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# 3D full hydrodynamic model for semiconductor optoelectronic devices: Stability of thermal equilibrium states

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**Abstract**

In this paper, we study a three-dimensional full hydrodynamic model in a bounded domain with insulating and adiabatic boundary. The model takes the form of nonisentropic Euler-Poisson system and incorporates recombination/generation terms, describing the bipolar transport of hot carriers in semiconductor optoelectronic devices. Of particular concern are the existence, uniqueness and exponential stability of thermal equilibrium states to the model, since these mathematical results are rendered useful in numerical simulation and physical theory of semiconductors. They are rigorously proved by the perturbation argument and energy method.


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## 1. Introduction

As a result of the rapid expansion of the semiconductor industry, the technology of devices combining optical and electrical ports has expanded enormously in the past few years. Solar cells, light emitting diodes (LEDs) and laser diodes are of some prevailing semiconductor optoelectronic devices for commercial use. In the field of simulation of semiconductor optoelectronic devices [2,43,45], the following three-dimensional (3D) full hydrodynamic (FHD) model, which is used to describe the *bipolar* transport of charge carriers (*electrons* and *holes*), becomes increasingly popular:

$$\begin{cases} n_{it} + \operatorname{div}(n_i \boldsymbol{u}_i) = -R, & \text{(a)} \\ (n_i \boldsymbol{u}_i)_t + \operatorname{div}(n_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla(n_i \theta_i) = (-1)^{i-1} n_i \nabla\phi - \dfrac{n_i \boldsymbol{u}_i}{\tau_m}, & \text{(b)} \\ n_i \theta_{it} + n_i \boldsymbol{u}_i \cdot \nabla\theta_i + \dfrac{2}{3} n_i \theta_i \operatorname{div}\boldsymbol{u}_i - \dfrac{2}{3}\Delta\theta_i = \dfrac{2\tau_e - \tau_m}{3\tau_m \tau_e} n_i |\boldsymbol{u}_i|^2 - \dfrac{n_i(\theta_i - \theta_L)}{\tau_e}, & \text{(c)} \\ \Delta\phi = n_1 - n_2 - D(x), \quad x \in \Omega \subset \mathbb{R}^3, \quad t > 0, \quad i = 1, 2, & \text{(d)} \end{cases} \tag{1.1}$$

in which the $i$-notation is adopted, namely, for electrons $i = 1$ and for holes $i = 2$; the unknowns $n_i(t,x)$, $\boldsymbol{u}_i(t,x)$, $\theta_i(t,x)$ and $\phi(t,x)$ represent the *density, velocity, temperature of carriers* and the *built-in electrostatic potential*; the physics parameters $\tau_m$, $\tau_e$ and $\theta_L$ stand for the *momentum relaxation time, energy relaxation time* and *lattice temperature*, all assumed to be positive constants, usually $\tau_m \le \tau_e$; the given function $D(x)$ is called *doping profile*, modeling the distribution of dopants in semiconductors; the smooth bounded domain $\Omega$ represents the *device geometry*, that is the bulk of semiconductor devices; the quantity $R$ is referred to as the *net recombination rate*, describing the balance between carrier recombination and generation in semiconductor devices, and it can be regarded as a certain kind of collision term in continuity equations (1.1a). As for the collision terms in momentum equations (1.1b) and energy equations (1.1c), however, we only consider the momentum relaxation time approximation and energy relaxation time approximation, respectively. As a matter of fact, the validity of introducing the recombination/generation effects into the collision terms in momentum and energy equations remains to be confirmed by semiconductor physics.

Compared to the classic drift-diffusion (DD) model for simulation of semiconductors, the merit of the FHD model is that the information provided by the model makes it possible to describe the behavior of semiconductor optoelectronic devices in which hot-carrier effects play a significant role, see [4,6].

In semiconductors, carrier generation and carrier recombination are processes by which electrons and holes are created and eliminated. More precisely, carrier generation describes the processes by which electrons gain energy and move from the valence band to the conduction band, producing electron-hole pairs; while carrier recombination describes processes by which conduction band electrons lose energy and reoccupy the energy state of holes in the valence band. Carrier generation and recombination processes are fundamental to the operation of semiconductor optoelectronic devices, and are always happening in these devices both optically and thermally.

Engineers and physicists in the field of semiconductors have already found multiple recombination/generation mechanisms, for more details, we refer the interested reader to [6,37,45,46]. Below, we merely list several common types of net recombination rates and indicate phenomenologically their physical mechanisms responsible for recombination/generation:

- *Photon transition or radiative recombination/optical generation*,

$$R^{OPT}(n_1, n_2) := Q\left(n_1 n_2 - \bar{n}^2\right), \quad x \in \Omega, \tag{1.2}$$

  where the positive constant $Q$ is the capture-emission rate; the positive constant $\bar{n}$ is the electron concentration in intrinsic semiconductors at the equilibrium.
- *Phonon transition or Shockley-Read-Hall recombination/generation*,

$$R^{SRH}(n_1, n_2) := \frac{n_1 n_2 - \bar{n}^2}{\iota_1(n_1 + \bar{n}_1) + \iota_2(n_2 + \bar{n}_2)}, \quad x \in \Omega, \tag{1.3}$$

  where the positive constants $\iota_1$ and $\iota_2$ are carrier lifetimes; the physical parameters $\bar{n}_1$ and $\bar{n}_2$ are carrier concentrations if the trap energy coincided with the Fermi energy, satisfying $\bar{n}_1\bar{n}_2 = \bar{n}^2$.
- *Three particle transitions or Auger recombination/generation*,

$$R^{AU}(n_1, n_2) := (c_1 n_1 + c_2 n_2)\left(n_1 n_2 - \bar{n}^2\right), \quad x \in \Omega, \tag{1.4}$$

  where the positive constants $c_1$ and $c_2$ are Auger capture coefficients.
- In addition to recombination/generation in the bulk of semiconductor device, electrons and holes may also be recombined/generated at a surface $S \subset \Omega \cup \partial\Omega$. For the purpose of modeling the *surface recombination/generation*, one usually assumes a formula which is structurally equivalent to the Shockley-Read-Hall expression for bulk recombination/generation,

$$R^{SURF}(n_1, n_2) := \frac{n_1 n_2 - \bar{n}^2}{\frac{1}{s_1}(n_1 + \bar{n}_1) + \frac{1}{s_2}(n_2 + \bar{n}_2)}, \quad x \in S, \tag{1.5}$$

  where the positive constants $s_1$ and $s_2$ denote the surface recombination velocities for electrons and holes, respectively.
- *Impact ionization or avalanche generation*, this effect is a pure generation process which is seen to be microscopically identical to the Auger generation process,

$$R^{II}(n_1, n_2, \boldsymbol{u}_1, \boldsymbol{u}_2) := -\left(o_1|n_1\boldsymbol{u}_1| + o_2|n_2\boldsymbol{u}_2|\right), \quad x \in \Omega, \tag{1.6}$$

  where the positive constants $o_1$ and $o_2$ are the ionization rates for electrons and holes defined as generated electron-hole pairs per unit length of travel and per electron and hole, respectively. For instance, an electron generates over a distance $1/o_1$ one electron-hole pair on average.

Note that the device geometry $\Omega$ is a bounded domain, so we have to propose some physically-motivated boundary conditions for analytical and simulating use. To this end, we split the boundary $\partial\Omega$ into two pieces,

$$\partial\Omega = \partial\Omega_O \cup \partial\Omega_I, \tag{1.7}$$

where $\partial\Omega_O$ denotes the part of the boundary corresponding to *ohmic contacts*; $\partial\Omega_I$ is the interface to *adiabatic and insulating material*.

At ohmic contacts between semiconductors and metals, we have Dirichlet boundary conditions for the electrostatic potential, carrier densities and carrier temperatures: for all $t > 0$ and $x \in \partial\Omega_O$,

$$\phi(t,x) = \Psi(x), \tag{1.8a}$$

$$n_i(t,x) = \frac{\sqrt{D^2(x) + 4\bar{n}^2} + (-1)^{i-1} D(x)}{2}, \quad i = 1, 2, \tag{1.8b}$$

$$\theta_i(t,x) = \theta_L, \quad i = 1, 2, \tag{1.8c}$$

where the given function $\Psi(x)$ represents the applied bias; the boundary conditions for carrier densities and temperatures mean that carriers keep the thermal equilibrium and vanishing space charge state at ohmic contacts, which corresponds to infinite surface recombination velocities (namely, $s_1 = +\infty$ and $s_2 = +\infty$ on $\partial\Omega_O$). The boundary conditions (1.8) are called *voltage-driven ohmic contact boundary condition*. The other type of boundary conditions at ohmic contacts is *current-driven ohmic contact boundary condition*, given in integral form, which is of the nonlocal boundary condition.

At the interfaces between semiconductors and adiabatic insulators, we have Neumann boundary conditions for the electrostatic potential and carrier temperatures, and the nonlinear boundary conditions for carrier densities and carrier velocities: for all $t > 0$ and $x \in \partial\Omega_I$,

$$\nabla\phi(t,x) \cdot \nu(x) = 0, \tag{1.9a}$$

$$\nabla\theta_i(t,x) \cdot \nu(x) = 0, \quad i = 1, 2, \tag{1.9b}$$

$$(n_i \boldsymbol{u}_i)(t,x) \cdot \nu(x) = (-1)^i R^{SURF}(n_1, n_2), \quad i = 1, 2, \tag{1.9c}$$

which corresponds to Gauss's law for the electrostatic potential, Fourier's law for carrier temperatures, and the surface recombination/generation mechanism for carrier densities and velocities, respectively. Here, the notation $\nu$ denotes the unit outer normal vector to the boundary $\partial\Omega$. Quite often the existence of surface recombination is simply ignored by assuming vanishing surface recombination velocities (namely, $s_1 = s_2 = 0$ on $\partial\Omega_I$) which leads to the boundary conditions: for all $t > 0$ and $x \in \partial\Omega_I$,

$$\boldsymbol{u}_i(t,x) \cdot \nu(x) = 0, \quad i = 1, 2. \tag{1.10}$$

Under certain circumstances, this reduced boundary conditions (1.10) can be justified by physical experiment on semiconductor optoelectronic devices. The boundary conditions (1.9) are called *adiabatic and insulating boundary condition*. For more general applications and details about the boundary conditions, see [5,45].

Except for the physical reasoning of these boundary conditions, their applicability remains to be justified by mathematical reasoning. That is one point of interest in the present paper and the future study. Together with the other point of interest, how the recombination/generation channels affect the bipolar transport of hot carriers, the main focus of this paper will be on the following initial-boundary value problem:

$$\begin{cases} n_{it} + \operatorname{div}(n_i \boldsymbol{u}_i) = -R^{OPT}(n_1, n_2), & \text{(a)} \\ (n_i \boldsymbol{u}_i)_t + \operatorname{div}(n_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla(n_i \theta_i) = (-1)^{i-1} n_i \nabla\phi - n_i \boldsymbol{u}_i, & \text{(b)} \\ n_i \theta_{it} + n_i \boldsymbol{u}_i \cdot \nabla\theta_i + \frac{2}{3} n_i \theta_i \operatorname{div}\boldsymbol{u}_i - \frac{2}{3}\Delta\theta_i = \frac{1}{3} n_i |\boldsymbol{u}_i|^2 - n_i(\theta_i - 1), & \text{(c)} \\ \Delta\phi = n_1 - n_2 - D(x), \quad x \in \Omega \subset \mathbb{R}^3, \quad t > 0, \quad i = 1, 2, & \text{(d)} \end{cases} \tag{1.11}$$

subject to the initial conditions

$$(n_i, \boldsymbol{u}_i, \theta_i)(0, x) = (n_{i0}, \boldsymbol{u}_{i0}, \theta_{i0})(x), \quad i = 1, 2, \quad \forall x \in \Omega, \tag{1.12}$$

and boundary conditions

$$\boldsymbol{u}_i \cdot \nu|_{\partial\Omega} = 0, \quad \nabla\theta_i \cdot \nu|_{\partial\Omega} = 0, \quad i = 1, 2, \quad \nabla\phi \cdot \nu|_{\partial\Omega} = 0, \quad \int_\Omega \phi(t, x)\mathrm{d}x = 0, \quad \forall t > 0, \tag{1.13}$$

where we have chosen the photon transition mechanism to describe the carrier recombination and generation processes in the bulk of semiconductor optoelectronic device, leaving the other recombination/generation regimes to the future study; we have also adopted the adiabatic and insulating boundary conditions on the entire boundary, assuming that there are no defects on the whole interface; the integral form condition for the electrostatic potential in (1.13) is the *charge neutrality condition*. Without loss of generality, we have assumed that $\tau_m = \tau_e = 1$ and $\bar{n} = \theta_L = 1$ in the system (1.11), in turn, we will use the following net recombination rate hereafter,

$$R^{OPT}(n_1, n_2) = Q\,(n_1 n_2 - 1)\,. \tag{1.14}$$

Thermal equilibrium is an important concept of semiconductor physics, which implies that no external forces such as voltages, electric fields, magnetic fields, or temperature gradients are acting on the semiconductor devices. All properties of the semiconductor devices will be independent of time in this case, and devices themselves are electrically neutral as well. Under thermal equilibrium conditions, the total current in semiconductor devices is of course identically zero. Because electron and hole activity is totally decoupled in thermal equilibrium, electron and hole velocities must also independently vanish. Therefore, thermal equilibrium states to the initial-boundary value problem (1.11)∼(1.13) are a special classification of steady states $(\tilde{n}_1, \tilde{\boldsymbol{u}}_1, \tilde{\theta}_1, \tilde{n}_2, \tilde{\boldsymbol{u}}_2, \tilde{\theta}_2, \tilde{\phi})(x)$ with vanishing carrier velocities $\tilde{\boldsymbol{u}}_1 = \boldsymbol{0}$ and $\tilde{\boldsymbol{u}}_2 = \boldsymbol{0}$. Then thermal equilibrium states are supposed to satisfy the time-invariant system

$$\begin{cases} 0 = -Q(\tilde{n}_1 \tilde{n}_2 - 1), & \text{(a)} \\ \nabla(\tilde{n}_i \tilde{\theta}_i) = (-1)^{i-1} \tilde{n}_i \nabla\tilde{\phi}, & \text{(b)} \\ -\frac{2}{3}\Delta\tilde{\theta}_i = -\tilde{n}_i(\tilde{\theta}_i - 1), \quad i = 1, 2, & \text{(c)} \\ \Delta\tilde{\phi} = \tilde{n}_1 - \tilde{n}_2 - D(x), \quad x \in \Omega, & \text{(d)} \end{cases} \tag{1.15}$$

and Neumann boundary conditions

$$\nabla \tilde{\theta}_i \cdot \nu|_{\partial\Omega} = 0, \quad i = 1, 2, \tag{1.16a}$$

$$\nabla \tilde{\phi} \cdot \nu|_{\partial\Omega} = 0, \quad \int_\Omega \tilde{\phi}(x)\mathrm{d}x = 0. \tag{1.16b}$$

Note that the partially decoupled system (1.15) consists of algebraic-partial differential equations (PDEs). From the algebraic equation (1.15a), it can be easily seen that the product of the electron and hole densities at thermal equilibrium is a constant ($\tilde{n}_1\tilde{n}_2 \equiv 1$), maintained by the net recombination rate $R^{OPT}$ vanishing. Experimental results of semiconductor optoelectronic devices obtained in [3] indicate that when there is a surplus of carriers (i.e. $n_1 n_2 > 1 \equiv \tilde{n}_1\tilde{n}_2$), the recombination process dominates the generation process, driving the system back towards thermal equilibrium in a very short time. Likewise, when there is a deficit of carriers (i.e. $n_1 n_2 < 1 \equiv \tilde{n}_1\tilde{n}_2$), the generation process overwhelms the recombination process, again driving the system back towards thermal equilibrium in a very short time. To mathematically verify this experimental observation is one motivation, in this paper, for establishing the existence, uniqueness and exponential stability of thermal equilibrium states to the initial-boundary value problem (1.11)∼(1.13). The other motivation in doing so is derived from our interest in the mathematical theory itself of semiconductor PDEs. In order to clarify this motivation, in what follows we will review necessary mathematical results in the existing literature concerning semiconductor PDEs.

Over the past three decades, major advances in the mathematical theory of semiconductor PDEs have been made by a lot of mathematicians. Due to the growing body of literature on this issue, we can not walk the reader through all the aspects. Instead, we will only present those results pertaining closely to the object of this paper. The earliest hydrodynamic (HD) models studied systematically are of so-called *unipolar* category, modeling the transport of just single type of charge carrier, such as $n$- or $p$-type semiconductors. For unipolar isothermal, isentropic and non-isentropic HD models, there are many mathematical results on the well-posedness, asymptotic limits with respect to small physical parameters, large time behaviors of smooth and/or weak solutions to Cauchy problem, periodic boundary value problem and/or initial-boundary value problem, see for instance [8,9,15,17–21,24–26,30,33,38,39] and references therein.

As for bipolar HD models in which the recombination/generation effects are not taken into account, there are relatively few mathematical outcomes due to the strong coupling caused by carrier drift-current terms $(-1)^{i-1} n_i \nabla\phi$ (for $i = 1, 2$) in momentum equations. The essential difficulty consists in the weakening of dissipative properties of bipolar models by this kind of strong coupling structure. However, via more or less restrictive conditions, some researchers did contribute interesting findings to this field. For example, Gasser et al. [13] discovered the nonlinear diffusion phenomenon of Cauchy problem of the one-dimensional (1D) isentropic HD model with the zero doping profile (i.e. $D(x) \equiv 0$) and two identical pressure functions, and proved that the global smooth solutions algebraically decay to a self-similar solution as time goes to infinity. Different from the previous studies in [13], Donatelli et al. [10] realized that the asymptotic profiles of Cauchy problem of the 1D isentropic HD model with two different pressure functions and a non-flat doping profile are stationary waves rather than the diffusion waves. Subsequently, Hu et al. [22] generalized the investigations in [10] to an initial-boundary value problem of the 1D FHD model over a bounded interval subject to the voltage-driven ohmic contact boundary condition, however, the doping profile has to be flat. Furthermore, Li et al. [27] considered the periodic boundary value problem of the 3D nonisentropic HD models without temperature diffusion terms and showed that the non-constant steady state is time-asymptotically stable; since the model is posed on the spatial periodic domain $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, the constraints on doping profile can be relaxed to being non-flat but possessing a positive lower bound. Up to now, the bipolar models without net recombination rates are still a subject of active research. Very recently, Peng et al. [41,42] confirmed the global quasi-neutral limit of classical solutions to Cauchy problem of the isentropic HD model, however, the results are still confined to the zero doping profile and 1D settings.

So far as semiconductor PDEs including net recombination rates are concerned, the relevant mathematical analyses are few and far between. Besides, a majority of pioneering and earlier results prefer handling the traditional DD models (see [34,35,12,47]) to dealing with more effective HD models. Regarding the HD models, Zhu et al. [50] first obtained the global existence of smooth solutions to Cauchy problem of the 1D isentropic HD model with $R^{OPT}$. Recently, Hu et al. [23] and Wu et al. [49] extended the studies in [50] to Cauchy problem of the 1D FHD model with $R^{OPT}$ and the initial-boundary value problem of the 1D isothermal HD model with $R^{OPT}$, respectively. For more physical backgrounds and mathematical results of semiconductor PDEs, the interested reader could refer to the monograph [31] and references therein.

In the final analysis, we form an intuition that the net recombination rates should be indispensable for the mathematical analysis of hydrodynamics-based bipolar models. We guess that the net recombination rate $R$, apart from $R^{II}$, could compensate for the loss of dissipative properties of bipolar HD models (as put forth above), thereby enabling us to investigate the 3D bipolar FHD model in a bounded domain with physical boundary conditions. The aim of this paper is to substantiate this guess, taking the net recombination rate $R^{OPT}$ and the insulating and adiabatic boundary condition as the first attempt. The discussion about the rest types of net recombination rates and physical boundary conditions will be left to the future studies.

We are now in a position to state the main results of the present paper.

**Theorem 1.1** *(Existence and uniqueness of thermal equilibrium states). Suppose that $D \in H^2(\Omega)$, for arbitrary constant $d \in \mathbb{R}$, there exist two positive constants $\delta_0$ and $C$ such that if $\|D - d\|_{H^2(\Omega)} \le \delta_0$, then the boundary value problem (1.15)&(1.16) has a unique solution $(\tilde{n}_1, \tilde{\theta}_1, \tilde{n}_2, \tilde{\theta}_2, \tilde{\phi}) \in \left[H^4(\Omega)\right]^5$ satisfying the estimates*

$$\|(\tilde{n}_1 - n_{1d},\ \tilde{n}_2 - n_{2d},\ \tilde{\phi})\|_{H^4(\Omega)} \le C\|D - d\|_{H^2(\Omega)}, \tag{1.17a}$$

$$0 < b \le \tilde{n}_1(x),\ \tilde{n}_2(x) \le B, \quad \tilde{\theta}_1(x) = \tilde{\theta}_2(x) \equiv 1, \quad \forall x \in \overline{\Omega}, \tag{1.17b}$$

*where the positive constants $n_{1d}$, $n_{2d}$, $b$ and $B$ are defined by*

$$n_{1d} := \frac{d + \sqrt{d^2+4}}{2}, \quad n_{2d} := \frac{1}{n_{1d}}, \quad b := \frac{\min\{n_{1d}, n_{2d}\}}{2}, \quad B := 2\max\{n_{1d}, n_{2d}\}. \tag{1.18}$$

*In particular, $(\tilde{n}_1, \mathbf{0}, 1, \tilde{n}_2, \mathbf{0}, 1, \tilde{\phi})(x)$ is exactly the unique thermal equilibrium state to the initial-boundary value problem (1.11)∼(1.13).*

**Remark 1.1.** In fact, Theorem 1.1 also holds true for the periodic domain $\Omega = \mathbb{T}^3$. The proof closely parallels that of Theorem 1.1, so we will not repeat it in this paper.

**Theorem 1.2** *(Exponential stability of thermal equilibrium states). Let $(\tilde{n}_1, \mathbf{0}, 1, \tilde{n}_2, \mathbf{0}, 1, \tilde{\phi})$ be the thermal equilibrium state in Theorem 1.1. Assume that the initial data $(n_{i0}, \boldsymbol{u}_{i0}, \theta_{i0})$ for $i = 1, 2$ satisfy $n_{i0} - \tilde{n}_i \in H^3(\Omega)$, $\boldsymbol{u}_{i0} \in H^3(\Omega; \mathbb{R}^3)$, $\theta_{i0} - 1 \in H^4(\Omega)$ and are compatible with the boundary conditions (1.13) (see Lemma 4.1). Then there exists a positive constant $\delta_1$ such that if $\|D - d\|^2_{H^2(\Omega)} \le \delta_1^2$ and*

$$\sum_{i=1}^{2}\left(\sum_{j=0}^{3}\|(\partial_t^j(n_i-\tilde{n}_i), \partial_t^j\boldsymbol{u}_i)(0)\|^2_{H^{3-j}(\Omega)}+\sum_{j=0}^{1}\|\partial_t^j(\theta_i-1)(0)\|^2_{H^{4-j}(\Omega)}+\|\theta_{itt}(0)\|^2_{H^1(\Omega)}\right)\le\delta_1^2, \tag{1.19}$$

*then the initial-boundary value problem (1.11)∼(1.13) has a unique global solution $(n_i, \boldsymbol{u}_i, \theta_i, \phi)$ satisfying $n_i - \tilde{n}_i \in \bigcap_{j=0}^3 C^j([0,\infty); H^{3-j}(\Omega))$, $\boldsymbol{u}_i \in \bigcap_{j=0}^3 C^j([0,\infty); H^{3-j}(\Omega; \mathbb{R}^3))$, $\theta_i - 1 \in \bigcap_{j=0}^1 C^j([0,\infty); H^{4-j}(\Omega))$, $\theta_{itt} \in C([0,\infty); H^1(\Omega)) \cap L^2([0,\infty); H^2(\Omega))$ for $i = 1, 2$ and $\phi - \tilde{\phi} \in \bigcap_{j=0}^3 C^j([0,T]; H^{3-j+2}(\Omega))$. Moreover, there are positive constants $\gamma$ and $C$ such that for all $t \in [0,\infty)$,*

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\sum_{j=0}^{3}\|(\partial_t^j(n_i-\tilde{n}_i), \partial_t^j\boldsymbol{u}_i)(t)\|^2_{H^{3-j}(\Omega)}+\sum_{j=0}^{1}\|\partial_t^j(\theta_i-1)(t)\|^2_{H^{4-j}(\Omega)}+\|\theta_{itt}(t)\|^2_{H^1(\Omega)}\right)\\
&+\sum_{j=0}^{3}\|\partial_t^j(\phi-\tilde{\phi})(t)\|^2_{H^{5-j}(\Omega)}\\
\le& C\left[\sum_{i=1}^{2}\left(\sum_{j=0}^{3}\|(\partial_t^j(n_i-\tilde{n}_i), \partial_t^j\boldsymbol{u}_i)(0)\|^2_{H^{3-j}(\Omega)}\right.\right.\\
&\left.\left.+\sum_{j=0}^{1}\|\partial_t^j(\theta_i-1)(0)\|^2_{H^{4-j}(\Omega)}+\|\theta_{itt}(0)\|^2_{H^1(\Omega)}\right)\right]e^{-\gamma t}.
\end{aligned} \tag{1.20}$$

**Remark 1.2.** The same result in Theorem 1.2 applies to the periodic domain $\Omega = \mathbb{T}^3$ as well. The proof of Theorem 1.2 covers the case of periodic boundary conditions, and will become simpler in this case because there is no need to control the boundary integrals.

We conclude this section by illustrating the main ideas in the proofs of Theorems 1.1 and 1.2. Though both the proofs are lengthy, the basic ideas are easily comprehensible.

First of all, we elucidate the proof strategy of Theorem 1.1. As mentioned above, the thermal equilibrium system (1.15) is of the partially decoupled form. By using an illuminating derivation, the boundary value problem (1.15)&(1.16) can be reduced to the Neumann boundary value problem of the *Poisson-Boltzmann (PB) equation*, see (3.2). Although there have already been some results about Neumann problems of the PB equation, such as Looker [28] proved the unique solvability of (3.2) by the monotone operator theory, and then Allaire et al. [1] reconsidered this problem in the variational framework. However, these results can not provide us with a suitable flatness estimate of thermal equilibrium states in terms of the flatness level of the doping profile. What matters is that the flatness estimate will play a crucial role in the study of large time behavior of transient solutions. To get around this difficulty we will choose the so-called *perturbation argument* (see [11]), which is based on Banach's Fixed Point Theorem, as an alternative method to establish the well-posedness of boundary value problem (3.2). Meanwhile, the flatness estimate can be directly derived by applying this novel approach to the PB equation, see Lemma 3.1.

The proof of Theorem 1.2 is completed by using the standard *continuation principle* (see [29]). To carry through the proof process, we first establish the local existence of time-dependent solutions by combining the *energy method* and a standard *contraction mapping argument*. Next, to extend the local-in-time solution to the global-in-time one, we have to close the uniform a priori estimate in an appropriate *Sobolev smooth norm* (see (5.2) for definition) by making the most of the enhanced dissipation mechanism of the system (1.11). In truth, by the refined energy method, we found that the recombination/generation terms compensate for the dissipation loss engendered by the bipolar drift-current terms.

The organization of this paper is as follows. Some useful preliminaries are listed in Section 2. The proof of Theorem 1.1 is given in Section 3. The proof of Theorem 1.2 is given in Sections 4 and 5, including the local existence lemma, a series of lemmas for energy estimates, and the continuation of local-in-time solutions.

## 2. Preliminaries

In this section we gather some miscellaneous results that are more or less standard, but these preliminaries will be used repeatedly hereafter.

Firstly, we introduce the standard $L^2$-theory of the linear elliptic Neumann problem, which is excerpted from Grisvard's monograph [14].

**Proposition 2.1** *($L^2$-theory of the linear elliptic Neumann problem). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^m$. Then for every $f \in H^k(\Omega)$ with $k \geq 0$ an integer number, and $\lambda > 0$, there exists a unique solution $u \in H^{k+2}(\Omega)$ solving the Neumann boundary value problem*

$$\begin{cases} -\Delta u + \lambda u = f, \ in\ \Omega, & \text{(a)} \\ \dfrac{\partial u}{\partial \nu} = 0, \ on\ \partial\Omega, & \text{(b)} \end{cases} \tag{2.1}$$

*and satisfying the elliptic estimate*

$$\|u\|_{H^{k+2}(\Omega)} \leq C(k, \lambda, \Omega)\|f\|_{H^k(\Omega)}. \tag{2.2}$$

In addition, Moser [36] proved the following result concerning composition in Sobolev spaces.

**Proposition 2.2** *(Composition in Sobolev spaces). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^m$, $k \geq 1$ an integer number and $1 \leq p < \infty$. If $u \in W^{k,p}(\Omega)$, with $kp > m$ and $\Phi \in C^k(\mathbb{R})$, then $\Phi \circ u \in W^{k,p}(\Omega)$ and its weak derivatives could be calculated by Faa di Bruno's formula*

$$\partial_x^\alpha (\Phi \circ u) = \sum_{j=1}^{|\alpha|} \sum_{\substack{\beta_1+\cdots+\beta_j=\alpha \\ |\beta_l|\neq 0}} c_{\alpha,j,\beta_1,\cdots,\beta_j} \left(\Phi^{(j)} \circ u\right) \partial_x^{\beta_1} u \cdots \partial_x^{\beta_j} u, \quad a.e.\ x \in \Omega, \tag{2.3}$$

*where* $\alpha \in \mathbb{N}^m$ *with* $1 \le |\alpha| \le k$.

For convenience, we also state a new explicit formula for higher order derivatives of a quotient. For more details about this useful formula, we refer the reader to [16].

**Proposition 2.3** *(Quotient rule). Let* $u$ *and* $v$ *be* $n$ *times differentiable functions of a single variable* $t$*, for any* $n \in \mathbb{N}$ *and at each point where* $v \neq 0$*, we have that*

$$\left(\frac{u}{v}\right)^{(n)} = n! \sum_{l=0}^{n} \frac{u^{(n-l)}}{(n-l)!} \sum_{k=1}^{P(l)} \frac{c_k}{v^{1+\sum_{q=1}^{l} p_{kq}}} \prod_{q=1}^{l} \left(\frac{v^{(q)}}{q!}\right)^{p_{kq}}, \tag{2.4}$$

*where the vector* $(p_{k1}, \cdots, p_{kl})$ *is called the* $k$*-th partition of the nonnegative integer* $l$ *provided*

$$p_{k1} + 2p_{k2} + \cdots + lp_{kl} = \sum_{q=1}^{l} q p_{kq} = l,$$

*and note that* $0 \le p_{kq} \le l$ *while* $1 \le q \le l$*;the number of all possible partitions of an integer* $l$ *is denoted by* $P(l)$ *and hence* $1 \le k \le P(l)$*;the combinatorial coefficient is denoted by* $c_k$ *and its exact expression is described in [16].*

Next, the following div-curl lemma can be found in [48], and in effect has been widely used in references [7,15,40].

**Proposition 2.4** *(Div-curl lemma). Let* $\Omega$ *be a smooth bounded domain in* $\mathbb{R}^3$*,* $\nu$ *the unit outer normal vector of the boundary* $\partial\Omega$*,* $k \ge 0$ *an integer number and* $1 < p < \infty$*. Then for* $\boldsymbol{u} \in W^{k+1,p}(\Omega; \mathbb{R}^3)$ *with* $\boldsymbol{u} \cdot \nu = 0$ *on* $\partial\Omega$*, there exists a constant* $C = C(k, p, \Omega) > 0$ *such that*

$$\|\boldsymbol{u}\|_{W^{k+1,p}(\Omega;\mathbb{R}^3)} \le C \left( \|\mathrm{div}\boldsymbol{u}\|_{W^{k,p}(\Omega)} + \|\mathrm{curl}\boldsymbol{u}\|_{W^{k,p}(\Omega;\mathbb{R}^3)} + \|\boldsymbol{u}\|_{L^p(\Omega;\mathbb{R}^3)} \right) \tag{2.5}$$

Finally, we give a technical inequality that follows from Sobolev's embedding theorem. It will be utilized many times in the proof of nonlinear stability.

**Proposition 2.5.** *Let* $\Omega$ *be a smooth bounded domain in* $\mathbb{R}^3$*. Then for all* $u_1, u_2 \in H^1(\Omega)$*, there exists a constant* $C = C(\Omega) > 0$ *such that*

$$\|u_1 u_2\|_{L^2(\Omega)} \le C \|u_1\|_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)} \tag{2.6}$$

## 3. Thermal equilibrium states

This section is committed to proving Theorem 1.1. To this end, we first note that the boundary value problem (1.15c)&(1.16b) can be decoupled from the boundary value problem (1.15)&(1.16) whenever $\tilde{n}_i > 0$ is solved a priori. In that case, it is trivially verified that $\tilde{\theta}_i(x) \equiv 1$ is the unique solution to the boundary value problem (1.15c)&(1.16b), which in turn implies from the algebraic equation (1.15a) and gradient equation (1.15b) that

$$\tilde{n}_1 = e^{\tilde{\phi}+C} > 0, \quad \tilde{n}_2 = \tilde{n}_1^{-1} = e^{-(\tilde{\phi}+C)} > 0, \tag{3.1}$$

where $C$ is the arbitrary additive constant. For simplicity of notation, write $z(x)$ instead of $\tilde{\phi}(x) + C$. Substituting (3.1) into the boundary value problem (1.15d)&(1.16b) yields

$$\begin{cases} \Delta z = e^z - e^{-z} - D(x), \text{ in } \Omega, & \text{(a)} \\ \dfrac{\partial z}{\partial \nu} = 0, \text{ on } \partial\Omega, & \text{(b)} \end{cases} \tag{3.2}$$

which is the Neumann problem of the Poisson-Boltzmann equation. Once the unique solvability of the boundary value problem (3.2) is in hand, owing to the condition $\int_\Omega \tilde{\phi}(x)dx = 0$, the arbitrary additive constant $C$ can be uniquely determined, that is, $C = z_\Omega := \frac{1}{|\Omega|}\int_\Omega z(x)dx$. Therefore, $(\tilde{n}_1, \tilde{\theta}_1, \tilde{n}_2, \tilde{\theta}_2, \tilde{\phi})(x) := (e^z, 1, e^{-z}, 1, z - z_\Omega)(x)$ is exactly the unique solution to the boundary value problem (1.15)&(1.16). Furthermore, it is clear that $(\tilde{n}_1, \mathbf{0}, 1, \tilde{n}_2, \mathbf{0}, 1, \tilde{\phi})(x)$ is the unique thermal equilibrium state to the initial-boundary value problem (1.11)∼(1.13).

Next, it remains for us to prove that the boundary value problem (3.2) is well-posed under an appropriate restriction on the doping profile $D(x)$. The corresponding result is summarized in the following lemma.

**Lemma 3.1.** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^3$. Suppose that $d \in \mathbb{R}$ is an arbitrary real number, and the doping profile $D \in L^2(\Omega)$. There exist two positive constants $\delta_2$ and $C = C(d, \Omega)$ such that if $\|D - d\|_{L^2(\Omega)} \le \delta_2$, then the boundary value problem (3.2) admits a unique strong solution $z \in H^2(\Omega)$ satisfying the basic estimate*

$$\|z - z_d\|_{H^2(\Omega)} \le C(d, \Omega)\|D - d\|_{L^2(\Omega)}, \tag{3.3}$$

*where the constant $z_d$ is defined by*

$$z_d := \ln\left(\frac{d + \sqrt{d^2+4}}{2}\right). \tag{3.4}$$

*Furthermore, if the doping profile $D \in H^k(\Omega)$ for any integer $k > 0$ and $\|D - d\|_{H^{k-1}(\Omega)}$ is small enough, then the strong solution possesses the higher order regularity $z \in H^{k+2}(\Omega)$ and satisfies the higher order estimate*

$$\|z - z_d\|_{H^{k+2}(\Omega)} \le C(d, \Omega, k)\|D - d\|_{H^k(\Omega)}, \tag{3.5}$$

*where $C = C(d, \Omega, k)$ is a positive constant.*

**Proof.** The main idea of the proof is the application of Banach's Fixed Point Theorem by the perturbation argument. For clarity, we divide the proof into three steps below.

*Step 1. Background solution.* Let us temporarily fix the doping profile $D(x) \equiv d$ in equation (3.2a), where $d \in \mathbb{R}$ is an arbitrary real number. It is obvious that the boundary value problem (3.2) now has a unique constant solution $z_d$ satisfying the equation

$$e^{z_d} - e^{-z_d} - d = 0. \tag{3.6}$$

Solving (3.6) directly, we get

$$z_d = \ln\left(\frac{d+\sqrt{d^2+4}}{2}\right), \tag{3.7}$$

which in turn serves as the background solution of the perturbation argument in the next step.

*Step 2. Perturbation argument.* It is well-known that the perturbation argument based on Banach's Fixed Point Theorem usually casts a small nonlinear modification as a contraction mapping. The hallmark of such arguments is the occurrence of a parameter which must be taken small enough to ensure the strict contraction property (see [11]). Here we choose the following quantity

$$\delta := \|D-d\|_{L^2(\Omega)} \tag{3.8}$$

as the corresponding parameter. This main step is divided into four pieces:

Firstly, the perturbation argument is carried out in the complete metric space

$$\mathbb{B} := \left\{ v \in H^2(\Omega) \;\middle|\; \|v\|_{H^2(\Omega)} \le N\delta \right\}, \tag{3.9}$$

with the induced metric

$$\rho(v_1, v_2) := \|v_1 - v_2\|_{H^2(\Omega)}, \quad \forall v_1, v_2 \in \mathbb{B}, \tag{3.10}$$

where the positive constant $N$ will be determined later, see (3.15). Let the solution operator $\mathcal{A}$ be defined as follows. Given a function $u \in \mathbb{B}$, set $f(x) := h(u(x))$, $x \in \Omega$, and the function $h(u)$ is defined by

$$h(u) := 2\left[\sinh(u+z_d) - \sinh(z_d) - \sinh'(z_d)u\right], \tag{3.11}$$

where the function $\sinh(z) := \frac{1}{2}(e^z - e^{-z})$ is the hyperbolic sine. In light of Proposition 2.2, we see $f \in H^2(\Omega)$. Consequently, the $L^2$-theory set forth in Proposition 2.1 ensures that the linear elliptic Neumann problem

$$\begin{cases} \Delta w - \lambda_d w = f(x) - (D(x)-d), \ \text{in } \Omega, & \text{(a)} \\ \dfrac{\partial w}{\partial \nu} = 0, \ \text{on } \partial\Omega, & \text{(b)} \end{cases} \tag{3.12}$$

where $\lambda_d := 2\cosh(z_d) > 0$, has a unique solution $w \in H^2(\Omega)$ satisfying the elliptic estimate

$$\|w\|_{H^2(\Omega)} \le C(\lambda_d, \Omega)\|f - (D-d)\|_{L^2(\Omega)}. \tag{3.13}$$

Define $\mathcal{A} : \mathbb{B} \to H^2(\Omega)$ by setting $\mathcal{A}[u] = w$.

Secondly, we now claim that there exists a positive constant $N$ which only depends on $\lambda_d$ and $\Omega$ such that if $\delta$ is small enough, then $\mathcal{A}$ is a mapping from $\mathbb{B}$ to itself. To prove this, it suffices to further estimate the right-hand side of (3.13), that is

$$\begin{aligned}\|\mathcal{A}[u]\|_{H^2(\Omega)} =&\|w\|_{H^2(\Omega)}\\ \leq& C(\lambda_d,\Omega)\|f-(D-d)\|_{L^2(\Omega)}\\ \leq& C_1\|f\|_{L^2(\Omega)}+C_1\|D-d\|_{L^2(\Omega)}\\ =&C_1\|h(u)\|_{L^2(\Omega)}+C_1\delta\\ =&C_1\|\sinh(z_d+\xi_1 u)u^2\|_{L^2(\Omega)}+C_1\delta\\ =&C_1O(z_d)|\Omega|\|u\|^2_{C^{0,\frac{1}{2}}(\overline{\Omega})}+C_1\delta\\ \leq& C_2N^2\delta^2+C_1\delta, \end{aligned} \tag{3.14}$$

where the positive constants $C_1, C_2$ only depend on $z_d$ and $\Omega$. There $\xi_1 \in (0, 1)$ because of Taylor's formula. We have also used Sobolev's Embedding Theorem in the second-last line of (3.14). Define

$$N := 2C_1, \tag{3.15}$$

and let $\delta < \frac{1}{4C_1C_2}$, consequently $\mathcal{A}[u] \in \mathbb{B}$.

Thirdly, we can further claim that if $\delta$ is small enough, then $\mathcal{A} : \mathbb{B} \to \mathbb{B}$ is a strict contraction mapping. To prove this, choose $u_1, u_2 \in \mathbb{B}$, and define $w_1 = \mathcal{A}[u_1]$, $w_2 = \mathcal{A}[u_2]$ as above. Consequently, for $i = 1, 2$, we know that $w_i$ verifies (3.12) for $f_i(x) = h(u_i(x))$. We consider the difference $w_1 - w_2$, which satisfies the boundary value problem

$$\begin{cases}\Delta(w_1-w_2)-\lambda_d(w_1-w_2)=h(u_1(x))-h(u_2(x)),\ \text{in}\ \Omega, & \text{(a)}\\ \dfrac{\partial(w_1-w_2)}{\partial\nu}=0,\ \text{on}\ \partial\Omega. & \text{(b)}\end{cases} \tag{3.16}$$

Utilizing the elliptic estimate (2.2) when taking the value $k = 0$, we can then calculate from (3.16) that

$$\begin{aligned}&\|\mathcal{A}[u_1]-\mathcal{A}[u_2]\|_{H^2(\Omega)} = \|w_1-w_2\|_{H^2(\Omega)}\\ &\leq C(\lambda_d,\Omega)\|h(u_1)-h(u_2)\|_{L^2(\Omega)}\\ &=2C(\lambda_d,\Omega)\left\|[\sinh(u_1+z_d)-\sinh(u_2+z_d)]-\sinh'(z_d)(u_1-u_2)\right\|_{L^2(\Omega)}\\ &=2C(\lambda_d,\Omega)\,\|[\sinh(z_d+\xi_3(u_2+\xi_2(u_1-u_2)))(u_2+\xi_2(u_1-u_2))(u_1-u_2)\|_{L^2(\Omega)}\\ &\leq C_3\delta\|u_1-u_2\|_{L^2(\Omega)}, \end{aligned} \tag{3.17}$$

where the positive constant $C_3$ only depends on $z_d$ and $\Omega$. There $\xi_2, \xi_3 \in (0, 1)$ because we have used Taylor's formula twice in the calculations of (3.17). Thus $\mathcal{A}$ is a strict contraction, provided $\delta > 0$ is so small that $C_3\delta < 1$.

Finally, set

$$\delta_2 := \frac{1}{2}\min\left\{\frac{1}{4C_1C_2}, \frac{1}{C_3}\right\} > 0. \tag{3.18}$$

Given the doping profile $D \in L^2(\Omega)$ satisfying $\|D-d\|_{L^2(\Omega)} \le \delta_2$, we can then apply Banach's Fixed Point Theorem to the solution operator $\mathcal{A}$ to find a unique strong solution $\eta \in \mathbb{B}$ of the semi-linear elliptic Neumann problem

$$\begin{cases} \Delta\eta - \lambda_d \eta = h(\eta) - (D(x)-d), \text{ in } \Omega, & \text{(a)} \\ \dfrac{\partial \eta}{\partial \nu} = 0, \text{ on } \partial\Omega, & \text{(b)} \end{cases} \tag{3.19}$$

which is obviously equivalent to the boundary value problem (3.2) after defining $z(x) := \eta(x) + z_d$.

*Step 3. Higher order regularity.* To prove the higher order estimate (3.5), we shall use the induction on the order $k \ge 1$.

First we consider the base case $k = 1$, namely, $D \in H^1(\Omega)$. Let $\eta = z - z_d$, and then the right-hand side $h(\eta) - (D-d)$ of (3.19a) belongs to $H^1(\Omega)$. It follows from Proposition 2.1 that the unique solution $\eta \in H^3(\Omega)$ satisfies the elliptic estimate

$$\begin{aligned} \|z-z_d\|_{H^3(\Omega)} =& \|\eta\|_{H^3(\Omega)} \\ \le& C(1,\lambda_d,\Omega)\|h(\eta)-(D-d)\|_{H^1(\Omega)} \\ \le& C\|h(\eta)\|_{L^2(\Omega)} + C\sum_{|\alpha|=1}\|h'(\eta)\partial_x^\alpha \eta\|_{L^2(\Omega)} + C(1,\lambda_d,\Omega)\|D-d\|_{H^1(\Omega)} \\ \le& C\|\eta\|_{H^1(\Omega)} + C(1,\lambda_d,\Omega)\|D-d\|_{H^1(\Omega)} \\ \le& C\|D-d\|_{L^2(\Omega)} + C(1,\lambda_d,\Omega)\|D-d\|_{H^1(\Omega)} \\ \le& C(d,\Omega,1)\|D-d\|_{H^1(\Omega)}, \end{aligned} \tag{3.20}$$

where we have used the same technique as in (3.14) to deal with $\|h(\eta)\|_{L^2(\Omega)}$, and also appealed to the smallness of $\delta = \|D-d\|_{L^2(\Omega)}$.

Now assume inductively that $k > 1$, that is, $D \in H^k(\Omega)$ and we have already proven the fact that $\eta \in H^{(k-1)+2}(\Omega)$ and satisfies the inequality (3.5) for orders $k-1$. We shall use a similar argument to the preceding one. Precisely, Proposition 2.2 now implies $h(\eta) \in H^{k+1}(\Omega)$, which in turn yields the right-hand side $h(\eta) - (D-d)$ of (3.19a) is of class $H^k(\Omega)$. Hence, by Proposition 2.1, we obtain that $\eta \in H^{k+2}(\Omega)$ and satisfies the elliptic estimate

$$\begin{aligned} \|z-z_d\|_{H^{k+2}(\Omega)} =& \|\eta\|_{H^{k+2}(\Omega)} \\ \le& C(k,\lambda_d,\Omega)\|h(\eta)-(D-d)\|_{H^k(\Omega)} \\ \le& C\|h(\eta)\|_{L^2(\Omega)} + C\sum_{1\le|\alpha|\le k}\|\partial_x^\alpha(h(\eta))\|_{L^2(\Omega)} + C(k,\lambda_d,\Omega)\|D-d\|_{H^k(\Omega)} \\ \le& C\|h(\eta)\|_{L^2(\Omega)} \\ &+ C\sum_{1\le|\alpha|\le k}\left\|\sum_{j=1}^{|\alpha|}\sum_{\substack{\beta_1+\cdots+\beta_j=\alpha \\ |\beta_l|\neq 0}} c_{\alpha,j,\beta_1,\cdots,\beta_j}\left(h^{(j)}\circ\eta\right)\partial_x^{\beta_1}\eta\cdots\partial_x^{\beta_j}\eta\right\|_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&+C(k,\lambda_d,\Omega)\|D-d\|_{H^k(\Omega)}\\
\leq& C\|\eta\|_{H^k(\Omega)}+C(k,\lambda_d,\Omega)\|D-d\|_{H^k(\Omega)}\\
\leq& C(d,\Omega,k-1)\|D-d\|_{H^{k-1}(\Omega)}+C(k,\lambda_d,\Omega)\|D-d\|_{H^k(\Omega)}\\
\leq& C(d,\Omega,k)\|D-d\|_{H^k(\Omega)},
\end{aligned}
\tag{3.21}
$$

where we have utilized the formula (2.3) and the smallness of $\|D-d\|_{H^{k-1}(\Omega)}$. □

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Drawing upon the equivalent analysis at the very beginning of this section and the well-posedness result in Lemma 3.1, we may easily prove all the conclusions in Theorem 1.1 by directly using the transformation $(\tilde{n}_1,\tilde{\theta}_1,\tilde{n}_2,\tilde{\theta}_2,\tilde{\phi})=(e^z,1,e^{-z},1,z-z_\Omega)$ and the similar calculations to that in Step 3 of the proof of Lemma 3.1. □

## 4. Reformulation and local existence

In what follows, the main focus is on the suitably small global-in-time solutions to the initial-boundary value problem (1.11)∼(1.13), approaching the thermal equilibrium state $(\tilde{n}_1,\mathbf{0},1,\tilde{n}_2,\mathbf{0},1,\tilde{\phi})$ constructed in Theorem 1.1. For convenience, we denote the perturbation by

$$
\psi_i:=n_i-\tilde{n}_i,\quad \boldsymbol{u}_i:=\boldsymbol{u}_i-\mathbf{0},\quad \chi_i:=\theta_i-1,\quad \sigma:=\phi-\tilde{\phi},\quad i=1,2,
\tag{4.1}
$$

and introduce a vector $\boldsymbol{w}:=(\psi_1,\boldsymbol{u}_1,\chi_1,\psi_2,\boldsymbol{u}_2,\chi_2)$, thereby equivalently rewriting the original problem (1.11)∼(1.13) into the following form which is more amenable to energy method:

$$
\begin{cases}
\psi_{it}+\operatorname{div}(\tilde{n}_i\boldsymbol{u}_i)+Q(\tilde{n}_2\psi_1+\tilde{n}_1\psi_2)=f_i(\boldsymbol{w}), & \text{(a)}\\
\boldsymbol{u}_{it}+\boldsymbol{u}_i+\nabla(\tilde{n}_e\psi_i)+\tilde{n}_e\nabla\tilde{n}_i\chi_i+\nabla\chi_i+(-1)^i\nabla\sigma=\boldsymbol{g}_i(\boldsymbol{w}), & \text{(b)}\\
\tilde{n}_i\chi_{it}+\dfrac{2}{3}\tilde{n}_i\operatorname{div}\boldsymbol{u}_i-\dfrac{2}{3}\Delta\chi_i+\tilde{n}_i\chi_i=h_i(\boldsymbol{w}), & \text{(c)}\\
\Delta\sigma=\psi_1-\psi_2,\quad i=1,2, & \text{(d)}
\end{cases}
\tag{4.2}
$$

subjected to the initial conditions

$$
(\psi_i,\boldsymbol{u}_i,\chi_i)(0,x)=(\psi_{i0},\boldsymbol{u}_{i0},\chi_{i0})(x):=(n_{i0}-\tilde{n}_i,\boldsymbol{u}_{i0},\theta_{i0}-1),\quad x\in\Omega,
\tag{4.3}
$$

and boundary conditions

$$
\boldsymbol{u}_i\cdot\nu|_{\partial\Omega}=0,\quad \nabla\chi_i\cdot\nu|_{\partial\Omega}=0,\quad \nabla\sigma\cdot\nu|_{\partial\Omega}=0,\quad \int_\Omega\sigma(t,x)dx=0,\quad t>0,
\tag{4.4}
$$

where the nonlinear right-hand side terms $f_i(\boldsymbol{w})$, $\boldsymbol{g}_i(\boldsymbol{w})$ and $h_i(\boldsymbol{w})$ are defined by

$$
f_i(\boldsymbol{w}):=-\operatorname{div}(\psi_i\boldsymbol{u}_i)-Q\psi_1\psi_2,
\tag{4.5a}
$$

$$\boldsymbol{g}_i(\boldsymbol{w}) := \left[ \frac{\psi_i \chi_i}{n_i \tilde{n}_i} \nabla \tilde{n}_i - \frac{\psi_i^2}{n_i \tilde{n}_i^2} \nabla \tilde{n}_i + \frac{Q(\tilde{n}_2 \psi_1 + \tilde{n}_1 \psi_2 + \psi_1 \psi_2)}{n_i} \boldsymbol{u}_i \right] + \left( \frac{\psi_i}{n_i \tilde{n}_i} \nabla \psi_i - \frac{\chi_i}{n_i} \nabla \psi_i \right) - \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i, \tag{4.5b}$$

$$h_i(\boldsymbol{w}) := -\psi_i \chi_{it} - n_i \boldsymbol{u}_i \cdot \nabla \chi_i - \frac{2}{3}(\psi_i \chi_i + \psi_i + \tilde{n}_i \chi_i) \mathrm{div} \boldsymbol{u}_i + \frac{1}{3} n_i |\boldsymbol{u}_i|^2 - \psi_i \chi_i. \tag{4.5c}$$

We shall adopt the continuation principle to prove the global existence in the next section. As we all know, this principle consists of two underlying ingredients: existence of local-in-time solutions and uniform a priori estimates for the local-in-time solutions. Therefore, we first state the local existence result in the following lemma. The proof will be omitted because it is reminiscent of that in [32,44].

**Lemma 4.1** *(Local existence). Suppose that the initial data satisfy* $\partial_t^j \psi_i(0) \in H^{3-j}(\Omega)$ $(j = 0, 1, 2, 3)$, $\partial_t^j \boldsymbol{u}_i(0) \in H^{3-j}(\Omega; \mathbb{R}^3)$ $(j = 0, 1, 2, 3)$, $\partial_t^j \chi_i(0) \in H^{4-j}(\Omega)$ $(j = 0, 1)$, $\partial_t^2 \chi_i(0) \in H^1(\Omega)$ *and the compatibility condition* $\int_\Omega (\psi_{10} - \psi_{20} dx) = 0$, $\partial_t^j \boldsymbol{u}_i(0) \cdot \nu|_{\partial\Omega} = 0$ $(j = 0, 1, 2)$, $\nabla \partial_t^j \chi_i(0) \cdot \nu|_{\partial\Omega} = 0$ $(j = 0, 1)$, *here* $\partial_t^j \psi_i(0)$, $\partial_t^j \boldsymbol{u}_i(0)$ *and* $\partial_t^j \chi_i(0)$ *are the* $j^{th}$ *time derivative at* $t = 0$ *of any solutions to the initial-boundary value problem* (4.2)∼(4.4)*, and their expressions in terms of* $(\psi_{i0}, \boldsymbol{u}_{i0}, \chi_{i0})$ *could be directly calculated out of* (4.2)*. Then there exists a unique solution* $(\psi_i, \boldsymbol{u}_i, \chi_i)$ *to the initial-boundary value problem* (4.2)∼(4.4)*, defined on a maximal interval of existence* $[0, T_{\max})$, $T_{\max} \le \infty$ *such that for any* $T \in [0, T_{\max})$,

$$\psi_i \in \bigcap_{j=0}^{3} C^j([0, T]; H^{3-j}(\Omega)), \quad \boldsymbol{u}_i \in \bigcap_{j=0}^{3} C^j([0, T]; H^{3-j}(\Omega; \mathbb{R}^3)),$$
$$\chi_i \in \bigcap_{j=0}^{1} C^j([0, T]; H^{4-j}(\Omega)), \quad \chi_{itt} \in C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad i = 1, 2. \tag{4.6}$$

*Furthermore, if*

$$\sup_{t \in [0, T_{\max})} \left[ \sum_{i=1}^{2} \left( \sum_{j=0}^{3} \|(\partial_t^j \psi_i, \partial_t^j \boldsymbol{u}_i)(t)\|_{H^{3-j}(\Omega)}^2 + \sum_{j=0}^{1} \|\partial_t^j \chi_i(t)\|_{H^{4-j}(\Omega)}^2 + \|\chi_{itt}(t)\|_{H^1(\Omega)}^2 \right) \right] < \infty \tag{4.7a}$$

*and*

$$\sup_{(t,x) \in [0, T_{\max}) \times \overline{\Omega}} \sum_{i=1}^{2} \left| (\psi_i, \boldsymbol{u}_i, \chi_i)(t, x) \right| < \frac{1}{2} \min\{b, 1\}, \tag{4.7b}$$

*then* $T_{\max} = \infty$.

## 5. Global existence

In this section, we start establishing the uniform estimates of the local solutions, which is the key step to obtain the global existence put forth in Theorem 1.2. Careful examination of the linear part of the perturbed system (4.2) reveals that the thermal equilibrium state is linearly stable. That means the recombination/generation terms do provide the bipolar structure of the system (4.2) with a certain dissipative property. As a matter of fact, the case will become more complicated when we wonder whether the dissipative mechanism of the system (4.2) is strong enough to guarantee that the thermal equilibrium state is non-linearly stable as well. In what follows, we shall figure out how the dissipative structure of the system (4.2) ensures the exponentially non-linear stability of the thermal equilibrium state, and display the corresponding estimates in a series of lemmas. Based on these estimates, the proof of Theorem 1.2 will be given at the end of this section.

For convenience of notation, we introduce a small parameter

$$\epsilon := \|D - d\|_{H^2(\Omega)}, \tag{5.1}$$

and the Sobolev smooth norm

$$|||\boldsymbol{w}(t)|||^2 := \sum_{i=1}^{2}\left(\sum_{j=0}^{3}\|(\partial_t^j\psi_i, \partial_t^j\boldsymbol{u}_i)(t)\|^2_{H^{3-j}(\Omega)} + \sum_{j=0}^{1}\|\partial_t^j\chi_i(t)\|^2_{H^{4-j}(\Omega)} + \|\chi_{itt}(t)\|^2_{H^1(\Omega)}\right). \tag{5.2}$$

Due to the boundary conditions, the spatial derivatives of local-in-time solutions are unknown on the boundary $\partial\Omega$. However, inspired by an idea put forth in [15,40], we may estimate the time derivatives instead. The feasibility of this strategy lies in Proposition 2.4 and the structural features of the system (4.2).

Firstly, we estimate the divergence of the velocity field $\boldsymbol{u}_i$ for $i = 1, 2$ and the higher order time derivatives of their divergence in the following lemma.

**Lemma 5.1.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for $j = 0, 1, 2$,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{2}\int_\Omega \frac{1}{2}\tilde{n}_i\left|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right|^2\mathrm{d}x + \int_\Omega\frac{1}{2}\left(\partial_t^j\psi_1 - \partial_t^j\psi_2\right)^2\mathrm{d}x - \frac{j(j-1)}{2}N(t)\right] \\
&+\frac{1}{4}b\sum_{i=1}^{2}\left\|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right\|^2_{L^2(\Omega)} \\
&-\sum_{i=1}^{2}\int_\Omega\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\cdot\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x - \sum_{i=1}^{2}\int_\Omega\nabla\partial_t^j\chi_i\cdot\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x \\
\le\,& C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\boldsymbol{u}_{it}, \partial_t^j\boldsymbol{u}_i, \nabla\partial_t^j\boldsymbol{u}_i, \nabla\partial_t^j\chi_i, \partial_t^j\chi_i\right)\right\|^2_{L^2(\Omega)} + \frac{j(j-1)}{2}C|||\boldsymbol{w}(t)|||\sum_{i=1}^{2}\left\|\partial_t^3\chi_i\right\|^2_{L^2(\Omega)} \\
&+C|||\boldsymbol{w}(t)|||^3,
\end{aligned}\tag{5.3}$$

*where the index* $e := \frac{2}{i}$ *corresponding to* $i = 1, 2$*; the quantity* $N(t)$ *is defined by*

$$
\begin{aligned}
N(t) :=& \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2n_i^2} (\psi_i - \tilde{n}_i \chi_i) |\nabla \psi_{itt}|^2 \mathrm{d}x \\
&+ \sum_{i=1}^{2} \int_{\Omega} \Bigg\{ \Bigg[ \left(\frac{\chi_i}{n_i}\right)_{tt} - \left(\frac{\psi_i}{\tilde{n}_i n_i}\right)_{tt} \Bigg] \nabla \psi_i \\
&+ 2 \Bigg[ \left(\frac{\chi_i}{n_i}\right)_{t} - \left(\frac{\psi_i}{\tilde{n}_i n_i}\right)_{t} \Bigg] \nabla \psi_{it} \Bigg\} \cdot \nabla \mathrm{div}(\tilde{n}_i \boldsymbol{u}_{it}) \mathrm{d}x,
\end{aligned}
$$

*which in turn satisfies the following inequality*

$$
|N(t)| \leq C \|\boldsymbol{w}(t)\|^3. \tag{5.4}
$$

**Remark 5.1.** From the estimate (5.3), we found that the partial energy $\int_\Omega \frac{1}{2}(\partial_t^j \psi_1 - \partial_t^j \psi_2)^2 \mathrm{d}x$ gleaned from the bipolar structure of the system (4.2) is inoperative in comparison with the one in (5.57). However, it is harmless because of the nonnegativity. This is also the reason why we manage to gain the effective energy for the perturbed densities $\psi_1$ and $\psi_2$ from their continuity equations later (see Lemma 5.6).

**Proof.** First of all, in the following way

$$
\sum_{i=1}^{2} \int_{\Omega} \partial_t^j (4.2\mathrm{b}) \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x, \quad \text{for } j = 0, 1, 2, \tag{5.5}
$$

we calculate

$$
\begin{aligned}
&\sum_{i=1}^{2} \int_{\Omega} \partial_t^j \boldsymbol{u}_{it} \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} \partial_t^j \boldsymbol{u}_i \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x \\
&+ \sum_{i=1}^{2} \int_{\Omega} \nabla (\tilde{n}_e \partial_t^j \psi_i) \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} \tilde{n}_e \nabla \tilde{n}_i \partial_t^j \chi_i \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x \\
&+ \sum_{i=1}^{2} \int_{\Omega} \nabla \partial_t^j \chi_i \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} (-1)^i \nabla \partial_t^j \sigma \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x \\
&= \sum_{i=1}^{2} \int_{\Omega} \partial_t^j \boldsymbol{g}_i(\boldsymbol{w}) \cdot \left[ -\nabla \mathrm{div} \left( \tilde{n}_i \partial_t^j \boldsymbol{u}_i \right) \right] \mathrm{d}x, \quad \text{for } j = 0, 1, 2,
\end{aligned} \tag{5.6}
$$

In addition, we scrutinize (5.6) one integral at a time. To start with, we consider the first integral on the left-hand side of (5.6). After integrating by parts, we obtain for $j = 0, 1, 2$,

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\partial_t^j \boldsymbol{u}_{it}\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\mathrm{div}\partial_t^j\boldsymbol{u}_{it}\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\tilde{n}_i\left|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right|^2\mathrm{d}x\right)+\sum_{i=1}^{2}\int_{\Omega}\mathrm{div}\partial_t^j\boldsymbol{u}_{it}\nabla\tilde{n}_i\cdot\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
\geq&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\tilde{n}_i\left|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right|^2\mathrm{d}x\right)-C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\boldsymbol{u}_i,\partial_t^j\boldsymbol{u}_{it},\nabla\partial_t^j\boldsymbol{u}_i\right)\right\|_{L^2(\Omega)}^2,
\end{aligned}\tag{5.7}$$

where we have used the flatness estimate (1.17a) of the steady-state densities, Sobolev's Embedding Theorem and the Cauchy-Schwarz inequality to further control the integral remainder. Similarly, we proceed to estimate the second and the fourth integrals on the left-hand side of (5.6) below, for $j=0,1,2$, we have

$$\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\boldsymbol{u}_i\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\geq\frac{b}{2}\sum_{i=1}^{2}\left\|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2-C\epsilon\sum_{i=1}^{2}\left\|\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2,\tag{5.8}$$

and

$$\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_e\nabla\tilde{n}_i\partial_t^j\chi_i\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\geq-C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\boldsymbol{u}_i,\mathrm{div}\partial_t^j\boldsymbol{u}_i,\partial_t^j\chi_i,\nabla\partial_t^j\chi_i\right)\right\|_{L^2(\Omega)}^2.\tag{5.9}$$

And as for the third and the fifth integrals on the left-hand side of (5.6), we keep them both unchanged because they can be canceled out by the corresponding integral appearing respectively in another two estimates later on, see (5.33) and (5.40). Concerning the last integral on the left-hand side of (5.6), we have to pay more attention to it due to the influence of bipolar structure and net recombination rate. Precisely, integrating by parts yields that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}(-1)^i\nabla\partial_t^j\sigma\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
=&\int_{\Omega}\Delta\partial_t^j\sigma\sum_{i=1}^{2}(-1)^i\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x\\
=&\int_{\Omega}\left(\partial_t^j\psi_1-\partial_t^j\psi_2\right)\sum_{i=1}^{2}(-1)^i\left[-\partial_t^j\psi_{it}-Q\left(\tilde{n}_2\partial_t^j\psi_1+\tilde{n}_1\partial_t^j\psi_2\right)+\partial_t^j f_i(\boldsymbol{w})\right]\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\left(\partial_t^j\psi_1 - \partial_t^j\psi_2\right)^2 \mathrm{d}x - \int_\Omega \nabla\left(\partial_t^j\psi_1 - \partial_t^j\psi_2\right)\cdot\left[\partial_t^j(\psi_1\boldsymbol{u}_1) - \partial_t^j(\psi_2\boldsymbol{u}_2)\right]\mathrm{d}x \\
&\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\left(\partial_t^j\psi_1 - \partial_t^j\psi_2\right)^2 \mathrm{d}x - C|||\boldsymbol{w}(t)|||^3, \quad \text{for } j = 0, 1, 2, \qquad (5.10)
\end{aligned}
$$

where we have employed the equations $\partial_t^j$ (4.2a) and $\partial_t^j$ (4.2d) when obtaining the second equality of (5.10); and have utilized the expression (4.5a) of the nonlinearity term $f_i(\boldsymbol{w})$ and the slip boundary condition $\boldsymbol{u}_i \cdot \nu|_{\partial\Omega} = 0$ while establishing the third equality of (5.10).

Note that all the integrals calculated above are produced by the linear part of the velocity equation (4.2b), though there develops kind of the nonlinearity in (5.10). From now on, we shall face the genuinely nonlinear part of the velocity equation (4.2b), namely we take up estimating the integral on the right-hand side of (5.6). Firstly, the nonlinear terms of $\boldsymbol{g}_i(\boldsymbol{w})$ can be grouped into three sorts by their common feature manifested in the process of estimation,

$$
\begin{aligned}
\boldsymbol{g}_{i1}(\boldsymbol{w}) &:= \frac{\psi_i\chi_i}{n_i\tilde{n}_i}\nabla\tilde{n}_i - \frac{\psi_i^2}{n_i\tilde{n}_i^2}\nabla\tilde{n}_i + \frac{Q(\tilde{n}_2\psi_1 + \tilde{n}_1\psi_2 + \psi_1\psi_2)}{n_i}\boldsymbol{u}_i, \\
\boldsymbol{g}_{i2}(\boldsymbol{w}) &:= \frac{\psi_i}{n_i\tilde{n}_i}\nabla\psi_i - \frac{\chi_i}{n_i}\nabla\psi_i, \\
\boldsymbol{g}_{i3}(\boldsymbol{w}) &:= -\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i, \quad \text{for } i = 1, 2.
\end{aligned}
$$

Then we have got, for $j = 0, 1, 2$,

$$
\begin{aligned}
&\sum_{i=1}^2\int_\Omega \partial_t^j\boldsymbol{g}_i(\boldsymbol{w})\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x \\
&= \sum_{i=1}^2\int_\Omega \partial_t^j\boldsymbol{g}_{i1}(\boldsymbol{w})\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x + \sum_{i=1}^2\int_\Omega \partial_t^j\boldsymbol{g}_{i2}(\boldsymbol{w})\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x \\
&\quad + \sum_{i=1}^2\int_\Omega \partial_t^j\boldsymbol{g}_{i3}(\boldsymbol{w})\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x \\
&= I_1^{(j)} + I_2^{(j)} + I_3^{(j)}. \qquad (5.11)
\end{aligned}
$$

Since we observe that $\nabla\tilde{n}_i \cdot \nu|_{\partial\Omega} = 0$ and $\partial_t^j\boldsymbol{u}_i \cdot \nu|_{\partial\Omega} = 0$, the main property of the integral $I_1^{(j)}$ is that the boundary integral vanishes under integration by parts. To avoid redundancy, we only choose the first term in $I_1^{(2)}$ as an example to illustrate the necessary skills in estimating all the integrals $I_1^{(j)}$ for $j = 0, 1, 2$. We compute therefore by using integration by parts

$$
\sum_{i=1}^2\int_\Omega \partial_t^2\left(\frac{\psi_i\chi_i}{n_i\tilde{n}_i}\nabla\tilde{n}_i\right)\cdot\left[-\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\right]\mathrm{d}x
$$

$$
\begin{aligned}
&=\sum_{i=1}^{2}\int_{\Omega}\nabla\cdot\partial_t^2\left(\frac{\psi_i\chi_i}{n_i\tilde{n}_i}\nabla\tilde{n}_i\right)\operatorname{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\mathrm{d}x\\
&=\sum_{i=1}^{2}\int_{\Omega}\left[\frac{\tilde{n}_i\Delta\tilde{n}_i-|\nabla\tilde{n}_i|^2}{\tilde{n}_i^2}\partial_t^2\left(\frac{\psi_i\chi_i}{n_i}\right)+\frac{\nabla\tilde{n}_i}{\tilde{n}_i}\cdot\nabla\partial_t^2\left(\frac{\psi_i\chi_i}{n_i}\right)\right]\left(\tilde{n}_i\operatorname{div}\partial_t^2\boldsymbol{u}_i+\nabla\tilde{n}_i\cdot\partial_t^2\boldsymbol{u}_i\right)\mathrm{d}x\\
&=\sum_{i=1}^{2}\int_{\Omega}\left(\nabla\tilde{n}_i\cdot\frac{\chi_i}{n_i}\nabla\partial_t^2\psi_i\operatorname{div}\partial_t^2\boldsymbol{u}_i+\text{the remaining third order terms}\right)\mathrm{d}x+\text{lower order terms}\\
&\le C\sum_{i=1}^{2}\|\nabla\tilde{n}_i\|_{L^\infty(\Omega)}\|\chi_i\|_{L^\infty(\Omega)}\left\|\nabla\partial_t^2\psi_i\right\|_{L^2(\Omega)}\left\|\operatorname{div}\partial_t^2\boldsymbol{u}_i\right\|_{L^2(\Omega)}+\text{the remaining terms}\\
&\le C|||\boldsymbol{w}(t)|||^3 \qquad (5.12)
\end{aligned}
$$

provided $\epsilon+|||\boldsymbol{w}(t)|||\ll 1$. The integral $I_1^{(j)}$, as is the case in (5.12), can be easily estimated below

$$
I_1^{(j)}\le C|||\boldsymbol{w}(t)|||^3,\quad \text{for } j=0,1,2. \qquad (5.13)
$$

The strategy to handle the integral $I_2^{(j)}$ actually depends on the order of time derivative. For $j=0,1$, in much the same way as (5.13), the integral $I_2^{(j)}$ can be directly estimated below

$$
I_2^{(j)}\le C|||\boldsymbol{w}(t)|||^3,\quad \text{for } j=0,1. \qquad (5.14)
$$

For $j=2$, we walk one through computing merely the second term in the integral $I_2^{(2)}$ because it has extra information on temperature unknowns $\chi_i$ and the first term can be calculated similarly, that is

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\left(-\frac{\chi_i}{n_i}\nabla\psi_i\right)\cdot\left[-\nabla\operatorname{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\left[\left(\frac{\chi_i}{n_i}\right)_{tt}\nabla\psi_i+2\left(\frac{\chi_i}{n_i}\right)_t\nabla\psi_{it}\right]\cdot\partial_t\nabla\operatorname{div}(\tilde{n}_i\boldsymbol{u}_{it})\,\mathrm{d}x\\
&+\sum_{i=1}^{2}\int_{\Omega}\frac{\chi_i}{n_i}\nabla\psi_{itt}\cdot\partial_t^2\nabla\operatorname{div}(\tilde{n}_i\boldsymbol{u}_i)\,\mathrm{d}x\\
=&J_1+J_2. \qquad (5.15)
\end{aligned}
$$

And then, we compute

$$
J_1=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\left[\left(\frac{\chi_i}{n_i}\right)_{tt}\nabla\psi_i+2\left(\frac{\chi_i}{n_i}\right)_t\nabla\psi_{it}\right]\cdot\nabla\operatorname{div}(\tilde{n}_i\boldsymbol{u}_{it})\,\mathrm{d}x
$$

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_{\Omega}\left[\left(\frac{\chi_i}{n_i}\right)_{ttt}\nabla\psi_i+3\left(\frac{\chi_i}{n_i}\right)_{tt}\nabla\psi_{it}+2\left(\frac{\chi_i}{n_i}\right)_{t}\nabla\psi_{itt}\right]\cdot\nabla\mathrm{div}\,(\tilde{n}_i\boldsymbol{u}_{it})\,\mathrm{d}x\\
&\le\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\left[\left(\frac{\chi_i}{n_i}\right)_{tt}\nabla\psi_i+2\left(\frac{\chi_i}{n_i}\right)_{t}\nabla\psi_{it}\right]\cdot\nabla\mathrm{div}\,(\tilde{n}_i\boldsymbol{u}_{it})\,\mathrm{d}x+C|||\boldsymbol{w}(t)|||\sum_{i=1}^{2}\left\|\partial_t^3\chi_i\right\|_{L^2(\Omega)}^2\\
&\quad+C|||\boldsymbol{w}(t)|||^3,
\end{aligned}
\tag{5.16}
$$

where, in order to estimate the term in the second line of (5.16), we have used Proposition 2.5 and also carried out the following calculations by the quotient rule (2.4), for $m=1,2,3$,

$$
\begin{aligned}
\left|\partial_t^m\left(\frac{\chi_i}{n_i}\right)\right|&=\left|m!\sum_{l=0}^{m}\frac{\partial_t^{m-l}\chi_i}{(m-l)!}\sum_{k=1}^{P(l)}\frac{c_k}{n_i^{1+\sum_{q=1}^{l}p_{kq}}}\prod_{q=1}^{l}\left(\frac{\partial_t^q\psi_i}{q!}\right)^{p_{kq}}\right|\\
&\le\begin{cases}C|||\boldsymbol{w}(t)|||, & \text{if } m=1,\\ C|||\boldsymbol{w}(t)|||^2+C|||\boldsymbol{w}(t)|||\,|\psi_{itt}|+C\,|\chi_{itt}|, & \text{if } m=2,\\ C|||\boldsymbol{w}(t)|||^3+C|||\boldsymbol{w}(t)|||\,|(\psi_{ittt},\psi_{itt},\chi_{itt})|+C\,|\chi_{ittt}|, & \text{if } m=3,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla\mathrm{div}\,(\tilde{n}_i\boldsymbol{u}_{it})|&=|\tilde{n}_i\nabla\mathrm{div}\boldsymbol{u}_{it}+\nabla\tilde{n}_i\mathrm{div}\boldsymbol{u}_{it}+(\boldsymbol{u}_{it}\cdot\nabla)\,\nabla\tilde{n}_i+(\nabla\tilde{n}_i\cdot\nabla)\,\boldsymbol{u}_{it}+\nabla\tilde{n}_i\times(\nabla\times\boldsymbol{u}_{it})|\\
&\le C\epsilon\,|(\boldsymbol{u}_{it},\nabla\boldsymbol{u}_{it})|+C\,|\nabla\mathrm{div}\boldsymbol{u}_{it}|\,.
\end{aligned}
$$

Moreover, we calculate

$$
\begin{aligned}
J_2&=\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i\chi_i}{n_i}\nabla\psi_{itt}\cdot\partial_t^2\nabla\mathrm{div}\boldsymbol{u}_i\mathrm{d}x\\
&\quad+\sum_{i=1}^{2}\int_{\Omega}\frac{\chi_i}{n_i}\nabla\psi_{itt}\cdot\left[\nabla\tilde{n}_i\mathrm{div}\boldsymbol{u}_{itt}+(\boldsymbol{u}_{itt}\cdot\nabla)\,\nabla\tilde{n}_i+(\nabla\tilde{n}_i\cdot\nabla)\,\boldsymbol{u}_{itt}+\nabla\tilde{n}_i\times(\nabla\times\boldsymbol{u}_{itt})\right]\mathrm{d}x\\
&\le-\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i\chi_i}{n_i}\nabla\psi_{itt}\cdot\partial_t^2\nabla\left[\frac{\psi_{it}+\boldsymbol{u}_i\cdot\nabla(\psi_i+\tilde{n}_i)+Q(\tilde{n}_2\psi_1+\tilde{n}_1\psi_2+\psi_1\psi_2)}{n_i}\right]\mathrm{d}x\\
&\quad+C|||\boldsymbol{w}(t)|||^3\\
&\le-\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i\chi_i}{n_i^2}\nabla\psi_{itt}\cdot\partial_t^2\nabla(\psi_{it}+\boldsymbol{u}_i\cdot\nabla\psi_i)\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
&=-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i\chi_i}{2n_i^2}\,|\nabla\psi_{itt}|^2\,\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i}{2}\left(\frac{\chi_i}{n_i^2}\right)_t|\nabla\psi_{itt}|^2\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{2}\int_{\Omega}\operatorname{div}\left(\frac{\tilde{n}_i\chi_i\boldsymbol{u}_i}{2n_i^2}\right)|\nabla\psi_{itt}|^2\,\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
&\le-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i\chi_i}{2n_i^2}|\nabla\psi_{itt}|^2\,\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3, \qquad (5.17)
\end{aligned}
$$

where, in the third line of (5.17), we have used the continuity equation (4.2a) to deal with the term in the first line and applied the Cauchy-Schwarz inequality to the term in the second line. Substituting (5.16) and (5.17) into (5.15), we have

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\left(-\frac{\chi_i}{n_i}\nabla\psi_i\right)\cdot\left[-\nabla\operatorname{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
&\le\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\left\{-\int_{\Omega}\frac{\tilde{n}_i\chi_i}{2n_i^2}|\nabla\psi_{itt}|^2\,\mathrm{d}x+\int_{\Omega}\left[\left(\frac{\chi_i}{n_i}\right)_{tt}\nabla\psi_i+2\left(\frac{\chi_i}{n_i}\right)_t\nabla\psi_{it}\right]\cdot\nabla\operatorname{div}\left(\tilde{n}_i\boldsymbol{u}_{it}\right)\mathrm{d}x\right\}\\
&\quad+C|||\boldsymbol{w}(t)|||\sum_{i=1}^{2}\left\|\partial_t^3\chi_i\right\|_{L^2(\Omega)}^2+C|||\boldsymbol{w}(t)|||^3. \qquad (5.18)
\end{aligned}
$$

The same goes for the first term of $I_2^{(2)}$, thereby replacing $\chi_i$ by $-\frac{\psi_i}{\tilde{n}_i}$ in (5.18), we get

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\left(\frac{\psi_i}{n_i\tilde{n}_i}\nabla\psi_i\right)\cdot\left[-\nabla\operatorname{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
&\le\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\left\{\int_{\Omega}\frac{\psi_i}{2n_i^2}|\nabla\psi_{itt}|^2\,\mathrm{d}x-\int_{\Omega}\left[\left(\frac{\psi_i}{\tilde{n}_in_i}\right)_{tt}\nabla\psi_i+2\left(\frac{\psi_i}{\tilde{n}_in_i}\right)_t\nabla\psi_{it}\right]\cdot\nabla\operatorname{div}\left(\tilde{n}_i\boldsymbol{u}_{it}\right)\mathrm{d}x\right\}\\
&\quad+C|||\boldsymbol{w}(t)|||^3. \qquad (5.19)
\end{aligned}
$$

Summing up (5.18) and (5.19), together with (5.14), we now arrive at

$$
I_2^{(j)}\le\frac{j(j-1)}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}N(t)+C|||\boldsymbol{w}(t)|||\sum_{i=1}^{2}\left\|\partial_t^3\chi_i\right\|_{L^2(\Omega)}^2\right)+C|||\boldsymbol{w}(t)|||^3,\quad\text{for } j=0,1,2. \qquad (5.20)
$$

Based on the calculations and techniques we have developed, one can easily see that the estimate (5.4) is true.

Finally, we start dealing with the last nonlinear integral $I_3^{(j)}$. Similarly to $I_2^{(j)}$, we easily get

$$
I_3^{(j)}\le C|||\boldsymbol{w}(t)|||^3,\quad\text{for } j=0,1. \qquad (5.21)
$$

As far as $I_3^{(2)}$ is concerned, we calculate

$$
\begin{aligned}
I_3^{(2)} =&\sum_{i=1}^{2}\int_{\Omega} \partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^2\boldsymbol{u}_i\right)\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\nabla\mathrm{div}\partial_t^2\boldsymbol{u}_i\mathrm{d}x\\
&+\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\left[\nabla\tilde{n}_i\mathrm{div}\boldsymbol{u}_{itt}+\left(\boldsymbol{u}_{itt}\cdot\nabla\right)\nabla\tilde{n}_i+\left(\nabla\tilde{n}_i\cdot\nabla\right)\boldsymbol{u}_{itt}\right.\\
&\left.+\nabla\tilde{n}_i\times\left(\nabla\times\boldsymbol{u}_{itt}\right)\right]\mathrm{d}x\\
\leq&\sum_{i=1}^{2}\int_{\partial\Omega}\tilde{n}_i\mathrm{div}\partial_t^2\boldsymbol{u}_i\left[\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\nu\right]\mathrm{d}s-\sum_{i=1}^{2}\int_{\Omega}\mathrm{div}\left[\tilde{n}_i\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\right]\mathrm{div}\partial_t^2\boldsymbol{u}_i\mathrm{d}x\\
&+C\|\!|\boldsymbol{w}(t)|\!\|^3\\
\leq&J_{bdr}-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\partial_t^2\mathrm{div}\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\mathrm{div}\partial_t^2\boldsymbol{u}_i\mathrm{d}x+C\|\!|\boldsymbol{w}(t)|\!\|^3\\
=&J_{bdr}-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\partial_t^2\left[\boldsymbol{u}_i\cdot\nabla\mathrm{div}\boldsymbol{u}_i+\nabla\boldsymbol{u}_i:\left(\nabla\boldsymbol{u}_i\right)^T\right]\mathrm{div}\partial_t^2\boldsymbol{u}_i\mathrm{d}x+C\|\!|\boldsymbol{w}(t)|\!\|^3\\
\leq&J_{bdr}-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\left(\boldsymbol{u}_i\cdot\nabla\mathrm{div}\partial_t^2\boldsymbol{u}_i\right)\mathrm{div}\partial_t^2\boldsymbol{u}_i\mathrm{d}x+C\|\!|\boldsymbol{w}(t)|\!\|^3\\
=&J_{bdr}+\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\mathrm{div}(\tilde{n}_i\boldsymbol{u}_i)\left|\mathrm{div}\partial_t^2\boldsymbol{u}_i\right|^2\mathrm{d}x+C\|\!|\boldsymbol{w}(t)|\!\|^3\\
\leq&J_{bdr}+C\|\!|\boldsymbol{w}(t)|\!\|^3.
\end{aligned}
\tag{5.22}
$$

There remains a boundary integral $J_{bdr}$ still to be controlled. For this purpose, we borrow a useful trick from [15], which is further developed by introducing the tangent vector $\boldsymbol{u}^{\perp} := -\boldsymbol{u}\times\nu$ in [7]. The trick makes the most of the slip boundary condition $\boldsymbol{u}_i\cdot\nu|_{\partial\Omega}=0$ and also enables us, on the boundary $\partial\Omega$, to transfer one order of spatial derivative from the velocity field $\boldsymbol{u}_i(t,x)$ to the inwards-extended outer normal vector $\nu=\nu(x)$. More precisely, we have the following computation

$$
\begin{aligned}
J_{bdr}=&\sum_{i=1}^{2}\int_{\partial\Omega}\tilde{n}_i\mathrm{div}\partial_t^2\boldsymbol{u}_i\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\cdot\nu\right)\mathrm{d}s\\
=&-\sum_{i=1}^{2}\int_{\partial\Omega}\tilde{n}_i\partial_t^2\mathrm{div}\left(\boldsymbol{u}_i^{\perp}\times\nu\right)\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\nu\cdot\boldsymbol{u}_i\right)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \int_{\partial\Omega} \tilde{n}_i \partial_t^2 \left( \nabla \times \nu \cdot \boldsymbol{u}_i^{\perp} \right) \partial_t^2 \left( \boldsymbol{u}_i \cdot \nabla \nu \cdot \boldsymbol{u}_i \right) \mathrm{d}s \\
&\le C |||\boldsymbol{w}(t)|||^3, \qquad (5.23)
\end{aligned}
$$

where we have used the slip boundary condition $\boldsymbol{u}_i \cdot \nu|_{\partial\Omega} = 0$ in obtaining the second equality of (5.23). From the third equality of (5.23), one can see that there are at most second-order derivatives inside the boundary integral, thereby establishing the last inequality of (5.23) via the Trace Theorem in $H^1(\Omega)$. Substituting (5.23) into (5.22), in combination with (5.21), we now achieve

$$
I_3^{(j)} \le C |||\boldsymbol{w}(t)|||^3, \quad \text{for } j = 0, 1, 2. \tag{5.24}
$$

Combining the estimates (5.7)∼(5.11), (5.13), (5.20) and (5.24) together, we now obtain the desired estimate (5.3). □

**Remark 5.2.** It is worth mentioning that the difficulty of controlling the boundary integral is brought about by the fact that the convection term $\boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i$ encounters the slip boundary condition $\boldsymbol{u}_i \cdot \nu|_{\partial\Omega} = 0$. The same trouble, incidentally, also appears in the nonlinear stability analysis on the unipolar isentropic hydrodynamic model of semiconductors [15] and the compressible Naiver-Stokes equations [7]. Inspired by [15,7], the typical boundary integral here is handled in the same fashion. Also, it should be pointed out that this problem will not occur in the case that one considers the linear stability no matter what kind of boundary conditions one chooses, and in the case of the nonlinear stability under the periodic boundary conditions.

As we have mentioned before, in order to apply the Div-Curl Lemma (2.5) to complete the estimation of $\nabla \boldsymbol{u}_i$ for $i = 1, 2$ and their higher order time-space derivatives, the estimation of the vorticity $\mathrm{curl}\boldsymbol{u}_i$ for $i = 1, 2$ and their higher order time-space derivatives remains to be done. In the next lemma, we merely estimate the vorticity $\mathrm{curl}\boldsymbol{u}_i$ for $i = 1, 2$ and their higher order time derivatives. Plus we leave the necessary estimation about the time-space derivatives of the vorticity to Lemma 5.10 because we need to decide the necessary order of the derivatives according to what kind of the estimation for the perturbed densities we could have.

**Lemma 5.2.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for $j = 0, 1, 2$,*

$$
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2} \tilde{n}_i \left| \mathrm{curl} \partial_t^j \boldsymbol{u}_i \right|^2 \mathrm{d}x + \frac{b}{2} \sum_{i=1}^{2} \left\| \mathrm{curl} \partial_t^j \boldsymbol{u}_i \right\|_{L^2(\Omega)}^2 \le C\epsilon \sum_{i=1}^{2} \left\| \nabla \partial_t^j \chi_i \right\|_{L^2(\Omega)}^2 + C |||\boldsymbol{w}(t)|||^3. \tag{5.25}
$$

**Proof.** Conducting the following operation

$$
\sum_{i=1}^{2} \int_{\Omega} \mathrm{curl} \partial_t^j (4.2\mathrm{b}) \cdot \left[ \tilde{n}_i \mathrm{curl} \partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x, \quad \text{for } j = 0, 1, 2, \tag{5.26}
$$

we have for $j = 0, 1, 2$,

$$\sum_{i=1}^{2} \int_{\Omega} \operatorname{curl}\partial_t^j (\boldsymbol{u}_{it} + \boldsymbol{u}_i) \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x - \sum_{i=1}^{2} \int_{\Omega} \left[ \nabla \tilde{n}_i \times \nabla \left( \tilde{n}_e \partial_t^j \chi_i \right) \right] \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x$$
$$= \sum_{i=1}^{2} \int_{\Omega} \operatorname{curl}\partial_t^j \boldsymbol{g}_i(\boldsymbol{w}) \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x. \tag{5.27}$$

The first integral on the left-hand side of (5.27) is routinely dealt with as follows, for $j = 0, 1, 2$,

$$\sum_{i=1}^{2} \int_{\Omega} \operatorname{curl}\partial_t^j (\boldsymbol{u}_{it} + \boldsymbol{u}_i) \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x$$
$$\geq \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2} \tilde{n}_i \left| \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right|^2 \mathrm{d}x + b \sum_{i=1}^{2} \left\| \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right\|_{L^2(\Omega)}^2. \tag{5.28}$$

To dominate the other integral on the left-hand side of (5.27), we need the special properties of the density components of the equilibrium state, that is, for $i = 1, 2$,

$$\tilde{n}_i \tilde{n}_e \equiv 1, \text{ which in turn implies, } \nabla \tilde{n}_i \times \nabla \tilde{n}_e \equiv 0. \tag{5.29}$$

Then we calculate for $j = 0, 1, 2$,

$$- \sum_{i=1}^{2} \int_{\Omega} \left[ \nabla \tilde{n}_i \times \nabla \left( \tilde{n}_e \partial_t^j \chi_i \right) \right] \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x$$
$$= - \sum_{i=1}^{2} \int_{\Omega} \left[ (\nabla \tilde{n}_i \times \nabla \tilde{n}_e) \partial_t^j \chi_i + \left( \nabla \tilde{n}_i \times \nabla \partial_t^j \chi_i \right) \tilde{n}_e \right] \cdot \left[ \tilde{n}_i \operatorname{curl}\partial_t^j \boldsymbol{u}_i \right] \mathrm{d}x$$
$$= - \sum_{i=1}^{2} \int_{\Omega} \tilde{n}_i \tilde{n}_e \left( \nabla \tilde{n}_i \times \nabla \partial_t^j \chi_i \right) \cdot \operatorname{curl}\partial_t^j \boldsymbol{u}_i \mathrm{d}x$$
$$= \sum_{i=1}^{2} \int_{\Omega} \nabla \tilde{n}_i \cdot \left( \operatorname{curl}\partial_t^j \boldsymbol{u}_i \times \nabla \partial_t^j \chi_i \right) \mathrm{d}x$$
$$\geq - C\epsilon \sum_{i=1}^{2} \left\| \left( \operatorname{curl}\partial_t^j \boldsymbol{u}_i, \nabla \partial_t^j \chi_i \right) \right\|_{L^2(\Omega)}^2 \tag{5.30}$$

The nonlinear integral on the right-hand side of (5.27) can be dominated by $|||\boldsymbol{w}(t)|||^3$ because the order of time derivatives is at most 2. Actually, we have

$$\sum_{i=1}^{2}\int_{\Omega} \mathrm{curl}\partial_t^j \boldsymbol{g}_i(\boldsymbol{w})\cdot\left[\tilde{n}_i\mathrm{curl}\partial_t^j\boldsymbol{u}_i\right]\mathrm{d}x \le C|||\boldsymbol{w}(t)|||^3, \quad \text{for } j=0,1,2. \tag{5.31}$$

We only pick up the most complicated part $\boldsymbol{g}_{i3}(\boldsymbol{w})$ of the above nonlinear integral in the case $j=2$ to go into details, which is attributed to the convection terms. The remaining parts and cases can be estimated in much the same way but easier computations.

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega} \mathrm{curl}\partial_t^2 \boldsymbol{g}_{i3}(\boldsymbol{w})\cdot\left[\tilde{n}_i\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right]\mathrm{d}x\\
=&-\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\mathrm{curl}\left(\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\left[\tilde{n}_i\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right]\mathrm{d}x\\
=&-\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\mathrm{curl}\left(\frac{1}{2}\nabla|\boldsymbol{u}_i|^2-\boldsymbol{u}_i\times\mathrm{curl}\boldsymbol{u}_i\right)\cdot\left[\tilde{n}_i\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right]\mathrm{d}x\\
=&-\sum_{i=1}^{2}\int_{\Omega}\partial_t^2\left(\boldsymbol{u}_i\cdot\nabla\mathrm{curl}\boldsymbol{u}_i+\mathrm{div}\boldsymbol{u}_i\mathrm{curl}\boldsymbol{u}_i-\mathrm{curl}\boldsymbol{u}_i\cdot\nabla\boldsymbol{u}_i\right)\cdot\left[\tilde{n}_i\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right]\mathrm{d}x\\
\le&-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\boldsymbol{u}_i\cdot\nabla\mathrm{curl}\partial_t^2\boldsymbol{u}_i\cdot\mathrm{curl}\partial_t^2\boldsymbol{u}_i\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
=&-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\boldsymbol{u}_i\cdot\nabla\left(\frac{1}{2}\left|\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right|^2\right)\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
=&\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\mathrm{div}\left(\tilde{n}_i\boldsymbol{u}_i\right)\left|\mathrm{curl}\partial_t^2\boldsymbol{u}_i\right|^2\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
\le&C|||\boldsymbol{w}(t)|||^3.
\end{aligned}\tag{5.32}$$

Substituting (5.28), (5.30) and (5.31) into (5.27), together with the smallness of $\epsilon$, we get the desired estimate (5.25). $\square$

Motivated by the estimates in Lemmas 5.1 and 5.2, we set about proving the following two lemmas. These two lemmas together can achieve two purposes at one stroke because we can not only capture the energy for $\nabla\partial_t^j\chi_i$ and $\nabla\partial_t^j\psi_i$ but also construct the integral terms to cancel out the corresponding ones in Lemma 5.1.

**Lemma 5.3.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C>0$ such that for $j=0,1,2$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{3}{4}\tilde{n}_i\left|\nabla\partial_t^j\chi_i\right|^2\mathrm{d}x+\frac{3}{8}b\sum_{i=1}^{2}\left\|\nabla\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2+\frac{1}{2}\sum_{i=1}^{2}\left\|\Delta\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2$$

$$+\sum_{i=1}^{2}\int_{\Omega}\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\cdot\nabla\partial_t^j\chi_i\mathrm{d}x\le C\left(\epsilon+|||\boldsymbol{w}(t)|||\right)\sum_{i=1}^{2}\left\|\left(\partial_t^j\boldsymbol{u}_i,\partial_t^j\chi_i,\partial_t^j\chi_{it}\right)\right\|_{L^2(\Omega)}^2$$
$$+C|||\boldsymbol{w}(t)|||^3. \tag{5.33}$$

**Proof.** Careful observation of the estimate (5.3), we implement the following operation

$$\sum_{i=1}^{2}\int_{\Omega}\partial_t^j(4.2\mathrm{c})\left(-\frac{3}{2}\Delta\partial_t^j\chi_i\right)\mathrm{d}x,\quad \text{for } j=0,1,2, \tag{5.34}$$

which yields

$$-\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\tilde{n}_i\partial_t^j\chi_{it}\Delta\partial_t^j\chi_i\mathrm{d}x-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\Delta\partial_t^j\chi_i\mathrm{d}x+\sum_{i=1}^{2}\left\|\Delta\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2$$
$$-\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\tilde{n}_i\partial_t^j\chi_i\Delta\partial_t^j\chi_i\mathrm{d}x=\sum_{i=1}^{2}\int_{\Omega}\partial_t^j h_i(\boldsymbol{w})\left(-\frac{3}{2}\Delta\partial_t^j\chi_i\right)\mathrm{d}x,\quad \text{for } j=0,1,2. \tag{5.35}$$

And then we begin calculating the integrals of the linear principal part on the left-hand side of (5.35) one by one. For $j=0,1,2$, we have

$$\begin{aligned}&-\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\tilde{n}_i\partial_t^j\chi_{it}\Delta\partial_t^j\chi_i\mathrm{d}x\\&=\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\nabla\left(\tilde{n}_i\partial_t^j\chi_{it}\right)\cdot\nabla\partial_t^j\chi_i\mathrm{d}x\\&\ge\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{3}{4}\tilde{n}_i\left|\nabla\partial_t^j\chi_i\right|^2\mathrm{d}x-C\epsilon\sum_{i=1}^{2}\left\|\left(\nabla\partial_t^j\chi_i,\partial_t^j\chi_{it}\right)\right\|_{L^2(\Omega)}^2,\end{aligned} \tag{5.36}$$

and then we have to technically construct a specific integral form in processing the second integral on the left-hand side of (5.35), that is

$$\begin{aligned}&-\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\Delta\partial_t^j\chi_i\mathrm{d}x\\&=-\sum_{i=1}^{2}\int_{\Omega}\left[\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)-\nabla\tilde{n}_i\cdot\partial_t^j\boldsymbol{u}_i\right]\Delta\partial_t^j\chi_i\mathrm{d}x\\&\ge\sum_{i=1}^{2}\int_{\Omega}\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\cdot\nabla\partial_t^j\chi_i\mathrm{d}x-C\epsilon\sum_{i=1}^{2}\left\|\left(\Delta\partial_t^j\chi_i,\partial_t^j\boldsymbol{u}_i\right)\right\|_{L^2(\Omega)}^2,\end{aligned} \tag{5.37}$$

and now we turn to the last integral on the left-hand side of (5.35), namely

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\tilde{n}_i\partial_t^j\chi_i\Delta\partial_t^j\chi_i\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\frac{3}{2}\nabla\left(\tilde{n}_i\partial_t^j\chi_i\right)\cdot\nabla\partial_t^j\chi_i\mathrm{d}x\\
\geq&\frac{3}{4}b\sum_{i=1}^{2}\left\|\nabla\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2-C\epsilon\sum_{i=1}^{2}\left\|\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{5.38}
$$

Scrutinizing each terms in the expression (4.5c) of the nonlinearity $h_i(\boldsymbol{w})$, one can easily find that, by using the routine techniques repeatedly employed in Lemma 5.1, the integral on the right-hand side of (5.35) is able to be dominated as follows, for $j=0,1,2$,

$$
\sum_{i=1}^{2}\int_{\Omega}\partial_t^j h_i(\boldsymbol{w})\left(-\frac{3}{2}\Delta\partial_t^j\chi_i\right)\mathrm{d}x\leq C|||\boldsymbol{w}(t)|||\sum_{i=1}^{2}\left\|\left(\partial_t^j\chi_{it},\Delta\partial_t^j\chi_i\right)\right\|_{L^2(\Omega)}^2+C|||\boldsymbol{w}(t)|||^3.
\tag{5.39}
$$

Inserting (5.36)∼(5.39) into (5.35), and letting $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be small enough, we complete the proof of the desired estimate (5.33). □

**Lemma 5.4.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C>0$ such that for $j=0,1,2$,*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\tilde{n}_e\left|\nabla\partial_t^j\psi_i\right|^2\mathrm{d}x-\frac{j(j-1)}{2}M(t)\right]+Q\left\|\nabla\left(\tilde{n}_2\partial_t^j\psi_1+\tilde{n}_1\partial_t^j\psi_2\right)\right\|_{L^2(\Omega)}^2\\
&+\sum_{i=1}^{2}\int_{\Omega}\nabla\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\\
\leq& C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\psi_i,\partial_t^j\psi_{it},\nabla\partial_t^j\psi_i\right)\right\|_{L^2(\Omega)}^2+C|||\boldsymbol{w}(t)|||^3,
\end{aligned}
\tag{5.40}
$$

*where $M(t)$ is defined by*

$$
M(t):=\sum_{i=1}^{2}\int_{\Omega}\frac{\psi_i}{2\tilde{n}_i n_i}\left|\nabla\psi_{itt}\right|^2\mathrm{d}x,
$$

*which further satisfies the following inequality*

$$
|M(t)|\leq C|||\boldsymbol{w}(t)|||^3.
\tag{5.41}
$$

**Proof.** On closer examination of the estimate (5.3), we carry out the following procedure

$$\sum_{i=1}^{2}\int_{\Omega} \nabla\partial_t^j(4.2\text{a})\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x, \quad \text{for } j=0,1,2, \tag{5.42}$$

which gives

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j\psi_{it}\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\nabla\mathrm{div}(\tilde{n}_i\partial_t^j\boldsymbol{u}_i)\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\\&+Q\left\|\nabla\left(\tilde{n}_2\partial_t^j\psi_1+\tilde{n}_1\partial_t^j\psi_2\right)\right\|_{L^2(\Omega)}^2=\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j f_i(\boldsymbol{w})\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x, \quad \text{for } j=0,1,2.\end{aligned} \tag{5.43}$$

Note that $\nabla\tilde{n}_e\cdot\nu|_{\partial\Omega}=0$, then integration by parts gives the following estimate of the first integral on the left-hand side of (5.43) for $j=0,1,2$, that is

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j\psi_{it}\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\\&=\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j\psi_{it}\cdot\left(\tilde{n}_e\nabla\partial_t^j\psi_i+\nabla\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\tilde{n}_e\left|\nabla\partial_t^j\psi_i\right|^2\mathrm{d}x-\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_{it}\mathrm{div}\left(\nabla\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\\&\geq\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\tilde{n}_e\left|\nabla\partial_t^j\psi_i\right|^2\mathrm{d}x-C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\psi_i,\partial_t^j\psi_{it},\nabla\partial_t^j\psi_i\right)\right\|_{L^2(\Omega)}^2.\end{aligned} \tag{5.44}$$

The form of the second integral on the left-hand side of (5.43) stays the same and will be used to cancel out the counterpart in (5.3).

As regards the nonlinear integral on the right-hand side of (5.43), a straightforward computation yields

$$\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j f_i(\boldsymbol{w})\cdot\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\mathrm{d}x\leq C|||\boldsymbol{w}(t)|||^3, \quad \text{for } j=0,1. \tag{5.45}$$

From the point of view of skill, the handling of the case $j=2$ bears a resemblance to the estimate (5.17) of the integral $J_2$ in the proof of Lemma 5.1. More precisely, we calculate by integrating by parts together with $\nabla\tilde{n}_e\cdot\nu|_{\partial\Omega}=0$ that

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega} \nabla\partial_t^2 f_i(\boldsymbol{w})\cdot\nabla\left(\tilde{n}_e\partial_t^2\psi_i\right)\mathrm{d}x\\
=&-\sum_{i=1}^{2}\int_{\Omega} \nabla\partial_t^2\left[\operatorname{div}(\psi_i\boldsymbol{u}_i)+Q\psi_1\psi_2\right]\cdot\nabla\left(\tilde{n}_e\partial_t^2\psi_i\right)\mathrm{d}x\\
\le&-\sum_{i=1}^{2}\int_{\Omega} \nabla\partial_t^2\operatorname{div}(\psi_i\boldsymbol{u}_i)\cdot\nabla\left(\tilde{n}_e\partial_t^2\psi_i\right)\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
\le&-\sum_{i=1}^{2}\int_{\Omega} \tilde{n}_e\nabla\partial_t^2\psi_i\cdot\partial_t^2\nabla\operatorname{div}(\psi_i\boldsymbol{u}_i)\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3\\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\psi_i}{2\tilde{n}_i n_i}|\nabla\psi_{itt}|^2\,\mathrm{d}x+C|||\boldsymbol{w}(t)|||^3,
\end{aligned}\tag{5.46}
$$

where, to obtain the last inequality above, we have invoked (5.17) verbatim with $\frac{\chi_i}{n_i}$ replaced by $-\tilde{n}_e$ and with $\tilde{n}_i$ replaced by $\psi_i$. Plus, the estimate (5.41) is obviously true.

And lastly, substituting (5.44)~(5.46) into (5.43), we obtain the desired estimate (5.40). □

Although we have got the energy part for $\nabla\partial_t^j\psi_i$ in Lemma 5.4, the dissipation rate there is of the form $Q\|\nabla(\tilde{n}_2\partial_t^j\psi_1+\tilde{n}_1\partial_t^j\psi_2)\|^2_{L^2(\Omega)}$ due to the occurrence of the net recombination rate, which is ineffective for the purpose of closing the uniform estimate. Therefore, we manage to uncover the dissipative feature for $\nabla\partial_t^j\psi_i$ in the next lemma.

**Lemma 5.5.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C>0$ such that for $j=0,1,2$,*

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_i\operatorname{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x+\left\|\partial_t^j\psi_1-\partial_t^j\psi_2\right\|^2_{L^2(\Omega)}+\frac{b}{4}\sum_{i=1}^{2}\left\|\nabla\partial_t^j\psi_i\right\|^2_{L^2(\Omega)}\\
\le&C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\boldsymbol{u}_i,\partial_t^j\chi_i\right)\right\|^2_{L^2(\Omega)}+(\mu+C\epsilon)\sum_{i=1}^{2}\left\|\partial_t^j\psi_i\right\|^2_{L^2(\Omega)}\\
&+C_\mu\sum_{i=1}^{2}\left\|\left(\operatorname{div}\partial_t^j\boldsymbol{u}_i,\nabla\partial_t^j\chi_i\right)\right\|^2_{L^2(\Omega)}+C|||\boldsymbol{w}(t)|||^3,
\end{aligned}\tag{5.47}
$$

*where $\mu>0$ is arbitrarily small and $C_\mu$ is a positive constant dependent on $\mu$.*

**Proof.** Following the operation

$$
\sum_{i=1}^{2}\int_{\Omega}\partial_t^j(4.2\mathrm{b})\cdot\nabla\partial_t^j\psi_i\mathrm{d}x,\quad \text{for } j=0,1,2,\tag{5.48}
$$

we obtain

$$\sum_{i=1}^{2}\int_{\Omega}\partial_t^j \boldsymbol{u}_{it}\cdot\nabla\partial_t^j\psi_i\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\partial_t^j \boldsymbol{u}_{i}\cdot\nabla\partial_t^j\psi_i\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\nabla(\tilde{n}_e\partial_t^j\psi_i)\cdot\nabla\partial_t^j\psi_i\mathrm{d}x$$
$$+\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_e\nabla\tilde{n}_i\partial_t^j\chi_i\cdot\nabla\partial_t^j\psi_i\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j\chi_i\cdot\nabla\partial_t^j\psi_i\mathrm{d}x+\int_{\Omega}\nabla\partial_t^j\sigma\cdot\sum_{i=1}^{2}\Big[(-1)^i\nabla\partial_t^j\psi_i\Big]\mathrm{d}x$$
$$=\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\boldsymbol{g}_i(\boldsymbol{w})\cdot\nabla\partial_t^j\psi_i\mathrm{d}x,\quad \text{for } j=0,1,2. \tag{5.49}$$

It is the most complicated case to deal with the first integral on the left-hand side of (5.49) because we have to use the continuity equation (4.2a) to dominate it. This strategy will bring about a trouble because of the net recombination rate. Specifically, integrating by parts yields for $j=0,1,2$

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\partial_t^j \boldsymbol{u}_{it}\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\\
&=-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x+\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_{it}\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
&=-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
&\quad-\sum_{i=1}^{2}\int_{\Omega}\Big[\mathrm{div}(\tilde{n}_i\partial_t^j\boldsymbol{u}_i)+Q(\tilde{n}_2\partial_t^j\psi_1+\tilde{n}_1\partial_t^j\psi_2)-\partial_t^j f_i(\boldsymbol{w})\Big]\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
&\geq-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\psi_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x-C\epsilon\sum_{i=1}^{2}\left\|\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2\\
&\quad-\mu\sum_{i=1}^{2}\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2-C_\mu\sum_{i=1}^{2}\left\|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2-C|||\boldsymbol{w}(t)|||^3,
\end{aligned}\tag{5.50}$$

where we have also used Young's inequality. Similarly to (5.50), the four integrals in the middle of the left-hand side of (5.49) can be estimated in turn.

$$\sum_{i=1}^{2}\int_{\Omega}\partial_t^j \boldsymbol{u}_{i}\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\geq-\mu\sum_{i=1}^{2}\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2-C_\mu\sum_{i=1}^{2}\left\|\mathrm{div}\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2,\quad\text{for } j=0,1,2, \tag{5.51}$$

$$\sum_{i=1}^{2}\int_{\Omega}\nabla(\tilde{n}_e\partial_t^j\psi_i)\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\geq\frac{b}{2}\sum_{i=1}^{2}\left\|\nabla\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2-C\epsilon\sum_{i=1}^{2}\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2,\quad\text{for } j=0,1,2,\tag{5.52}$$

$$\sum_{i=1}^{2}\int_{\Omega}\tilde{n}_e\nabla\tilde{n}_i\partial_t^j\chi_i\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\geq-C\epsilon\sum_{i=1}^{2}\left\|\left(\partial_t^j\chi_i,\nabla\partial_t^j\psi_i\right)\right\|_{L^2(\Omega)}^2,\quad\text{for } j=0,1,2,\tag{5.53}$$

and

$$\sum_{i=1}^{2}\int_{\Omega}\nabla\partial_t^j\chi_i\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\geq-\mu\sum_{i=1}^{2}\left\|\nabla\partial_t^j\psi_i\right\|_{L^2(\Omega)}-C_\mu\sum_{i=1}^{2}\left\|\nabla\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2,\quad\text{for } j=0,1,2.\tag{5.54}$$

And the last integral on the left-hand side of (5.49) is actually influenced by the bipolar structure, so it is worth going through this integral. Integrating by parts and invoking the Poisson equation (4.2d) gives

$$\begin{aligned}&\int_{\Omega}\nabla\partial_t^j\sigma\cdot\sum_{i=1}^{2}\left[(-1)^i\nabla\partial_t^j\psi_i\right]\mathrm{d}x\\&=\int_{\Omega}\partial_t^j\Delta\sigma\left(\partial_t^j\psi_1-\partial_t^j\psi_2\right)\mathrm{d}x\\&=\int_{\Omega}\left(\partial_t^j\psi_1-\partial_t^j\psi_2\right)^2\mathrm{d}x\\&=\left\|\partial_t^j\psi_1-\partial_t^j\psi_2\right\|_{L^2(\Omega)}^2,\quad\text{for } j=0,1,2.\end{aligned}\tag{5.55}$$

As for the nonlinear integral on the right-hand side of (5.49), there is no need to introduce any new skill, and one can easily control it as follows.

$$\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\boldsymbol{g}_i(\boldsymbol{w})\cdot\nabla\partial_t^j\psi_i\mathrm{d}x\leq C|||\boldsymbol{w}(t)|||^3,\quad\text{for } j=0,1,2.\tag{5.56}$$

Inserting (5.50)∼(5.56) into (5.49), letting $\mu$ be suitably small, and allowing $\epsilon$ to be sufficiently small, we have now established the necessary estimate (5.47). □

Incidentally, our computation in Lemma 5.5 shows that the bipolar structure of the system (4.2) prevents the electric field term $(-1)^i\nabla\sigma$ in the velocity equation (4.2b) from producing the dissipation rate for $\partial_t^j\psi_i$, see (5.55). Compared with the unipolar case (see [15,17]), this is the reason why we claimed in the introduction section of the present paper that the bipolar structure would weaken the dissipative mechanism of the system. Fortunately, there is a chance to make up for the loss of the dissipative mechanism when constructing the energy part for $\partial_t^j\psi_i$

by the continuity equation (4.2a), due to the appearance of the net recombination rate. This novel finding will be summarized and proved in the following lemma.

**Lemma 5.6.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for $j = 0, 1, 2$,*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_e}{2Q}\left|\partial_t^j\psi_i\right|^2\mathrm{d}x + b^2\sum_{i=1}^{2}\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2 + \int_{\Omega}2\partial_t^j\psi_1\partial_t^j\psi_2\mathrm{d}x \\
&+\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x \le C|||\boldsymbol{w}(t)|||^3. \qquad (5.57)
\end{aligned}
$$

**Proof.** Implementing the operation below

$$
\sum_{i=1}^{2}\int_{\Omega}\partial_t^j(4.2\mathrm{a})\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\mathrm{d}x, \quad \text{for } j = 0, 1, 2 \qquad (5.58)
$$

gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_e}{2Q}\left|\partial_t^j\psi_i\right|^2\mathrm{d}x + \sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x \\
&+\int_{\Omega}Q\left(\tilde{n}_2\partial_t^j\psi_1 + \tilde{n}_1\partial_t^j\psi_2\right)\sum_{i=1}^{2}\left(\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\right)\mathrm{d}x \\
=&\sum_{i=1}^{2}\int_{\Omega}\partial_t^j f_i(\boldsymbol{w})\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\mathrm{d}x, \quad \text{for } j = 0, 1, 2, \qquad (5.59)
\end{aligned}
$$

where the first term on the left-hand side of (5.59) has been treated in the usual way. Also, we shall keep the second integral on the left-hand side of (5.59) unchanged for the purpose of cancellation with its counterpart in the next lemma. And the last integral on the left-hand side of (5.59) is impacted by the net recombination rate, producing an ill-behaved dissipative integral. More precisely,

$$
\begin{aligned}
&\int_{\Omega}Q\left(\tilde{n}_2\partial_t^j\psi_1 + \tilde{n}_1\partial_t^j\psi_2\right)\sum_{i=1}^{2}\left(\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\right)\mathrm{d}x \\
=&\int_{\Omega}\left(\tilde{n}_2\partial_t^j\psi_1 + \tilde{n}_1\partial_t^j\psi_2\right)^2\mathrm{d}x \\
=&\int_{\Omega}\left[\tilde{n}_2^2\left(\partial_t^j\psi_1\right)^2 + 2\tilde{n}_2\tilde{n}_1\partial_t^j\psi_1\partial_t^j\psi_2 + \tilde{n}_1^2\left(\partial_t^j\psi_2\right)^2\right]\mathrm{d}x
\end{aligned}
$$

$$\geq b^2 \sum_{i=1}^{2} \left\| \partial_t^j \psi_i \right\|_{L^2(\Omega)}^2 + \int_\Omega 2\partial_t^j \psi_1 \partial_t^j \psi_2 \mathrm{d}x, \quad \text{for } j = 0, 1, 2, \tag{5.60}$$

where we have made the most of the properties (1.17b) and (5.29) of the thermal equilibrium state in the last line.

The nonlinear integral on the right-hand side of (5.59) is a normal one, so we can easily in the usual manner obtain

$$\sum_{i=1}^{2} \int_\Omega \partial_t^j f_i(\boldsymbol{w}) \frac{\tilde{n}_e}{Q} \partial_t^j \psi_i \mathrm{d}x \leq C |||\boldsymbol{w}(t)|||^3, \quad \text{for } j = 0, 1, 2. \tag{5.61}$$

Substituting (5.60) and (5.61) into (5.59), we reach the estimate (5.57). □

**Remark 5.3.** For the time being, the plausible dissipation rate $b^2 \sum_{i=1}^2 \|\partial_t^j \psi_i\|_{L^2(\Omega)}^2$ alone is still useless as a direct result of the recombination-generation phenomenon. In fact, the net recombination rate leads to the occurrence of the extra and bad dissipative integral $\int_\Omega 2\partial_t^j \psi_1 \partial_t^j \psi_2 \mathrm{d}x$. However, if we still bear the estimate (5.47) in mind, then it will help us to find that the harmless dissipative integral $\|\partial_t^j \psi_1 - \partial_t^j \psi_2\|_{L^2(\Omega)}^2$ can be put to use. Precisely, adding the dissipative integral $\|\partial_t^j \psi_1 - \partial_t^j \psi_2\|_{L^2(\Omega)}^2$ brought about by the bipolar structure and the other one $\int_\Omega 2\partial_t^j \psi_1 \partial_t^j \psi_2 \mathrm{d}x$ caused by the net recombination rate, we can obtain an enhanced part $\sum_{i=1}^2 \|\partial_t^j \psi_i\|_{L^2(\Omega)}^2$ of the dissipation rate $b^2 \sum_{i=1}^2 \|\partial_t^j \psi_i\|_{L^2(\Omega)}^2$ without any ill-behaved dissipative integrals, as what we will do later in Lemma 5.9.

Up to now, we have not established the estimates for $\partial_t^j \boldsymbol{u}_i$ and $\partial_t^j \chi_i$ yet. The following two lemmas will be devoted to this task.

**Lemma 5.7.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for $j = 0, 1, 2$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( \int_\Omega \frac{1}{2Q} \left|\nabla \partial_t^j \sigma\right|^2 \mathrm{d}x + \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_i}{2Q} \left|\partial_t^j \boldsymbol{u}_i\right|^2 \mathrm{d}x \right) + \frac{b}{Q} \sum_{i=1}^{2} \left\| \partial_t^j \boldsymbol{u}_i \right\|_{L^2(\Omega)}^2$$
$$- \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_e}{Q} \partial_t^j \psi_i \mathrm{div}\left(\tilde{n}_i \partial_t^j \boldsymbol{u}_i\right) \mathrm{d}x - \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_i}{Q} \partial_t^j \chi_i \mathrm{div} \partial_t^j \boldsymbol{u}_i \mathrm{d}x \leq C |||\boldsymbol{w}(t)|||^3. \tag{5.62}$$

**Proof.** Executing the following computational process

$$\sum_{i=1}^{2} \int_\Omega \partial_t^j (4.2\mathrm{b}) \cdot \frac{\tilde{n}_i}{Q} \partial_t^j \boldsymbol{u}_i \mathrm{d}x, \quad \text{for } j = 0, 1, 2 \tag{5.63}$$

yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i}{2Q}\left|\partial_t^j \boldsymbol{u}_i\right|^2\mathrm{d}x+\frac{b}{Q}\sum_{i=1}^{2}\left\|\partial_t^j\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{2}\int_{\Omega}\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\cdot\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x$$
$$+\sum_{i=1}^{2}\int_{\Omega}\left(\tilde{n}_e\nabla\tilde{n}_i\partial_t^j\chi_i+\nabla\partial_t^j\chi_i\right)\cdot\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x+\int_{\Omega}\nabla\partial_t^j\sigma\cdot\sum_{i=1}^{2}(-1)^i\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x$$
$$\leq\sum_{i=1}^{2}\int_{\Omega}\partial_t^j\boldsymbol{g}_i(\boldsymbol{w})\cdot\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x,\quad\text{for } j=0,1,2, \tag{5.64}$$

where the first two terms on the left-hand side of (5.64) have already been adjusted by adopting the general practice. And after applying integration by parts to the third integral on the left-hand side of (5.64), we find with ease that

$$\sum_{i=1}^{2}\int_{\Omega}\nabla\left(\tilde{n}_e\partial_t^j\psi_i\right)\cdot\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x=-\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_e}{Q}\partial_t^j\psi_i\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\mathrm{d}x,\quad\text{for } j=0,1,2 \tag{5.65}$$

which is of the same form as the last term on the left-hand side of (5.57) but with the opposite sign. As for the fourth term on the left-hand side of (5.64), by the special property (5.29) and integration by parts, we calculate

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\left(\tilde{n}_e\nabla\tilde{n}_i\partial_t^j\chi_i+\nabla\partial_t^j\chi_i\right)\cdot\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\frac{1}{Q}\left[\tilde{n}_i\tilde{n}_e\partial_t^j\chi_i\nabla\tilde{n}_i\cdot\partial_t^j\boldsymbol{u}_i-\partial_t^j\chi_i\mathrm{div}\left(\tilde{n}_i\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
=&\sum_{i=1}^{2}\int_{\Omega}\frac{1}{Q}\left[\partial_t^j\chi_i\nabla\tilde{n}_i\cdot\partial_t^j\boldsymbol{u}_i-\left(\partial_t^j\chi_i\nabla\tilde{n}_i\cdot\partial_t^j\boldsymbol{u}_i+\partial_t^j\chi_i\tilde{n}_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\right)\right]\mathrm{d}x\\
=&-\sum_{i=1}^{2}\int_{\Omega}\frac{\tilde{n}_i}{Q}\partial_t^j\chi_i\mathrm{div}\partial_t^j\boldsymbol{u}_i\mathrm{d}x,\quad\text{for } j=0,1,2.\end{aligned} \tag{5.66}$$

Sometimes the impact of the net recombination rate disappears when it meets the bipolar effect, the process of computing the last term on the left-hand side of (5.64) gives the best example. More precisely, using integration by parts, the continuity equation (4.2a) and the Poisson equation (4.2d), we calculate

$$\begin{aligned}&\int_{\Omega}\nabla\partial_t^j\sigma\cdot\sum_{i=1}^{2}(-1)^i\frac{\tilde{n}_i}{Q}\partial_t^j\boldsymbol{u}_i\mathrm{d}x\\
=&\int_{\Omega}\frac{1}{Q}\partial_t^j\sigma\sum_{i=1}^{2}(-1)^i\left[\partial_t^j\psi_{it}+\partial_t^jR(n_1,n_2)+\mathrm{div}\partial_t^j\left(\psi_i\boldsymbol{u}_i\right)\right]\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= - \int_\Omega \frac{1}{Q} \partial_t^j \sigma \Delta \partial_t^j \sigma_t \mathrm{d}x + \int_\Omega \frac{1}{Q} \partial_t^j \sigma \sum_{i=1}^{2} (-1)^i \operatorname{div} \partial_t^j \left( \psi_i \boldsymbol{u}_i \right) \mathrm{d}x \\
&\geq \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2Q} \left| \nabla \partial_t^j \sigma \right|^2 \mathrm{d}x - C |||\boldsymbol{w}(t)|||^3, \quad \text{for } j = 0, 1, 2.
\end{aligned} \tag{5.67}
$$

Similarly to (5.61), using the standard method, we obtain for $j = 0, 1, 2$,

$$
\sum_{i=1}^{2} \int_\Omega \partial_t^j \boldsymbol{g}_i(\boldsymbol{w}) \cdot \frac{\tilde{n}_i}{Q} \partial_t^j \boldsymbol{u}_i \mathrm{d}x \leq C |||\boldsymbol{w}(t)|||^3. \tag{5.68}
$$

Substituting the estimates (5.65)∼(5.68) into (5.64), we get the desired estimate (5.62). □

**Lemma 5.8.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for $j = 0, 1, 2$,*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{2} \int_\Omega \frac{3\tilde{n}_i}{4Q} \left| \partial_t^j \chi_i \right|^2 \mathrm{d}x + \frac{3b}{2Q} \sum_{i=1}^{2} \left\| \partial_t^j \chi_i \right\|_{L^2(\Omega)}^2 + \frac{1}{Q} \sum_{i=1}^{2} \left\| \nabla \partial_t^j \chi_i \right\|_{L^2(\Omega)}^2 \\
&+ \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_i}{Q} \partial_t^j \chi_i \operatorname{div} \partial_t^j \boldsymbol{u}_i \mathrm{d}x \\
\leq& \frac{j(j-1)}{2} C |||\boldsymbol{w}(t)||| \sum_{i=1}^{2} \left\| \partial_t^3 \chi_i \right\|_{L^2(\Omega)}^2 + C |||\boldsymbol{w}(t)|||^3.
\end{aligned} \tag{5.69}
$$

**Proof.** Actually, one can easily check the estimate (5.69) as per the following strategy

$$
\sum_{i=1}^{2} \int_\Omega \partial_t^j (4.2\text{c}) \frac{3}{2Q} \partial_t^j \chi_i \mathrm{d}x, \quad \text{for } j = 0, 1, 2, \tag{5.70}
$$

together with the standard methods used repeatedly. Besides, the fourth integral on the left-hand side of (5.69) is directly generated by the second term on the left-hand side of the energy equation (4.2c) and stay put in order to cancel out its counterpart in (5.62). □

While proving the above lemmas, we have noted that some integral terms appearing in certain estimates could cancel each other out. Therefore, we are going to make this observation work in the next lemma.

**Lemma 5.9.** *Let $\alpha > 0$, $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then, for $j = 0, 1, 2$, there exist positive constants $\Upsilon_j > 0$ and $C > 0$ such that*

$$
\frac{\mathrm{d}}{\mathrm{d}t} E_j(t) + \Upsilon_j F_j(t) \leq C \left( \epsilon + |||\boldsymbol{w}(t)||| \right) \sum_{i=1}^{2} \left\| \left( \partial_t^j \psi_{it}, \partial_t^j \boldsymbol{u}_{it}, \partial_t^j \chi_{it} \right) \right\|_{L^2(\Omega)}^2 + C |||\boldsymbol{w}(t)|||^3 \tag{5.71}
$$

*where $E_j(t)$ and $F_j(t)$ are defined by*

$$
\begin{aligned}
E_j(t) :=& \sum_{i=1}^{2} \int_\Omega \frac{1}{2}\tilde{n}_i \left|\mathrm{div}\partial_t^j \boldsymbol{u}_i\right|^2 \mathrm{d}x + \int_\Omega \frac{1}{2}\left(\partial_t^j \psi_1 - \partial_t^j \psi_2\right)^2 \mathrm{d}x - \frac{j(j-1)}{2} N(t) \\
&+ \sum_{i=1}^{2} \int_\Omega \frac{1}{2}\tilde{n}_i \left|\mathrm{curl}\partial_t^j \boldsymbol{u}_i\right|^2 \mathrm{d}x + \sum_{i=1}^{2} \int_\Omega \frac{3}{4}\tilde{n}_i \left|\nabla \partial_t^j \chi_i\right|^2 \mathrm{d}x \\
&+ \sum_{i=1}^{2} \int_\Omega \frac{1}{2}\tilde{n}_e \left|\nabla \partial_t^j \psi_i\right|^2 \mathrm{d}x - \frac{j(j-1)}{2} M(t) \\
&- \alpha \sum_{i=1}^{2} \int_\Omega \partial_t^j \psi_i \mathrm{div}\partial_t^j \boldsymbol{u}_i \mathrm{d}x + \alpha \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_e}{2Q} \left|\partial_t^j \psi_i\right|^2 \mathrm{d}x \\
&+ \alpha \int_\Omega \frac{1}{2Q} \left|\nabla \partial_t^j \sigma\right|^2 \mathrm{d}x + \alpha \sum_{i=1}^{2} \int_\Omega \frac{\tilde{n}_i}{2Q} \left|\partial_t^j \boldsymbol{u}_i\right|^2 \mathrm{d}x \\
&+ \alpha \sum_{i=1}^{2} \int_\Omega \frac{3\tilde{n}_i}{4Q} \left|\partial_t^j \chi_i\right|^2 \mathrm{d}x,
\end{aligned}
$$

*and*

$$
F_j(t) := \sum_{i=1}^{2} \left( \left\| \left( \partial_t^j \psi_i, \partial_t^j \boldsymbol{u}_i, \partial_t^j \chi_i \right) \right\|_{H^1(\Omega)}^2 + \left\| \Delta \partial_t^j \chi_i \right\|_{L^2(\Omega)}^2 \right).
$$

**Proof.** It seems preferable, for clarity's sake, to merge all the estimates built in the previous lemmas into the unified one. Closer scrutiny of these estimates inspires us to do certain necessary calculations in which a suitably small positive constant $\alpha$ will be chosen to adjust the miscellaneous orders of magnitude among these estimates. The outline of calculations is listed below

$$
\Big[(5.3) + (5.25) + (5.33) + (5.40)\Big] + \alpha \Big[(5.47) + (5.57) + (5.62) + (5.69)\Big]. \tag{5.72}
$$

Avoiding unnecessary details, we merely demonstrate several main points:

1. During the computations in the first square bracket above, the last two integral terms on the left-hand side of (5.3) will be used to cancel out their counterparts in (5.33) and (5.40), respectively; similarly, among the calculations in the second square bracket above, the last two integral terms on the left-hand side of (5.62) will be separately counterbalanced by their counterparts in (5.57) and (5.69).
2. We could dispense with the ineffective dissipation rate $Q\|\nabla(\tilde{n}_2\partial_t^j \psi_1 + \tilde{n}_1 \partial_t^j \psi_2)\|_{L^2(\Omega)}^2$ on the left-hand side of (5.40) due to its positivity and the estimate (5.47).

3. Recalling what we have claimed in Remark 5.3, we can, in the computational process of (5.47) + (5.57), obtain the augmented dissipation rate for $j = 0, 1, 2$,

$$\begin{aligned}&\left\|\partial_t^j\psi_1-\partial_t^j\psi_2\right\|_{L^2(\Omega)}^2+b^2\sum_{i=1}^2\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2+\int_\Omega 2\partial_t^j\psi_1\partial_t^j\psi_2\mathrm{d}x\\&=\left(b^2+1\right)\sum_{i=1}^2\left\|\partial_t^j\psi_i\right\|_{L^2(\Omega)}^2,\end{aligned}\tag{5.73}$$

which can further absorb the corresponding term $(\mu + C\epsilon)\sum_{i=1}^2 \|\partial_t^j \psi_i\|_{L^2(\Omega)}^2$ on the right-hand side of (5.47) by fixing suitably small positive constants $\mu$ and $\epsilon$. In the meantime, the positive constant $C_\mu$ is fixed as well because the small positive constant $\mu$ has already been fixed. And then we can further opt for an appropriately small positive constant $\alpha$ to make the term $C_\mu\sum_{i=1}^2 \|(\mathrm{div}\partial_t^j \boldsymbol{u}_i, \nabla\partial_t^j \chi_i)\|_{L^2(\Omega)}^2$ on the right-hand side of (5.47) absorbed by the corresponding dissipation rates in the estimates (5.3) and (5.33), respectively.
4. Note that the constant $\alpha$ has already been determined, and therefore for $j = 0, 1, 2$ we are able to apply Proposition 2.4 with $(k, p) = (0, 2)$ to obtain the dissipation rate $\sum_{i=1}^2 \|\partial_t^j \boldsymbol{u}_i\|_{H^1(\Omega)}^2$ in the process of the calculation (5.3) + (5.25) + $\alpha$(5.62).

Keeping the essentials of the strategy (5.72) listed above in mind and using the straightforward but tedious computations, we get the predigested and unified estimate (5.71) provided both $\epsilon$ and $|||\boldsymbol{w}(t)|||$ are small enough. $\square$

Up to now, we have not yet derived the amply dissipative mechanism from the system (4.2) to close the uniform a priori estimate in terms of the Sobolev smooth norm $|||\boldsymbol{w}(t)|||^2$. Actually, it remains for us to establish some necessary estimates for the higher-order spatial derivatives and space-time mixed derivatives of the vorticity $\mathrm{curl}\boldsymbol{u}_i$ for $i = 1, 2$.

**Lemma 5.10.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exists a positive constant $C > 0$ such that for the integer pairs $(l, m) = (1, 0), (2, 0), (1, 1)$,*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^2\int_\Omega\frac12\left|\nabla^l\mathrm{curl}\partial_t^m\boldsymbol{u}_i\right|^2\mathrm{d}x+\frac12\sum_{i=1}^2\left\|\nabla^l\mathrm{curl}\partial_t^m\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2\\&\le C\epsilon\sum_{i=1}^2\left\|\nabla\partial_t^m\chi_i\right\|_{H^l(\Omega)}^2+C|||\boldsymbol{w}(t)|||^3.\end{aligned}\tag{5.74}$$

**Proof.** Similarly to the proof of Lemma 5.2 but somewhat complicatedly, we follow the computational procedure for integer pairs $(l, m) = (1, 0), (2, 0), (1, 1)$,

$$\sum_{i=1}^2\int_\Omega\nabla^l\mathrm{curl}\partial_t^m(4.2\mathrm{a}):\nabla^l\mathrm{curl}\partial_t^m\boldsymbol{u}_i\mathrm{d}x,\tag{5.75}$$

where the notation ":" represents the Frobenius inner product, to establish the desired estimate (5.74) without difficulty. Since there are no technical points need to be emphasized, so we do not lay out the details here. □

Based on the predigested estimate in Lemma 5.9 and the supplemented estimate in Lemma 5.10, we are now in a position to complete the final step of establishing the desired a priori estimate for local-in-time solutions.

**Lemma 5.11.** *Let $\epsilon$ and $|||\boldsymbol{w}(t)|||$ be sufficiently small. Then there exist positive constants $\Upsilon > 0$ and $C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) + \Upsilon F(t) \le C\,(\epsilon + |||\boldsymbol{w}(t)|||)\, R(t) + C|||\boldsymbol{w}(t)|||^3, \tag{5.76}$$

*where the total energy $E(t)$, the total dissipation rate $F(t)$ and the quadratic remaining term $R(t)$ are defined by*

$$E(t) := \sum_{j=0}^{2} E_j(t) + \sum_{\substack{(l,m)=(1,0),\\ (2,0),(1,1)}} \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2} \left| \nabla^l \mathrm{curl} \partial_t^m \boldsymbol{u}_i \right|^2 \mathrm{d}x,$$

$$F(t) := \sum_{j=0}^{2} F_j(t) + \sum_{\substack{(l,m)=(1,0),\\ (2,0),(1,1)}} \sum_{i=1}^{2} \left\| \nabla^l \mathrm{curl} \partial_t^m \boldsymbol{u}_i \right\|_{L^2(\Omega)}^2,$$

$$R(t) := \sum_{j=0}^{2} \sum_{i=1}^{2} \left\| \left( \partial_t^j \psi_{it}, \partial_t^j \boldsymbol{u}_{it}, \partial_t^j \chi_{it} \right) \right\|_{L^2(\Omega)}^2 + \sum_{\substack{(l,m)=(1,0),\\ (2,0),(1,1)}} \sum_{i=1}^{2} \left\| \nabla \partial_t^m \chi_i \right\|_{H^l(\Omega)}^2 .$$

*Moreover, there exist positive constants $C_4$, $C_5$, $C_6$ and $C_7$ such that $E(t)$, $F(t)$ and $R(t)$ satisfy the following estimates,*

$$R(t) \le C_4 F(t), \tag{5.77a}$$

$$C_5 |||\boldsymbol{w}(t)|||^2 \le E(t) \le C_6 |||\boldsymbol{w}(t)|||^2, \tag{5.77b}$$

$$|||\boldsymbol{w}(t)|||^2 \le C_7 F(t). \tag{5.77c}$$

**Proof.** Formally, the a priori estimate (5.76) can be easily built by implementing the following operation

$$\sum_{j=0}^{2} (5.71) + \sum_{\substack{(l,m)=(1,0),\\ (2,0),(1,1)}} (5.74). \tag{5.78}$$

Nevertheless, how the estimate (5.76) makes the continuation principle work significantly depends on the verification of the quantitative relation (5.77). To this end, we first define for simplicity

$$
\begin{aligned}
A(t) :=& \sum_{j=0}^{2}\sum_{i=1}^{2}\left\|\left(\partial_t^j\psi_i, \partial_t^j\boldsymbol{u}_i, \partial_t^j\chi_i\right)\right\|_{H^1(\Omega)}^2 + \sum_{\substack{(l,m)=(1,0),\\(2,0),(1,1)}}\sum_{i=1}^{2}\left\|\nabla^l \mathrm{curl}\partial_t^m\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2 \\
=& F(t) - \sum_{j=0}^{2}\sum_{i=1}^{2}\left\|\Delta\partial_t^j\chi_i\right\|_{L^2(\Omega)}^2 \\
\le& F(t).
\end{aligned}
\tag{5.79}
$$

Under the smallness assumption $|||\boldsymbol{w}(t)||| \ll 1$, we are also able to build some auxiliary estimates:

1. A careful comparison between the expressions of $E(t)$ and A(t) enables us to easily draw the equivalent relation

$$
E(t) \sim A(t), \tag{5.80}
$$

that is, there exist two positive constants $c$ and $C$ such that $cA(t) \le E(t) \le CA(t)$.

2. A thorough examination of the expressions for $R(t)$ and $F(t)$ makes us aware that the following terms in the expression of $R(t)$ are not formally included in $F(t)$: for $i = 1, 2$,

$$
\left\|\partial_t^3\psi_i\right\|_{L^2(\Omega)}^2, \ \left\|\partial_t^3\boldsymbol{u}_i\right\|_{L^2(\Omega)}^2, \ \left\|\partial_t^3\chi_i\right\|_{L^2(\Omega)}^2, \ \left\|\nabla^2\chi_i\right\|_{L^2(\Omega)}^2, \ \left\|\nabla^3\chi_i\right\|_{L^2(\Omega)}^2 \ \text{and} \ \left\|\nabla^2\chi_{it}\right\|_{L^2(\Omega)}^2. \tag{5.81}
$$

In what follows, we list the estimates for those terms in (5.81).

2a). From $\partial_t^2$ (4.2a), $\partial_t^2$ (4.2b) and $\frac{1}{\tilde{n}_i}\partial_t^2$ (4.2c), we have

$$
\begin{aligned}
&\sum_{i=1}^{2}\left\|\left(\partial_t^3\psi_i, \partial_t^3\boldsymbol{u}_i\right)\right\|_{L^2(\Omega)}^2 \le CA(t) \text{ and} \\
&\sum_{i=1}^{2}\left\|\partial_t^3\chi_i\right\|_{L^2(\Omega)}^2 \le C\left(A(t) + \left\|\Delta\partial_t^2\chi_i\right\|_{L^2(\Omega)}^2\right) \le CF(t).
\end{aligned}
\tag{5.82}
$$

2b). From $\partial_t^k$ (4.2c) for $k = 0, 1$, we derive

$$
\begin{cases}
-\dfrac{2}{3}\Delta\partial_t^k\chi_i + \tilde{n}_i\partial_t^k\chi_i = -\tilde{n}_i\partial_t^k\chi_{it} - \dfrac{2}{3}\tilde{n}_i\mathrm{div}\partial_t^k\boldsymbol{u}_i + \partial_t^k h_i(\boldsymbol{w}), \ \text{in } \Omega, & \text{(a)} \\
\nabla\partial_t^k\chi_i\cdot\nu = 0, \ \text{on } \partial\Omega, \quad \text{for } i = 1, 2, & \text{(b)}
\end{cases}
\tag{5.83}
$$

which by the elliptic estimate implies

$$
\sum_{k=0}^{1}\sum_{i=1}^{2}\left\|\partial_t^k\chi_i\right\|_{H^{4-k}(\Omega)}^2 \le C\left(A(t) + \sum_{i=1}^{2}\left\|\left(\nabla\mathrm{div}\boldsymbol{u}_i, \nabla\mathrm{div}\boldsymbol{u}_{it}, \nabla^2\mathrm{div}\boldsymbol{u}_i\right)\right\|_{L^2(\Omega)}^2\right). \tag{5.84}
$$

2c). From $\nabla^s\partial_t^k$ (4.2a) for $(s,k)=(1,0),(1,1),(2,0)$, we can solve out the exact expression of $\nabla^s \mathrm{div}\partial_t^k \boldsymbol{u}_i$, and then we obtain

$$\sum_{i=1}^{2}\left\|\left(\nabla\mathrm{div}\boldsymbol{u}_i,\nabla\mathrm{div}\boldsymbol{u}_{it},\nabla^2\mathrm{div}\boldsymbol{u}_i\right)\right\|_{L^2(\Omega)}^2\le C\left(A(t)+\sum_{i=1}^{2}\left\|\left(\nabla^2\psi_i,\nabla^2\psi_{it},\nabla^3\psi_i\right)\right\|_{L^2(\Omega)}^2\right). \tag{5.85}$$

2d). From $\nabla^s\partial_t^k$ (4.2b) for $(s,k)=(1,0),(1,1),(2,0)$, we can solve out the exact expression of $\nabla^{s+1}\partial_t^k\psi_i$, and then we have

$$\sum_{i=1}^{2}\left\|\left(\nabla^2\psi_i,\nabla^2\psi_{it},\nabla^3\psi_i\right)\right\|_{L^2(\Omega)}^2\le CA(t). \tag{5.86}$$

3. By using (5.84)∼(5.86), we can further confirm that the following equivalent relation,

$$A(t)\sim |||\boldsymbol{w}(t)|||^2, \tag{5.87}$$

is true.

Finally, according to (5.79)∼(5.82) and (5.84)∼(5.87), we know that the quantitative relations in (5.77) hold. □

**Remark 5.4.** It is supposed to be declared that some calculations in obtaining (5.76) are actually formal because of the inadequate regularity of the local-in-time solutions. However, the rigorous verification of (5.76) can be carried out with ease by employing the standard mollifier technique. Avoiding the tautological statement, we will not go into details here.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** From the estimates (5.76) and (5.77), it is not hard to deduce that there exists a positive constant $\gamma>0$ such that for the local-in-time solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\gamma E(t)\le 0, \tag{5.88}$$

provided there exists a sufficiently small positive constant $\delta_3>0$ such that $\epsilon\le\delta_3$ and $|||\boldsymbol{w}(t)|||\le\delta_3$. Then applying Gronwall's inequality to (5.88), we have

$$E(t)\le E(0)e^{-\gamma t}, \tag{5.89}$$

thereby showing that there is a positive constant $C_8>1$ such that

$$|||\boldsymbol{w}(t)|||^2\le C_8|||\boldsymbol{w}(0)|||^2e^{-\gamma t}, \tag{5.90}$$

where we have used the estimate (5.77b) again.

Based on the a priori estimate (5.90), we now set about completing the standard continuation argument. First of all, we define a time $T^*$ below

$$T^* := \sup\left\{T \in [0, T_{\max}) \mid |||\boldsymbol{w}(t)|||^2 \le \delta_3^2,\ \forall t \in [0, T]\right\}. \tag{5.91}$$

From here, we intend to show that $T^* = \infty$ under the assumption that the initial perturbation is small, namely $|||\boldsymbol{w}(0)||| \ll 1$. According to the definition of $T^*$, it is apparent that either $T^* < T_{\max}$ or $T^* = T_{\max}$. If the former case holds, then we take

$$\delta_1 := \frac{1}{\sqrt{2C_8}}\delta_3 < \delta_3 \tag{5.92}$$

which is independent of $\epsilon$ and $|||\boldsymbol{w}(0)|||$. Assume that $\epsilon \le \delta_1$ and $|||\boldsymbol{w}(0)||| \le \delta_1$, then it follows from the a priori estimate (5.90) that

$$|||\boldsymbol{w}(t)|||^2 \le \frac{1}{2}\delta_3^2 e^{-\gamma t}, \quad t \in [0, T^*]. \tag{5.93}$$

This contradicts the maximality of the time $T^*$. If the latter case holds, then the amplitude of the solution $|||\boldsymbol{w}(t)|||$ is bounded above by $\delta_3$ for all $t \in [0, T_{\max})$. Since $\delta_3$ can be chosen as small as we want, then it follows from Sobolev's Embedding Theorem and Lemma 4.1 that $T_{\max} = \infty$, thereby obtaining the inequality (5.90) for all $t \ge 0$. Meanwhile, by the elliptic estimate, we also have

$$\sum_{j=0}^{3} \|\partial_t^j \sigma(t)\|^2_{H^{5-j}(\Omega)} \le C \sum_{j=0}^{3} \|(\partial_t^j \psi_1, \partial_t^j \psi_2)(t)\|^2_{H^{3-j}(\Omega)} \le C|||\boldsymbol{w}(t)|||^2, \quad \forall t \in [0, \infty). \tag{5.94}$$

In a nutshell, if we take the small positive constant $\delta_1$ defined in (5.92), and let $\epsilon \le \delta_1$ and $|||\boldsymbol{w}(0)||| \le \delta_1$, then adding the estimates (5.90) and (5.94) for all $t \ge 0$, we show that the local-in-time solution can be uniquely extended to the global-in-time one and the exponential decay estimate (1.20) of the global-in-time solution holds for all $t \ge 0$. This completes the proof of Theorem 1.2. □

## Data availability

No data was used for the research described in the article.

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