partial^\alpha\overline{v}, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}\right\rangle\\&-\frac{\gamma-1}{2}\left\langle\overline{\sigma}\partial^\alpha\nabla\sigma_*, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{v}\right\rangle\\&-\left\langle\overline{v}\cdot\partial^\alpha\nabla\sigma_*, (1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}\right\rangle\\&+\left\langle(\frac{\gamma-1}{2}\overline{\sigma}+1)\partial^\alpha\overline{v}, \nabla(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}\right\rangle\\&+\frac{\gamma-1}{2}\left\langle\sigma_*\partial^\alpha\overline{v}, \nabla(1+\sigma_*+\Phi(\sigma_*))\partial^\alpha\overline{\sigma}\right\rangle\\&+\frac{\gamma-1}{2}\left\langle\overline{v}, \nabla(1+\sigma_*+\Phi(\sigma_*))(|\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&\triangleq \sum_{j=1}^{9} I_{1,j}(t),\\
&I_{\alpha,\beta}^{(\sigma)}(t) = \left\langle \partial^{\alpha-\beta}\overline{v}\cdot\nabla\partial^{\beta}\overline{\sigma}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{\sigma}\right\rangle\\
&+\frac{\gamma-1}{2}\left\langle \partial^{\alpha-\beta}\overline{\sigma}\partial^{\beta}\nabla\cdot\overline{v}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{\sigma}\right\rangle\\
&+\frac{\gamma-1}{2}\left\langle \partial^{\alpha-\beta}\sigma_*\nabla\cdot\partial^{\beta}\overline{v}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{\sigma}\right\rangle\\
&+\left\langle \partial^{\alpha-\beta}\overline{v}\partial^{\beta}\nabla\cdot\sigma_*, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{\sigma}\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{\alpha,\beta}^{(v)}(t) &= \left\langle \partial^{\alpha-\beta}\overline{v}\cdot\nabla\partial^{\beta}\overline{v}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}\right\rangle\\
&+\frac{\gamma-1}{2}\left\langle \partial^{\alpha-\beta}\overline{\sigma}\nabla\partial^{\beta}\overline{\sigma}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}\right\rangle\\
&+\frac{\gamma-1}{2}\left\langle \partial^{\alpha-\beta}\sigma_*\nabla\partial^{\beta}\overline{\sigma}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}\right\rangle\\
&+\left\langle \partial^{\alpha-\beta}\overline{v}\times\partial^{\beta}\overline{B}, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}\right\rangle\\
&+\frac{\gamma-1}{2}\left\langle \partial^{\alpha-\beta}\overline{\sigma}\nabla\partial^{\beta}\sigma_*, (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}\right\rangle.
\end{aligned}
$$

First, for the second term on the right-hand side of (3.11), we have

$$
\begin{aligned}
&\sum_{\beta<\alpha} C_{\alpha}^{\beta}\left\langle (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}, \partial^{\alpha-\beta}\left(\frac{1}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\right)\Delta\partial^{\beta}\overline{v}\right\rangle\\
&\le C\sum_{\beta<\alpha} C_{\alpha}^{\beta}\left\langle \left|\partial^{\alpha}\overline{v}\right|, \left|\partial^{\alpha-\beta}\left(\frac{1}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\right)\Delta\partial^{\beta}\overline{v}\right|\right\rangle\\
&\le C\|\partial^{\alpha-\beta}\overline{\sigma}\|_{L^3}\|\Delta\partial^{\beta}\overline{v}\|\|\partial^{\alpha}\overline{v}\|_{L^6} + C\|\partial^{\alpha-\beta}\sigma_*\|_{L^3}\|\Delta\partial^{\beta}\overline{v}\|\|\partial^{\alpha}\overline{v}\|_{L^6}\\
&\le C(\|\overline{\sigma}\|_s+\delta)\|\nabla\overline{v}\|_s^2.
\end{aligned}
$$

Similarly, for the last term on the left-hand side of (3.11), we obtain

$$
\begin{aligned}
&-\gamma^{-\frac{1}{2}}\left\langle (1+\sigma_*+\Phi(\sigma_*))\partial^{\alpha}\overline{v}, \frac{\Delta\partial^{\alpha}\overline{v}}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\right\rangle\\
&=\gamma^{-\frac{1}{2}}\|\nabla\partial^{\alpha}\overline{v}\|^2 - \left\langle \frac{\Phi(\sigma_*)-\overline{\sigma}-\Phi(\sigma_*+\overline{\sigma})}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}\partial^{\alpha}\overline{v}, \partial^{\alpha}\Delta\overline{v}\right\rangle\\
&\ge\gamma^{-\frac{1}{2}}\|\nabla\partial^{\alpha}\overline{v}\|^2 - C\|\overline{\sigma}\|_s\|\nabla\partial^{\alpha}\overline{v}\|^2.
\end{aligned}
$$

Then, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\langle 1+\sigma_*+\Phi(\sigma_*),\, |\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\Big\rangle+\gamma^{-\frac{1}{2}}\|\nabla\partial^\alpha\overline{v}\|^2 \\ &\quad+\gamma^{-\frac{1}{2}}\left\langle\partial^\alpha\overline{E},\,(1+\sigma_*+\Phi(\sigma_*))|\partial^\alpha\overline{v}|\right\rangle \\ &\quad\le C(\|\overline{\sigma}\|_s+\delta)\|\nabla\overline{v}\|_s^2+I_1(t)-\sum_{\beta<\alpha}C_\alpha^\beta I_{\alpha,\beta}^{(\sigma)}(t)-\sum_{\beta<\alpha}C_\alpha^\beta I_{\alpha,\beta}^{(v)}(t). \end{aligned} \tag{3.12}$$

We estimate the term $I_1(t)$ on the right-hand side as follows. When $|\alpha|=0$,

$$\begin{aligned} I_1(t)\le& C\|\nabla\cdot\overline{v}\|(\|\overline{\sigma}\|\|\overline{\sigma}\|_{L^\infty}+\|\overline{v}\|\|\overline{v}\|_{L^\infty}) \\ &+C\|\nabla\overline{\sigma}\|_{L^3}\|\overline{v}\|\|\overline{\sigma}\|_{L^6}+C\|\overline{v}\|_{L^\infty}^2\|\overline{B}\|\|1+\sigma_*+\Phi(\sigma_*)\| \\ &+C\|\nabla\sigma_*\|\|\overline{v}\|_{L^\infty}\|\overline{\sigma}\|+C\|\nabla\sigma_*\|\|\overline{v}\|_{L^6}\|\overline{\sigma}\|_{L^3} \\ &+C\|\overline{\sigma}\|\|\overline{v}\|_{L^\infty}\|\overline{\sigma}\|+C\|\nabla\sigma_*\|\|\overline{v}\|_{L^6}\|\overline{\sigma}\|_{L^3} \\ &+C\|\nabla\sigma_*\|\|\overline{v}\|_{L^6}\|\overline{\sigma}\|_{L^3}\|\sigma_*\|_{L^\infty}+C\|\nabla\sigma_*\|\|\overline{v}\|_{L^6}\|\overline{\sigma}\|_{L^3}\|\overline{\sigma}\|_{L^\infty} \\ &+C\|\overline{v}\|_{L^6}\|\nabla\sigma_*\|\|\overline{v}\|_{L^3}\|\overline{v}\|_{L^\infty} \\ \le& C(\|(\overline{\sigma},\overline{v},\overline{B})\|_s+\delta+\delta\|\overline{v}\|)(\|\overline{\sigma}\|_s^2+\|\nabla\overline{v}\|_s^2), \end{aligned}$$

which will be further bounded by the right-hand side of (3.10).

When $|\alpha|\ge 1$, we get

$$\begin{aligned} &I_{1,4}(t)+I_{1,7}(t)+I_{1,8}(t)+I_{1,9}(t) \\ &\quad\le C\|\nabla\sigma_*\|_{L^\infty}\|\partial^\alpha\overline{v}\|\|1+\sigma_*+\Phi(\sigma_*)\|_{L^\infty}\|\partial^\alpha\overline{\sigma}\|+C\|\partial^\alpha\overline{v}\|\|\partial^\alpha\overline{\sigma}\|\|\nabla\sigma_*\|_{L^\infty} \\ &\qquad+C\|\nabla\sigma_*\|_{L^\infty}\|\partial^\alpha\overline{v}\|\|\overline{\sigma}\|_{L^\infty}\|\partial^\alpha\overline{\sigma}\|+C\|\nabla\sigma_*\|_{L^\infty}\|\partial^\alpha\overline{v}\|\|\sigma_*\|_{L^\infty}\|\partial^\alpha\overline{\sigma}\| \\ &\qquad+C\|\overline{v}\|_{L^\infty}\|\nabla\sigma_*\|_{L^\infty}\|\nabla(\overline{\sigma},\overline{v})\|_{s-1}^2 \\ &\quad\le C(\delta+\delta\|(\overline{\sigma},\overline{v})\|_s)(\|\overline{\sigma}\|_s^2+\|\nabla\overline{v}\|_s^2), \\ &I_{1,1}(t)+I_{1,2}(t)\le C\|(\overline{\sigma},\overline{v})\|_s\|\nabla(\overline{\sigma},\overline{v})\|_{s-1}^2, \end{aligned}$$

and

$$I_{1,3}(t)+I_{1,5}(t)+I_{1,6}(t)\le C(\|\overline{B}\|_s+\delta)\|\nabla(\overline{\sigma},\overline{v})\|_{s-1}^2.$$

By combining the above three estimates, we have

$$I_1(t)\le C(\|(\overline{\sigma},\overline{v},\overline{B})\|_s+\delta+\delta\|(\overline{\sigma},\overline{v})\|_s)(\|\overline{\sigma}\|_s^2+\|\nabla\overline{v}\|_s^2),$$

which is bounded by the right-hand side terms of (3.10). On the other hand, since each term in $I_{\alpha,\beta}^{(\sigma)}(t)$ and $I_{\alpha,\beta}^{(v)}(t)$ is the integration of the four-terms product in which there is at least one term containing the derivative, we get

$$
\begin{aligned}
&-\sum_{\beta<\alpha} C_\alpha^\beta I_{\alpha,\beta}^{(\sigma)}(t) - \sum_{\beta<\alpha} C_\alpha^\beta I_{\alpha,\beta}^{(v)}(t) \\
&\le C\|\overline{\sigma}\|_s(\|\overline{\sigma}\|_s^2 + \|\nabla\overline{v}\|_s^2) + C\delta(\|\overline{\sigma}\|_s^2 + \|\nabla\overline{v}\|_s^2) \\
&\quad + C\|\overline{v}\|_s\|\nabla\overline{v}\|_s^2 + C\|\overline{B}\|_s\|\nabla\overline{v}\|_s^2 \\
&\le C(\|(\overline{\sigma}, \overline{v}, \overline{B})\|_s + \delta)(\|\overline{\sigma}\|_s^2 + \|\nabla\overline{v}\|_s^2),
\end{aligned}
$$

which is also bounded by the right-hand side of (3.10).

From (2.8), standard energy estimates on $\partial^\alpha \overline{E}$ and $\partial^\alpha \overline{B}$ with $|\alpha| \le s$ give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(\overline{E}, \overline{B})\|^2 - \gamma^{-\frac{1}{2}}\left\langle(1+\sigma_* + \Phi(\sigma_*))\partial^\alpha\overline{v}, \partial^\alpha\overline{E}\right\rangle \\
&= \gamma^{-\frac{1}{2}}\left\langle\partial^\alpha((\Phi(\overline{\sigma}+\sigma_*) - \Phi(\sigma_*))\overline{v}), \partial^\alpha\overline{E}\right\rangle + \gamma^{-\frac{1}{2}}\left\langle\partial^\alpha(\overline{\sigma}\overline{v}), \partial^\alpha\overline{E}\right\rangle \\
&\quad + \gamma^{-\frac{1}{2}}\sum_{\beta<\alpha} C_\alpha^\beta\left\langle\partial^{\alpha-\beta}(1+\sigma_*+\Phi(\sigma_*)\partial^\beta\overline{v}, \partial^\alpha\overline{E}\right\rangle \\
&\triangleq \sum_{j=1}^2 I_{2,j}(t) + \sum_{\beta<\alpha} C_\alpha^\beta I_{2,\beta}(t).
\end{aligned}
\tag{3.13}
$$

In a similar way as before, when $|\alpha| = 0$, it suffices to estimate

$$
\sum_{j=1}^2 I_{2,j}(t) \le C\|\overline{E}\|_1\|\nabla\overline{v}\|\|\overline{\sigma}\|.
$$

When $|\alpha| \ge 1$, these terms can be estimated as

$$
\sum_{j=1}^2 I_{2,j}(t) \le C\|\overline{E}\|_s\|\nabla\overline{v}\|_{s-1}\|\nabla\overline{\sigma}\|_{s-1},
$$

and

$$
\sum_{\beta<\alpha} C_\alpha^\beta I_{2,\beta}(t) \le C\delta\|\nabla\overline{v}\|_{s-1}\|\nabla\overline{E}\|_{s-2} \le C\delta\|\nabla\overline{v}\|_{s-1}^2 + C\delta\|\nabla\overline{E}\|_{s-2}^2.
$$

which will be further bounded by the right-hand side of (3.10).

Then, (3.10) follows by the summation of (3.12) and (3.13) over $|\alpha| \le s$. Then, the time evolution of the full instant energy $\|V(t)\|_s^2$ has been obtained but its dissipation rate only contains $\nabla\overline{v}$. By introducing the interactive functionals as follows, the dissipation from contributions of the rest components $\overline{\sigma}$, $\overline{E}$, and $\overline{B}$ can be obtained in turn.

Step 2. It holds that

$$\frac{d}{dt}\sum_{|\alpha|\le s-1}\left\langle\partial^\alpha\overline{v},\nabla\partial^\alpha\overline{\sigma}\right\rangle+\lambda\|\overline{\sigma}\|_s^2\le C(\|(\overline{\sigma},\overline{B})\|_s+\delta)(\|\nabla\overline{v}\|_s^2+\|\overline{\sigma}\|_s^2)+\|\nabla\overline{v}\|_s^2. \tag{3.14}$$

In fact, the first two equations of (2.8) can be rewritten as

$$\partial_t\overline{\sigma}+\nabla\cdot\overline{v}=\mathfrak{R}_1 \tag{3.15}$$

and

$$\partial_t\overline{v}+\nabla\overline{\sigma}+\gamma^{-\frac{1}{2}}\overline{E}-\gamma^{-\frac{1}{2}}\frac{\Delta\overline{v}}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}=\mathfrak{R}_2. \tag{3.16}$$

Here

$$\begin{cases}\mathfrak{R}_1=-\overline{v}\cdot\nabla\overline{\sigma}-\dfrac{\gamma-1}{2}\overline{\sigma}\nabla\cdot\overline{v}-\overline{v}\cdot\nabla\sigma_*-\dfrac{\gamma-1}{2}\sigma_*\nabla\cdot\overline{v},\\ \mathfrak{R}_2=-\overline{v}\cdot\nabla\overline{v}-\dfrac{\gamma-1}{2}\overline{\sigma}\nabla\overline{\sigma}-\overline{v}\times\overline{B}-\dfrac{\gamma-1}{2}\sigma_*\nabla\overline{\sigma}-\dfrac{\gamma-1}{2}\overline{\sigma}\nabla\sigma_*.\end{cases} \tag{3.17}$$

For $\alpha\in\mathbb{N}^3$ with $|\alpha|\le s-1$, applying ∂^α to (3.16), taking the inner product of the resulting equation with $\partial^\alpha\nabla\overline{\sigma}$ in $L^2(\mathbb{R}^3)$, and then using integration by parts and also the final equation of (2.8), replacing $\partial_t\bar{\sigma}$ from (3.15) imply that

$$\begin{aligned}&\frac{d}{dt}\left\langle\partial^\alpha\nabla\overline{v},\nabla\partial^\alpha\nabla\overline{\sigma}\right\rangle+\|\nabla\partial^\alpha\nabla\overline{\sigma}\|^2+\gamma^{-1}\|\partial^\alpha\nabla\overline{\sigma}\|^2\\&\quad=-\gamma^{-1}\left\langle\partial^\alpha(\Phi(\overline{\sigma}+\sigma_*)-\Phi(\sigma_*)),\partial^\alpha\overline{\sigma}\right\rangle+\left\langle\partial^\alpha\mathfrak{R}_2,\nabla\partial^\alpha\overline{\sigma}\right\rangle+\|\nabla\cdot\partial^\alpha\overline{v}\|^2\\&\qquad-\left\langle\partial^\alpha\mathfrak{R}_1,\nabla\cdot\partial^\alpha\overline{v}\right\rangle+\gamma^{-\frac{1}{2}}\left\langle\partial^\alpha(\frac{\Delta\overline{v}}{1+\sigma_*+\overline{\sigma}+\Phi(\sigma_*+\overline{\sigma})}),\nabla\partial^\alpha\overline{\sigma}\right\rangle.\end{aligned}$$

Then, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}&\frac{d}{dt}\left\langle\partial^\alpha\nabla\overline{v},\nabla\partial^\alpha\nabla\overline{\sigma}\right\rangle+\lambda(\|\nabla\partial^\alpha\overline{\sigma}\|^2+\|\partial^\alpha\nabla\overline{\sigma}\|^2)\\&\quad\le C(\|(\overline{\sigma},\overline{B})\|_s+\delta)(\|\nabla\overline{v}\|_s^2+\|\overline{\sigma}\|_s^2)+C\|\nabla\overline{v}\|_s^2,\end{aligned}$$

which implies (3.14) by taking summation of it over $|\alpha|\le s-1$.

Step 3. It holds that

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le s-2}\left\langle\partial^\alpha\nabla\times\overline{v},\partial^\alpha\nabla\times\overline{E}\right\rangle+\lambda\sum_{|\alpha|\le s-2}\|\partial^\alpha\nabla\overline{E}\|^2\\&\quad\le C\left(\|\nabla\overline{v}\|_s^2+\delta\|\nabla\overline{E}\|_{s-2}^2+\|\overline{\sigma}\|_s^2+\|\nabla^2\overline{B}\|_{s-3}^2\right)\\&\qquad+C(\|(\overline{\sigma},\overline{v},\overline{B})\|_s+\delta)(\|\nabla\overline{v}\|_s^2+\|\nabla\overline{E}\|_{s-2}^2).\end{aligned} \tag{3.18}$$

Taking the rotation of the equation of (3.16), we have

$$\partial_t \nabla \times \overline{v} + \gamma^{-\frac{1}{2}} \nabla \times \overline{E} = \nabla \times \mathfrak{R}_2 + \gamma^{-\frac{1}{2}} \nabla \times \left(\frac{\Delta \overline{v}}{1 + \sigma_* + \overline{\sigma} + \Phi(\sigma_* + \overline{\sigma})} \right).$$

For $\alpha \in \mathbb{N}^3$ with $|\alpha| \le s-2$, applying ∂^α to the equation above, taking the inner product of the resulted equation with $\partial^\alpha \nabla \times \overline{E}$ in $L^2(\mathbb{R}^3)$, using integration by parts and replacing $\partial_t \overline{E}$ from the third equation of (2.8) yield

$$\begin{aligned}
&\frac{d}{dt} \left\langle \partial^\alpha \nabla \times \overline{v}, \partial^\alpha \nabla \times \overline{E} \right\rangle + \gamma^{-\frac{1}{2}} \|\partial^\alpha \nabla \times \overline{E}\|^2 \\
&\quad = \gamma^{-\frac{1}{2}} \langle \partial^\alpha \nabla \times \overline{v}, \partial^\alpha \nabla \times \nabla \times \overline{B} \rangle + \gamma^{-\frac{1}{2}} \|\partial^\alpha \nabla \times \overline{v}\|^2 \\
&\qquad + \langle \partial^\alpha \nabla \times \mathfrak{R}_2, \partial^\alpha \nabla \times \overline{E} \rangle \\
&\qquad + \gamma^{-\frac{1}{2}} \left\langle \partial^\alpha \nabla \times \left(\frac{\Delta \overline{v}}{1 + \sigma_* + \overline{\sigma} + \Phi(\sigma_* + \overline{\sigma})} \right), \partial^\alpha \nabla \times \overline{E} \right\rangle \\
&\qquad + \gamma^{-\frac{1}{2}} \left\langle \partial^\alpha \nabla \times \overline{v}, \partial^\alpha \nabla \times ((\Phi(\sigma_* + \overline{\sigma}) + \sigma_* + \overline{\sigma})\overline{v}) \right\rangle,
\end{aligned}$$

which together with the Cauchy–Schwarz inequality further implies

$$\begin{aligned}
&\frac{d}{dt} \left\langle \partial^\alpha \nabla \times \overline{v}, \partial^\alpha \nabla \times \overline{E} \right\rangle + \gamma^{-\frac{1}{2}} \|\partial^\alpha \nabla \times \overline{E}\|^2 \\
&\quad \le C(\| \left(\overline{\sigma}, \overline{v}, \overline{B}\right) \|_s + \delta) \left(\|\nabla \overline{v}\|_s^2 + \|\nabla \overline{E}\|_{s-2}^2 \right) \\
&\qquad + C(\|\nabla \overline{v}\|_s^2 + \delta \|\nabla \overline{E}\|_{s-2}^2 + \|\overline{\sigma}\|_s^2 + \|\nabla^2 \overline{B}\|_{s-3}^2).
\end{aligned}$$

Thus, (3.18) follows by the last equation in (2.8) and taking summation of these estimates over $|\alpha| \le s-2$.

Step 4. It holds that

$$\begin{aligned}
&\frac{d}{dt} \sum_{1 \le |\alpha| \le s-2} \left\langle (-\partial^\alpha \nabla \times \overline{B}), \partial^\alpha \overline{E} \right\rangle + \lambda \sum_{1 \le |\alpha| \le s-2} \|\partial^\alpha \nabla \overline{B}\|^2 \\
&\quad \le C(\|\overline{\sigma}\|_s + \delta)(\|\nabla \overline{v}\|_s^2 + \|\nabla^2 \overline{B}\|_{s-3}^2) + C\|\nabla \overline{E}\|_{s-2}^2.
\end{aligned} \tag{3.19}$$

In fact, for $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s-2$, applying ∂^α to the third equation of (2.8), taking the inner product of the resulted equation with $-\partial^\alpha \nabla \times \overline{B}$ in $L^2(\mathbb{R}^3)$, and then using integration by parts and replacing $\partial_t \overline{B}$ from the fourth equation of (2.8), we get

$$\begin{aligned}
&\frac{d}{dt} \left\langle \partial^\alpha (-\nabla \times \overline{B}), \partial^\alpha \overline{E} \right\rangle + \gamma^{-\frac{1}{2}} \|\partial^\alpha \nabla \times \overline{B}\|^2 \\
&\quad = \gamma^{-\frac{1}{2}} \|\partial^\alpha \nabla \times \overline{E}\|^2 - \gamma^{-\frac{1}{2}} \left\langle \partial^\alpha \nabla \times \overline{B}, \partial^\alpha \overline{v} \right\rangle \\
&\qquad - \gamma^{-\frac{1}{2}} \langle \partial^\alpha \nabla \times \overline{B}, \partial^\alpha \left((\Phi(\overline{\sigma} + \sigma_*) - \Phi(\sigma_*))\overline{v} \right) \rangle \\
&\qquad - \gamma^{-\frac{1}{2}} \left\langle \partial^\alpha \nabla \times \overline{B}, \partial^\alpha \left((1 + \sigma_* + \Phi(\sigma_*))\overline{v} \right) \right\rangle - \gamma^{-\frac{1}{2}} \langle \partial^\alpha \nabla \times \overline{B}, \partial^\alpha (\overline{\sigma v}) \rangle
\end{aligned}$$

$$+\gamma^{-\frac{1}{2}}\langle\partial^\alpha\nabla\times\overline{B},\partial^\alpha\overline{v}\rangle,$$

which gives (3.19) by further using Cauchy–Schwarz inequality and taking summation over $1\le|\alpha|\le s-2$, where we also used

$$\|\partial^\alpha\partial_i\overline{B}\|=\|\partial_i\Delta^{-1}\nabla\times(\nabla\times\partial^\alpha\overline{B})\|\le\|\nabla\times\partial^\alpha\overline{B}\|,$$

for each $1\le i\le 3$, due to the fact $\partial_i\Delta^{-1}\nabla$ is bounded from L^p to itself for $1<p<\infty$.

Step 5. Now, following the four steps above, we are ready to prove (3.9). For

$$\begin{aligned}\mathfrak{E}_s(\overline{V}(t))=&\sum_{|\alpha|\le s}\langle 1+\sigma_*+\Phi(\sigma_*),|\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\rangle+\|(\overline{E},\overline{B})\|_s^2\\&+\mathfrak{K}_1\sum_{|\alpha|\le s-1}\left\langle\partial^\alpha\overline{v},\nabla\partial^\alpha\overline{\sigma}\right\rangle+\mathfrak{K}_2\sum_{|\alpha|\le s-2}\left\langle\partial^\alpha\nabla\times\overline{E},\partial^\alpha\nabla\times\overline{v}\right\rangle\\&+\mathfrak{K}_3\sum_{1\le|\alpha|\le s-2}\left\langle\partial^\alpha(-\nabla\times\overline{B}),\partial^\alpha\overline{E}\right\rangle,\end{aligned}$$

where constants $0<\mathfrak{K}_3\ll\mathfrak{K}_2\ll\mathfrak{K}_1\ll 1$ to be determined. Notice that as long as $0<\mathfrak{K}_i\ll 1$ is small enough for $i=1,2,3$, and $\sigma_*+\Phi(\sigma_*)$ depending only on x is sufficiently small compared with 1, then $\mathfrak{E}_s(\overline{V}(t))\sim\|\overline{V}(t)\|_s^2$ holds true. Moreover, letting $0<\mathfrak{K}_3\ll\mathfrak{K}_2\ll\mathfrak{K}_1\ll 1$ with $\mathfrak{K}_2^{3/2}\ll\mathfrak{K}_3$, the sum of (3.14) $\times\mathfrak{K}_1$, (3.18) $\times\mathfrak{K}_2$, (3.19) $\times\mathfrak{K}_3$ implies that there are $\lambda>0$, $C>0$ such that (3.9) holds true with $\mathfrak{D}_s(\cdot)$. Here, we have used the following Cauchy–Schwarz inequality:

$$2\mathfrak{K}_2\|\overline{v}\|_s\|\nabla\overline{B}\|_{s-2}\le\mathfrak{K}_2^{1/2}\|\overline{v}\|_s^2+\mathfrak{K}_2^{3/2}\|\nabla\overline{B}\|_{s-2}^2$$

Due to $\mathfrak{K}_2^{3/2}\ll\mathfrak{K}_3$, both terms on the right-hand side of the above inequality can be controlled. This completes the proof of Theorem 3.1. □

Since system (2.8) is quasi-linear symmetric hyperbolic–parabolic, the local time existence can be obtained as that in Feng et al. (2021). From Theorem 3.1 and the continuity argument, it is easy to see that $\mathfrak{E}_s(\overline{V}(t))$ is bounded uniformly in time under the assumptions that $\mathfrak{E}_s(\overline{V}_0)>0$ and $\|n_b-1\|_{W_0^{s+1,2}}$ are small enough. Therefore, the global existence of solutions satisfying (3.5) and (3.6) follows in the standard way; see also Duan (2012). This completes the proof of Proposition 3.1, namely, the global existence result in Theorem 1.1. □

4 Decay in time for the nonlinear system

In the following, we continue to study the time decay rate of the smooth solution which convergence to the steady state $(n_*,0,E_*,0)$. For this goal, let us introduce

$$\overline{\rho}=n-n_*,\quad\overline{u}=u,\quad E_1=E-E_*,\quad B_1=B,\quad\rho_*=n_*-1,$$

then $\overline{U} = (\overline{\rho}, \overline{u}, E_1, B_1)$ satisfies

$$\begin{cases} \partial_t \overline{\rho} + \mathrm{div}\overline{u} = g_1, \\ \partial_t \overline{u} + E_1 + \gamma \nabla \overline{\rho} - \Delta \overline{u} = g_2, \\ \partial_t E_1 - \nabla \times B_1 - \overline{u} = g_3, \\ \partial_t B_1 + \nabla \times E_1 = 0, \\ \mathrm{div} E_1 = -\bar{\rho}, \quad \mathrm{div} B_1 = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

with the initial condition

$$\overline{U}|_{t=0} = \overline{U}_0 := \left(\overline{\rho}_0, \overline{u}_0, E_{1,0}, B_{1,0}\right) = (n_0 - n_*, u_0, E_0 - E_*, B_0), \quad x \in \mathbb{R}^3, \tag{4.2}$$

which satisfies the compatibility conditions

$$\mathrm{div} E_{1,0} = -\overline{\rho}_0, \quad \mathrm{div} B_{1,0} = 0, \quad x \in \mathbb{R}^3. \tag{4.3}$$

Here the nonlinear source terms are

$$\begin{cases} g_1 = -\mathrm{div}\left((\overline{\rho} + \rho_*)\,\overline{u}\right), \\ g_2 = -\overline{u} \cdot \nabla \overline{u} - \overline{u} \times B_1 - \gamma \left((\overline{\rho} + \rho_* + 1)^{\gamma-2} - 1\right) \nabla \overline{\rho} \\ \qquad - \gamma \left((\overline{\rho} + \rho_* + 1)^{\gamma-2} - (\rho_* + 1)^{\gamma-2}\right) \nabla \rho_* - \left(1 - \dfrac{1}{\overline{\rho} + \rho_* + 1}\right) \Delta \overline{u}, \\ g_3 = (\overline{\rho} + \rho_*)\,\overline{u}. \end{cases} \tag{4.4}$$

Next, we investigate the linearized homogeneous equations

$$\begin{cases} \partial_t \rho + \mathrm{div} u = 0, \\ \partial_t u + E + \gamma \nabla \rho - \Delta u = 0, \\ \partial_t E - \nabla \times B - u = 0, \\ \partial_t B + \nabla \times E = 0, \\ \mathrm{div} E = -\rho, \quad \mathrm{div} B = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.5}$$

with the initial condition

$$U|_{t=0} = \overline{U}_0 := \left(\overline{\rho}_0, \overline{u}_0, E_{1,0}, B_{1,0}\right), \quad x \in \mathbb{R}^3, \tag{4.6}$$

also satisfying the compatibility conditions (4.3).

For the above linearized equations, the $L^p - L^q$ time decay property was proved by Duan Duan (2012). We list only some special $L^p - L^q$ time decay properties in the following Proposition.

Proposition 4.1 *Assume that $U(t) = e^{tL}\overline{U}_0$ is the solution to the initial value problem with the initial data $\overline{U}_0 = \left(\overline{\rho}_0, \overline{u}_0, E_{1,0}, B_{1,0}\right)$ which satisfies (4.3). Then, $U = (\rho, u, E, B)$ satisfies the following time decay property:*

$$
\begin{cases}
\|\rho(t)\| \le C(1+t)^{-\frac{5}{4}}\|(\overline{\rho}_0, \overline{u}_0)\|_{L^1\cap L^2}, \\
\|u(t)\| \le C(1+t)^{-\frac{5}{4}}\|\overline{\rho}_0\|_{L^1\cap L^2} + C(1+t)^{-\frac{3}{4}}\|(\overline{u}_0, E_{1,0})\|_{L^1\cap L^2} \\
\quad + C(1+t)^{-\frac{5}{8}}\|B_{1,0}\|_{L^1\cap L^2}, \\
\|E(t)\| \le C(1+t)^{-\frac{3}{4}}\|\overline{u}_0\|_{L^1\cap L^2} + C(1+t)^{-\frac{3}{4}}(\|E_{1,0}\|_{L^1\cap L^2} + \|\nabla^2 E_{1,0}\|) \\
\quad + C(1+t)^{-\frac{9}{8}}(\|B_{1,0}\|_{L^1\cap L^2} + \|\nabla^3 B_{1,0}\|), \\
\|B(t)\| \le C(1+t)^{-\frac{5}{8}}\|\overline{u}_0\|_{L^1\cap L^2} + C(1+t)^{-\frac{9}{8}}(\|E_{1,0}\|_{L^1\cap L^2} + \|\nabla^3 E_{1,0}\|) \\
\quad + C(1+t)^{-\frac{3}{8}}(\|B_{1,0}\|_{L^1\cap L^2} + \|\nabla B_{1,0}\|), \\
\|\nabla B(t)\| \le C(1+t)^{-\frac{7}{8}}\|\overline{u}_0\|_{L^1\cap L^2} + C(1+t)^{-\frac{5}{8}}(\|(E_{1,0}, B_{1,0})\|_{L^1\cap L^2} \\
\quad + \|\nabla^3(E_{1,0}, B_{1,0})\|).
\end{cases}
\tag{4.7}
$$

In the sequence, due to the fact that we shall apply the linear $L^p - L^q$ time decay property of the homogeneous system, the mild form of the nonlinear initial value problem is needed here. From now on, we usually denote $\overline{U} = (\overline{\rho}, \overline{u}, E_1, B_1)$ as the solution to the nonlinear initial value problem. Therefore, by the Duhamel's principle, the solution $\overline{U}$ can be formally written as

$$
\overline{U}(t) = e^{tL}\overline{U}_0 + \int_0^t e^{(t-\tau)L}\,(g_1(\tau), g_2(\tau), g_3(\tau), 0)\,d\tau,
$$

where $e^{tL}\overline{U}_0$ denotes the solution to the linearized homogeneous initial value problem.

For establishing the time decay rates of the perturbed solution, the vital task lies in dealing with the linear nonhomogeneous terms. In order to solve this problem, we make iteration for the following two lemmas which give the full and high-order energy estimates.

Lemma 4.1 *Suppose $\overline{V} = (\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})$ to be the solution of problem (2.8)-(2.9) with initial data $\overline{V}_0 = (\overline{\sigma}_0, \overline{v}_0, \overline{E}_0, \overline{B}_0)$ which satisfies (2.10). If $\mathfrak{E}_s(\overline{V}_0)$ and $\|n_b(x) - 1\|_{W_0^{s+1,2}}$ are sufficiently small, then*

$$
\frac{d}{dt}\mathfrak{E}_s(\overline{V}(t)) + \lambda\mathfrak{D}_s(\overline{V}(t)) \le 0,
\tag{4.8}
$$

holds for any $t > 0$, where $\mathfrak{E}_s(\overline{V}(t))$, $\mathfrak{D}_s(\overline{V}(t))$ are defined in the form of (3.1) and (3.3), respectively.

Proof It can be seen directly from the proof of Theorem 3.1. □

Lemma 4.2 *Assume that $\overline{V} = (\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})$ is the solution to problem (2.8)-(2.9) with initial data $\overline{V}_0 = (\overline{\sigma}_0, \overline{v}_0, \overline{E}_0, \overline{B}_0)$ satisfying (2.10). If $\mathfrak{E}_s(\overline{V}_0)$ and $\|n_b(x)-1\|_{W_0^{s+1,2}}$ are small enough, then there exist the high-order instant energy functional $\mathfrak{E}_s^h(\cdot)$ and the corresponding dissipation rate $\mathfrak{D}_s(\cdot)$ such that, for any $t > 0$,*

$$\frac{d}{dt}\mathfrak{E}_s^h(\overline{V}(t)) + \lambda\mathfrak{D}_s^h(\overline{V}(t)) \le 0. \tag{4.9}$$

Proof The proof can be done by slightly modifying the proof of Theorem 3.1. In fact, by letting the energy estimates made only on the high-order derivatives, then corresponding to (3.10), (3.14), (3.18) and (3.19). we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\sum_{1\le|\alpha|\le s}\langle 1+\sigma_*+\Phi(\sigma_*), |\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\rangle+\|\nabla(\overline{E},\overline{B})\|_{s-1}^2\right)\\
&\quad+\gamma^{-\frac{1}{2}}\sum_{1\le|\alpha|\le s}\|\nabla\partial^\alpha\overline{v}\|^2\\
&\quad\le C(\|\nabla(\overline{\sigma},\overline{v})\|_{s-1}+\delta)(\|\nabla^2\overline{v}\|_{s-1}^2+\|\overline{\sigma}\|_{s-1}^2)\\
&\qquad+C(\|\nabla(\overline{v},\overline{E},\overline{B})\|_{s-1}+\delta)\|\nabla(\overline{\sigma},\overline{v},\overline{B})\|_{s-1}^2,\\
&\frac{d}{dt}\sum_{1\le|\alpha|\le s-1}\left\langle\partial^\alpha\overline{v},\nabla\partial^\alpha\overline{\sigma}\right\rangle+\lambda\|\nabla\overline{\sigma}\|_{s-1}^2\\
&\quad\le C(\|\nabla\,(\overline{\sigma},\overline{v})\,\|_{s-1}+\delta)(\|\nabla^2\overline{v}\|_{s-1}^2+\|\overline{\sigma}\|_{s-1}^2)\\
&\qquad+C\|\overline{\sigma}\|_{s-1}\|\overline{v}\|_{s-1}\|\overline{B}\|_{s-1}+C\|\nabla^2\overline{v}\|_{s-1}^2,\\
&\frac{d}{dt}\sum_{1\le|\alpha|\le s-2}\left\langle\partial^\alpha\nabla\times\overline{v},\partial^\alpha\nabla\times\overline{E}\right\rangle+\lambda\|\nabla^2\overline{E}\|_{s-3}^2\\
&\quad\le C(\|\left(\overline{\sigma},\overline{v},\overline{B}\right)\|_{s-1}+\delta)(\|\nabla^2\overline{v}\|_{s-1}^2+\|\nabla\overline{\sigma}\|_{s-1}^2+\|\nabla^2\overline{E}\|_{s-3}^2)\\
&\qquad+C\|\nabla\overline{v}\|_{s-1}\|\nabla\overline{B}\|_{s-1}\|\nabla\overline{E}\|_{s-1}\\
&\qquad+C\|\nabla^3\overline{B}\|_{s-4}^2+C\delta\|\nabla\overline{v}\|_{s-1}^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\sum_{2\le|\alpha|\le s-2}\left\langle(-\partial^\alpha\nabla\times\overline{B}),\partial^\alpha\overline{E}\right\rangle+\lambda\|\nabla^3\overline{B}\|_{s-4}^2\\
&\quad\le C(\|\,(\overline{\sigma},\overline{v})\,\|_{s-1}+\delta)(\|\nabla\overline{\sigma}\|_{s-1}^2+\|\nabla^3\overline{B}\|_{s-4}^2+\|\nabla^2\overline{v}\|_{s-1}^2)\\
&\qquad+C(1+\delta)(\|\nabla^2\overline{v}\|_{s-1}^2+\|\nabla^2\overline{E}\|_{s-3}^2).
\end{aligned}$$

Here, the details of proof are omitted for simplicity. Now, similar to (3.1), set

$$\mathfrak{E}_s^h(\overline{V}(t)) = \sum_{1\le|\alpha|\le s}\langle 1+\sigma_*+\Phi(\sigma_*), |\partial^\alpha\overline{\sigma}|^2+|\partial^\alpha\overline{v}|^2\rangle+\|\nabla\left(\overline{E},\overline{B}\right)\|_{s-1}^2$$

$$
\begin{aligned}
&+\mathfrak{K}_1 \sum_{1\le|\alpha|\le s-1} \left\langle \partial^\alpha \overline{v}, \partial^\alpha \nabla \overline{\sigma} \right\rangle + \mathfrak{K}_2 \sum_{1\le|\alpha|\le s-2} \left\langle \partial^\alpha \nabla \times \overline{E}, \partial^\alpha \nabla \times \overline{v} \right\rangle \\
&-\mathfrak{K}_3 \sum_{2\le|\alpha|\le s-2} \left\langle \partial^\alpha (-\nabla \times \overline{B}), \partial^\alpha \overline{E} \right\rangle. \qquad (4.10)
\end{aligned}
$$

Similarly, due to the fact that $\sigma_* + \Phi(\sigma_*)$ depending only on x is sufficiently small compared with 1, we choose $0 < \mathfrak{K}_3 \ll \mathfrak{K}_2 \ll \mathfrak{K}_1 \ll 1$ with $\mathfrak{K}_2^{3/2} \ll \mathfrak{K}_3$ such that $\mathfrak{E}_s^h(\overline{V}(t)) \sim \|\nabla \overline{V}(t)\|_{s-1}^2$. Furthermore, the linear combination of previously obtained four estimates with coefficients corresponding to (4.10) yields (4.9) with $\mathfrak{D}_s^h(\cdot)$. This completes the proof of Lemma 4.2. □

Recalling the definition of g_1, g_2, g_3, we have

$$
\begin{cases}
g_1 = -\mathrm{div}\left((\overline{\rho} + \rho_*)\, \overline{u}\right), \\
g_2 \sim \overline{u} \cdot \nabla \overline{u} + \overline{u} \times B_1 + \overline{\rho} \cdot \nabla \overline{\rho} + \rho_* \nabla \overline{\rho} + \overline{\rho} \nabla \rho_* + \overline{\rho} \Delta \overline{u} + \rho_* \Delta \overline{u}, \\
g_3 = (\overline{\rho} + \rho_*)\, \overline{u}.
\end{cases}
$$

We begin with the time-weighted estimate and iteration for the Lyapunov inequality. To this end, let us define

$$
\mathfrak{X}(t) = \sup_{0\le\tau\le t} (1+\tau)^{\frac{3}{4}} \mathfrak{E}_s(\overline{V}(\tau)), \quad t \ge 0.
$$

In fact, we have

Lemma 4.3 *If $\|\overline{V}_0\|_{L^1\cap H^{s+1}}$ is sufficiently small, then*

$$
\sup_{t\ge 0} \mathfrak{X}(t) \le C \|\overline{V}_0\|_{L^1\cap H^{s+1}}^2. \qquad (4.11)
$$

Proof By using the smallness assumption of $\|\overline{V}_0\|_{L^1\cap H^{s+1}}$, we have

$$
\frac{d}{dt} \mathfrak{E}_s(\overline{V}(t)) + \lambda \mathfrak{D}_s(\overline{V}(t)) \le 0,
$$

for any $t \ge 0$. This is the beginning point to deduce (4.11). In fact, fix a constant $\epsilon > 0$ small enough. Then, the further time weighted estimate on (4.8) gives

$$
\begin{aligned}
&(1+t)^{\frac{3}{4}+\epsilon} \mathfrak{E}_s(\overline{V}(t)) + \lambda \int_0^t (1+\tau)^{\frac{3}{4}+\epsilon} \mathfrak{D}_s(\overline{V}(\tau)) d\tau \le \mathfrak{E}_s(\overline{V}_0) \\
&\quad + \left(\frac{3}{4} + \epsilon\right) \int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon} \mathfrak{E}_s(\overline{V}(\tau)) d\tau.
\end{aligned}
$$

It follows from

$$
\mathfrak{E}_s(\overline{V}) \sim \|\overline{V}\|_s^2 \le \mathfrak{D}_{s+1}(\overline{V}) + \|(\overline{v}, \overline{E}, \overline{B})\|^2 + \|\nabla \overline{B}\|^2
$$

and

$$\int_0^t \mathfrak{D}_{s+1}(\overline{V}(\tau))d\tau \le C\mathfrak{E}_s(\overline{V}_0)$$

that

$$\begin{aligned}
&(1+t)^{\frac{3}{4}+\epsilon}\mathfrak{E}_s(\overline{V}(t)) + \lambda\int_0^t (1+\tau)^{\frac{3}{4}+\epsilon}\mathfrak{D}_s(\overline{V}(\tau))d\tau \\
&\le \mathfrak{E}_s(\overline{V}_0) + C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\mathfrak{D}_{s+1}(\overline{V}(\tau))d\tau \\
&\quad + C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|(\overline{v},\overline{E},\overline{B})(\tau)\|^2 d\tau \\
&\quad + C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|\nabla\overline{B}(\tau)\|^2 d\tau \\
&\le \mathfrak{E}_{s+1}(\overline{V}_0) + C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|(\overline{v},\overline{E},\overline{B})(\tau)\|^2 d\tau \\
&\quad + C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|\nabla\overline{B}(\tau)\|^2 d\tau.
\end{aligned} \tag{4.12}$$

Due to Proposition 4.1, we have

$$\begin{aligned}
\|B_1\| \le& C(1+t)^{-\frac{3}{8}}\left(\|\overline{u}_0\|_{L^1\cap L^2} + \|E_{1,0}\|_{L^1\cap L^2\cap \dot{H}^3} + \|B_{1,0}\|_{L^1\cap H^1}\right) \\
&+C\int_0^t (1+t-\tau)^{-\frac{5}{8}}\|g_2(\tau)\|_{L^1\cap L^2}d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{9}{8}}(\|g_3(\tau)\|_{L^1\cap L^2} + \|\nabla^3 g_3(\tau)\|)d\tau, \\
\|E_1\| \le& C(1+t)^{-\frac{3}{4}}\left(\|\overline{u}_0\|_{L^1\cap L^2} + \|E_{1,0}\|_{L^1\cap L^2\cap \dot{H}^2} + \|B_{1,0}\|_{L^1\cap L^2\cap \dot{H}^3}\right) \\
&+C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|g_2(\tau)\|_{L^1\cap L^2}d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|g_3(\tau)\|_{L^1\cap L^2} + \|\nabla^2 g_3(\tau)\|)d\tau, \\
\|\overline{u}\| \le& C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap L^2} + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|g_1(\tau)\|_{L^1\cap L^2}d\tau \\
&+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|(g_2(\tau),g_3(\tau))\|_{L^1\cap L^2}d\tau
\end{aligned}$$

and

$$
\begin{aligned}
\|\nabla B_1\| \leq & C(1+t)^{-\frac{5}{8}}\left(\|\overline{u}_0\|_{L^1\cap L^2}+\|(E_{1,0},B_{1,0})\|_{L^1\cap L^2\cap \dot{H}^3}\right)\\
& +C\int_0^t (1+t-\tau)^{-\frac{7}{8}}\|g_2(\tau)\|_{L^1\cap L^2}d\tau\\
& +C\int_0^t (1+t-\tau)^{-\frac{5}{8}}(\|g_3(\tau)\|_{L^1\cap L^2}+\|\nabla^3 g_3(\tau)\|)d\tau.
\end{aligned}
$$

It is straightforward to verify

$$
\left\|(g_1(\tau),g_2(\tau),g_3(\tau))\|_{L^1\cap L^2}+\|g_3(\tau)\right\|_3 \leq C\delta\mathfrak{E}_s^{\frac{1}{2}}(\overline{U})+C\mathfrak{E}_s(\overline{U}).
$$

In view of

$$
\mathfrak{E}_s(\overline{U}(\tau)) \leq C\mathfrak{E}_s(\overline{V}(\gamma^{\frac{1}{2}}\tau)),
$$

we have

$$
\mathfrak{E}_s(\overline{V}(\gamma^{\frac{1}{2}}\tau)) \leq (1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}\mathfrak{X}(\gamma^{\frac{1}{2}}t).
$$

Then, it follows that

$$
\begin{aligned}
&\|(g_1(\tau),g_2(\tau),g_3(\tau))\|_{L^1\cap L^2}+\|g_3(\tau)\|_3 \leq C\delta(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{8}}\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\\
&\quad +C(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}\mathfrak{X}(\gamma^{\frac{1}{2}}t).
\end{aligned}
$$

Then, $\|B_1\|$ is estimated by

$$
\begin{aligned}
\|B_1\| \leq & C(1+t)^{-\frac{3}{8}}\|(\overline{u}_0,E_{1,0},B_{1,0})\|_{L^1\cap H^3}\\
& +C\int_0^t (1+t-\tau)^{-\frac{5}{8}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}d\tau\mathfrak{X}(\gamma^{\frac{1}{2}}t)\\
& +C\delta\int_0^t (1+t-\tau)^{-\frac{5}{8}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{8}}d\tau\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\\
\leq & C(1+t)^{-\frac{3}{8}}\left(\|(\overline{u}_0,E_{1,0},B_{1,0})\|_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\right),
\end{aligned}
$$

and in the same way, it holds that

$$
\|(\overline{u},E_1,B_1)\| \leq C(1+t)^{-\frac{3}{8}}\left(\|\overline{V}_0\|_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\right),
$$

and

$$
\begin{aligned}
\|\nabla B_1\| \leq & C(1+t)^{-\frac{5}{8}}\|(\overline{u}_0, E_{1,0}, B_{1,0})\|_{L^1\cap H^3} \\
& + C\int_0^t (1+t-\tau)^{-\frac{5}{8}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}d\tau\left(\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\right) \\
\leq & C(1+t)^{-\frac{3}{8}}\left(\|(\overline{u}_0, E_{1,0}, B_{1,0})\|_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{X}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\right).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
& \int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|(\overline{u}(\tau), E_1(\tau), B_1(\tau))\|^2 d\tau \\
\leq & C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}(1+\tau)^{-\frac{3}{4}}d\tau\left(\|\overline{V}_0\|^2_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)^2+\delta\mathfrak{X}(\gamma^{\frac{1}{2}}t)\right) \\
\leq & C(1+t)^{\epsilon}\left(\|\overline{V}_0\|^2_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)^2+\delta\mathfrak{X}(\gamma^{\frac{1}{2}}t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}\|\nabla B_1(\tau)\|^2 d\tau \\
& \leq C\int_0^t (1+\tau)^{-\frac{1}{4}+\epsilon}(1+\tau)^{-\frac{3}{4}}d\tau\left(\|\overline{V}_0\|^2_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)^2+\delta\mathfrak{X}(\gamma^{\frac{1}{2}}t)\right) \\
& \leq C(1+t)^{\epsilon}\left(\|\overline{V}_0\|^2_{L^1\cap H^3}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)^2+\delta\mathfrak{X}(\gamma^{\frac{1}{2}}t)\right).
\end{aligned}
$$

Since $\|\overline{B}(t)\| \leq C\|B_1(\gamma^{-\frac{1}{2}}t)\|$ and $(\overline{\rho}, \overline{u}, E_1, B_1)$ is equivalent with $(\overline{\sigma}, \overline{v}, \overline{E}, \overline{B})$ up to a positive constant, we have

$$
\begin{aligned}
& (1+t)^{\frac{3}{4}+\epsilon}\mathfrak{E}_s(\overline{V}(t))+\lambda\int_0^t (1+\tau)^{\frac{3}{4}+\epsilon}\mathfrak{D}_s(\overline{V}(\tau))d\tau \\
& \quad \leq C(1+t)^{\epsilon}\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}+\mathfrak{X}(t)^2+\delta\mathfrak{X}(t)\right),
\end{aligned}
\tag{4.13}
$$

which implies that

$$
(1+t)^{\frac{3}{4}}\mathfrak{E}_s(\overline{V}(t) \leq C\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}+\mathfrak{X}(t)^2+\delta\mathfrak{X}(t)\right).
$$

Therefore,

$$
\mathfrak{X}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{3}{4}}\mathfrak{E}_s(\overline{V}(\tau)) \leq C\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}+\mathfrak{X}(t)^2+\delta\mathfrak{X}(t)\right),
$$

and then

$$
\mathfrak{X}(t) \leq C\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}+\mathfrak{X}(t)^2\right).
$$

Since $\|\overline{V}_0\|_{L^1\cap H^{s+1}}$ is sufficiently small, we have $\mathfrak{X}(t)$ is bounded uniformly in time, and we also obtain

$$\sup_{t\geq 0}\mathfrak{X}(t) \leq C\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}$$

holds true, which gives (1.6) in Theorem 1.1. □

Next we start with the high-order energy inequality time-weighted estimate and iteration for the Lyapunov inequality. For this purpose, let us define

$$\mathfrak{Y}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{5}{4}}\mathfrak{E}_s^h(\overline{V}(\tau)), \quad \mathfrak{F}_0(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{5}{4}}\|(\overline{\rho},\overline{u})\|^2.$$

Similar to obtain the uniform-in-time bound of $\mathfrak{X}(t)$ in Lemma 4.3, we have the following result to show the boundedness of $\mathfrak{Y}(t)$ for all $t \geq 0$.

Lemma 4.4 *If $\|\overline{V}_0\|_{L^1\cap H^{s+2}}$ is sufficiently small, then*

$$\sup_{t\geq 0}\mathfrak{Y}(t) \leq C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}. \tag{4.14}$$

Proof Under smallness assumption of $\|\overline{V}_0\|_{L^1\cap H^{s+2}}$, we obtain that (4.9) holds true for any $t \geq 0$. Fix $\epsilon > 0$ small enough, multiplying (4.9) by $(1+t)^{\frac{5}{4}+\epsilon}$ and integrating the resulting equation over $[0, t]$, we have

$$\begin{aligned}(1+t)^{\frac{5}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(t))+\lambda\int_0^t(1+\tau)^{\frac{5}{4}+\epsilon}\mathfrak{D}_s^h(\overline{V}(\tau))d\tau \leq \mathfrak{E}_s^h(\overline{V}_0)\\ + C\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(\tau))d\tau.\end{aligned} \tag{4.15}$$

Because

$$\mathfrak{E}_s^h(\overline{V}) \sim \|\nabla\overline{V}\|^2_{s-1} \leq \mathfrak{D}_{s+1}(\overline{V}) + \|\nabla\overline{B}\|^2, \tag{4.16}$$

we have

$$\begin{aligned}&(1+t)^{\frac{5}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(t)) + \lambda\int_0^t(1+\tau)^{\frac{5}{4}+\epsilon}\mathfrak{D}_s^h(\overline{V}(\tau))d\tau\\ &\leq \mathfrak{E}_s^h(\overline{V}_0) + C\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}\mathfrak{D}_{s+1}(\overline{V}(\tau))d\tau + C\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}\|\nabla\overline{B}(\tau)\|^2 d\tau.\end{aligned}$$

From (4.13), we have

$$\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}\mathfrak{D}_{s+1}(\overline{V}(\tau))d\tau \leq C(1+t)^{\epsilon}\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}, \tag{4.17}$$

which implies that

$$(1+t)^{\frac{5}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(t))+\lambda\int_0^t(1+\tau)^{\frac{5}{4}+\epsilon}\mathfrak{D}_s^h(\overline{V}(\tau))d\tau \\ \leq C\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}\|\nabla\overline{B}(\tau)\|^2d\tau+C(1+t)^{\epsilon}\|\overline{V}_0\|_{L^1\cap H^{s+2}}^2.$$

From Proposition 4.1, we have

$$\|\nabla B_1\| \leq C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap H^3}+C\int_0^t(1+t-\tau)^{-\frac{7}{8}}\|g_2(\tau)\|_{L^1\cap L^2}d\tau \\ +C\int_0^t(1+t-\tau)^{-\frac{5}{8}}(\|g_3(\tau)\|_{L^1\cap L^2}+\|\nabla^3 g_3(\tau)\|)d\tau.$$

It is easy to verify

$$\|(g_2(\tau),g_3(\tau))\|_{L^1\cap L^2}+\|\nabla^3 g_3(\tau)\| \\ \leq C\mathfrak{E}_s(\overline{U})+C\delta\sqrt{\mathfrak{E}_s^h(\overline{U})}+C\delta\|(\overline{\rho},\overline{u})\| \\ \leq C(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}\mathfrak{X}(\gamma^{\frac{1}{2}}t)+C\delta(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{5}{8}}\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)+C\delta(1+\tau)^{-\frac{5}{8}}\mathfrak{F}_0^{\frac{1}{2}}(t).$$

Then,

$$\|\nabla B_1\| \\ \leq C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap H^3}+C\int_0^t(1+t-\tau)^{-\frac{5}{8}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}d\tau\mathfrak{X}(\gamma^{\frac{1}{2}}t) \\ +C\delta\int_0^t(1+t-\tau)^{-\frac{5}{8}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{5}{8}}d\tau\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t) \\ +C\delta\int_0^t(1+t-\tau)^{-\frac{5}{8}}(1+\tau)^{-\frac{5}{8}}d\tau\mathfrak{F}_0^{\frac{1}{2}}(t) \\ \leq (1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap H^{N+1}}+(1+t)^{-\frac{5}{8}}\mathfrak{X}(\gamma^{\frac{1}{2}}t) \\ +\delta(1+t)^{-\frac{5}{8}}\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)+\delta(1+t)^{-\frac{5}{8}}\mathfrak{F}_0^{\frac{1}{2}}(t) \\ \leq C(1+t)^{-\frac{5}{8}}(\|\overline{U}_0\|_{L^1\cap H^{s+1}}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{F}_0^{\frac{1}{2}}(t)).$$

Next, we begin to estimate $\mathfrak{F}_0(t)$. From (4.7), we have

$$\|\overline{\rho}\| \leq C(1+t)^{-\frac{5}{4}}\|(\overline{\rho}_0,\overline{u}_0)\|_{L^1\cap L^2}+\int_0^t(1+t-\tau)^{-\frac{5}{4}}\|(g_1,g_2)(\tau)\|_{L^1\cap L^2}d\tau,$$

and

$$\begin{aligned}\|\overline{u}\| \leq & C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap L^2}+\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|g_1(\tau)\|_{L^1\cap L^2}d\tau \\ & +\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|(g_2, g_3)(\tau)\|_{L^1\cap L^2}d\tau.\end{aligned}$$

It is straightforward to get

$$\begin{aligned}&\|(g_1, g_2, g_3)(\tau)\|_{L^1\cap L^2} \\ &\quad \leq C\mathfrak{E}_s(\overline{U}) + C\delta\sqrt{\mathfrak{E}_s^h(\overline{U})} + C\delta\|(\overline{\rho}, \overline{u})\| \\ &\quad \leq C(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}\mathfrak{X}(\gamma^{\frac{1}{2}}t) + C\delta(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{5}{8}}\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t) + C\delta(1+\tau)^{-\frac{5}{8}}\mathfrak{F}_0^{\frac{1}{2}}(t).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|(\overline{\rho}, \overline{u})\| \\ &\quad \leq C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap L^2} + \int_0^t (1+t-\tau)^{-\frac{3}{4}}\|(g_1, g_2, g_3)(\tau)\|_{L^1\cap L^2}d\tau \\ &\quad \leq C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap L^2} + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{3}{4}}d\tau\,\mathfrak{X}(\gamma^{\frac{1}{2}}t) \\ &\qquad +C\delta\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\gamma^{\frac{1}{2}}\tau)^{-\frac{5}{8}}ds\,\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t) \\ &\qquad +C\delta\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{5}{8}}d\tau\,\mathfrak{F}_0^{\frac{1}{2}}(t) \\ &\quad \leq C(1+t)^{-\frac{5}{8}}\|\overline{U}_0\|_{L^1\cap L^2}+C(1+t)^{-\frac{3}{4}}\mathfrak{X}(\gamma^{\frac{1}{2}}t)+C\delta(1+t)^{-\frac{5}{8}}\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t) \\ &\qquad +C\delta(1+t)^{-\frac{5}{8}}\mathfrak{F}_0^{\frac{1}{2}}(t) \\ &\quad \leq C(1+t)^{-\frac{5}{8}}\left(\|\overline{U}_0\|_{L^1\cap L^2} + \mathfrak{X}(\gamma^{\frac{1}{2}}t) + \delta\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t) + \delta\mathfrak{F}_0^{\frac{1}{2}}(t)\right).\end{aligned}$$

Then, by the definition of $\mathfrak{F}_0(t)$, we have

$$\mathfrak{F}_0(t) \leq C\|\overline{U}_0\|^2_{L^1\cap L^2} + C\mathfrak{X}^2(\gamma^{\frac{1}{2}}t) + C\delta^2\mathfrak{Y}(\gamma^{\frac{1}{2}}t) + C\delta^2\mathfrak{F}_0(t),$$

which further implies that

$$\mathfrak{F}_0(t) \leq C\|\overline{U}_0\|^2_{L^1\cap L^2} + C\mathfrak{X}^2(\gamma^{\frac{1}{2}}t) + C\delta^2\mathfrak{Y}(\gamma^{\frac{1}{2}}t). \tag{4.18}$$

Then, we have

$$
\begin{aligned}
&\|\nabla B_1\| \\
&\quad \le C(1+t)^{-\frac{5}{8}}\left(\delta\left(\|\overline{U}_0\|^2_{L^1\cap L^2}+\mathfrak{X}^2(\gamma^{\frac{1}{2}}t)+\delta^2\mathfrak{Y}(\gamma^{\frac{1}{2}}t)\right)^{\frac{1}{2}}\right.\\
&\qquad \left.+\delta\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)+\mathfrak{X}(\gamma^{\frac{1}{2}}t)\|\overline{U}_0\|_{L^1\cap H^{s+1}}\right)\\
&\quad \le C(1+t)^{-\frac{5}{8}}\left(\|\overline{U}_0\|_{L^1\cap H^{s+1}}+\mathfrak{X}(\gamma^{\frac{1}{2}}t)+\delta\mathfrak{Y}^{\frac{1}{2}}(\gamma^{\frac{1}{2}}t)\right).
\end{aligned}
$$

Due to the fact that $\|\nabla\overline{B}\| \sim \|\nabla B_1\|$, we have

$$
\|\nabla\overline{B}\| \le C(1+t)^{-\frac{5}{8}}\left(\|\overline{V}_0\|_{L^1\cap H^{s+1}}+\mathfrak{X}(t)+\delta\mathfrak{Y}^{\frac{1}{2}}(t)\right).
$$

The inequality above together with (4.16) and (4.17) give

$$
\begin{aligned}
&\int_0^t (1+\tau)^{\frac{1}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(\tau))d\tau\\
&\quad \le C(1+t)^{\epsilon}\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}\\
&\qquad + C\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+1}}+\mathfrak{X}^2(t)+\delta^2\mathfrak{Y}(t)\right)\int_0^t(1+\tau)^{\frac{1}{4}+\epsilon}(1+\tau)^{-\frac{5}{4}}d\tau\\
&\quad \le C(1+t)^{\epsilon}\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}+\mathfrak{X}^2(t)+\delta^2\mathfrak{Y}(t)\right).
\end{aligned}
$$

Substitute it into (4.15), we obtain

$$
\begin{aligned}
&(1+t)^{\frac{5}{4}+\epsilon}\mathfrak{E}_s^h(\overline{V}(t))+\lambda\int_0^t(1+\tau)^{\frac{5}{4}+\epsilon}\mathfrak{D}_s^h(\overline{V}(\tau))d\tau\\
&\quad \le C\mathfrak{E}_s^h(\overline{V}_0)+C(1+t)^{\epsilon}\left(\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}+\mathfrak{X}^2(t)+\delta^2\mathfrak{Y}(t)\right).
\end{aligned}
$$

Taking the limit $\epsilon \to 0$, we have

$$
\begin{aligned}
&(1+t)^{\frac{5}{4}}\mathfrak{E}_s^h(\overline{V}(t))+\lambda\int_0^t(1+\tau)^{\frac{5}{4}}\mathfrak{D}_s^h(\overline{V}(\tau))d\tau \le C\mathfrak{E}_s(\overline{V}_0)\\
&\quad +C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}+C\mathfrak{X}^2(t)+C\delta^2\mathfrak{Y}(t),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\mathfrak{Y}(t) &= \sup_{0\le\tau\le t}(1+\tau)^{\frac{5}{4}}\mathfrak{E}_s^h(\overline{V}(\tau))\\
&\le C\mathfrak{E}_s(\overline{V}_0)+C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}+C\mathfrak{X}^2(t)+C\delta^2\mathfrak{Y}(t),
\end{aligned}
$$

and then

$$\mathfrak{Y}(t) \le C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}} + C\mathfrak{X}^2(t).$$

It follows from lemma 4.3 that

$$\mathfrak{Y}(t) \le C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}} + C\|\overline{V}_0\|^4_{L^1\cap H^{s+1}} \le C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}}.$$

Therefore, (4.14) holds true since $\|\overline{V}_0\|_{L^1\cap H^{s+2}}$ is sufficiently small. This completes the proof of Lemma 4.4. □

Due to the fact that $\mathfrak{E}^h_s(\overline{V}(t)) \sim \|\nabla\overline{V}\|^2_{s-1}$, it follows from (4.14) that

$$\|\nabla\overline{V}\|_{s-1} \le (1+t)^{-\frac{5}{8}}\|\overline{V}_0\|_{L^1\cap H^{s+2}}, \tag{4.19}$$

which yields (1.7) in Theorem 1.1. On the other hand, by (4.11), (4.14) and (4.18), we have

$$\mathfrak{F}_0(t) \le C\|\overline{V}_0\|^2_{L^1\cap L^2} + C\|\overline{V}_0\|^4_{L^1\cap H^{s+1}} + C\delta^2\|\overline{V}_0\|^2_{L^1\cap H^{s+2}} \le C\|\overline{V}_0\|^2_{L^1\cap H^{s+2}},$$

which further gives

$$\|(\overline{\rho}, \overline{u})\| \le C(1+t)^{-\frac{5}{8}}\|\overline{V}_0\|_{L^1\cap H^{s+2}},$$

then (1.8) follows. We have completed the proof of Theorem 1.1. □

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