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RELAXATION TIME LIMITS OF SUBSONIC STEADY STATES FOR MULTIDIMENSIONAL HYDRODYNAMIC MODEL OF SEMICONDUCTORS*

YUE-HONG FENG†, HAIFENG HU‡, MING MEI§, GANTUMUR TSOGTGEREL¶, AND GUOJING ZHANG‖

Abstract. This paper is concerned with the relaxation-time limits to a multidimensional radial steady hydrodynamic model of semiconductors in the form of Euler–Poisson equations with sonic or nonsonic boundary as the relaxation time $\tau \to \infty$ and $\tau \to 0^+$, respectively, where the sonic boundary is the critical and difficult case, because of the degeneracy at the boundary and the formation of boundary layers. For the case of $\tau \to \infty$, after showing the boundedness of the density by using the divergence form, we prove the convergence of the solutions to their nontrivial asymptotic states with the convergence order $O(\tau^{-\frac{1}{2}})$ in the L^∞-sense. In order to overcome the degeneracy caused by the critical sonic boundary, we introduce an inverse transform as a technical tool to remove the second-order degeneracy, and observe the advantage of a first-order degeneracy due to the monotonicity of this transformation. Moreover, when $\tau \to 0^+$ with different boundary values, where the boundary layers appear, we show the strong convergence order $O(\tau)$ or $O(\tau^{1-\varepsilon})$ for different boundary cases. In order to overcome the difficulty caused by the boundary layer, we propose a new technique in asymptotic limit analysis and identify the width of the boundary layers as $O(\tau)$. These new proposed methods develop and improve upon the existing studies. Finally, a series of numerical simulations are conducted, which corroborate our theoretical analysis, particularly regarding the formation of boundary layers.

Key words. multidimensional Euler–Poisson equations, sonic boundary, interior subsonic solutions, relaxation time limit

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†School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100022, China (fyh@bjut.edu.cn).

‡Corresponding author. School of Mathematics and Statistics, Changchun University, Changchun 130022, China (huhf@ccu.edu.cn).

§School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi, 330022, China, and Department of Mathematics, Champlain College Saint-Lambert, Quebec, J4P 3P2, Canada, and Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada, and School of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China (ming.mei@mcgill.ca).

¶Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada, and Department of Physics, National University of Mongolia, Ulan Bator 14201, Mongolia (gantumur.tsogtgerel@mcgill.ca).

‖Corresponding author. School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China (zhanggj100@nenu.edu.cn).

1. Introduction and the main results. *Modeling equations.* The hydrodynamic model was first derived by Bløtekjær [3] for electrons in a semiconductor. After appropriate simplifications, the multidimensional time-dependent system in the isentropic case reads

$$\begin{cases} \tilde{\rho}_t + \operatorname{div}(\tilde{\rho}\tilde{u}) = 0, \\ (\tilde{\rho}\tilde{u})_t + \operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + \nabla P = \tilde{\rho}\tilde{E} - \dfrac{\tilde{\rho}\tilde{u}}{\tau}, \\ \lambda^2 \operatorname{div}\tilde{E} = \tilde{\rho} - \tilde{d}(x), \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, n=2,3, \end{cases} \tag{1.1}$$

where $\tilde{\rho}(x,t)\in\mathbb{R}$, $\tilde{u}(x,t)\in\mathbb{R}^n$, and $\tilde{E}(x,t)\in\mathbb{R}^n$ denote the electron density, velocity, and electric field, respectively. The function $P = P(\tilde{\rho}) = T\tilde{\rho}^\gamma$ is the pressure, where $T > 0$ is Boltzmann's constant and $\gamma \geq 1$ is the adiabatic exponent. The constant parameter $\tau > 0$ is the momentum relaxation time. The physical parameter $\lambda > 0$ represents the scaled Debye length. The given background density $\tilde{d}(x) > 0$ is called the doping profile standing for a background fixed charge of ions in the semiconductor crystal. The hydrodynamic model (1.1) is also called Euler–Poisson equation with semiconductor effect. For more details, we refer to treatises [39, 53] and references therein.

In the present paper, we consider the isothermal case, and set $T = 1$ and $\lambda = 1$ without loss of generality, i.e., $P(\tilde{\rho}) = \tilde{\rho}$. Throughout this paper, we consider the steady-state solutions of (1.1) in an annulus domain

$$\mathcal{A} := \{x\in\mathbb{R}^n | r_0 < |x| < r_1\}, \quad 0 < r_0 < r_1,$$

with the inner boundary

$$\Gamma_0 := \{x\in\mathbb{R}^n : |x| = r_0\},$$

and the outer boundary

$$\Gamma_1 := \{x\in\mathbb{R}^n : |x| = r_1\}.$$

Its closure is denoted by

$$\bar{\mathcal{A}} := \Gamma_0 \cup \mathcal{A} \cup \Gamma_1.$$

By noting that $\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) = \tilde{\rho}(\tilde{u}\cdot\nabla)\tilde{u} + \operatorname{div}(\tilde{\rho}\tilde{u})\tilde{u}$, then the corresponding stationary equations of (1.1) can be written as

$$\begin{cases} \operatorname{div}(\tilde{\rho}\tilde{u}) = 0, \\ (\tilde{u}\cdot\nabla)\tilde{u} + \dfrac{\nabla\tilde{\rho}}{\tilde{\rho}} = \tilde{E} - \dfrac{\tilde{u}}{\tau}, \\ \operatorname{div}\tilde{E} = \tilde{\rho} - \tilde{d}(x), \quad x\in\mathcal{A}. \end{cases} \tag{1.2}$$

The aim of our work is to investigate the zero-/infinite relaxation time limits of (1.2) in two- and three-dimensional annulus domains with degenerate boundary, sonic or nonsonic boundaries, and to study various analytical features including the requirement of the doping profile and the adopted methods in the proofs by comparing with the one-dimensional case [16].

Moreover, we call $M := \frac{|\tilde{u}|}{c(\tilde{\rho})}$ the Mach number for $c(\tilde{\rho}) := \sqrt{P'(\tilde{\rho})} = 1$. Here, $c(\tilde{\rho})$ is called the local sound speed. Depending on the size of M, the analytic features of (1.2) vary: if $M > 1$, the stationary flow is called supersonic; if $M < 1$, the corresponding flow is called subsonic; otherwise, $M = 1$ is the sonic state.

In the following, we suppose that d is in $L^\infty(r_0,r_1)$ such that $\tilde{d}(x) := d(r)$ in $\bar{\mathcal{A}}$, and we denote

$$(\tilde{\rho},\tilde{u},\tilde{E})(x) := \left(\rho(r), u(r)\frac{x}{r}, E(r)\frac{x}{r}\right), \tag{1.3}$$

where $r=|x|$, and the boundary conditions are stated as follows:

$$(\tilde{\rho}|_{\Gamma_0}, \tilde{\rho}|_{\Gamma_1}, \tilde{\rho}\tilde{u}|_{\Gamma_0}) = \left(\rho(r_0), \rho(r_1), \rho(r_0)u(r_0)\frac{x}{r}\right) = \left(a, b, c\frac{x}{r}\right) \tag{1.4}$$

for positive constants (a,b,c). Therefore, (1.2) and (1.4) are reduced to

$$\begin{cases} (r^{n-1}\rho u)_r = 0, \\ \left(r^{n-1}\rho u^2\right)_r + r^{n-1}\rho_r = r^{n-1}\rho\left(E - \dfrac{u}{\tau}\right), \\ (r^{n-1}E)_r = r^{n-1}(\rho - d(r)), \\ (\rho(r_0), \rho(r_1), u(r_0)) = \left(a, b, \dfrac{c}{a}\right), \end{cases} \tag{1.5}$$

so that the sonic state is redefined by $|u| = M = 1$. Clearly, each pair of the solution (ρ,u,E) to system (1.5) always corresponds to a solution $(\tilde{\rho},\tilde{u},\tilde{E})$ to (1.2) and (1.4).

Definition 1.1 (radial subsonic/supersonic solution). *We call* $(\tilde{\rho},\tilde{u},\tilde{E})$ *with* $M<1 (M>1)$ *in* $\bar{\mathcal{A}}$ *radial subsonic (correspondingly, supersonic) to systems* (1.2) *and* (1.4) *if the corresponding solution* (ρ,u,E) *of* (1.5) *satisfies* $|u|<1 (|u|>1)$ *over* (r_0,r_1).

We now focus on (1.5). Let $J := r^{n-1}\rho u$. Without loss of generality, let us also take $J>0$. From the first equation of (1.5) we have

$$J = \text{ constant} = r_0^{n-1}c, \quad r\in[r_0,r_1], \tag{1.6}$$

which implies

$$r^{n-1}\rho u^2 = \frac{J^2}{r^{n-1}\rho}. \tag{1.7}$$

By substituting (1.7) into the second equation of (1.5) and dividing the resulting equation by ρ, we obtain

$$\left(\frac{r^{n-1}}{\rho} - \frac{J^2}{r^{n-1}\rho^3}\right)\rho_r = r^{n-1}E + (n-1)J^2\frac{1}{r^n\rho^2} - \frac{J}{\tau\rho}. \tag{1.8}$$

By differentiating (1.8) with respect to r and substituting the third equation of (1.5) into the resulting equation, problem (1.5) becomes for $r\in(r_0,r_1)$,

$$\begin{cases} \left(\left(\dfrac{r^{n-1}}{\rho} - \dfrac{J^2}{r^{n-1}\rho^3}\right)\rho_r\right)_r = r^{n-1}(\rho-d) - \dfrac{n(n-1)J^2}{r^{n+1}\rho^2} + \left(\dfrac{J}{\tau\rho^2} - 2\dfrac{(n-1)J^2}{r^n\rho^3}\right)\rho_r, \\ \rho(r_0) = a, \quad \rho(r_1) = b, \end{cases} \tag{1.9}$$

where $r_0^{n-1}a$, $r_1^{n-1}b \ge J$. Moreover, by noting (1.6), it follows that

$$a \ge c \quad \text{and} \quad b \ge \left(\frac{r_0}{r_1}\right)^{n-1} c. \tag{1.10}$$

Moreover, from (1.5), we also get

$$\begin{cases} r^{n-1}E = \dfrac{r^{n-1}}{\rho}\left(1-\left(\dfrac{J}{r^{n-1}\rho}\right)^2\right)\rho_r + \dfrac{J}{\tau\rho} - (n-1)\dfrac{J^2}{r^n\rho^2}, \\ (r^{n-1}E)_r = r^{n-1}(\rho - d(r)). \end{cases} \tag{1.11}$$

It follows from definition of J and Definition 1.1 that (ρ, u) is a pair of subsonic solutions to (1.9) provided that

$$r^{n-1}\rho = \frac{J}{u} > J. \tag{1.12}$$

We assume that the doping profile $d(r)$ is of class $L^\infty(0,1)$. For simplicity of notation, its infimum and supremum over $[r_0, r_1]$ is denoted by

$$\underline{d} := \inf_{r\in[r_0,r_1]} d(r) \quad \text{and} \quad \bar{d} := \sup_{r\in[r_0,r_1]} d(r),$$

respectively.

The existence and uniqueness of the subsonic solution to the sonic boundary value problem of system (1.9) was obtained by Chen et al. in [5].

Regarding the relaxation time limit $\tau \to \infty$ or $\tau \to 0^+$, let us denote the solutions of (1.9) and (1.11) by $(\rho_\tau, E_\tau)(r)$ with respect to τ. In what follows, we consider limit problems about relaxation time τ in (1.9). On the one hand, when $\tau \to \infty$, let us set $\bar{\rho}(r) = \lim_{\tau\to\infty}\rho_\tau(r)$ and $\bar{E}(r) = \lim_{\tau\to\infty} E_\tau(r)$, which formally satisfy

$$\begin{cases} \left(\left(\dfrac{r^{n-1}}{\bar{\rho}} - \dfrac{J^2}{r^{n-1}\bar{\rho}^3}\right)\bar{\rho}_r\right)_r = r^{n-1}(\bar{\rho}-d) - \dfrac{n(n-1)J^2}{r^{n+1}\bar{\rho}^2} - 2\dfrac{(n-1)J^2}{r^n\bar{\rho}^3}\bar{\rho}_r, r\in(r_0,r_1), \\ \bar{\rho}(r_0) = a, \quad \bar{\rho}(r_1) = b, \end{cases} \tag{1.13}$$

and

$$\begin{cases} r^{n-1}\bar{E} = \dfrac{r^{n-1}}{\bar{\rho}}\left(1-\left(\dfrac{J}{r^{n-1}\bar{\rho}}\right)^2\right)\bar{\rho}_r - (n-1)\dfrac{J^2}{r^n\bar{\rho}^2}, \\ (r^{n-1}\bar{E})_r = r^{n-1}(\bar{\rho} - d). \end{cases} \tag{1.14}$$

We point out that there is no boundary layer in this case.

On the other hand, when $\tau \to 0^+$, we let $\underline{\rho}(r) = \lim_{\tau\to 0^+}\rho_\tau(r)$, which formally satisfies that

$$\underline{\rho} = \text{ constant}, \quad r \in (r_0, r_1), \tag{1.15}$$

which deduces that the boundary layers will appear in this case.

Background of studies. We now draw a picture of the progress on the studies of well-posedness for hydrodynamic model of semiconductors. There are major advances in the mathematical theory of steady-state Euler–Poisson equations with/without the semiconductor effect. For the subsonic steady-state flows, Degond and Markowich [8] first proved the existence of the subsonic solution to the one-dimensional steady-state Euler–Poisson with the semiconductor effect when its boundary states belong to the subsonic region. Subsequently, Degond and Markowich [9] further showed the existence and local uniqueness of irrotational subsonic flows to the three-dimensional steady-state semiconductor hydrodynamic model. After that, the steady-state subsonic flows were investigated in various physical boundary conditions and different dimensions [2, 14, 24, 42]. As for the supersonic steady-state flows, Peng and Violet [45] established the existence and uniqueness of the supersonic solutions with the semiconductor effect. Lately, Donatelli and Juhász [10] and Donatelli and Marcati [11] investigated the oscillations and defect measures for the quasi-neutral limit and the primitive equations. Regarding transonic steady states, Ascher et al. [1] first examined the existence of the transonic solution to the one-dimensional isentropic Euler–Poisson equations, and then Rosini [49] extended this work to the nonisentropic case. When the doping profile is nonconstant, Gamba [19, 20] studied the one-dimensional and two-dimensional transonic solutions with shocks, respectively. Luo et al. [35] and Luo and Xin [36] further considered the one-dimensional Euler–Poisson equations without the semiconductor effect, a comprehensive analysis on the structure and classification of steady states was carried out in [36]. Meanwhile, both structural and dynamical stability of steady transonic shock solutions was obtained in [35]. And then, He and Huang [26] and Huang et al. [28] studied the nonlinear stability of large amplitude viscous shock wave and the stability of transonic contact discontinuity for two-dimensional steady compressible Euler flows in a finitely long nozzle. Recently, Li et al. [33, 34] explored the one-dimensional semiconductor Euler–Poisson equations with the sonic boundary condition. Motivated by their pioneering works [33, 34], there is a series of interesting generalizations into the transonic doping profile case in [4], the case of transonic C^∞-smooth steady states in [51], the multidimensional cases in [5, 6], and even the bipolar case [41]. Lately, Feng, Mei, and Zhang [18] demonstrated the structural stability of these smooth transonic steady-states by the local singularity analysis. See also [15, 27, 52, 54] for the structural stability in the sonic boundary case.

Moreover, in addition to these results on the well-posedness, a series of studies were concerned with the asymptotic limits in the hydrodynamic model, such as the Newtonian limits in the speed of light for the relativistic Euler–Poisson equations [37, 38, 40], the quasi-neutral limits [7, 12, 13, 29, 43, 44, 47, 50], the zero-electron-mass limits [22, 23, 32], and the zero-relaxation-time limits [17, 25, 30, 31, 46, 48], for instance. These investigations are important and amazing, but do not involve the degeneracy, to the best of our knowledge. Recently, the quasi-neutral limit for subsonic-sonic solution of system (1.5) with the degenerate sonic boundary was investigated by Chen et al. [6]. Very recently, Feng et al. [16] consider the relaxation time limits problem to the one-dimensional Euler–Poisson equations, and get the strong convergence of the approximate solutions to their asymptotic profiles in L^∞ norm.

However, by noting the difficulties caused by the degeneracy and boundary layers, the relaxation-time limits for subsonic steady-state solution of system (1.5) with sonic or nonsonic boundary values are still open and challenging. Hence, the goal of this paper is to answer this question in two directions.

Main results. The main results of the paper are stated as follows.

THEOREM 1.2 (infinite-relaxation-time limits). *Assume that the doping profile $d \in L^\infty[r_0, r_1]$ is subsonic such that $\inf_{r\in[r_0,r_1]} \left\{r^{n-1}d(r)\right\} - (n-2)(n-1) > J$. Let $(\rho_\tau, E_\tau)(r)$ be the interior subsonic solution of system* (1.5) *corresponding to the doping profile $d(x)$. Then problem* (1.5) *converges to* (1.13) *with* (1.14) *as $\tau \to \infty$ uniformly in the sense that*

$$\|\rho_\tau - \bar{\rho}\|_{L^\infty(r_0,r_1)} \le C\tau^{-\frac{1}{2}} \quad \text{and} \quad \|E_\tau - \bar{E}\|_{L^\infty(r_0,r_1)} \le C\tau^{-\frac{1}{2}}, \tag{1.16}$$

where $C > 0$ is a constant independent of $\tau > 0$.

THEOREM 1.3 (zero-relaxation-time limits). *Assume that the doping profile $d \in L^\infty[r_0, r_1]$ is subsonic such that $\inf_{r\in[r_0,r_1]}\{r^{n-1}d(r)\} - (n-2)(n-1) > J$. Let $(\rho_\tau, E_\tau)(r)$ be the interior subsonic solution of system* (1.5) *corresponding to the doping profile $d(r)$. Then the following two results hold:*

(I) *If $\rho_\tau(r_0) = \rho_\tau(r_1) = a \geq \frac{J}{r_0^{n-1}}$, then $(\rho_\tau, E_\tau)(r)$ converges to its asymptotic state*

$$(\underline{\rho}, \underline{E}) = \left(a, \frac{a}{n} r - \frac{1}{r^{n-1}} \left(\frac{a}{n} r_0^n + \int_{r_0}^r s^{n-1} d(s) ds + (n-1) \frac{J^2}{r_0^n a^2} \right)\right) \tag{1.17}$$

as $\tau \to 0^+$, without boundary layer for the density $\rho_\tau(r)$, but with a huge gap $\frac{J}{\tau r^{n-1} a}$ between the electric field E_τ and its asymptotic state $\underline{E}$ over the entire interval $[r_0, r_1]$. Namely, there exist two constants $C > 0$ and $0 < \tau_0 \ll 1$ such that for all $0 < \tau \leq \tau_0$, the following error estimates hold:

$$\|\rho_\tau(r) - a\|_{L^\infty[r_0,r_1]} \leq C\tau, \tag{1.18}$$

$$\left\| E_\tau - \underline{E} - \frac{J}{\tau r^{n-1} a} \right\|_{L^\infty[r_0,r_1]} \leq C\tau, \tag{1.19}$$

and

$$\left\| \frac{dE_\tau}{dr} - \left(a - d - \frac{(n-1)J}{\tau r^n a} - \frac{n-1}{r} \underline{E} \right) \right\|_{L^\infty[r_0,r_1]} \leq C\tau. \tag{1.20}$$

(II) *If $\rho_\tau(r_0) = a$, $\rho_\tau(r_1) = b$ with $b > a \geq \frac{J}{r_0^{n-1}}$, then the density $\rho_\tau(r)$ converges to its asymptotic state $\underline{\rho}(r) \equiv a$ outside the boundary layer, and the width of boundary layer becomes thinner as $\tau \to 0^+$; the electric field $E_\tau(r)$ converges to its asymptotic state*

$$\underline{E}(x) = \frac{a}{n} r - \frac{1}{r^{n-1}} \left(\frac{a}{n} r_0^n + \int_{r_0}^r s^{n-1} d(s) ds + (n-1) \frac{J^2}{r_0^n a^2} \right)$$

with a huge correction $\frac{J}{\tau r^{n-1} a}$ over the whole interval $[r_0, r_1]$ as $\tau \to 0^+$. Namely, there exist two constants $C > 0$ and $0 < \tau_1 \ll 1$ such that for all $0 < \tau \leq \tau_1$ and $0 < \varepsilon < 1/2$, the following error estimates outside the boundary layer hold:

$$|\rho_\tau(r) - a| \leq C\tau \quad \forall r \in [r_0, r_1 - \alpha\tau^{1-\varepsilon}], \tag{1.21}$$

$$\left\| E_\tau - \underline{E} - \frac{J}{\tau r^{n-1} a} \right\|_{L^\infty[r_0,r_1]} \leq C\tau^{1-\varepsilon}, \tag{1.22}$$

and

$$\left\| \frac{dE_\tau}{dr} - \left(a - d - \frac{(n-1)J}{\tau r^n a} - \frac{n-1}{r} \underline{E} \right) \right\|_{L^\infty[r_0,r_1]} \leq C\tau^{1-\varepsilon}. \tag{1.23}$$

Remark 1.4. If we choose $n = 1$ and $r_0 = 0$ in the asymptotic state (1.17), we get

$$(\underline{\rho}, \underline{E}) = \left(a, ar - \int_0^r d(s) ds \right),$$

which is exactly the asymptotic state of one-dimensional Euler–Poisson equations (see [16]). That is to say that our multidimensional results here contain the one-dimensional case.

Remark 1.5. It should be pointed out that the different forms of (1.5) are very important in the investigations of different problems. At first, we use the equivalent form (1.9), which is a divergence form, to establish the boundedness of density ρ_τ (see Proposition 2.4). After that, we introduce a new unknown $m_\tau(r) = r^{n-1}\rho_\tau(r)$ and use another equivalent form (3.2) to study the limit problem as $\tau \to \infty$ (see section 3). And then, the third equivalent form (4.1) is used for investigating the limit problem as $\tau \to 0+$ (see section 4).

Difficulties and strategies. Now, let us give a brief sketch of the proof of our main results and show the main difference in techniques between the one-dimensional and multidimensional Euler–Poisson equations. In the first part, we study the relaxation time limits problem as $\tau \to \infty$. Due to the boundary degeneracy, the study of the infinite-relaxation time limit problems of interior subsonic solutions over $[r_0, r_1]$ appears challenging. If we use the usual method as that in [52], then we cannot remove the difficulty caused by boundary degeneracy, and therefore, the uniform estimates about the error function $\rho_\tau - \bar{\rho}$ on τ cannot be established. In order to overcome this difficulty, we use an inverse transform (see (3.4)) to turn the second-order degeneracy into first-order degeneracy. Fortunately, this remaining first-order degeneracy is a good term since the transform used here is monotonically increasing (see (3.9)). Then we efficiently overcome the degenerate effect.

In the second part, we continue to study the relaxation time limit problem when $\tau \to 0^+$. In detail, we study the limit problems with equivalent form (4.1) in two cases according to different boundary values. To this end, we first establish two comparison principles and show the estimates on $\frac{d\rho_\tau}{dr}$ in Lemma 4.2. It should be pointed out (4.8) containing the second-order term $(\frac{d\rho_\tau}{dr})^2$ which is very different from the corresponding equation for the one-dimensional Euler–Poisson system (see [16]). Therefore, the techniques used there does not work. We introduce new methods to remove this difficulty (see the proof of Lemma 4.2).

Case 1. $\rho_\tau(r_0) = \rho_\tau(r_1) = a \geq \frac{J}{r_0^{n-1}}$. There is no boundary layer effect for this case. First, by using comparison principles on $\frac{d\rho_\tau}{dr}$ and the proof by contradiction, we establish the upper and lower bounds of $\rho_\tau(r) - \rho_\tau(r_0)$ (see (4.47) and (4.51), respectively). After that, we study the zero-relaxation time limit for E_τ. Based on (1.11), we introduce a new unknown $\Xi = r^{n-1}E_\tau$. Then, we consider the initial value problem (4.54) for the ordinary differential equation of Ξ. After careful estimating, we get $\left\|\Xi(r) - \left(M\left(\frac{1}{\tau}, a, r_0\right) + L(a,r) - D(r)\right)\right\|_{L^\infty[r_0,r_1]} \leq C\tau$ (see (4.60)). Furthermore, in view of the boundedness of r, we obtain the estimate for E_τ (see (4.64)). That is to say $E_\tau \to \frac{1}{r^{n-1}}(\frac{J}{\tau a} - (n-1)\frac{J^2}{r_0^n a^2}) + \frac{a}{n}(r - \frac{r_0^n}{r^{n-1}}) - \frac{1}{r^{n-1}}\int_{r_0}^r s^{n-1}d(s)ds =: \mathscr{E}(r)$ in $L^\infty[r_0, r_1]$ with the converging rate τ. Finally, from the second equation of (1.11) and the estimate for E_τ, we get the converging property for $\frac{dE_\tau}{dr}$ (see (4.68)).

Case 2. $\rho_\tau(r_0) = a$, $\rho_\tau(r_1) = b$, and $a \neq b$ with $b > a \geq \frac{J}{r_0^{n-1}}$. In this case, the boundary layer must appear since $a \neq b$. Here, we use a new method to look at the width of the boundary layer. For $r \in [r_0, r_1]$, we first establish the lower bound as $\rho_\tau(r) - a \geq -(3\frac{\rho_+}{\bar{B}} + 1)\tau$ (see (4.69)). Meanwhile, we prove the upper bound like $\rho_\tau(r) - b \leq (3\frac{\bar{d}+H}{\bar{B}} + 1)\tau$ (see (4.73)). Furthermore, we investigate the boundary layer near the right endpoint $r = r_1$. For any $r \in [r_0, r_1 - \alpha\tau^{1-\varepsilon}]$, we obtain the estimate as $\rho_\tau(r) - a \leq (3\frac{\bar{d}+H}{\bar{B}} + 1)\tau$ (see (4.77)). Moreover, we prove the estimates for E_τ and

$\frac{dE_\tau}{dr}$ like (4.86) and (4.87), respectively. In fact, we find that $E_\tau \to \mathscr{E}(r)$ in $L^\infty[r_0,r_1]$ with the different converging rate $\tau^{1-\varepsilon}$.

We conclude this section by stating the arrangement of the rest of this paper. In section 2, we give the important preliminaries such as boundedness and regularity of subsonic solutions. In section 3, we analyze the infinite-relaxation-time limits of subsonic steady states and prove Theorem 1.2. In section 4, we investigate zero-relaxation-time limits of subsonic steady states and finish the proof of Theorem 1.3. In section 5, we carry out some numerical simulations in different cases, which perfectly validate our theoretical studies in Theorems 1.2 and 1.3.

2. Preliminaries.

In this section we give the important preliminaries for later use. First, we recall the definition of the interior subsonic solution.

DEFINITION 2.1. *We say ρ_τ is an interior subsonic solution of the boundary value problem* (1.9) *if $r_0^{n-1}a = r_0^{n-1}b = J$ and $r^{n-1}\rho_\tau > J$ for $r \in (r_0,r_1)$, and $(r^{n-1}\rho_\tau - J)^2 \in H_0^1(r_0,r_1)$, and it holds that*

$$\begin{aligned}&\int_{r_0}^{r_1}\left(\left(\frac{r^{n-1}}{\rho}-\frac{J^2}{r^{n-1}\rho^3}\right)\rho_r+2\frac{(n-1)J^2}{r^n\rho^3}-\frac{J}{\tau\rho^2}\right)\varphi_r dr\\&\quad+\int_{r_0}^{r_1}\left(r^{n-1}(\rho-d)-\frac{n(n-1)J^2}{r^{n+1}\rho^2}\right)\varphi dr=0,\end{aligned}\tag{2.1}$$

for any $\varphi\in H_0^1(r_0,r_1)$.

In addition, we continue to recall the existence and uniqueness of interior subsonic solutions, which is excerpted from Theorem 1.4 in [5].

PROPOSITION 2.2 (existence [5]). *Suppose that the doping profile d is subsonic such that $r^{n-1}d(r)\in L^\infty(r_0,r_1)$. Then the following two results hold:*

1. *For $n=2$, if*

$$\bar{\mathfrak{B}} := \sup_{r\in[r_0,r_1]}\left\{r^{n-1}d(r)\right\}+\frac{1}{\tau}>J,$$

and

$$\inf_{r\in[r_0,r_1]}\left\{r^{n-1}d(r)\right\}+\frac{J}{\tau\bar{\mathfrak{B}}}>J,$$

then for all $\tau\in(0,\infty]$ the boundary value problem (1.9) *admits a unique interior subsonic solution ρ_τ over* $[r_0,r_1]$.

2. *For $n=3$, if*

$$\bar{\mathscr{B}} := \sup_{r\in[r_0,r_1]}\left\{r^{n-1}d(r)+\frac{2r}{\tau}-2\right\}>J,$$

and

$$\min_{r\in[r_0,r_1]}\left\{r^{n-1}d(r)+\frac{2rJ}{\tau\bar{\mathscr{B}}}-2\right\}>J,$$

then for all $\tau\in(0,\infty]$ the boundary value problem (1.9) *admits a unique interior subsonic solution ρ_τ over* $[r_0,r_1]$.

Thus, the subsonic condition $\inf_{r\in[r_0,r_1]}\left\{r^{n-1}d(r)\right\}-(n-2)(n-1)>J$ for doping profile in Theorems 1.2 and 1.3, can ensure the existence and uniqueness of subsonic or sonic-subsonic solution to the boundary value problem (1.9) for any $\tau>0$.

Furthermore, from [5], the regularity of ρ_τ is stated as follows.

PROPOSITION 2.3 (regularity). *For $1 \le p < 2$, the subsonic solution ρ_τ satisfies the following properties:*

$$\rho_\tau \in C^{\frac{1}{2}}(r_0, r_1) \quad \text{and} \quad \rho_\tau \in W^{1,p}(r_0, r_1). \tag{2.2}$$

Next, let us show the boundedness of ρ_τ.

PROPOSITION 2.4 (boundedness of ρ_τ). *If ρ_τ is an interior subsonic solution to problem* (1.9), *then there exist constants $\rho_-, \rho_+ > 0$ independent of τ, such that*

$$\rho_- \le \rho_\tau \le \rho_+ \quad \text{over} \quad r \in [r_0, r_1]. \tag{2.3}$$

Proof. It follows from (1.12) that

$$\rho_\tau \ge \frac{J}{r^{n-1}} \ge \frac{J}{r_1^{n-1}} := \rho_\ell > 0. \tag{2.4}$$

Next, let us introduce

$$\rho_+ = \max\left\{a, b, \bar{d} + \frac{n(n-1)J^2}{r_0^{2n}\rho_\ell^2}\right\} + 1. \tag{2.5}$$

Then it is not difficult to observe that the zero-order terms $r^{n-1}(\rho_\tau - d) - \frac{n(n-1)J^2}{r^{n+1}\rho_\tau^2}$ in (1.9) are positive over $\Omega_+ := \{r \in [r_0, r_1] | \rho_\tau(r) > \rho_+\}$. In fact,

$$\begin{aligned} r^{n-1}(\rho_\tau - d) - \frac{n(n-1)J^2}{r^{n+1}\rho_\tau^2} &\ge r^{n-1}(\rho_\tau - \bar{d}) - \frac{n(n-1)J^2}{r_0^{n+1}\rho_\ell^2} \\ &> r^{n-1}\frac{n(n-1)J^2}{r_0^{2n}\rho_\ell^2} - \frac{n(n-1)J^2}{r_0^{n+1}\rho_\ell^2} \\ &\ge r_0^{n-1}\frac{n(n-1)J^2}{r_0^{2n}\rho_\ell^2} - \frac{n(n-1)J^2}{r_0^{n+1}\rho_\ell^2} = 0. \end{aligned} \tag{2.6}$$

After that, for the coefficient of the first-order term in (1.9), we get

$$\left|\frac{J}{\tau\rho_\tau^2} - 2\frac{(n-1)J^2}{r^n\rho_\tau^3}\right| \le \frac{J}{\tau\rho_\tau^2} + 2\frac{(n-1)J^2}{r_0^n\rho_\tau^3} \le \frac{J}{\tau\rho_\ell^2} + 2\frac{(n-1)J^2}{r_0^n\rho_\ell^3} < C, \tag{2.7}$$

namely,

$$\left\|\frac{J}{\tau\rho_\tau^2} - 2\frac{(n-1)J^2}{r^n\rho_\tau^3}\right\|_{L^\infty(r_0,r_1)} < C. \tag{2.8}$$

Later, for the coefficient of the second-order term of (1.9) in the divergence form, by (1.6), (1.10), and (2.5), we obtain

$$\begin{aligned} \frac{r^{n-1}}{\rho_\tau} - \frac{J^2}{r^{n-1}\rho_\tau^3} &= \frac{r^{n-1}}{\rho_\tau}\left(1 - \frac{J^2}{(r^{n-1}\rho_\tau)^2}\right) \\ &\ge \frac{r^{n-1}}{\rho_\tau}\left(1 - \frac{J^2}{(r_0^{n-1}\rho_\tau)^2}\right) \\ &\ge \frac{r^{n-1}}{\rho_\tau}\left(1 - \frac{J^2}{(r_0^{n-1}(a+1))^2}\right) \\ &\ge \frac{r^{n-1}}{\rho_\tau}\left(1 - \frac{J^2}{(r_0^{n-1}(c+1))^2}\right) \\ &= \frac{r^{n-1}}{\rho_\tau}\left(1 - \frac{J^2}{J^2 + 2r_0^{2(n-1)}c + r_0^{2(n-1)}}\right) \\ &\ge C > 0, \quad \text{as} \quad r \in \Omega_+. \end{aligned} \tag{2.9}$$

Eventually, by combining (2.6), (2.8)–(2.9), and the weak maximum principle (see Theorem 8.1 in [21]), we have

$$\rho_\tau \le \rho_+, \tag{2.10}$$

which implies that the right-hand side of (2.3) holds.

Of course, ρ_ℓ is a lower bound of ρ_τ. We give another lower bound of it below. Let us set

$$\rho_- = \min\{a, b, \underline{d}\}. \tag{2.11}$$

Then the zero-order terms in (1.9) are negative on $\Omega_- := \{r \in [r_0, r_1] | \rho_\tau(r) < \rho_-\}$. Namely,

$$r^{n-1}(\rho_\tau - d) - \frac{n(n-1)J^2}{r^{n+1}\rho_\tau^2} < 0 \quad \text{as} \quad r \in \Omega_-.$$

Therefore, by the weak maximum principle again, it holds that

$$\rho_\tau \ge \rho_-. \tag{2.12}$$

This is the left-hand side of (2.3). The proof of Proposition 2.4 is completed. ∎

Moreover, we can establish the boundedness for $\bar\rho$ as follows.

PROPOSITION 2.5. *If $\bar\rho$ is an interior subsonic solution to problem* (1.13)*, then*

$$\rho_- \le \bar\rho \le \rho_+ \quad \text{over} \quad r \in [r_0, r_1], \tag{2.13}$$

where $\rho_-, \rho_+ > 0$ are the same constants in Proposition 2.5 *independent of τ.*

Proof. The proof is very similar to that in the proof of Proposition 2.4. We omit it here for simplicity. ∎

3. Relaxation limits as $\tau \to \infty$.

This section is devoted to proving our main result when $\tau \to \infty$. For any constants $a > \frac{J}{r_0^{n-1}}$ and $b > \frac{J}{r_1^{n-1}}$, by using the following method, we can get the similar results as (1.16) in Theorem 1.2. Here, we only consider the case in which $a = \frac{J}{r_0^{n-1}}$ and $b = \frac{J}{r_1^{n-1}}$ for the sake of simplicity.

For the convenience of study in this section, we make a transformation for (1.9). In detail, let us set

$$m_\tau(r) = r^{n-1}\rho_\tau(r), \tag{3.1}$$

then

$$r^{n-1}\rho_\tau u^2 = \frac{J^2}{r^{n-1}\rho_\tau} = \frac{J^2}{m_\tau}.$$

Substituting this into the second equation of (1.5), we have

$$\left(1 - \frac{J^2}{m_\tau^2}\right)\frac{dm_\tau}{dr} = m_\tau\left(E_\tau + \frac{n-1}{r}\right) - \frac{J}{\tau}.$$

Then problem (1.5) becomes

$$\begin{cases} \left(1 - \frac{J^2}{m_\tau^2}\right)\frac{dm_\tau}{dr} = m_\tau\left(E_\tau + \frac{n-1}{r}\right) - \frac{J}{\tau}, \quad r \in (r_0, r_1), \\ (r^{n-1}E_\tau)_r = m_\tau - r^{n-1}d(r), \\ (m_\tau(r_0), m_\tau(r_1)) = (r_0^{n-1}a, r_1^{n-1}b). \end{cases} \tag{3.2}$$

Dividing the first equation of (3.2) by m_τ, then multiplying the resulting equation by r^{n-1}, and then differentiating the resulting equation with respect to r, and substituting the second equation of (3.2) into the resulting equation, we obtain

$$\tag{3.3} \begin{cases} \left(r^{n-1}(\omega_\tau)_r\right)_r = m_\tau - r^{n-1}d(r) + (n-1)(n-2)r^{n-3} - \left(\dfrac{r^{n-1}J}{m_\tau \tau}\right)_r, \quad r \in (r_0, r_1), \\ (m_\tau(r_0), m_\tau(r_1)) = (r_0^{n-1}a, r_1^{n-1}b), \end{cases}$$

where

$$\tag{3.4} \omega_\tau = \omega(m_\tau(r)) := \ln m_\tau + \frac{J^2}{2m_\tau^2}.$$

When $\tau \to \infty$, it follows that

$$\bar{m} := \lim_{\tau\to\infty} m_\tau = \lim_{\tau\to\infty} r^{n-1}\rho_\tau = r^{n-1}\bar{\rho},$$

which satisfies

$$\tag{3.5} \begin{cases} \left(r^{n-1}\bar{\omega}_r\right)_r = \bar{m} - r^{n-1}d(r) + (n-1)(n-2)r^{n-3}, r \in (r_0, r_1), \\ (\bar{m}(r_0), \bar{m}(r_1)) = (r_0^{n-1}a, r_1^{n-1}b). \end{cases}$$

Here,

$$\bar{\omega} = \omega(\bar{m}(r)) := \ln \bar{m} + \frac{J^2}{2\bar{m}^2}.$$

Proof of Theorem 1.2. Let us set $V = \bar{\omega} - \omega_\tau$. By taking the difference between (3.3) and (3.5), we obtain

$$\tag{3.6} \begin{cases} \left(r^{n-1}\left(\bar{\omega} - \omega_\tau\right)_r\right)_r = \bar{m} - m_\tau + \left(\dfrac{r^{n-1}J}{m_\tau \tau}\right)_r, \quad r \in (r_0, r_1), \\ V|_{r=r_0} = V|_{r=r_1} = 0. \end{cases}$$

In view of Propositions 2.4 and 2.5, we get

$$\tag{3.7} J \le \bar{m}, \quad m_\tau \le r_1^{n-1}\rho_+, \quad r \in (r_0, r_1).$$

Multiplying the first equation of (3.6) by V and integrating the resulting equation over $[r_0, r_1]$, and then by integration by parts, we get

$$\tag{3.8} \int_{r_0}^{r_1} r^{n-1}|V_r|^2 dr + \int_{r_0}^{r_1} (\bar{m} - m_\tau)V dx = \frac{1}{\tau}\int_{r_0}^{r_1} \frac{r^{n-1}J}{m_\tau} V_r dr.$$

Noting the monotonicity of the function $\omega_\tau = \omega(m_\tau)$, we obtain

$$\tag{3.9} \int_{r_0}^{r_1} (\bar{m} - m_\tau)V dr = \int_{r_0}^{r_1} (\bar{m} - m_\tau)(\bar{\omega} - \omega_\tau) dr \ge 0.$$

Then it follows from (3.7)–(3.9) and the Hölder inequality that

$$
\begin{aligned}
r_0^{n-1}\int_{r_0}^{r_1}|V_r|^2dr \leq & \int_{r_0}^{r_1} r^{n-1}|V_r|^2dr \\
\leq & \frac{1}{\tau}\int_{r_0}^{r_1}\left|\frac{r^{n-1}J}{m_\tau}\right||V_r|\,dr \\
\leq & \frac{r_1^{n-1}}{\tau}\int_{r_0}^{r_1}|V_r|\,dr \\
\leq & \frac{r_1^{n-1}}{\tau}(r_1-r_0)^{\frac{1}{2}}\left(\int_{r_0}^{r_1}|V_r|^2\,dr\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.10}
$$

Hence, we have

$$
\|V_r\|_{L^2[r_0,r_1]} \leq (r_1-r_0)^{\frac{1}{2}}\left(\frac{r_1}{r_0}\right)^{n-1}\tau^{-1}. \tag{3.11}
$$

The Poincaré inequality implies that

$$
\|V\|_{L^2[r_0,r_1]} \leq C\|V_r\|_{L^2[r_0,r_1]} \leq C\tau^{-1}. \tag{3.12}
$$

Then it follows from the Sobolev imbedding theorems that

$$
\|V\|_{L^\infty[r_0,r_1]} \leq C\|V\|_{H^1[r_0,r_1]} \leq C\tau^{-1}. \tag{3.13}
$$

By (3.2) and the Taylor series of ω_τ and $\bar{\omega}$ at $m_\tau = J$ and $\bar{m} = J$, respectively, we have

$$
V = \bar{\omega} - \omega_\tau = \left(\ln\bar{m} + \frac{J^2}{2\bar{m}^2}\right) - \left(\ln m_\tau + \frac{J^2}{2m_\tau^2}\right) = \frac{(\bar{m}-m_\tau)(\bar{m}+m_\tau-2J)}{J^2} + \cdots. \tag{3.14}
$$

Then in view of (3.7), we obtain

$$
|V| = |\bar{\omega}-\omega_\tau| \geq \begin{cases} |m_\tau - \bar{m}|^2 & \text{as} \quad m_\tau \to J, \bar{m} \to J, \\ |m_\tau - \bar{m}| & \text{otherwise.} \end{cases} \tag{3.15}
$$

Hence, by combining (3.13) and (3.15), we get

$$
\|m_\tau - \bar{m}\|_{L^\infty(r_0,r_1)} \leq C\tau^{-\frac{1}{2}}, \tag{3.16}
$$

which implies

$$
\|\rho_\tau - \bar{\rho}\|_{L^\infty(r_0,r_1)} = \left\|\frac{1}{r^{n-1}}(m_\tau - \bar{m})\right\|_{L^\infty(r_0,r_1)} \leq C\tau^{-\frac{1}{2}}. \tag{3.17}
$$

This is the first part of (1.16).

Next, we study the infinity-relaxation time limit for E. Dividing the first equation of (3.2) by m_τ, then multiplying the resulting equation by r^{n-1}, and noting the definitions of ω_τ and $\bar{\omega}$, we obtain

$$
\begin{cases} r^{n-1}\bar{\omega}_r = r^{n-1}\bar{E} + (n-1)r^{n-2}, \\ r^{n-1}(\omega_\tau)_r = r^{n-1}E_\tau + (n-1)r^{n-2} - \dfrac{Jr^{n-1}}{\tau m_\tau}. \end{cases} \tag{3.18}
$$

Then, by noting (3.7) and (3.11), we get

(3.19)
$$\left\|r^{n-1}\bar{E}-r^{n-1}E_\tau\right\|_{L^2[r_0,r_1]}\le\left\|r^{n-1}(\bar{\omega}-\omega_\tau)_r\right\|_{L^2[r_0,r_1]}+\frac{J}{\tau}\left\|\frac{r^{n-1}}{m_\tau}\right\|_{L^2[r_0,r_1]}\le C\tau^{-1}.$$

Moreover, by the second equation of (3.2), we have

$$(r^{n-1}\bar{E})_r=\bar{m}-r^{n-1}d(r)\quad\text{and}\quad(r^{n-1}E_\tau)_r=m_\tau-r^{n-1}d(r). \tag{3.20}$$

Then, by the first part of (1.16) and (3.20), we get

$$\|(r^{n-1}(E_\tau-\bar{E}))_r\|_{L^\infty(r_0,r_1)}=\|m_\tau-\bar{m}\|_{L^\infty(r_0,r_1)}\le C\tau^{-\frac{1}{2}}. \tag{3.21}$$

This, together with (3.19), implies

$$\|r^{n-1}(E_\tau-\bar{E})\|_{H^1(r_0,r_1)}\le C\tau^{-\frac{1}{2}}. \tag{3.22}$$

Then, from the Sobolev imbedding theorems, we obtain

$$\|E_\tau-\bar{E}\|_{L^\infty(r_0,r_1)}=\left\|\frac{1}{r^{n-1}}\left(r^{n-1}(E_\tau-\bar{E})\right)\right\|_{L^\infty(r_0,r_1)}\le C\tau^{-\frac{1}{2}}, \tag{3.23}$$

which is the second part of (1.16). The proof of Theorem 1.2 is completed. ∎

4. Relaxation limits as $\tau\to 0^+$. The main task of this section is to prove our main result when $\tau\to 0^+$. From (1.15), we obtain that $\underline{\rho}=\lim_{\tau\to 0^+}\rho_\tau$ is a constant over the interval $[r_0,r_1]$. This, together with the second equation of (1.9), implies $\underline{\rho}(r)=a$ or b $\forall r\in[r_0,r_1]$. Hence, the boundary layers must appear when $a\neq b$. Here, we use a new method to look at the width of the boundary layer.

For the convenience of investigations in the following, we shall take another transformation of (1.9). In detail, we divide the first equation of (1.9) by r^{n-1} and then the resulting equation by $\frac{1}{\rho_\tau}-\frac{J^2}{r^{2n-2}\rho_\tau^3}$, problem (1.9) is equivalent to, for $r\in(r_0,r_1)$,

$$\begin{cases}\rho_{rr}=\dfrac{1}{\delta(\rho,r)}\left(\rho-d-h(\rho,r)+\left(\dfrac{J}{\tau r^{n-1}\rho^2}-f(\rho,r)\right)\rho_r+g(\rho,r)\,(\rho_r)^2\right),\\ \rho(r_0)=a,\quad \rho(r_1)=b,\end{cases} \tag{4.1}$$

where $\rho=\rho_\tau$ for simplicity, and

$$\begin{aligned}\delta(\rho,r)&=\frac{1}{\rho_\tau}-\frac{J^2}{r^{2n-2}\rho_\tau^3}, & h(\rho,r)&=\frac{n(n-1)J^2}{r^{2n}\rho_\tau^2},\\ f(\rho,r)&=3\frac{(n-1)J^2}{r^{2n-1}\rho_\tau^3}+\frac{n-1}{r\rho_\tau}, & g(\rho,r)&=\frac{1}{\rho_\tau^2}-\frac{3J^2}{r^{2n-1}\rho_\tau^4}.\end{aligned} \tag{4.2}$$

It follows from (1.12) and (2.3) in Proposition 2.4 that

$$J<r^{n-1}\rho_\tau\le r_1^{n-1}\rho_+\quad\text{over}\quad r\in(r_0,r_1).$$

Then there exist positive constants

$$\frac{1}{M}=\frac{r_1^{n-1}}{J^3}\left((r_1^{n-1}\rho_+)^2-J^2\right),\quad H=\frac{n(n-1)}{r_0^2},\quad \tilde{B}=\frac{J}{r_1^{n-1}\rho_+^2},$$

and

$$B = \frac{1}{\rho_-}, \quad F = 4(n-1)\frac{r_1^{n-2}}{J}, \quad A = \frac{1}{\rho_-^2}\left(1+\frac{3}{r_0}\right),$$

independent of τ such that

$$0 < \delta(\rho,r) = \frac{1}{\rho_\tau}\left(1-\frac{J^2}{(r^{n-1}\rho_\tau)^2}\right) = \frac{r^{n-1}}{(r^{n-1}\rho_\tau)^3}\left((r^{n-1}\rho_\tau)^2 - J^2\right) \le \frac{1}{M}, \tag{4.3}$$

$$0 < h(\rho,r) = \frac{n(n-1)J^2}{r^{2n}\rho_\tau^2} = \frac{n(n-1)J^2}{r^2(r^{n-1}\rho_\tau)^2} \le \frac{n(n-1)}{r^2} \le H, \tag{4.4}$$

$$0 < \frac{\tilde{B}}{\tau} \le \frac{J}{\tau r^{n-1}\rho_\tau^2} = \frac{1}{\tau}\frac{J}{r^{n-1}\rho_\tau}\frac{1}{\rho_\tau} \le \frac{B}{\tau}, \tag{4.5}$$

$$0 < f(\rho,r) = 3\frac{(n-1)J^2}{r^{2n-1}\rho_\tau^3} + \frac{n-1}{r\rho_\tau} = 3\frac{(n-1)J^2 r^{n-2}}{(r^{n-1}\rho_\tau)^3} + \frac{(n-1)r^{n-2}}{r^{n-1}\rho_\tau} \le F, \tag{4.6}$$

and

(4.7)
$$|g(\rho,r)| = \left|\frac{1}{\rho_\tau^2} - \frac{3J^2}{r^{2n-1}\rho_\tau^4}\right| = \frac{1}{\rho_\tau^2}\left|1-\frac{3J^2}{r(r^{n-1}\rho_\tau)^2}\right| \le \frac{1}{\rho_\tau^2}\left(1+\frac{3J^2}{r(r^{n-1}\rho_\tau)^2}\right) \le A.$$

Remark 4.1. It should be pointed out that (4.1) is not degenerate on any subset $[\alpha,\beta] \subset [r_0,r_1]$ since $J < r^{n-1}\rho_\tau$ over $r \in [\alpha,\beta]$. This implies that $\rho_\tau \in W^{2,\infty}([\alpha,\beta])$ and then $\rho_\tau \in C^{1,1}([\alpha,\beta])$. Hence, the regularity of ρ_τ is very good over any subset $[\alpha,\beta] \subset [r_0,r_1]$.

Let us consider the following initial value problem, for $r \in (r_0,r_1)$,

$$\begin{cases} \dfrac{d^2\rho_\tau}{dr^2} = \dfrac{1}{\delta(\rho_\tau,r)}\left(\rho_\tau - d - h(\rho_\tau,r) + \left(\dfrac{J}{\tau r^{n-1}\rho_\tau^2} - f(\rho_\tau,r)\right)\dfrac{d\rho_\tau}{dr} + g(\rho_\tau,r)\left(\dfrac{d\rho_\tau}{dr}\right)^2\right), \\ \left.\dfrac{d\rho_\tau}{dr}\right|_{r=r_*} = \rho_*, \end{cases} \tag{4.8}$$

where the initial point $r_* \in (r_0,r_1)$ and the initial value ρ_* is given.

First, the properties of $\frac{d\rho_\tau}{dr}$ are shown as follows.

LEMMA 4.2 (estimate on $\frac{d\rho_\tau(r)}{dr}$). *Assume that the doping profile $d \in L^\infty[r_0,r_1]$ is subsonic. Let $\frac{d\rho_\tau}{dr}$ be the solution to the initial problem* (4.8). *Then, for $r \in [r_*,r_1)$, the following properties hold:*

$$\frac{d\rho_\tau(r)}{dr} \ge \min\left\{\left(e^{\frac{\tilde{B}M}{3\tau}(r-r_*)} + 3\frac{\bar{d}+H}{\tilde{B}}\right)\tau, \frac{\tilde{D}}{\tau}\right\} \quad \text{as} \quad \rho_* > \left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau, \tag{4.9}$$

and

$$\frac{d\rho_\tau(r)}{dr} \le \max\left\{-\left(e^{-\frac{\tilde{B}M}{3\tau}(r-r_*)} + \frac{3\rho_+}{\tilde{B}}\right)\tau, -\frac{\tilde{D}}{\tau}\right\} \quad \text{as} \quad \rho_* < -\left(\frac{3\rho_+}{\tilde{B}}+1\right)\tau, \tag{4.10}$$

where $\tilde{D} = \frac{\tilde{B}}{3A}$ is a positive constant independent of τ.

Proof. By noting $\frac{d\rho_\tau(r_*)}{dr} > 0$ (or < 0) and the continuity of $\frac{d\rho_\tau(r)}{dr}$, there exists a neighborhood of $r = r_*$, such that $\frac{d\rho_\tau(r)}{dr} > 0$ (or < 0) for any r in this neighborhood.

First, for $\frac{d\rho_\tau(r_*)}{dr} > 0$, we want to prove (4.9). By (4.4)–(4.7) and the fact that $\frac{d\rho_\tau(r)}{dr} > 0$ on a neighborhood of $r = r_*$, we have

$$(4.11)\quad \begin{aligned} &\rho_\tau - d - h(\rho_\tau, r) + \left(\frac{J}{\tau r^{n-1}\rho_\tau^2} - f(\rho_\tau, r)\right)\frac{d\rho_\tau}{dr} + g(\rho_\tau, r)\left(\frac{d\rho_\tau}{dr}\right)^2 \\ &\geq -(\bar{d} + H) - A\left(\frac{d\rho_\tau}{dr}\right)^2 + \left(\frac{\tilde{B}}{\tau} - F\right)\frac{d\rho_\tau}{dr} \\ &= \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} - (\bar{d} + H) + \left(\frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} - A\left(\frac{d\rho_\tau}{dr}\right)^2\right) + \left(\frac{\tilde{B}}{3\tau} - F\right)\frac{d\rho_\tau}{dr}. \end{aligned}$$

Furthermore, we suppose that

$$(4.12)\quad 0 < \frac{d\rho_\tau}{dr} \leq \frac{\tilde{B}}{3A\tau} = \frac{\tilde{D}}{\tau},$$

where

$$\tilde{D} = \frac{\tilde{B}}{3A}.$$

Then, it holds that

$$(4.13)\quad \frac{\tilde{B}}{3\tau} - A\frac{d\rho_\tau}{dr} \geq 0,$$

and for $0 < \tau \ll 1$,

$$(4.14)\quad \frac{\tilde{B}}{3\tau} - F \geq 0.$$

This, together with (4.11) and the fact that $\frac{d\rho_\tau}{dr} > 0$, gives

$$(4.15)\quad \rho_\tau - d - h(\rho_\tau, r) + \left(\frac{J}{\tau r^{n-1}\rho_\tau^2} - f(\rho_\tau, r)\right)\frac{d\rho_\tau}{dr} + g(\rho_\tau, r)\left(\frac{d\rho_\tau}{dr}\right)^2 \geq \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} - (\bar{d} + H).$$

Then it follows from (4.3) and the first equation of (4.8) and (4.15) that

$$(4.16)\quad \frac{d^2\rho_\tau}{dr^2} \geq \frac{1}{\delta(\rho_\tau, r)}\left(\frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} - (\bar{d} + H)\right).$$

Moreover, for $0 < \tau \ll 1$, we further assume that

$$(4.17)\quad 3\frac{\bar{d} + H}{\tilde{B}}\tau \leq \frac{d\rho_\tau}{dr} \leq \frac{\tilde{D}}{\tau},$$

which implies

$$(4.18)\quad \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} - (\bar{d} + H) \geq 0.$$

Therefore, by combining (4.3), (4.18), and (4.16), we obtain

$$\frac{d^2\rho_\tau}{dr^2} \geq \frac{\tilde{B}M}{3\tau}\frac{d\rho_\tau}{dr} - M(\bar{d}+H). \tag{4.19}$$

Now, based on the process above, we choose the initial value $\frac{d\rho_\tau}{dr}\Big|_{r=r_*} = \rho_*$ which satisfies

$$3\frac{\bar{d}+H}{\tilde{B}}\tau < \rho_* < \frac{\tilde{D}}{\tau}. \tag{4.20}$$

Then, by the continuity of $\frac{d\rho_\tau(r)}{dr}$, there exists constant $\eta > 0$, such that

$$3\frac{\bar{d}+H}{\tilde{B}}\tau \leq \frac{d\rho_\tau(r)}{dr} \leq \frac{\tilde{D}}{\tau} \quad \text{for any} \quad r \in (r_*, r_*+\eta).$$

Furthermore, it follows from (4.19) that

$$\begin{cases} \dfrac{d^2\rho_\tau}{dr^2} \geq \dfrac{\tilde{B}M}{3\tau}\dfrac{d\rho_\tau}{dr} - M(\bar{d}+H), \quad r \in (r_*, r_*+\eta), \\ \dfrac{d\rho_\tau}{dr}\Big|_{r=r_*} = \rho_* \in \left(3\dfrac{\bar{d}+H}{\tilde{B}}\tau, \dfrac{\tilde{D}}{\tau}\right). \end{cases} \tag{4.21}$$

For proving (4.9), let us consider the following problem:

$$\begin{cases} u_r = \dfrac{\tilde{B}M}{3\tau}u - M(\bar{d}+H), \quad r \in (r_*, r_*+\eta), \\ u|_{r=r_*} = \rho_*. \end{cases} \tag{4.22}$$

By taking the difference between (4.21) and (4.22), we have

$$\begin{cases} \dfrac{d\left(\frac{d\rho_\tau}{dr} - u\right)}{dr} \geq \dfrac{\tilde{B}M}{3\tau}\left(\dfrac{d\rho_\tau}{dr} - u\right), \quad r \in (r_*, r_*+\eta), \\ \left(\dfrac{d\rho_\tau}{dr} - u\right)_{r=r_*} = 0. \end{cases} \tag{4.23}$$

We introduce

$$V = \frac{d\rho_\tau}{dr} - u, \quad r \in (r_*, r_*+\eta). \tag{4.24}$$

Then (4.23) turns into

$$\begin{cases} V_r \geq \dfrac{\tilde{B}M}{3\tau}V, \quad r \in (r_*, r_*+\eta), \\ V(r_*) = 0. \end{cases} \tag{4.25}$$

Multiplying (4.25) by $e^{-\frac{\tilde{B}M}{3\tau}(r-r_*)}$, we get

$$\begin{cases} \left(e^{-\frac{\tilde{B}M}{3\tau}(r-r_*)}V\right)_r \geq 0, \quad r \in (r_*, r_*+\eta), \\ \left(e^{-\frac{\tilde{B}M}{3\tau}(r-r_*)}V\right)(r_*) = 0, \end{cases} \tag{4.26}$$

which shows that

$$V \geq 0, \quad r \in (r_*, r_* + \eta). \tag{4.27}$$

This is equivalent to

$$\frac{d\rho_\tau}{dr} \geq u, \qquad r \in (r_*, r_* + \eta). \tag{4.28}$$

On the other hand, from (4.22), we get

$$\begin{aligned} u(r) &= e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}\rho_* - \int_{r_*}^{r} e^{\frac{\tilde{B}M}{3\tau}(r-s)} M(\bar{d}+H)ds \\ &= e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}\rho_* - 3\frac{(\bar{d}+H)\tau}{\tilde{B}} e^{\frac{\tilde{B}M}{3\tau}(r-r_*)} + 3\frac{\bar{d}+H}{\tilde{B}}\tau. \end{aligned} \tag{4.29}$$

This, together with (4.28), gives

$$\frac{d\rho_\tau}{dr} \geq e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}\left(\rho_* - 3\frac{\bar{d}+H}{\tilde{B}}\tau\right) + 3\frac{\bar{d}+H}{\tilde{B}}\tau, \qquad r \in (r_*, r_* + \eta). \tag{4.30}$$

For ρ_* satisfying (4.20), we further assume that

$$\left(3\frac{\bar{d}+H}{\tilde{B}} + 1\right)\tau < \rho_* < \frac{\tilde{D}}{\tau}. \tag{4.31}$$

Then, from (4.30), we get

$$\frac{d\rho_\tau(r)}{dr} \geq e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}\tau + 3\frac{\bar{d}+H}{\tilde{B}}\tau, \qquad r \in (r_*, r_* + \eta). \tag{4.32}$$

By (4.11), we know $\frac{d^2\rho_\tau}{dr^2}(s) > 0$ for $\frac{d\rho_\tau(s)}{dr}$ around the value $\frac{\tilde{D}}{\tau}$, and thus when $\frac{d\rho_\tau(s)}{dr} = \frac{\tilde{D}}{\tau}$, it holds that $\frac{d\rho_\tau(r)}{dr} \geq \frac{\tilde{D}}{\tau}$ for all $r \in [s, r_1)$. Thus (4.9) holds on $[r_*, r_1)$ for ρ_* satisfying (4.31). And for $\rho_* \geq \frac{\tilde{D}}{\tau}$, it naturally holds that $\frac{d\rho_\tau(r)}{dr} \geq \frac{\tilde{D}}{\tau}$ for all $r \in [r_*, r_1)$, and thus (4.9) also holds on $[r_*, r_1)$.

Next, for $\frac{d\rho_\tau}{dr} < 0$, we take the similar analysis. In detail, by Proposition 2.4, we have

$$\begin{aligned} &\rho_\tau - d - h(\rho_\tau, r) + \left(\frac{J}{\tau r^{n-1}\rho_\tau^2} - f(\rho_\tau, r)\right)\frac{d\rho_\tau}{dr} + g(\rho_\tau, r)\left(\frac{d\rho_\tau}{dr}\right)^2 \\ &\leq \rho_+ + A\left(\frac{d\rho_\tau}{dr}\right)^2 + \left(\frac{\tilde{B}}{\tau} - F\right)\frac{d\rho_\tau}{dr} \\ &= \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} + \rho_+ + \left(A\left(\frac{d\rho_\tau}{dr}\right)^2 + \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr}\right) + \left(\frac{\tilde{B}}{3\tau} - F\right)\frac{d\rho_\tau}{dr}. \end{aligned} \tag{4.33}$$

Furthermore, we suppose that

$$-\frac{\tilde{D}}{\tau} = -\frac{\tilde{B}}{3A\tau} \leq \frac{d\rho_\tau}{dr} < 0. \tag{4.34}$$

Then, for $0 < \tau \ll 1$, it holds that

$$A\frac{d\rho_\tau}{dr} + \frac{\tilde{B}}{3\tau} \geq 0 \quad \text{and} \quad \frac{\tilde{B}}{3\tau} - F \geq 0. \tag{4.35}$$

This, together with (4.33) and the fact that $\frac{d\rho_\tau}{dr} < 0$, gives

(4.36)
$$\rho_\tau - d - h(\rho_\tau, r) + \left(\frac{J}{\tau r^{n-1}\rho_\tau^2} - f(\rho_\tau, r)\right)\frac{d\rho_\tau}{dr} + g(\rho_\tau, r)\left(\frac{d\rho_\tau}{dr}\right)^2 \le \frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} + \rho_+.$$

Then it follows from (4.3) and the first equation of (4.8) and (4.36) that

$$\frac{d^2\rho_\tau}{dr^2} \le \frac{1}{\delta(\rho_\tau, r)}\left(\frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} + \rho_+\right). \tag{4.37}$$

Moreover, for $0 < \tau \ll 1$, we further assume that

$$-\frac{\tilde{D}}{\tau} \le \frac{d\rho_\tau}{dr} \le -3\frac{\rho_+}{\tilde{B}}\tau, \tag{4.38}$$

which implies

$$\frac{\tilde{B}}{3\tau}\frac{d\rho_\tau}{dr} + \rho_+ \le 0. \tag{4.39}$$

Therefore, by combining (4.3), (4.37), and (4.39), we obtain

$$\frac{d^2\rho_\tau}{dr^2} \le \frac{\tilde{B}M}{3\tau}\frac{d\rho_\tau}{dr} + M\rho_+. \tag{4.40}$$

Now, based on the process above, we assume the initial value $\frac{d\rho_\tau}{dr}|_{r=r_*} = \rho_*$ satisfies

$$-\frac{\tilde{D}}{\tau} < \rho_* < -3\frac{\rho_+}{\tilde{B}}\tau. \tag{4.41}$$

Then, by the continuity of $\frac{d\rho_\tau(r)}{dr}$, there exists constant $\eta > 0$, such that

$$-\frac{\tilde{D}}{\tau} \le \frac{d\rho_\tau(r)}{dr} \le -3\frac{\rho_+}{\tilde{B}}\tau \quad \text{for any} \quad r \in (r_*, r_* + \eta).$$

It follows from (4.40) and (4.41) that

$$\begin{cases} \dfrac{d^2\rho_\tau}{dr^2} \le \dfrac{\tilde{B}M}{3\tau}\dfrac{d\rho_\tau}{dr} + M\rho_+, & r \in (r_*, r_* + \eta), \\ \left.\dfrac{d\rho_\tau}{dr}\right|_{r=r_*} = \rho_* \in \left(-\dfrac{\tilde{D}}{\tau}, -3\dfrac{\rho_+}{\tilde{B}}\tau\right). \end{cases} \tag{4.42}$$

In order to prove (4.10), we consider the following problem:

$$\begin{cases} v_r = \dfrac{\tilde{B}M}{3\tau}v + M\rho_+, & r \in (r_*, r_* + \eta), \\ v(r_*) = \rho_*. \end{cases} \tag{4.43}$$

From (4.42), (4.43) and the comparison principle of the ordinary differential equations, we get

$$\frac{d\rho_\tau(r)}{dr} \le v(r) = e^{\frac{\tilde{B}M}{3\tau}(r - r_*)}\left(\rho_* + 3\frac{\rho_+}{\tilde{B}}\tau\right) - 3\frac{\rho_+}{\tilde{B}}\tau, \qquad r \in (r_*, r_* + \eta). \tag{4.44}$$

For ρ_* satisfying (4.41), we further assume that

$$-\frac{\tilde{D}}{\tau} < \rho_* < -\left(3\frac{\rho_+}{\tilde{B}} + 1\right)\tau. \tag{4.45}$$

Then, from (4.44), we get

$$\frac{d\rho_\tau(r)}{dr} \le -e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}\tau - 3\frac{\rho_+}{\tilde{B}}\tau, \qquad r \in (r_*, r_* + \eta). \tag{4.46}$$

Hence, when $r \in (r_*, r_* + \eta)$, (4.10) follows by combining (4.45) and (4.46) for ρ_* satisfying (4.45). Noticing that $\frac{d^2\rho_\tau}{dr^2}(s) < 0$ for $\frac{d\rho_\tau(s)}{dr}$ around the value $-\frac{\tilde{D}}{\tau}$, which can be seen from (4.33), we can deduce that (4.10) holds on $[r_*, r_1)$ for ρ_* satisfying (4.45). And for $\rho_* \le -\frac{\tilde{D}}{\tau}$, we can directly have $\frac{d\rho_\tau(s)}{dr} \le -\frac{\tilde{D}}{\tau}$ on $[r_*, r_1)$. The proof of Lemma 4.2 is finished. ∎

Next, based on Lemma 4.2, we begin to study the relaxation time limit of (4.1).

Proof of Theorem 1.3. Now, we study the zero relaxation limit problems in two cases. We assume $r_1 \le 1$ without loss of generality.

Case 1. $\rho_\tau(r_0) = \rho_\tau(r_1) = a \ge \frac{J}{r_0^{n-1}}$.

We claim that

$$\rho_\tau(r) - \rho_\tau(r_0) \le \left(3\frac{\bar{d} + H}{\tilde{B}} + 1\right)\tau. \tag{4.47}$$

In fact, if (4.47) is false, then there exists $r \in (r_0, r_1)$ such that

$$\rho_\tau(r) - \rho_\tau(r_0) > \left(3\frac{\bar{d} + H}{\tilde{B}} + 1\right)\tau. \tag{4.48}$$

It follows from the Mean Value Theorem that there exists $r_* \in (r_0, r)$ such that

$$\left.\frac{d\rho_\tau}{dr}\right|_{r=r_*} = \frac{\rho_\tau(r) - \rho_\tau(r_0)}{r - r_0} > \left(3\frac{\bar{d} + H}{\tilde{B}} + 1\right)\tau. \tag{4.49}$$

In view of (4.9), for any $r \in [r_*, r_1)$, we obtain

$$\frac{d\rho_\tau(r)}{dr} \ge \min\left\{\left(e^{\frac{\tilde{B}M}{3\tau}(r-r_*)} + 3\frac{\bar{d} + H}{\tilde{B}}\right)\tau, \frac{\tilde{D}}{\tau}\right\} > 0, \tag{4.50}$$

which contradicts the fact that $\rho_\tau(r_0) = \rho_\tau(r_1) = a$. Hence, the claim (4.47) is correct.

Moreover, by (4.10), in a similar way to the proof for (4.47), it follows that

$$\rho_\tau(r) - \rho_\tau(r_0) \ge -\left(3\frac{\rho_+}{\tilde{B}} + 1\right)\tau. \tag{4.51}$$

This, together with (4.47), gives

$$|\rho_\tau(r) - \rho_\tau(r_0)| \le C\tau, \tag{4.52}$$

which implies (1.18).

Next, we consider the zero-relaxation limit for $E_\tau = E_\tau(r)$. Let us introduce

$$\Xi = r^{n-1}E_\tau. \tag{4.53}$$

Then from (1.11), we have

$$\begin{cases} \Xi_r = r^{n-1}(\rho_\tau - d), \\ \Xi(r)|_{r=r_0} = \beta^\tau(r_0) + M\left(\frac{1}{\tau}, a, r_0\right), \end{cases} \tag{4.54}$$

where

$$\beta^\tau(r_0) = \frac{r_0^{n-1}}{a}\left(1 - \left(\frac{J}{r_0^{n-1}a}\right)^2\right)\frac{d\rho_\tau}{dr}\Big|_{r=r_0}, \quad M\left(\frac{1}{\tau}, a, r_0\right) = \frac{J}{\tau a} - (n-1)\frac{J^2}{r_0^n a^2}. \tag{4.55}$$

From (4.52), it is easy to get

$$|\beta^\tau(r_0)| \leq C\tau. \tag{4.56}$$

It follows from (4.54) that

$$\begin{aligned} \Xi(r) &= \Xi(r_0) + \int_{r_0}^r \Xi_r(s)ds \\ &= M\left(\frac{1}{\tau}, a, r_0\right) + \beta^\tau(r_0) + \int_{r_0}^r s^{n-1}(a + \rho_\tau - a - d)ds \\ &= M\left(\frac{1}{\tau}, a, r_0\right) + L(a,r) - D(r) + \int_{r_0}^r s^{n-1}(\rho_\tau - a)ds + \beta^\tau(r_0), \end{aligned} \tag{4.57}$$

where

$$L(a,r) = \int_{r_0}^r s^{n-1}a ds = \frac{a}{n}(r^n - r_0^n), \quad D(r) = \int_{r_0}^r s^{n-1}d(s)ds. \tag{4.58}$$

For the term $\int_{r_0}^r s^{n-1}(\rho_\tau - a)ds$ in (4.57), by (4.52), we have

$$\left|\int_{r_0}^r s^{n-1}(\rho_\tau - a)ds\right| \leq C\tau \quad \text{for} \quad 0 < \tau \ll 1. \tag{4.59}$$

By combining (4.56), (4.57), and (4.59), we obtain

$$\left\|\Xi(r) - \left(M\left(\frac{1}{\tau}, a, r_0\right) + L(a,r) - D(r)\right)\right\|_{L^\infty[r_0,r_1]} \leq C\tau \quad \text{for} \quad 0 < \tau \ll 1. \tag{4.60}$$

After setting

$$R = \Xi(r) - \left(M\left(\frac{1}{\tau}, a, r_0\right) + L(a,r) - D(r)\right), \tag{4.61}$$

we get from (4.53) that

$$E_\tau = \frac{\Xi}{r^{n-1}} = \bar{M}\left(\frac{1}{\tau}, a, r_0\right) + \bar{L}(a,r) - \bar{D}(r) + \bar{R}, \tag{4.62}$$

where

$$\begin{aligned} &\bar{M}\left(\frac{1}{\tau}, a, r_0\right) = \frac{1}{r^{n-1}}M\left(\frac{1}{\tau}, a, r_0\right), \quad \bar{D}(r) = \frac{1}{r^{n-1}}D(r), \\ &\bar{L}(a,r) = \frac{1}{r^{n-1}}L(a,r) = \frac{a}{n}\left(r - \frac{r_0^n}{r^{n-1}}\right), \quad \bar{R} = \frac{1}{r^{n-1}}R. \end{aligned} \tag{4.63}$$

Then it follow from (4.60) and (4.62) that

$$\left\|E_\tau-\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right\|_{L^\infty[r_0,r_1]}=\|\bar{R}\|_{L^\infty[r_0,r_1]}\le C\tau, \tag{4.64}$$

for $0<\tau\ll 1$. Hence, (1.19) follows.

Next, it follows from the second equation of (1.11) that

$$r^{n-1}\frac{dE_\tau}{dr}=r^{n-1}(\rho_\tau-d)-(n-1)r^{n-2}E_\tau, \tag{4.65}$$

which implies

$$\begin{aligned}\frac{dE_\tau}{dr}&=(\rho_\tau-d)-\frac{n-1}{r}E_\tau\\&=(\rho_\tau-a)+(a-d)-\frac{n-1}{r}\left(E_\tau-\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right)\\&\quad-\frac{n-1}{r}\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right).\end{aligned} \tag{4.66}$$

Furthermore, we have

$$\begin{aligned}&\frac{dE_\tau}{dr}-\left((a-d)-\frac{n-1}{r}\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right)\\&\quad=(\rho_\tau-a)-\frac{n-1}{r}\left(E_\tau-\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right).\end{aligned} \tag{4.67}$$

This, together with (4.64) and (4.52), gives

$$\begin{aligned}&\left\|\frac{dE_\tau}{dr}-\left((a-d)-\frac{n-1}{r}\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right)\right\|_{L^\infty[r_0,r_1]}\\&\quad\le\|(\rho_\tau-a)\|_{L^\infty[r_0,r_1]}+\left\|\frac{n-1}{r}\left(E_\tau-\left(\bar{M}\left(\frac{1}{\tau},a,r_0\right)+\bar{L}(a,r)-\bar{D}(r)\right)\right)\right\|_{L^\infty[r_0,r_1]}\\&\quad\le C\tau\quad\text{for}\quad 0<\tau\ll 1,\end{aligned} \tag{4.68}$$

which implies (1.20).

Case 2. $\rho_\tau(r_0)=a$, $\rho_\tau(r_1)=b$, and $a\neq b$. Here, $a\ge\frac{J}{r_0^{n-1}}, b\ge\frac{J}{r_1^{n-1}}$.

In what follows, we assume that $a<b$ without loss of generality. We begin to establish the estimate for $\rho_\tau(r)-\rho_\tau(r_0)$. First, we claim that

$$\rho_\tau(r)-\rho_\tau(r_0)\ge-\left(3\frac{\rho_+}{\tilde{B}}+1\right)\tau\quad\forall r\in[r_0,r_1]. \tag{4.69}$$

Indeed, if (4.69) is not correct, then there is $r_2\in(r_0,r_1)$ such that

$$\rho_\tau(r_2)-\rho_\tau(r_0)<-\left(3\frac{\rho_+}{\tilde{B}}+1\right)\tau. \tag{4.70}$$

It follows from the Mean Value Theorem that there exists $\xi_2\in(r_0,r_2)$ such that

$$\left.\frac{d\rho_\tau}{dr}\right|_{r=\xi_2}=\frac{\rho_\tau(r_2)-\rho_\tau(r_0)}{r_2-r_0}<-\frac{1}{r_2}\left(3\frac{\rho_+}{\tilde{B}}+1\right)\tau<-\left(3\frac{\rho_+}{\tilde{B}}+1\right)\tau. \tag{4.71}$$

In view of (4.10) and (4.71), for any $r\in[\xi_2,r_1)$, we obtain

$$\frac{d\rho_\tau}{dr}\le \max\left\{-\left(e^{-\frac{\tilde{B}M}{3\tau}(r-r_*)}+\frac{3\rho_+}{\tilde{B}}\right)\tau,-\frac{\tilde{D}}{\tau}\right\}<0, \tag{4.72}$$

which contradicts the fact that $\rho_\tau(r_1)=b>a=\rho_\tau(r_0)$. Then, the lower bound estimate (4.69) holds.

Next, we continue to claim that

$$\rho_\tau(r)-\rho_\tau(r_1)\le\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau\quad\forall r\in[r_0,r_1]. \tag{4.73}$$

In fact, if (4.73) does not hold, then there exists $r_3\in(r_0,r_1)$ such that

$$\rho_\tau(r_3)-\rho_\tau(r_1)>\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau. \tag{4.74}$$

Then it follows from the Mean Value Theorem that there is $\xi_3\in(0,r_3)$ such that

$$\left.\frac{d\rho_\tau}{dr}\right|_{r=\xi_3}=\frac{\rho_\tau(r_3)-\rho_\tau(r_1)}{r_3-y_0}>\frac{1}{r_3-y_0}\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau>\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau, \tag{4.75}$$

where $y_0\in(r_0,r_1)$ is the last point before $r=r_3$ such that $\rho_\tau(y_0)=b=\rho_\tau(r_1)$. By noting (4.9), for any $r\in[\xi_3,r_1)$, it follows that

$$\frac{d\rho_\tau}{dr}\ge\min\left\{\left(e^{\frac{\tilde{B}M}{3\tau}(r-r_*)}+3\frac{\bar{d}+H}{\tilde{B}}\right)\tau,\frac{\tilde{D}}{\tau}\right\}>0. \tag{4.76}$$

This is impossible because $\rho_\tau(r_1)=b$. Hence, we get the upper bound estimate (4.73).

Furthermore, we investigate the boundary layer near the right endpoint $r=r_1$. We assume that the width of the boundary layer is $\alpha\tau^{1-\varepsilon}$, where $\alpha>0$ and $0<\varepsilon<\frac{1}{2}$ are positive constants. We claim that

$$\rho_\tau(r)-\rho_\tau(r_0)\le\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau\quad\forall r\in[r_0,r_1-\alpha\tau^{1-\varepsilon}]. \tag{4.77}$$

Indeed, if (4.77) is not correct, then there is $r_4\in[r_0,r_1-\alpha\tau^{1-\varepsilon}]$ such that

$$\rho_\tau(r_4)-\rho_\tau(r_0)>\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau. \tag{4.78}$$

It follows from the Mean Value Theorem that there exists $\xi_4\in(0,r_4)$ such that

$$\left.\frac{d\rho_\tau}{dr}\right|_{r=\xi_4}=\frac{\rho_\tau(r_4)-\rho_\tau(r_0)}{r_4-r_0}>\frac{1}{r_4-r_0}\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau>\left(3\frac{\bar{d}+H}{\tilde{B}}+1\right)\tau. \tag{4.79}$$

Then in view of (4.9), for any $r\in[\xi_4,r_1)$, we obtain

$$\frac{d\rho_\tau}{dr}\ge\min\left\{\left(e^{\frac{\tilde{B}M}{3\tau}(r-\xi_4)}+3\frac{\bar{d}+H}{\tilde{B}}\right)\tau,\frac{\tilde{D}}{\tau}\right\}. \tag{4.80}$$

And for $r\in[r_1-\frac{\alpha}{2}\tau^{1-\varepsilon},r_1)$, it holds that

$$e^{\frac{\tilde{B}M}{3\tau}(r-\xi_4)}\tau>e^{\frac{\tilde{B}M}{3\tau}\cdot\frac{\alpha}{2}\tau^{1-\varepsilon}}\tau=e^{\frac{\tilde{B}M\alpha}{6\tau^{\varepsilon}}}\tau>\frac{\tilde{D}}{\tau}, \tag{4.81}$$

for $\tau \ll 1$. That means $\frac{d\rho_\tau}{dr} \geq \frac{\tilde{D}}{\tau}$ for $r \in [r_1 - \frac{\alpha}{2}\tau^{1-\varepsilon}, r_1)$. Therefore,

$$\rho_\tau(r_1) - \rho_\tau(\xi_4) \geq \int_{r_1 - \frac{\alpha}{2}\tau^{1-\varepsilon}}^{r_1} \frac{d\rho_\tau}{dr} dr \geq \frac{\tilde{D}}{\tau} \cdot \frac{\alpha}{2}\tau^{1-\varepsilon} = \frac{\tilde{D}\alpha}{2\tau^\varepsilon} \to \infty \quad \text{as} \quad \tau \to 0. \tag{4.82}$$

This is impossible and then (4.77) follows. Then, by combining (4.77) and (4.69), we have

$$|\rho_\tau(r) - \rho_\tau(r_0)| \leq C\tau \quad \forall r \in [r_0, r_1 - \alpha\tau^{1-\varepsilon}] \quad \text{if} \quad 0 < \tau \ll 1, \tag{4.83}$$

which implies (1.21).

Next, we consider the zero-relaxation limit for $E_\tau = E_\tau(r)$.

For the term $\int_{r_0}^{r_1} s^{n-1}(\rho_\tau - a)ds$ in (4.57), different from (4.59), by (4.83), we have

$$\begin{aligned} \left|\int_{r_0}^{r} s^{n-1}(\rho_\tau - a)ds\right| &= \left|\left(\int_{r_0}^{r-\tau^{1-\varepsilon}} + \int_{r-\tau^{1-\varepsilon}}^{r}\right) s^{n-1}(\rho_\tau - a)ds\right| \\ &\leq \int_{r_0}^{r-\tau^{1-\varepsilon}} \left|s^{n-1}(\rho_\tau - a)\right| ds + \int_{r-\tau^{1-\varepsilon}}^{r} \left|s^{n-1}(\rho_\tau - a)\right| ds \\ &\leq C\tau + C\tau^{1-\varepsilon} \\ &\leq C\tau^{1-\varepsilon} \quad \text{for} \quad 0 < \tau \ll 1. \end{aligned} \tag{4.84}$$

By combining (4.56), (4.57), and (4.84), we obtain

$$\left\|\Xi(r) - \left(M\left(\frac{1}{\tau}, a, r_0\right) + L(a,r) - D(r)\right)\right\|_{L^\infty[r_0,r_1]} \leq C\tau^{1-\varepsilon} \quad \text{for} \quad 0 < \tau \ll 1. \tag{4.85}$$

Then it follow from (4.85) and (4.62) that

$$\left\|E_\tau - \left(\bar{M}\left(\frac{1}{\tau}, a, r_0\right) + \bar{L}(a,r) - \bar{D}(r)\right)\right\|_{L^\infty[r_0,r_1]} \leq C\tau^{1-\varepsilon} \quad \text{for} \quad 0 < \tau \ll 1. \tag{4.86}$$

Hence, (1.22) follows. Next, by combining (4.67), (4.83), and (4.86), we have

$$\begin{aligned} &\left\|\frac{dE_\tau}{dr} - \left((a-d) - \frac{n-1}{r}\left(\bar{M}\left(\frac{1}{\tau}, a, r_0\right) + \bar{L}(a,r) - \bar{D}(r)\right)\right)\right\|_{L^\infty[r_0,r_1]} \\ &\leq \|(\rho_\tau - a)\|_{L^\infty[r_0,r_1]} + \left\|\frac{n-1}{r}\left(E_\tau - \left(\bar{M}\left(\frac{1}{\tau}, a, r_0\right) + \bar{L}(a,r) - \bar{D}(r)\right)\right)\right\|_{L^\infty[r_0,r_1]} \\ &\leq C\tau^{1-\varepsilon} \quad \text{for} \quad 0 < \tau \ll 1, \end{aligned} \tag{4.87}$$

which gives (1.23). The proof of Theorem 1.3 is finished. ◻

5. Numerical simulations. In this section, we engage in the numerical verification of our theoretical results. Because there is no essential difference between the two and three dimensions, we will exclusively simulate the situation of $n = 3$. For both Theorems 1.2 and 1.3, we make the unified numerical settings as follows.

- Take values $r_0 = 1$, $r_1 = 2$, and $J = 1$, for simplicity.
- Doping profile is set as

$$d(r) := 4 + \sin(\pi(r-1)) \quad \forall r \in [1,2],$$

which manifestly satisfies the subsonic condition

$$\inf_{r\in[r_0,r_1]} \left(r^{n-1} d(r)\right) - (n-2)(n-1) > J.$$

Based on these simplified settings, the sonic boundary data $a = \frac{J}{r_0^{n-1}}$ and $b = \frac{J}{r_1^{n-1}}$ reduce to

$$a = 1, \quad b = \frac{1}{4} \quad \text{(sonic)},$$

and the subsonic boundary data become accordingly

$$a > 1, \quad b > \frac{1}{4} \quad \text{(subsonic)}.$$

First, Theorem 1.2 indicates that the interior subsonic solution to the problem (1.9) with (1.11) uniformly converges toward the one to problems (1.13) with (1.14) as the relaxation time τ tends to $+\infty$. As mentioned before, Theorem 1.2 holds regardless of whether the boundary data are sonic or subsonic. To highlight the nature of Theorem 1.2, we only numerically verify the critical case here, in which we take the sonic boundary values at both two endpoints $r = r_0 = 1$ and $r = r_1 = 2$. Namely, we set

- $a = 1$ and $b = 0.25$ in simulations for Theorem 1.2; see Figure 1.

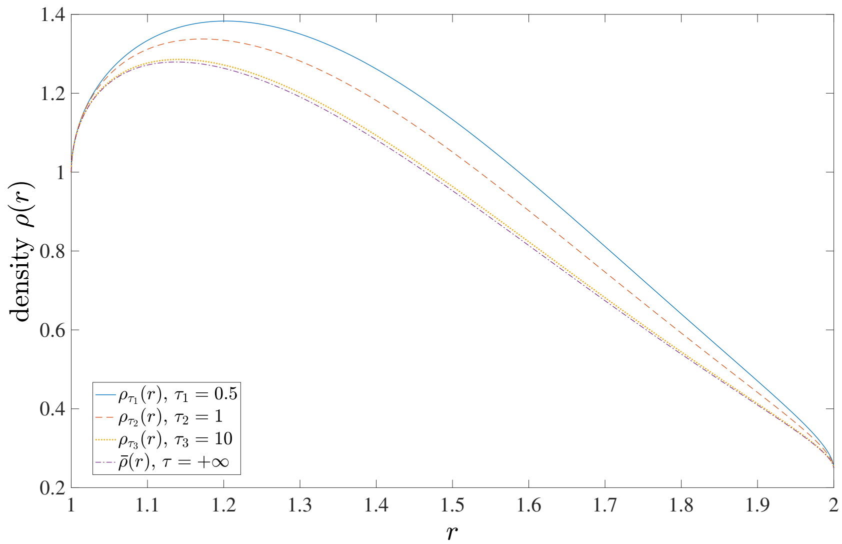
We also set up the finite approximation sequence of relaxation times as $\tau_1 = 0.5$, $\tau_2 = 1$, and $\tau_3 = 10$; and the limiting relaxation time τ is $+\infty$. Numerically, the radial density $\rho_\tau(r)$ is in close proximity to the limiting density $\bar{\rho}(r)$ whenever the relaxation time $\tau \geq 10$, which is prominently displayed in Figure 1(a), and so is the radial electric field $E_\tau(r)$; see Figure 1(b).

We now proceed to numerically check Theorem 1.3 which demonstrates that whether the boundary data a and b take the same value will make the zero-relaxation-time limit results very different. More precisely, if $\frac{J}{r_0^{n-1}} \leq a = b$, then the boundary layer does not occur between $\rho_\tau(r)$ and its asymptotic profile $\underline{\rho}(r)$ as $\tau \to 0^+$; if $\frac{J}{r_0^{n-1}} \leq a < b$, then the boundary layer $(r_1 - \tau^{1-\varepsilon}, r_1]$ appears near the right endpoint $r = r_1$ with a rough width $\tau^{1-\varepsilon}$. For numerical purpose, we set

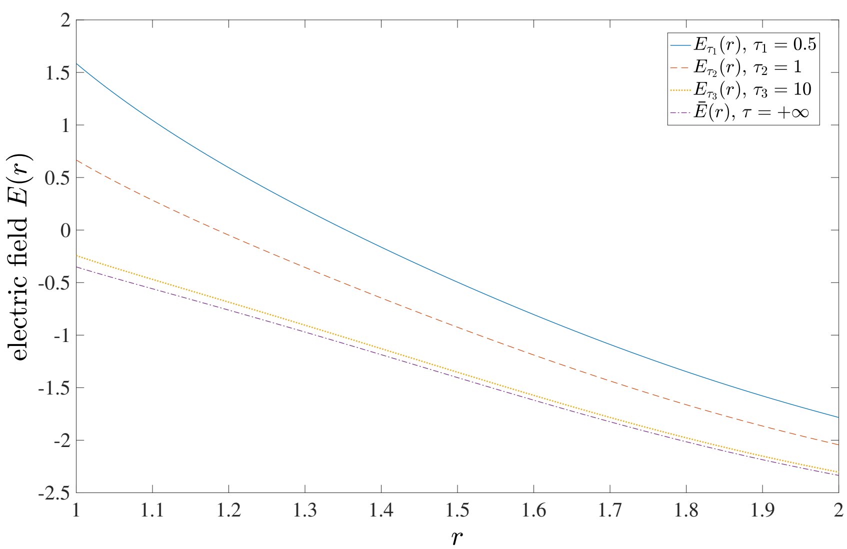
- $a = b = 1$ in simulations for part (I) of Theorem 1.3; refer to Figure 2.
- $a = 1$ and $b = 1.5$ in simulations for part (II) of Theorem 1.3; see Figure 3.

In both cases above, we opt for the finite approximation sequence of relaxation times as $\tau_1 = 0.1$, $\tau_2 = 0.01$, and $\tau_3 = 0.001$; and the limiting relaxation time τ at the moment is 0. Compared with Figure 2(a), we can easily observe in Figure 3(a) that the boundary layer of densities occurs near the right endpoint $r = 2$ provided the relaxation time τ is small enough, and the width of boundary layer gets thinner and thinner as the relaxation time τ goes to 0. However, in either case (see Figures 2(b) and 3(b)), the electric field $E_\tau(r)$ always keeps a huge gap $\frac{1}{r^2\tau}$ from its asymptotic profile $\underline{E}(r)$ over the entire interval $[1,2]$.

In summary, all these numerical simulations conducted in this section perfectly support our theoretical results obtained in Theorems 1.2 and 1.3.

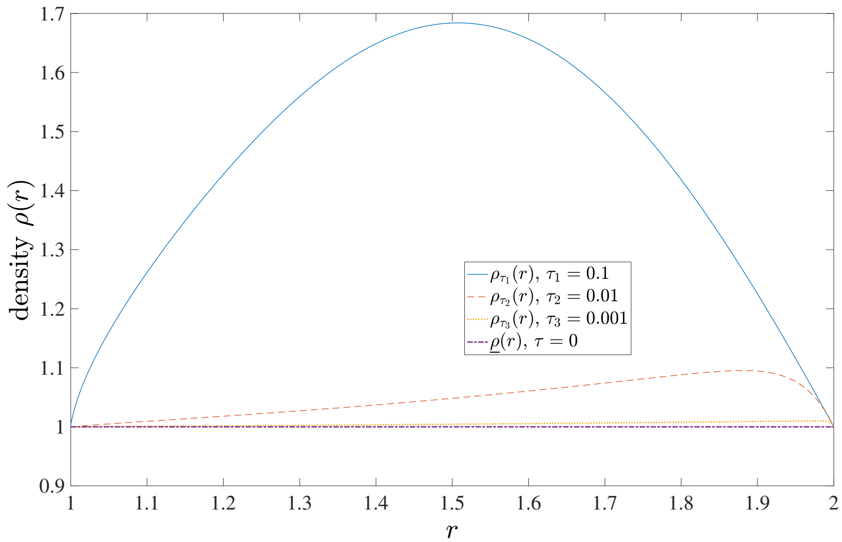


(a) $\|\rho_\tau - \bar{\rho}\|_{L^\infty(1,2)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

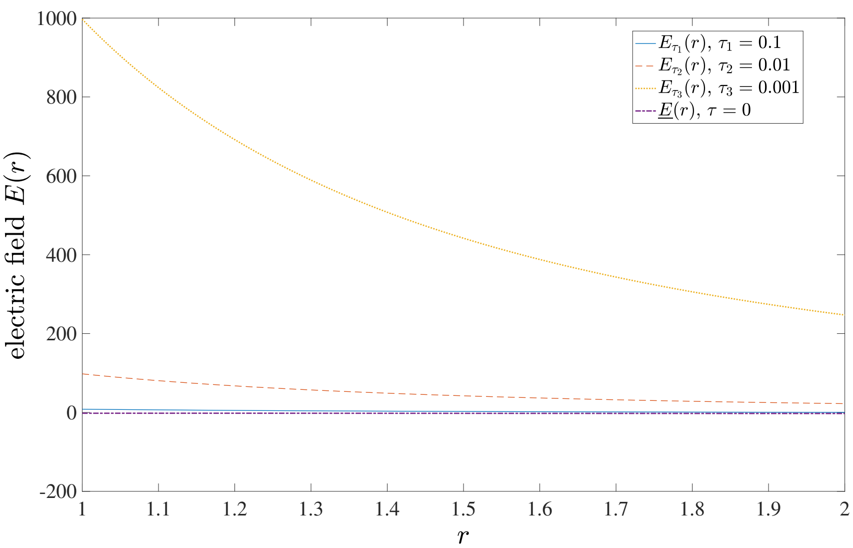


(b) $\|E_\tau - \bar{E}\|_{L^\infty(1,2)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

FIG. 1. *Theorem 1.2: $a = 1$ and $b = 0.25$, both of which are sonic boundary data.*

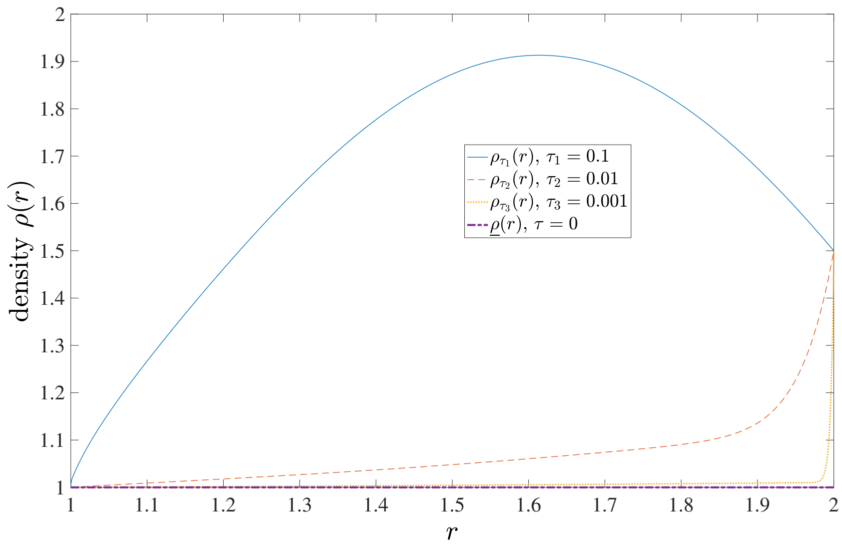


(a) $\|\rho_\tau - \underline{\rho}\|_{L^\infty(1,2)} \leq C\tau$, as $\tau \to 0^+$.

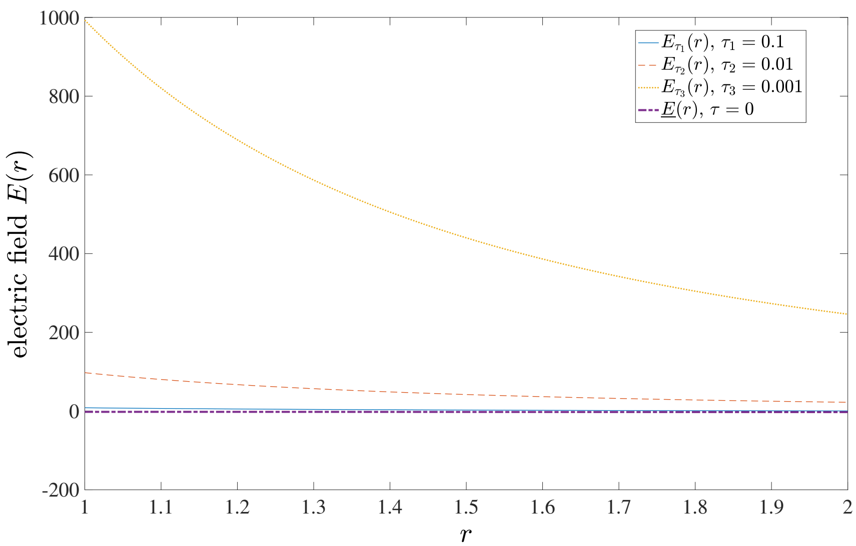


(b) $\|E_\tau - \underline{E} - \frac{J}{\tau r^{n-1} a}\|_{L^\infty(1,2)} \leq C\tau$, as $\tau \to 0^+$.

FIG. 2. *Part* (I) *of Theorem* 1.3*:* $a = b = 1$*, there is no boundary layer of densities.*



(a) $|\rho_\tau(r) - a| \leq C\tau, \ \ 1 \leq r \leq 2 - \tau^{1-\varepsilon}, \ \ \text{as } \tau \to 0^+.$



(b) $\|E_\tau - \underline{E} - \frac{J}{\tau r^{n-1} a}\|_{L^\infty(1,2)} \leq C\tau^{1-\varepsilon}, \ \ \text{as } \tau \to 0^+.$

FIG. 3. *Part* (II) *of Theorem* 1.3*:* $a = 1$ *and* $b = 1.5$*, the boundary layer of densities occurs.*

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