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TRANSONIC STEADY-STATES OF EULER–POISSON EQUATIONS FOR SEMICONDUCTOR MODELS WITH SONIC BOUNDARY*

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Abstract. In this paper, we mainly focus on radial transonic solutions for the steady hydrodynamic model of semiconductors represented by Euler–Poisson equations with sonic boundary in n dimensions. In an annulus domain, given constant electronic current j_0 at the inner boundary and the sonic data of boundary electronic density (ρ_0, ρ_1), we present that the Euler–Poisson system possesses infinitely many transonic shock solutions when the relaxation time is large and infinitely many C^1-smooth transonic solutions when the relaxation time is small and the doping profile is continuous. To study the structure of shock-transonic steady-states, the approach is the constructive method based on the entropy condition and Rankine–Hugoniot jump condition. To show the existence of C^1-smooth solutions, the adopted approach is the local continuation method, and the derivation on C^1-regularity of transonic steady-states is based on a new defined iterative approximation combining the singularity analysis on the critical point. The difficulties caused by the high dimensions for the system and the nonconstant doping both are essential. In short, the concept of local analysis throughout the entire proof is the basis and the core for solving this nonautonomous system.

Key words. hydrodynamic model of semiconductors, Euler–Poisson equations, transonic shock solutions, C^1-smooth transonic solutions, local analysis

AMS subject classifications. 35L60, 35Q35, 35B40, 35C06

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1. Introduction.

In this paper, we investigate the transonic steady-states of the hydrodynamic model for semiconductors with sonic boundary, represented in n-dimensional stationary Euler–Poisson equations [11]:

$$
\begin{cases}
\operatorname{div}(\rho \mathbf{u}) = 0, \\
(\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{\nabla P}{\rho} = \nabla \Phi - \dfrac{\mathbf{u}}{\tau}, \quad x \in \mathbb{R}^n, \quad n = 2, 3, \\
\Delta \Phi = \rho - b(x).
\end{cases} \tag{1.1}
$$

Here $\rho(x)$ is the density of electrons, $\mathbf{u}(x)$ presents the average particle velocity at location x, and $\Phi(x)$ denotes the electrostatic potential of electrons. The pressure $P = P(\rho)$ is the pressure-density relation, where we assume in present paper that $P = T\rho$ for the isothermal flow with the constant temperature $T > 0$, the constant

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$\tau > 0$ represents the momentum relaxation time, and the known function $b(x) > 0$ is the doping profile standing for a background density of changed ions.

The main objective of this paper is to study the structure of radial transonic solutions to (1.1) in an annulus domain which is defined by

$$\mathcal{A} := \{x \in \mathbb{R}^n | r_0 < |x| < r_1\} \quad \text{with fixed constants} \quad 0 < r_0 < r_1,$$

where the inner boundary is given by

$$\Gamma_0 := \{x \in \mathbb{R}^n : |x| = r_0\}$$

and the outer boundary is given by

$$\Gamma_1 := \{x \in \mathbb{R}^n : |x| = r_1\}.$$

The closure of $\mathcal{A}$ is denoted by

$$\overline{\mathcal{A}} := \Gamma_0 \cup \mathcal{A} \cup \Gamma_1.$$

From the terminology of fluid dynamics, we call $c(\rho) := \sqrt{P'(\rho)} = \sqrt{T}$ the *local sound speed* and $M := \frac{|\mathbf{u}|}{c(\rho)}$ the *Mach number*. It is well known that the system (1.1) is a mixed-type system, that is, an elliptic system for the subsonic flow ($M < 1$) and a hyperbolic-elliptic system for the supersonic flow ($M > 1$), and the sonic state for $M = 1$. The radial transonic solutions to (1.1), introduced first in [6], are defined as follows.

Definition 1.1 (radial transonic solution of the Euler–Poisson system). *Suppose that there exists a curve/surface $\Gamma_s = \{|x| = s\}$, where $r_0 < s < r_1$; then the domain $\mathcal{A}$ is divided into two connected subdomains, a supersonic region $\mathcal{A}^-$ and a subsonic region $\mathcal{A}^+$, respectively. That is, $M > 1$ in $\mathcal{A}^- = \{r_0 < |x| < s\}$ and $M < 1$ in $\mathcal{A}^+ = \{s < |x| < r_1\}$. Hereinto, M always changes from the state of $M > 1$ to the state of $M < 1$ across Γ_s. Let $\mathbf{n}_s$ be the unit normal vector field on Γ_s oriented into $\mathcal{A}^+$ and $\{\mathbf{v}_s^{(i)}\}_{i=1}^{n-1}$ be the tangent vector fields on Γ_s with $\{\mathbf{v}_s^{(i)}\}_{i=1}^{n-1}$ being linearly independent at each point on Γ_s.*

1. *Radial transonic shock solutions: The solution $(\rho, \mathbf{u}, \Phi) \in [C(\overline{\mathcal{A}^\pm}) \cap C^1(\mathcal{A}^\pm)]^{1+n} \times [C^1(\overline{\mathcal{A}^\pm}) \cap W^{2,\infty}(\mathcal{A}^\pm)]$ is called a radial transonic shock solution to* (1.1) *in $\mathcal{A}$ if $(\rho, \mathbf{u}, \Phi)$ satisfies* (1.1) *and*

$$(\rho, \mathbf{u}, \Phi)(x) = (\rho, \mathbf{u}, \Phi)(|x|) \tag{1.2}$$

pointwise in $\mathcal{A}^\pm$, respectively, and satisfies the extended Rankine–Hugoniot jump conditions

$$[\rho \mathbf{u} \cdot \mathbf{n}_s]_{\Gamma_s} = [\rho(\mathbf{u} \cdot \mathbf{n}_s)\mathbf{u} + T\rho \mathbf{n}_s]_{\Gamma_s} = [\nabla \Phi \cdot \mathbf{n}_s]_{\Gamma_s} = 0 \tag{1.3}$$

and the entropy condition on Γ_s,

$$0 < \mathbf{u}_+ \cdot \mathbf{n}_s < \mathbf{u}_- \cdot \mathbf{n}_s.$$

Here $[Q]_{\Gamma_s}$ is denoted by $[Q]_{\Gamma_s} := Q|_{\Gamma_s^-} - Q|_{\Gamma_s^+}$ and $\mathbf{u}_\pm$ denote $\mathbf{u}$ restricted on $\mathcal{A}^\pm$, respectively.

2. *Radial C^1-smooth transonic solutions: On the other hand, the solution* $(\rho, \mathbf{u}, \Phi) \in [C(\overline{\mathcal{A}}) \cap C^1(\mathcal{A})]^{1+n} \times [C^1(\overline{\mathcal{A}}) \cap W^{2,\infty}(\mathcal{A})]$ *is called a radial C^1-smooth transonic solution to* (1.1) *in $\mathcal{A}$ if* $(\rho, \mathbf{u}, \Phi)$ *satisfies* (1.1)–(1.3) *in $\mathcal{A}$ and further it holds that*

$$\mathbf{u}_+ \cdot \mathbf{n}_s = \mathbf{u}_- \cdot \mathbf{n}_s \quad \text{and} \quad \nabla \mathbf{u}_+ \cdot \mathbf{n}_s = \nabla \mathbf{u}_- \cdot \mathbf{n}_s \quad \text{on} \quad \Gamma_s.$$

Our purpose of this paper is to show the existence results about radial transonic solutions to system (1.1) with sonic boundary. Precisely, given a sonic density ρ_0 and prescribed constant current j_0 at the inner boundary Γ_0, and a sonic density ρ_1 at the outer boundary Γ_1, we show that there exist infinitely many radial transonic shock steady-states of (1.1) with a large relaxation time and infinitely many radial C^1-smooth transonic steady-states of (1.1) with a small relaxation time.

Before stating our main theoretical results, let us provide some background studies of this topic. System (1.1) is physically originated from the hydrodynamic model of semiconductors [24, 30],

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \bigotimes \mathbf{u}) + \nabla P = \rho \nabla \Phi - \dfrac{\rho \mathbf{u}}{\tau}, \\ \Delta \Phi = \rho(t, x) - b(x), \end{cases}$$

which, first introduced by Bløtekjær [7], is fundamentally developed to simulate the motion of electrons [17, 23] or plasmas [31]. More specifically, the evolution of this model is dictated by conservation laws for mass, momentum, and energy coupled with the Poisson equation for the electric potential, the so-called Euler–Poisson equations. A great deal of mathematical analysis has been proposed to study the above equations with different suitable conditions.

Regarding the stationary Euler–Poisson system (1.1), Degond and Markowich [10] first considered one-dimensional (1.1) in a bounded domain with a fully subsonic background (small electronic current and heavy electronic density on boundary, and subsonic doping profile) and proved the existence and uniqueness of the subsonic solution. They further considered the three-dimensional potential flow case in [11]. Since then, subsonic steady-states in different cases have been further investigated by Markowich [25], Amster et al. [2], and Bae, Duan, and Xie [3, 4]. On the other hand, when the system is set with strongly supersonic background (large electronic current and supersonic doping profile), Peng and Violet [28] proved the existence of the unique subsonic steady-state in the one-dimensional case, and then Bae et al. [5] obtained the supersonic steady-state in the two-dimensional case. When the doping profile $b(x)$ is a supersonic constant, even when the boundary of the system is still subsonic, Ascher et al. [1] first observed some transonic steady-states via a phase-plane analysis in the one-dimensional case, which was then extended by Rosini [29] to the nonisentropic flow. Furthermore, when the doping profile $b(x)$ is supersonic and non-constant, Gamba [13] and Gamba and Morawetz [14] constructed one- and two-dimensional transonic steady-states with shocks by the artificial viscosity approximation, respectively. When the boundary of the system is set as one part in subsonic region and the other in supersonic region, Luo and Xin [22] thoroughly analyzed the existence, nonexistence, uniqueness, and nonuniqueness of transonic solutions with a constant doping profile. See also the recent study by Duan and Zhou [12]. Regarding the stability of steady-state solutions and the time-asymptotic behavior of smooth solutions, we refer to [15, 16, 18, 21, 26, 27] and references therein.

When the boundary is sonic, which is the critical case, the situation becomes more complicated and challenging as we know, and our research group recently paid more attention to this topic. When the doping profile is set to be subsonic or supersonic, the classification of all types of subsonic/supersonic/transonic/sonic solutions was first studied by Li et al. in the series of two pioneering papers [19, 20]. That is, once the doping profile is subsonic, the Euler–Poisson system admits a unique subsonic steady-state, at least one supersonic steady-state, and infinitely many shock-transonic steady-states when the semiconductor effect is weak ($\tau \gg 1$), and infinitely many C^1-smooth transonic steady-states when the semiconductor effect is strong ($\tau \ll 1$), while, once the doping profile is supersonic and far from the sonic line, there is not any physical (subsonic/supersonic/transonic) solution. The supersonic steady-state and many shock-transonic steady-states exist only when the doping profile is sufficiently close to the sonic line. When the doping profile is transonic, based on the different case of the subsonic-dominated doping profile or supersonic-dominated doping profile, we [8] investigated the structure of all subsonic/supersonic/shock-transonic solutions. In particular, once the doping profile is a constant, Wei et al. [32] first observed two C^∞-smooth transonic solutions for a one-dimensional stationary Euler–Poisson system when the doping profile is a subsonic constant, or when the doping profile is supersonic constant but the relaxation time is sufficiently small. Furthermore, in the multiple-dimensional case, we [9] showed the criteria of the existence and nonexistence of the radial subsonic/supersonic steady-states in some sense of the subsonic-dominated doping profile and supersonic-dominated doping profile. But the existence and structure of transonic solutions are still unknown. To answer this question will be the main target of the present paper.

In this paper, we continue to study the multiple-dimensional Euler–Poisson system (1.1) and look for the transonic steady-states. Analogously to [19], the radial transonic solutions are also divided in two types, discontinuous transonic steady-states and continuous transonic steady-states, the so-called shock-transonic steady-states and smooth transonic steady-states. As we know, it is quite significant to study the structure of all physical solutions for the multiple-dimensional Euler–Poisson system subjected on a *general domain*, but this case is really challenging and difficult and is almost not related. So, here we are going to consider the system (1.1) in an *annulus domain* and look into the structure of the shock/smooth transonic solutions in the *radial form*. We set the electronic current j_0 on the inner boundary and the electronic density (ρ_0, ρ_1) to be sonic on the inner and outer boundaries, respectively. We will show that the Euler–Poisson system subjected to the sonic boundaries possesses infinitely many transonic shock steady-states when the relaxation time is large and infinitely many C^1-smooth transonic steady-states when the relaxation time is small and the doping profile is continuous. To investigate the existence and the structure of shock-transonic steady-states, our approach is the constructive method based on the entropy condition and Rankine–Hugoniot jump condition. When the relaxation time is sufficiently large, we note that the norm of the electric field $\nabla\Phi$ at boundary actually keeps away from zero, which is crucial for us to employ the above constructive method. On the other hand, to prove the existence of smooth transonic solutions, the adopted approach is the local continuation method, and the derivation on C^1-regularity of transonic steady-states is based on a locally iterative approximation combining the singularity analysis on the critical point. The difficulties caused by the high dimensions for the system and the nonconstant doping both are essential. Note that the local analysis throughout the entire proof is the basis and the core for solving this nonautonomous system.

For a sake of convenience, let us set the electric field

$$E := \nabla\Phi$$

and assume

$$b(x) := \tilde{b}(r) \quad \text{in } \overline{\mathcal{A}} \quad \text{with} \quad \tilde{b} \in L^\infty(r_0, r_1).$$

In the polar coordinate, from (1.2), we denote

$$(\rho, \mathbf{u}, E)(x) := (\tilde{\rho}(r), \tilde{u}(r)\vec{\mathbf{e}}, \tilde{E}(r)\vec{\mathbf{e}}), \tag{1.4}$$

where $r = |x|$ and $\vec{\mathbf{e}} = \frac{x}{r}$. For positive constants (ρ_0, ρ_1, j_0), system (1.1) is supplemented by the following boundary conditions:

$$(\rho|_{\Gamma_0}, \rho|_{\Gamma_1}, \rho\mathbf{u}|_{\Gamma_0}) = (\rho_0, \rho_1, j_0\vec{\mathbf{e}}). \tag{1.5}$$

Thus by (1.4) and (1.5), the system (1.1) is transformed into

$$\begin{cases} (r^{n-1}\tilde{\rho}\tilde{u})_r = 0, \\ (r^{n-1}\tilde{\rho}\tilde{u}^2)_r + r^{n-1}T\tilde{\rho}_r = r^{n-1}\tilde{\rho}(\tilde{E} - \frac{\tilde{u}}{\tau}), \\ (r^{n-1}\tilde{E})_r = r^{n-1}(\tilde{\rho} - \tilde{b}(r)), \\ (\tilde{\rho}(r_0), \tilde{\rho}(r_1), \tilde{u}(r_0)) = (\rho_0, \rho_1, j_0/\rho_0). \end{cases} \quad \text{for} \quad r_0 < r < r_1, \tag{1.6}$$

Denote $\tilde{J} := \tilde{\rho}\tilde{u}$. Without loss of generality, let us also take

$$\tilde{J} > 0 \quad \text{and} \quad T = 1.$$

Then the first equation of (1.6) yields from (1.5) that

$$\tilde{J}(r) = j_0 r_0^{n-1}/r^{n-1} \quad \text{for} \quad r \in [r_0, r_1], \tag{1.7}$$

and the second equation of (1.6) becomes

$$\left(1 - \frac{\tilde{J}^2}{\tilde{\rho}^2}\right)\tilde{\rho}_r + \frac{n-1}{r}\frac{\tilde{J}^2}{\tilde{\rho}} = \tilde{\rho}\tilde{E} - \frac{\tilde{J}}{\tau}; \tag{1.8}$$

thus the sonic boundary conditions to (1.6) are proposed by

$$\rho_0 = j_0 \quad \text{and} \quad \rho_1 = j_0 \cdot \frac{r_0^{n-1}}{r_1^{n-1}}. \tag{1.9}$$

Actually, it follows from (1.7) that (1.9) corresponds to the sonic state $M = \tilde{u} = \frac{\tilde{J}}{\tilde{\rho}} = 1$, and the singularities of equation (1.8) exist at the boundary where the boundary value problem (1.6) is degenerate.

To simplify the equations (1.6), we define a new variable

$$m(r) := r^{n-1}\tilde{\rho}(r), \quad r \in [r_0, r_1],$$

and a parameter

$$\mathcal{J} := j_0 r_0^{n-1} > 0.$$

Clearly, it holds that

$$\tilde{J} = \frac{\mathcal{J}}{r^{n-1}}, \quad \tilde{u} = \frac{\mathcal{J}}{m}, \quad \text{and} \quad \tilde{\rho} = \frac{m}{r^{n-1}},$$

so that the system (1.6)–(1.9) can be reduced to

$$\begin{cases} \left(1 - \dfrac{\mathcal{J}^2}{m^2}\right) m_r = m\left(\tilde{E} + \dfrac{n-1}{r}\right) - \dfrac{\mathcal{J}}{\tau}, \\ (r^{n-1}\tilde{E})_r = m - B, \\ m(r_0) = m(r_1) = \mathcal{J}, \end{cases} \tag{1.10}$$

where the function B is denoted by

$$B(r) := r^{n-1}\tilde{b}(r) \quad \text{over} \quad [r_0, r_1].$$

Throughout this paper we denote

$$\underline{B} := \operatorname*{essinf}_{r\in[r_0,r_1]} B(r) \quad \text{and} \quad \overline{B} := \operatorname*{esssup}_{r\in[r_0,r_1]} B(r).$$

Note that for the new system (1.10), $m = \mathcal{J}$ is the sonic state corresponding to $M = 1$. So $m > \mathcal{J}$ implies the flow of (1.10) to be subsonic, and $m < \mathcal{J}$ is equivalent to the supersonic flow of (1.10). Consequently, to look for the solution of (1.1), (1.5), and (1.9) is equivalent to solving (1.10). So, corresponding to Definition 1.1, we have the following equivalent definition of transonic solutions for (1.10).

Definition 1.2.

1. $(m, \tilde{E})(r)$ *is called a transonic shock solution of system* (1.10) *if* $m(r_0) = m(r_1) = \mathcal{J}$ *and it is separated by a point* $x_0 \in (r_0, r_1)$ *in the form*

$$(m, \tilde{E})(r) = \begin{cases} (m_{sup}, \tilde{E}_{sup})(r), & r \in [r_0, x_0), \\ (m_{sub}, \tilde{E}_{sub})(r), & r \in (x_0, r_1], \end{cases}$$

where $0 < m_{sup}(r) < \mathcal{J}$ *and* $m_{sub}(r) > \mathcal{J}$ *satisfy the entropy condition at* x_0*,*

$$0 < m_{sup}(x_0^-) < \mathcal{J} < m_{sub}(x_0^+), \tag{1.11}$$

and the Rankine–Hugoniot condition at x_0*,*

$$m_{sup}(x_0^-) + \frac{\mathcal{J}^2}{m_{sup}(x_0^-)} = m_{sub}(x_0^+) + \frac{\mathcal{J}^2}{m_{sub}(x_0^+)}, \quad \tilde{E}_{sup}(x_0^-) = \tilde{E}_{sub}(x_0^+). \tag{1.12}$$

Thus, by a direct calculation, the first equation of (1.12) *reduces to*

$$m_- m_+ = \mathcal{J}^2,$$

where $m_- := m_{sup}(x_0^-)$ *and* $m_+ := m_{sub}(x_0^+)$*.*

2. $(m, \tilde{E})(r)$ *is called a* C^1*-smooth transonic solution of system* (1.10) *if* $m(r_0) = m(r_1) = \mathcal{J}$ *and there exists a point* $x_0 \in (r_0, r_1)$ *such that*

$$(m, \tilde{E})(r) = \begin{cases} (m_{sup}, \tilde{E}_{sup})(r), & r \in [r_0, x_0], \\ (m_{sub}, \tilde{E}_{sub})(r), & r \in [x_0, r_1], \end{cases}$$

where $0 < m_{sup}(r) < \mathcal{J}$ *and* $m_{sub}(r) > \mathcal{J}$ *satisfy the smoothness conditions at* x_0*,*

$$m_- = m_+ = \mathcal{J} \quad \text{and} \quad m'_{sup}(x_0^-) = m'_{sub}(x_0^+). \tag{1.13}$$

Our main results for the existence of shock/C^1-smooth transonic steady-states for (1.10) are stated as follows.

THEOREM 1.3 (two-dimensional case: $n = 2$). *Assume that* $B \in L^\infty(r_0, r_1)$ *satisfies* $\mathcal{J} < \underline{B} \le B \le \overline{B}$*; then the following hold:*

1. *Once the relaxation time* τ *is large enough, then* (1.10) *has infinitely many transonic shock solutions* $(m_{tran}, \tilde{E}_{tran})(r)$ *over* $[r_0, r_1]$*, satisfying the entropy condition* (1.11) *and the Rankine–Hugoniot condition* (1.12) *at a jump point* $x_0 \in (r_0, r_1)$*.*
2. *Once the relaxation time* τ *is small enough and the doping profile satisfies* $B \in C[r_0, r_1]$*, then there exist infinitely many* C^1*-smooth transonic solutions to* (1.10)*, satisfying the smoothness conditions* (1.13) *at the continuous point* x_0 *with the condition* $|x_0 - r_0| \ll 1$*. Furthermore, there is no transonic shock solution to* (1.10) *in this case.*

THEOREM 1.4 (three-dimensional case: $n = 3$). *Assume that* $B \in L^\infty(r_0, r_1)$ *satisfies* $\underline{B} > \mathcal{J} + 2$*; then the following hold:*

1. *Once the relaxation time* τ *is large enough, the system* (1.10) *has infinitely many transonic shock solutions* $(m_{tran}, \tilde{E}_{tran})(r)$ *over* $[r_0, r_1]$*.*
2. *Once the relaxation time* τ *is small enough and the doping profile satisfies* $B \in C[r_0, r_1]$*, the system* (1.10) *has infinitely many* C^1*-smooth transonic solutions. No transonic shock solutions exist.*

Remark 1.5.

1. To prove the existence of C^1-smooth transonic solutions, the smoothness condition $B \in C[r_0, r_1]$ is necessary in the proofs of both Theorems 1.3 and 1.4. In addition, if the function $B(r)$ has a higher regularity in part 2 of Theorem 1.3, it can be proved that the regularity of smooth transonic solutions can be raised up.
2. In Theorem 1.3 and Theorem 1.4, the condition $|x_0 - r_0| \ll 1$ is crucial in the proofs. In fact, this condition indicates that the subsonic part of transonic solutions is dominating. In particular, it is necessary if $\tau \ll 1$ for C^1-smooth transonic solutions.
3. In Theorem 1.3, the condition of $B > \mathcal{J}$ is necessary in some sense for the existence of transonic solutions. However, when $B \le \mathcal{J}$, inspired by [20, 8], it can be proved that there exist transonic solutions only for $|B - \mathcal{J}| \ll 1$, and no transonic solutions exist once $B \ll \mathcal{J}$.
4. Note that the requirement $B > \mathcal{J} + 2$ for the doping profile in Theorem 1.4 is stronger than the condition $B > \mathcal{J}$ in Theorem 1.3. This is somewhat consistent to what we showed in [8].

The paper is organized as follows. Section 2 is devoted to getting the result of transonic shock solutions to (1.10) in two space dimensions when $\tau \gg 1$. Section 3 explores the existence of C^1-smooth transonic solutions to (1.10) for $n = 2$ and $\tau \ll 1$ by locally analyzing the trajectories of a nonautonomous system. What is more, section 4 briefly sketches the conclusions about transonic shock solutions and C^1-smooth transonic solutions of (1.10) in three space dimensions.

2. Infinitely many radial transonic shock solutions. In this section, inspired by [19], we are ready to study the existence of transonic shock solutions of (1.10) with a large relaxation time (namely, $\tau \gg 1$). First let's consider the two-dimensional system (1.10) with sonic boundary on a bounded domain when the semiconductor effect vanishes (namely, $\frac{1}{\tau} = 0$). For convenience, set $\hat{E} = \tilde{E} + 1/r$; then (1.10) becomes

$$\begin{cases} \left(1 - \dfrac{\mathcal{J}^2}{m^2}\right) m_r = m\hat{E}, \\ (r\hat{E})_r = m - B, \\ m(r_0) = m(r_0 + L) = \mathcal{J}, \end{cases} \tag{2.1}$$

where $L > \frac{r_1 - r_0}{4}$ is a positive constant. We note that the function B, both here and below, has been extended periodically to $[r_0, +\infty)$. As we showed in the proof of Theorem 3.2 [9], when $\underline{B} > \mathcal{J}$, system (2.1) has an interior supersonic solution $(m_L, \hat{E}_L)(r)$ on $[r_0, r_0 + L]$ satisfying

$$0 < m_L(r) < \mathcal{J} \quad \text{over} \quad (r_0, r_0 + L).$$

Then we show the following estimates with respect to the interior supersonic solution $(m_L, \hat{E}_L)$.

LEMMA 2.1. *Assume that* $\underline{B} \le B \le \overline{B}$ *and* $\underline{B} > \mathcal{J}$ *and that* $(m_L, \hat{E}_L)$ *are interior supersonic solutions of* (2.1). *Then*

$$\hat{E}_L(r_0) \ge \alpha_1(L), \quad \hat{E}_L(r_0 + L) \le -\alpha_2(L),$$

where $\alpha_1(L)$ *and* $\alpha_2(L)$ *are positive constants.*

Proof. Denoting the solution $(m_L, \hat{E}_L)$ by $(m, \hat{E})$, we get for $r \in [r_0, r_0 + L]$,

$$\frac{d(r\hat{E})}{dm} = \frac{(m - B)(m^2 - \mathcal{J}^2)}{m^3 \hat{E}}. \tag{2.2}$$

For system (2.1), we briefly analyze the monotonicity properties of the trajectories in the phase-plane $(m, \hat{E})$. Note that $0 < m < \mathcal{J}$ and $\underline{B} > \mathcal{J}$; then from (2.1), the solution m is decreasing with respect to r if $\hat{E} > 0$, and m is increasing in r if $\hat{E} < 0$. Furthermore, $\hat{E}$ is always decreasing over $[r_0, r_0 + L]$.

Therefore, there exists a number $z_0 \in [r_0, r_0 + L]$ such that $m(r)$ reaches the minimum value at the point z_0. Indeed, we have

$$m(z_0) = \min_{r \in [r_0, r_0 + L]} m(r) =: m_0, \quad m'(z_0) = 0, \quad \text{and} \quad \hat{E}(z_0) = 0. \tag{2.3}$$

Now, we are going to estimate m_0. It follows from (2.1) that $\tilde{u} = \frac{\mathcal{J}}{m}$ satisfies $\tilde{u} \ge 1$ and

$$\left[r \left(\tilde{u} - \frac{1}{\tilde{u}} \right) \tilde{u}_r \right]_r = \frac{\mathcal{J}}{\tilde{u}} - B, \quad \tilde{u}(r_0) = \tilde{u}(r_0 + L) = 1. \tag{2.4}$$

It is noted that $(m - \mathcal{J})^2 \in H_0^1(r_0, r_0 + L)$; hence, $(\tilde{u} - 1)^2 \in H_0^1(r_0, r_0 + L)$. Then multiplying (2.4) by $(\tilde{u} - 1)^2$, we obtain

$$\int_{r_0}^{r_0 + L} \frac{r(\tilde{u} + 1)}{2\tilde{u}} \left| \left[(\tilde{u} - 1)^2 \right]_r \right|^2 dr = \int_{r_0}^{r_0 + L} \left(B - \frac{\mathcal{J}}{\tilde{u}} \right) (\tilde{u} - 1)^2 \, dr, \tag{2.5}$$

where we have used

$$2\int_{r_0}^{r_0+L} r\left(\tilde{u}-\frac{1}{\tilde{u}}\right)(\tilde{u}-1)(\tilde{u}_r)^2 dr = \int_{r_0}^{r_0+L} \frac{r(\tilde{u}+1)}{2\tilde{u}}\left|\left[(\tilde{u}-1)^2\right]_r\right|^2 dr.$$

Similarly to the proof of Lemma 4.1 in [19], it is concluded that

$$\begin{aligned}\int_{r_0}^{r_0+L}\left(B-\frac{\mathcal{J}}{\tilde{u}}\right)(\tilde{u}-1)^2\,dr &\le \frac{r_0}{4L^2}\int_{r_0}^{r_0+L}(\tilde{u}-1)^4\,dr + \frac{L^3\overline{B}^2}{r_0}\\ &\le \frac{r_0}{4}\int_{r_0}^{r_0+L}\left|\left[(\tilde{u}-1)^2\right]_r\right|^2 dr + \frac{L^3\overline{B}^2}{r_0},\end{aligned}$$

where we have used Young's inequality and Poincaré's inequality. Hence, this indicates by (2.5) that

$$\left\|\left[(\tilde{u}-1)^2\right]_r\right\|_{L^2(r_0,r_0+L)} \le \frac{2\overline{B}L\sqrt{L}}{r_0},$$

so that for any $\phi \in H_0^1(r_0, r_0+L)$, it follows from the inequality $\|\phi\|_{L^\infty} \le \sqrt{L}\|\phi_r\|_{L^2}$ that

$$(\tilde{u}-1)^2 \le \frac{2\overline{B}L^2}{r_0}.$$

Furthermore, it holds that

$$\tilde{u} \le 1 + \sqrt{2\overline{B}/r_0}\cdot L$$

such that we estimate m_0 by

$$m_0 = \frac{\mathcal{J}}{\tilde{u}(z_0)} \ge \frac{\mathcal{J}}{1+\sqrt{2\overline{B}/r_0}\cdot L} =: \beta_1(L,\overline{B}).$$

The following proof is divided into two cases.

Case 1. Assume that the minimum point z_0 satisfies $z_0 - r_0 \ge \frac{L}{2}$. Thus, by integrating the second equation of (2.1) over $[r_0, z_0]$, one can see that

$$\hat{E}(r_0) = -\frac{1}{r_0}\int_{r_0}^{z_0}(m-B)ds \ge \frac{\underline{B}-\mathcal{J}}{r_0}\cdot(z_0-r_0) \ge \frac{L(\underline{B}-\mathcal{J})}{2r_0} =: \alpha_1(L).$$

Afterwards, a direct computation yields from (2.1) that

$$m_{rr} = \frac{m^3}{m+\mathcal{J}}\left[\frac{1}{m^2(\mathcal{J}-m)}\left(\frac{3\mathcal{J}^2}{m^2}-1\right)|m_r|^2 - \frac{m_r}{r} + \frac{B-m}{r(\mathcal{J}-m)}\right] \quad \text{on} \quad [r_0, r_0+L].$$

Since $m_r(r) < 0$ over $[r_0, z_0]$ and $\underline{B} \ge \mathcal{J} \ge m$, we immediately conclude that

$$\begin{aligned} m_{rr} &\ge \frac{m^3}{m+\mathcal{J}}\left[\frac{1}{m^2(\mathcal{J}-m)}\left(\frac{3\mathcal{J}^2}{m^2}-1\right)|m_r|^2 + \frac{\underline{B}-m}{(r_0+L)(\mathcal{J}-m)}\right]\\ &\ge \frac{m_0^3}{2\mathcal{J}(r_0+L)} \ge \frac{\beta_1^3(L,\overline{B})}{2\mathcal{J}(r_0+L)} \quad \text{on} \quad [r_0, z_0].\end{aligned}$$

By Taylor expansion,

$$m(r_0) = m(z_0) - m'(z_0)(z_0 - r_0) + m''(\xi)(z_0 - r_0)^2/2 \quad \text{with} \quad \xi \in [r_0, z_0];$$

it then follows that

$$m_0 \le \mathcal{J} - \frac{L^2\beta_1^3(L,\overline{B})}{16\mathcal{J}(r_0+L)} \le \mathcal{J} - \frac{\mathcal{J}^2L^2}{16(r_0+L)} \cdot \frac{1}{\left(1+\sqrt{2\overline{B}/r_0}\cdot L\right)^3} =: \beta_2(L,\overline{B}).$$

Using (2.2) and (2.3), we derive

$$\begin{aligned}\frac{(r_0+L)^2}{2r_0}\cdot \hat{E}^2(r_0+L) &\ge \int_{z_0}^{r_0+L} \frac{(m-\underline{B})(m^2-\mathcal{J}^2)}{m^3}\\ &= \frac{(2\mathcal{J}-\underline{B})}{2} + \mathcal{J} - \underline{B}\ln\mathcal{J}\\ &\quad - \left[\frac{\mathcal{J}^2(2m_0-\underline{B})}{2m_0^2} + m_0 - \underline{B}\ln m_0\right]\\ &= f(m_0;\underline{B}),\end{aligned}$$

where

$$\begin{aligned}f(\ell;\underline{B}) := \mathcal{J} + \frac{(2\mathcal{J}-\underline{B})(\ell-\mathcal{J})^2 + \mathcal{J}(2\mathcal{J}-2\underline{B})(\ell-\mathcal{J})}{2\ell^2}\\ - \ell + \underline{B}\ln\ell - \underline{B}\ln\mathcal{J}\end{aligned} \tag{2.6}$$

for $\ell \in (0,\mathcal{J})$. One finds that $f(\mathcal{J};\underline{B}) = 0$ and $f'(\ell;\underline{B}) = -\frac{(\underline{B}-\ell)(\mathcal{J}^2-\ell^2)}{\ell^3} < 0$ for $\ell \in (0,\mathcal{J})$. Thus, we get

$$\hat{E}^2(r_0+L) \ge \frac{2r_0 f(m_0;\underline{B})}{(r_0+L)^2} \ge \frac{2r_0 f(\beta_2;\underline{B})}{(r_0+L)^2} =: \alpha_2^2(L,\underline{B}).$$

Here $\alpha_2(L,\underline{B})$ is a positive constant. Integrating the second equation of (2.1) over $[z_0, r_0+L]$, we have

$$\hat{E}(r_0+L) = \frac{1}{r_0+L}\int_{z_0}^{r_0+L}(m-B)dr \le \frac{(r_0+L-z_0)(\mathcal{J}-\underline{B})}{r_0+L} < 0.$$

Hence, it is easy to check that $\hat{E}(r_0+L) \le -\alpha_2(L,\underline{B}) < 0$.

Case 2. Suppose that $z_0 - r_0 \le \frac{L}{2}$, that is, $r_0 + L - z_0 \ge \frac{L}{2}$; then we directly obtain

$$\hat{E}(r_0+L) = \frac{1}{r_0+L}\int_{z_0}^{r_0+L}(m-B)dr \le \frac{(r_0+L-z_0)(\mathcal{J}-\underline{B})}{r_0+L} \le \frac{L(\mathcal{J}-\underline{B})}{2(r_0+L)} =: -\alpha_2(L,\underline{B}) < 0, \tag{2.7}$$

where the constant $\alpha_2(L,\underline{B})$ is also positive. On the other hand, we estimate $\hat{E}(r_0)$ by arguing about the size of m_0. In fact, if $m_0 > \mathcal{J} - \mu$ with a positive constant μ, then

$$\frac{(r_0+L)^2}{2r_0}\cdot\hat{E}^2(r_0+L) \le \int_{z_0}^{r_0+L}\frac{(m-\overline{B})(m^2-\mathcal{J}^2)}{m^3} \le f(m_0;\overline{B}) < f(\mathcal{J}-\mu;\overline{B}),$$

where the function f is denoted by (2.6). From the above inequality, we get

$$\hat{E}^2(r_0+L) < \frac{2r_0 f(\mathcal{J}-\mu;\overline{B})}{(r_0+L)^2}$$

and further choose a constant $\mu(L,\underline{B})$ to satisfy

$$f(\mathcal{J}-\mu;\overline{B}) \le \frac{L^2(\underline{B}-\mathcal{J})^2(r_0+L)^2}{4r_0},$$

which gives $\hat{E}(r_0+L) > -\alpha_2(L,\underline{B})$. This is a contradiction to (2.7). Therefore, $m_0 \le \mathcal{J}-\mu$ for the fixed constant $\mu(L,\underline{B})$; then

$$\frac{r_0\hat{E}^2(r_0)}{2} \ge f(m_0;\underline{B}) \ge f(\mathcal{J}-\mu;\overline{B}).$$

Here we determine $\alpha_1(L) = \sqrt{\frac{2f(\mathcal{J}-\mu;\underline{B})}{r_0}} > 0$ to satisfy $\hat{E}(r_0) \ge \alpha_1(L)$. The proof is finished. $\square$

Now we are ready to prove the existence of transonic shock solutions of the following two-dimensional system:

$$\begin{cases} \left(1-\dfrac{\mathcal{J}^2}{m^2}\right)m_r = m\hat{E} - \dfrac{\mathcal{J}}{\tau}, \\ (r\hat{E})_r = m - B, \\ m(r_0) = m(r_1) = \mathcal{J}. \end{cases} \tag{2.8}$$

Since a transonic solution consists of the supersonic and subsonic parts, the goal of Lemma 2.1 is to present a transonic shock solution where the supersonic part is dominating. Furthermore, we may ignore the singularity of the subsonic part of this solution, because the subsonic part is extremely small near the sonic state. Hence, from this point of view, we show the following theorem.

THEOREM 2.2. *Let $B \in L^\infty(r_0,r_1)$ satisfy $\mathcal{J} < \underline{B} \le B \le \overline{B}$, and assume further that τ is large enough; then* (2.8) *has a transonic shock solution $(m_{tran},\hat{E}_{tran})(r)$ over $[r_0,r_1]$, satisfying the entropy condition* (1.11) *and the Rankine–Hugoniot condition* (1.12) *at a point $x_0 \in (r_0,r_1)$. Because of the arbitrary choices of x_0, the transonic shock solutions are infinitely many.*

Proof. This proof is split into three steps.

Step 1. First we know that there exists a supersonic solution $(m_L,\hat{E}_L)(r)$ over $[r_0,r_0+L]$ to (2.1) such that $\hat{E}_L(r_0) \ge \alpha_1(L) > 0$ and $\hat{E}_L(r_0+L) \le -\alpha_2(L) < 0$, which is given in Lemma 2.1. Here the constants α_1 and α_2 only depend on $\underline{B}$, $\overline{B}$, and L. Then, in the case of $\tau \gg 1$, we consider the boundary value problem as follows:

$$\begin{cases} \left(1-\dfrac{\mathcal{J}^2}{m^2}\right)m_r = m\hat{E} - \dfrac{\mathcal{J}}{\tau}, \\ (r\hat{E})_r = m - B, \\ m(r_0) = m(r_0+L) = \mathcal{J}. \end{cases} \tag{2.9}$$

Of course, from the proof of Theorem 2.1 in [9], one finds that (2.9) has a supersonic solution $(m_1,\hat{E}_1)(r)$ on $[r_0,r_0+L]$, and we claim that

$$\hat{E}_1(r_0) \ge \frac{\alpha_1(L)}{2} > 0 \quad \text{and} \quad \hat{E}_1(r_0+L) \le -\frac{\alpha_2(L)}{2} < 0. \tag{2.10}$$

That is, we only need to prove that if $\tau \gg 1$,

$$|\hat{E}_1(r) - \hat{E}_L(r)| \le C\left(\frac{1}{\tau}\right) \quad \text{on} \quad [r_0, r_0 + L],$$

which actually is a problem of structural stability to (2.1) under a perturbation term $\frac{\mathcal{J}}{\tau}$. Thus, by a standard energy estimate and Gronwall's inequality, noting that $(m_1 - \mathcal{J}) \in C^{1/2}[r_0, r_0 + L]$ and $(m_L - \mathcal{J}) \in C^{1/2}[r_0, r_0 + L]$, we show that for $r \in [r_0, r_0 + L]$,

$$|m_1(r) - m_L(r)|^2 + |\hat{E}_1(r) - \hat{E}_L(r)|^2 \le \frac{C}{\tau^2} \quad \text{with a constant } C.$$

The claim is proved.

Step 2. Let η be a small number, to be determined later, satisfying $0 < \eta \ll 1$. It can be presented that there exists a last number x_1 at which m_1 attains to the line $\mathcal{J} - \eta$. Indeed, as in the proof of Theorem 4.2 [19], it follows that

$$|\hat{E}_1(x_1) - \hat{E}_1(r_0 + L)| \le C\eta, \quad m_1(x_1) = \mathcal{J} - \eta, \quad \text{and} \quad |r_0 + L - x_1| \le C\eta^2, \tag{2.11}$$

where C is a positive constant independent of η. Now let's focus on the ODE system

$$\begin{cases} \left(1 - \dfrac{\mathcal{J}^2}{m^2}\right) m_r = m\hat{E} - \dfrac{\mathcal{J}}{\tau}, \\ (r\hat{E})_r = m - B, \\ (m(r_0), \hat{E}(r_0)) = (\mathcal{J}, \hat{E}_1(r_0)) \end{cases} \tag{2.12}$$

and construct a transonic solution of (2.12) on an interval $[r_0, x_2]$ with

$$m(r_0) = m(x_2) = \mathcal{J}$$

in the form

$$(m, \hat{E})(r) = \begin{cases} (m_{sup}, \hat{E}_{sup})(r), & r \in [r_0, x_1), \\ (m_{sub}, \hat{E}_{sub})(r), & r \in (x_1, x_2]. \end{cases}$$

Obviously, $(m_{sup}, \hat{E}_{sup})(r) = (m_1, \hat{E}_1)(r)$ on $[r_0, x_1]$. Set $m_- = m_{sup}(x_1) = \mathcal{J} - \eta$ and $\hat{E}_- = \hat{E}_{sup}(x_1)$, and we take $x_1 \in (0, x_2)$ as a jump point. Hence, $m_+ = \mathcal{J}^2/m_- = \frac{\mathcal{J}^2}{\mathcal{J}-\eta} > \mathcal{J}$, $\hat{E}_+ = \hat{E}_-$. Then we prepare to prove that (2.12) has a subsonic solution $(m_{sub}, \hat{E}_{sub})(r)$ on $[x_1, x_2]$ with the initial data $(m_+, \hat{E}_+)$ satisfying

$$m_{sub}(x_2) = \mathcal{J}, \quad |\hat{E}_{sub}(x_2) - \hat{E}_+| \le C\eta, \quad \text{and} \quad x_2 - x_1 \le C\eta.$$

In fact, if $C\eta \le \frac{\alpha_2(L)}{4}$, we derive from (2.10) and (2.11) that

$$m_+\hat{E}_+ - \frac{\mathcal{J}}{\tau} \le \frac{\mathcal{J}^2}{\mathcal{J} - \eta}\left(\hat{E}_1(r_0 + L) + C\eta\right) \le \frac{\mathcal{J}^2}{\mathcal{J} - \eta}\left(-\frac{\alpha_2(L)}{2} + C\eta\right) \le -\frac{\mathcal{J}\alpha_2(L)}{4}.$$

Thus, from the first equation of (2.12), one can easily see that the subsonic solution m_{sub} is decreasing near x_1^+. Then in the neighborhood of x_1^+, if $\eta \leq \mathcal{J} - \frac{\mathcal{J}^2}{\underline{B}}$ and $C\eta \leq \frac{\alpha_2(L)}{4}$, we get

$$\begin{aligned}\hat{E}_{sub}(r) &= \frac{x_1}{r}\hat{E}_+ + \frac{1}{r}\int_{x_1}^{r}(m_{sub} - B)dr\\ &\leq \frac{x_1}{r}\left(\hat{E}_1(r_0+L) + C\eta\right) + \frac{1}{r}\int_{x_1}^{r}\left(\frac{\mathcal{J}^2}{\mathcal{J}-\eta} - \underline{B}\right)dr\\ &\leq -\frac{x_1\alpha_2(L)}{4r} < 0.\end{aligned}$$

Note that the function $g(s) := \frac{s^3}{s^2-\mathcal{J}^2}$ is monotone decreasing on $[\mathcal{J}, \sqrt{3}\mathcal{J}]$; then if $\eta \leq \min\{\frac{\alpha_2(L)}{4C}, \frac{\mathcal{J}}{2}\}$,

$$m'_{sub} = \frac{m_{sub}\hat{E}_{sub} - \frac{\mathcal{J}}{r}}{1 - \frac{\mathcal{J}^2}{(m_{sub})^2}} \leq \frac{m_+^3\hat{E}_+}{m_+^2 - \mathcal{J}^2} \leq \frac{\mathcal{J}^4\hat{E}_+}{\eta(\mathcal{J}-\eta)(2\mathcal{J}-\eta)} \leq -\frac{\mathcal{J}\alpha_2(L)}{4}, \tag{2.13}$$

which indicates that m_{sub} keeps decreasing and arrives the line $\mathcal{J}$ at the end point x_2. Hence, we derive from (2.13) that

$$x_2 - x_1 = \frac{\mathcal{J} - \frac{\mathcal{J}^2}{\mathcal{J}-\eta}}{\int_0^1 m'_{sub}(sx_2 + (1-s)x_1)ds} \leq C\eta. \tag{2.14}$$

So we construct the transonic solution of (2.12) on $[r_0, x_2]$, which satisfies the boundary conditions

$$m_{sup}(r_0) = m_{sub}(x_2) = \mathcal{J},$$

the entropy condition

$$0 < m_- < \mathcal{J} < m_+,$$

and the Rankine–Hugoniot condition (1.12) at the jump point x_1. Moreover, one can get from (2.11) and (2.14) that

$$|x_2 - L - r_0| \leq |x_2 - x_1| + |x_1 - L - r_0| \leq C\eta$$

and

$$\hat{E}_{sub}(x_2) \geq \frac{x_1\hat{E}_1(x_1)}{x_2} - \frac{\overline{B}(x_2 - x_1)}{x_2} \geq \hat{E}_+ - C\eta.$$

Step 3. We argue it by the continuity method. Let $L = \frac{r_1 - r_0}{2}$ and define the interior supersonic solution of (2.9) by $(m_1^{(1)}, \hat{E}_1^{(1)})$ and the corresponding transonic shock solution of (2.12) by $(m_2, \hat{E}_2)$. From Steps 1–2, we deduce that there exists a number x_3 such that

$$\left|x_3 - \frac{r_0 + r_1}{2}\right| \leq C\eta, \quad m_2(r_0) = m_2(x_3) = \mathcal{J}, \quad \text{and} \quad \left|\hat{E}_2(x_3) - \hat{E}_1^{(1)}\left(\frac{r_0+r_1}{2}\right)\right| \leq C\eta.$$

Furthermore, there must be a jump point z_1 satisfying $m_2(z_1^-) = \mathcal{J} - \eta$, $m_2(z_1^+) = \frac{\mathcal{J}^2}{\mathcal{J}-\eta} > \mathcal{J}$ and $\hat{E}_2(z_1) < 0$. Therefore, (2.12) has a transonic solution on $[r_0, x_3]$ with $L = \frac{r_1 - r_0}{2}$ as follows:

$$(m_2, \hat{E}_2)(x) = \begin{cases} (m_{sup}, \hat{E}_{sup})(x), & x \in [r_0, z_1), \\ (m_{sub}, \hat{E}_{sub})(x), & x \in (z_1, x_3]. \end{cases}$$

Here, the Rankine–Hugoniot condition and the entropy condition are satisfied at jump point z_1.

Similarly, we set $L = 2(r_1 - r_0)$ and denote the corresponding supersonic solutions of (2.9) and the transonic shock solution of (2.12) by $(m_1^{(2)}, \hat{E}_1^{(2)})$ and $(m_3, \hat{E}_3)$, respectively. Then we prove that there exists a number x_4 satisfying

$$|x_4 - (2r_1 - r_0)| \le C\eta$$

such that

$$m_3(r_0) = m_3(x_4) = \mathcal{J} \quad \text{and} \quad \left|\hat{E}_3(x_4) - \hat{E}_1^{(2)}(2r_1 - r_0)\right| \le C\eta.$$

Additionally, it is shown that there exists a jump point $z_2 \in [r_0, x_4]$ such that

$$m_3(z_2^-) = \mathcal{J} - \eta < \mathcal{J} < m_3(z_2^+) = \frac{\mathcal{J}^2}{\mathcal{J} - \eta},$$

where the Rankine–Hugoniot condition (1.11) and the entropy condition (1.12) hold.

After this, one can see that the transonic solution of (2.12) continuously depends on the length of the solution and the initial value $\hat{E}(r_0)$, respectively. Therefore, choosing η sufficiently small such that the solution length satisfies $r_1 - r_0 \in [x_3, x_4]$, we then look for a parameter $\hat{E}(r_0)$ as the corresponding initial value. A continuity argument certainly verifies the existence of the initial value, so we have a transonic solution $(m_{tran}, \hat{E}_{tran})$ to system (2.8). Furthermore, one can fix a number η_0, and owing to the arbitrary choice of $0 < \eta < \eta_0$, we have infinitely many transonic solutions to (2.8). This proof is complete. ∎

3. Infinitely many radial C^1-smooth transonic solutions. In this section, we are going to construct C^1-smooth transonic solutions by analyzing the properties of the interior subsonic and supersonic solutions on the boundary when the relaxation time is small. Since the doping profile is not a constant and the ODE system is non-autonomous on a bounded domain by the effect of radial high-dimensional space, the phase-plane analysis, shown in [19], can't be directly applied to deal with the system (1.10). Actually, by a mathematical experience, the essential features of the transonic solutions for this system are consistent with those of the one-dimensional system. Therefore, inspired by this, we first focus on the following two-dimensional problem:

$$\begin{cases} \left(1 - \dfrac{\mathcal{J}^2}{m^2}\right) m_r = m\left(\tilde{E} + \dfrac{1}{r}\right) - \dfrac{\mathcal{J}}{\tau}, \\ (r\tilde{E})_r = m - B, \\ m(r_0) = m(r_1) = \mathcal{J}. \end{cases} \tag{3.1}$$

Then one of the primary theorems in this section is stated as follows.

THEOREM 3.1. *Assume that $B \in L^\infty(r_0, r_1)$ satisfies $\mathcal{J} < \underline{B} \le B \le \overline{B}$ and that there exists a constant $\tau_0(\overline{B}, r_0)$ such that for $0 < \tau < \tau_0$; then system* (3.1) *has infinitely many continuous transonic solutions $(m_{tran}, \tilde{E}_{tran})$ in the form of*

$$(m_{tran}, \tilde{E}_{tran})(r) = \begin{cases} (m_{sup}, \tilde{E}_{sup})(r) & \text{for} \quad r \in [r_0, x_0], \\ (m_{sub}, \tilde{E}_{sub})(r) & \text{for} \quad r \in [x_0, r_1]. \end{cases}$$

Indeed,

$$m_{sub}(x_0) = m_{sup}(x_0) = \mathcal{J} \quad \text{and} \quad \tilde{E}_{sub}(x_0) = \tilde{E}_{sup}(x_0) = \frac{1}{\tau} - \frac{1}{x_0} > 0$$

at the continuous points x_0. Here x_0 satisfies $|x_0 - r_0| \le C(\tau)$ with a constant C positively depending on τ. Moreover, there is no transonic shock solution to system (3.1)*.*

For convenience, we set

$$F := r\left(\tilde{E} + \frac{1}{r} - \frac{\mathcal{J}}{\tau m}\right) \quad \text{and} \quad w := m - \mathcal{J};$$

then (3.1) can be transformed into

$$\begin{cases} w_r = \dfrac{(w+\mathcal{J})^3}{r(w+2\mathcal{J})} \cdot \dfrac{F}{w}, \\ F_r = w + \mathcal{J} - B + \dfrac{\mathcal{J}(w+\mathcal{J})}{\tau(w+2\mathcal{J})} \cdot \dfrac{F}{w} - \dfrac{\mathcal{J}}{\tau(w+\mathcal{J})} \end{cases} \tag{3.2}$$

with the boundary conditions $w(r_0) = w(r_1) = 0$. Here $B(r)$ has been extended periodically to $[r_0, +\infty)$. Although system (3.2) is nonautonomous, all trajectories still can be presented in the phase-plane (w, F), satisfying

$$\frac{dF}{dw} = r\left(\frac{(w+\mathcal{J}-B)(w+2\mathcal{J})}{(w+\mathcal{J})^3} \cdot \frac{w}{F} + \frac{\mathcal{J}}{\tau(w+\mathcal{J})^2} - \frac{\mathcal{J}(w+2\mathcal{J})}{\tau(w+\mathcal{J})^4} \cdot \frac{w}{F}\right). \tag{3.3}$$

Obviously, from (3.3), one can easily see that some trajectories may intersect at some different points. Now before proving Theorem 3.1, a definition is given below.

DEFINITION 3.2. *If $(m, \tilde{E})$ is called an interior subsonic* (*resp., interior supersonic*) *solution to* (3.1)*, the corresponding trajectory $\tilde{E} = \tilde{E}(m; r)$ in the phase-plane $(m, \tilde{E})$ is called an interior subsonic* (*resp., interior supersonic*) *trajectory to* (3.1)*. Furthermore, the transformed function $F(w; r)$ denotes an interior positive* (*resp., interior negative*) *trajectory to* (3.2) *in the (w, F) plane.*

From the above definition, it's easy to see that every interior subsonic (resp., interior supersonic) trajectory of (3.1) always corresponds to an interior positive (resp., interior negative) trajectory to (3.2). To seek a transonic solution of (3.1), we turn to analyzing the structure of solutions to system (3.2). For clarity, we first show the following three lemmas.

LEMMA 3.3. *Let $B \in L^\infty(r_0, r_1)$ satisfy $\mathcal{J} < \underline{B} \le B \le \overline{B}$, and let there exist a positive constant τ_0 depending on $(\overline{B}, r_0)$; then for any $r_0 < x_0 < r_1$ and $0 < \tau < \tau_0$, all interior positive trajectories to system* (3.2) *over (x_0, r_1) start from the point $(0, 0)$.*

Proof. First suppose that a trajectory starts from the point $(0,h)$ with a positive constant h, and because $F(0)=h>0$ and $\frac{dF}{dw}\big|_{w(x_0)=0}=\frac{x_0}{\tau\mathcal{J}}>0$, we here focus on the region $F\geq 0$. According to Lemma 5.2 in [19], and similarly to (3.2), we define an autonomous ODE system

$$\begin{cases} w_r=\dfrac{(w+\mathcal{J})^3}{x_0(w+2\mathcal{J})}\cdot\dfrac{F_1}{w}, \\ (F_1)_r=w+\mathcal{J}-\overline{B}+\dfrac{\mathcal{J}(w+\mathcal{J})}{\tau(w+2\mathcal{J})}\cdot\dfrac{F_1}{w}-\dfrac{\mathcal{J}}{\tau(w+\mathcal{J})}\end{cases} \quad \text{for} \quad r\geq x_0, \tag{3.4}$$

with $F_1(0)=h/2$. Therefore,

$$\begin{aligned}\frac{dF_1}{dw}&=x_0\left(\frac{(w+\mathcal{J}-\overline{B})(w+2\mathcal{J})}{(w+\mathcal{J})^3}\cdot\frac{w}{F_1}+\frac{\mathcal{J}}{\tau(w+\mathcal{J})^2}-\frac{\mathcal{J}(w+2\mathcal{J})}{\tau(w+\mathcal{J})^4}\cdot\frac{w}{F_1}\right)\\ &=: x_0\cdot H(w,F_1;\overline{B}).\end{aligned} \tag{3.5}$$

Hereinto the equation $H(w,F_1;\overline{B})=0$ determines a function in the $w-F$ coordinate system as follows:

$$\overline{\Xi}(w)=\frac{(w+2\mathcal{J})w}{(w+\mathcal{J})^2}-\frac{\tau(w+\mathcal{J}-\overline{B})(w+2\mathcal{J})w}{\mathcal{J}(w+\mathcal{J})}=:\frac{(w+2\mathcal{J})w}{(w+\mathcal{J})^2}+\Psi(w). \tag{3.6}$$

Clearly, all critical points of the trajectories to system (3.5) are located at the curve $\overline{\Xi}(w)$. For (3.6), a straightforward calculation shows that

$$\Psi(0)=0,\quad \Psi(\overline{B}-\mathcal{J})=0,$$

$$\Psi'(w)=-\frac{\tau}{\mathcal{J}}\left(2w+2\mathcal{J}-\overline{B}-\frac{\mathcal{J}^2\overline{B}}{(w+\mathcal{J})^2}\right)\quad\text{for}\quad w\geq 0, \tag{3.7}$$

$$\Psi''(w)=-\frac{2\tau}{\mathcal{J}}\left(1+\frac{\mathcal{J}^2\overline{B}}{(w+\mathcal{J})^3}\right)<0\quad\text{for}\quad w\geq 0,$$

$$\Psi'(0)=\frac{2\tau}{\mathcal{J}}\left(\overline{B}-\mathcal{J}\right)>0,\quad\text{and}\quad \Psi'(\overline{B}-\mathcal{J})=-\frac{\tau}{\mathcal{J}}\left(\overline{B}-\frac{\mathcal{J}^2}{\overline{B}}\right)<0,$$

which implies that Ψ is concave on $[0,\infty)$. Further, we have

$$\overline{\Xi}(0)=0,\quad \overline{\Xi}(w)\geq\Psi(w)\geq 0\quad\text{for}\quad w\in[0,\overline{B}-\mathcal{J}] \tag{3.8}$$

and

$$\overline{\Xi}'(w)=\frac{2\mathcal{J}^2}{(w+\mathcal{J})^3}+\Psi'(w)\quad\text{for}\quad w\in[0,\overline{B}-\mathcal{J}]. \tag{3.9}$$

Hereafter it follows from (3.5) and (3.6) that

$$\begin{aligned}\frac{dF_1}{dw}&=x_0\left(-\frac{\mathcal{J}\overline{\Xi}}{\tau(w+\mathcal{J})^2F_1}+\frac{\mathcal{J}}{\tau(w+\mathcal{J})^2}\right)\\ &=\frac{x_0\mathcal{J}}{\tau(w+\mathcal{J})^2}\cdot\left(\frac{F_1-k\overline{\Xi}}{F_1}+\frac{(k-1)\overline{\Xi}}{F_1}\right),\end{aligned} \tag{3.10}$$

where $k > 0$ is a constant to be determined later. Thanks to (3.7)–(3.10), we obtain

(3.11)
$$\begin{aligned}(F_1^2 - k^2\overline{\Xi}^2)' &= 2F_1F_1' - 2k^2\overline{\Xi}\overline{\Xi}' \\ &= 2\mathcal{J}\overline{\Xi}\left[\frac{x_0(k-1)}{\tau(w+\mathcal{J})^2} - \frac{2k^2\mathcal{J}}{(w+\mathcal{J})^3} + \frac{k^2\tau}{\mathcal{J}^2}\left(2w + 2\mathcal{J} - \overline{B} - \frac{\mathcal{J}^2\overline{B}}{(w+\mathcal{J})^2}\right)\right] \\ &\quad + \frac{2x_0\mathcal{J}(F_1 - k\overline{\Xi})}{\tau(w+\mathcal{J})^2} \\ &= 2\mathcal{J}\overline{\Xi}\cdot I(w;x_0,\overline{B}) + (F_1^2 - k^2\overline{\Xi}^2)\cdot\frac{2x_0\mathcal{J}}{\tau(w+\mathcal{J})^2(F_1 + k\overline{\Xi})},\end{aligned}$$

where

$$I(w;x_0,\overline{B}) := \frac{x_0(k-1)}{\tau(w+\mathcal{J})^2} - \frac{2k^2\mathcal{J}}{(w+\mathcal{J})^3} + \frac{k^2\tau}{\mathcal{J}^2}\left(2w + 2\mathcal{J} - \overline{B} - \frac{\mathcal{J}^2\overline{B}}{(w+\mathcal{J})^2}\right).$$

We next determine k such that $I > 0$ for $w \in [0, \overline{B} - \mathcal{J}]$. To this end, we set $k = \frac{r_0}{8\tau}$, so that if $\tau < \min\{\frac{2\mathcal{J}^2}{(\overline{B}^3 + \mathcal{J}^2\overline{B})}, \frac{r_0}{16}\}$, we have

$$\begin{aligned}I &= \frac{1}{\tau(w+\mathcal{J})^2}\cdot\left[x_0(k-1) - \frac{2k^2\tau\mathcal{J}}{w+\mathcal{J}} + \frac{k^2\tau^2}{\mathcal{J}^2}\left(2(w+\mathcal{J})^3 - \overline{B}(w+\mathcal{J})^2 - \mathcal{J}^2\overline{B}\right)\right] \\ &> \frac{1}{\tau(w+\mathcal{J})^2}\cdot\left[(r_0k - 2k^2\tau) - r_0 - \frac{k^2\tau^2}{\mathcal{J}^2}(\overline{B}^3 + \mathcal{J}^2\overline{B})\right] \\ &= \frac{1}{\tau(w+\mathcal{J})^2}\cdot\left[k(r_0 - 2k\tau - \frac{r_0\tau}{8\mathcal{J}^2}(\overline{B}^3 + \mathcal{J}^2\overline{B})) - r_0\right] \\ &> \frac{1}{\tau(w+\mathcal{J})^2}\cdot\left(\frac{r_0k}{2} - r_0\right) = \frac{r_0}{\tau(w+\mathcal{J})^2}\cdot\left(\frac{r_0}{16\tau} - 1\right) > 0,\end{aligned}$$

which in combination with (3.8), (3.11), and $F_1^2(0) - k^2\overline{\Xi}^2(0) = h^2/4 > 0$ indicates that

$$F_1(w) \ge k\overline{\Xi}(w) \ge 0 \quad \text{for} \quad w \in [0, \overline{B} - \mathcal{J}]. \tag{3.12}$$

One can see that $k = \frac{r_0}{8\tau} > 1$ and $\overline{\Xi}(\overline{B} - \mathcal{J}) = \frac{(\overline{B}+\mathcal{J})(\overline{B}-\mathcal{J})}{\overline{B}^2} > 0$, and if $\tau \le \frac{r_0(\overline{B}+\mathcal{J})(\overline{B}-\mathcal{J})}{8\overline{B}^2}$, it yields that

$$F_1(\overline{B} - \mathcal{J}) \ge \frac{r_0(\overline{B}+\mathcal{J})(\overline{B}-\mathcal{J})}{8\tau\overline{B}^2} > 1. \tag{3.13}$$

Thus, it implies by (3.5) that for $w \ge \overline{B} - \mathcal{J}$,

$$\begin{aligned}\frac{dF_1}{dw} &\ge x_0\left(\frac{\mathcal{J}}{\tau(w+\mathcal{J})^2} - \frac{\mathcal{J}(w+2\mathcal{J})}{\tau(w+\mathcal{J})^4}\cdot\frac{w}{F_1}\right) \\ &= \frac{x_0\mathcal{J}}{\tau(w+\mathcal{J})^2F_1}\left(F_1 - 1 + \frac{\mathcal{J}^2}{(w+\mathcal{J})^2}\right) > 0.\end{aligned} \tag{3.14}$$

Therefore, the trajectories of (3.4) starting from $(0, h)$ go to infinity but cannot go back to the line $w = 0$.

For any $r \in (x_0, r_1)$, we next claim that if $F(0) = h > 0$,

$$\frac{dF(w)}{dw} > 0 \quad \text{for} \quad w \geq 0. \tag{3.15}$$

First denote the solution of (3.2) by (w, F) and the solution of (3.4) by (w_1, F_1). In fact, there exist constants ε_0, $\bar{r}_1$, and $\hat{r}_1$ such that $\varepsilon_0 < \overline{B} - \mathcal{J}$, $w(\bar{r}_1) = \varepsilon_0$, and $w_1(\hat{r}_1) = \varepsilon_0$. For $w \in [0, \varepsilon_0]$, defining $\bar{F} := F - F_1$, and by (3.3) and (3.5), one finds that

$$\begin{aligned} \frac{d\bar{F}}{dw} &= r \cdot H(w, F; B) - x_0 \cdot H(w, F_1; \overline{B}) \\ &= r \cdot H(w, F; \overline{B}) - x_0 \cdot H(w, F_1; \overline{B}) + \frac{r}{F} \cdot K(w), \end{aligned} \tag{3.16}$$

where the function H is defined in (3.5) and

$$K(w) := \frac{(\overline{B} - B)(w + 2\mathcal{J})w}{(w + \mathcal{J})^3}.$$

Obviously, note that $F(0) - F_1(0) = h/2 > 0$; then take ε_0 small enough so that $F(w) - F_1(w) > 0$ for $0 < w \leq \varepsilon_0$; further we see that $K(w) \geq 0$ and

$$H(w, F; \overline{B}) \geq H(w, F_1; \overline{B}) = \frac{1}{x_0} \cdot \frac{dF_1}{dw} > 0 \quad \text{on} \quad (0, \varepsilon_0).$$

Here the last inequality is proved by (3.10)–(3.12). Thus, it's easy to check that $\frac{d\bar{F}}{dw}(w) > 0$ over $(0, \varepsilon_0]$. Since $\frac{dF_1(w)}{dw} > 0$ on $[0, \overline{B} - \mathcal{J}]$ in the case of $\tau \ll 1$, the result is easily extended up to $w = \overline{B} - \mathcal{J}$, that is,

$$\frac{dF}{dw} \geq \frac{dF_1}{dw} > 0 \quad \text{over} \quad [0, \overline{B} - \mathcal{J}].$$

Hereinto we assume that there exist constants $\bar{r}_2$ and $\hat{r}_2$ such that $w(\bar{r}_2) = w_1(\hat{r}_2) = \overline{B} - \mathcal{J}$. Now if $\bar{r}_2 < r_1$, we continue to show that $\frac{dF}{dw} > 0$ for $r \in [\bar{r}_2, r_1]$. As in (3.16), and by (3.3) and (3.14), one can repeat the above argument to derive the result up to $r = r_1$. As a consequence, (3.15) holds such that the claim is verified.

On the other hand, a trajectory cannot start from $(0, -h)$ by (3.2). Accordingly, if $0 < \tau < \tau_0(\overline{B}, r_0) := \min\{\frac{2\mathcal{J}^2}{\overline{B}^3 + \mathcal{J}^2\overline{B}}, \frac{r_0(\overline{B}+\mathcal{J})(\overline{B}-\mathcal{J})}{8\overline{B}^2}, \frac{r_0}{16}\}$, this lemma is proved. ∎

LEMMA 3.4. *Let all assumptions of Lemma* 3.3 *hold, and let there exist a constant* $\tau_0(\overline{B}, r_0)$ *such that for any* $0 < \tau < \tau_0$ *and* $r_0 < x_0 < r_1$, *satisfying* $x_0 - r_0 \leq C(\tau)$, *all interior negative trajectories over* (r_0, x_0) *to system* (3.2) *end at the point* $(0, 0)$.

Proof. We first consider another ODE system as follows:

$$\begin{cases} w_r = \dfrac{(w + \mathcal{J})^3}{r_0(w + 2\mathcal{J})} \cdot \dfrac{F_2}{w}, \\ (F_2)_r = w + \mathcal{J} - \overline{B} + \dfrac{\mathcal{J}(w + \mathcal{J})}{\tau(w + 2\mathcal{J})} \cdot \dfrac{F_2}{w} - \dfrac{\mathcal{J}}{\tau(w + \mathcal{J})} \end{cases} \quad \text{for} \quad r \geq r_0,$$

and

$$\frac{dF_2}{dw} = r_0 \cdot H(w, F_2; \overline{B}). \tag{3.17}$$

Then the critical curve is also denoted by (3.6), and note that
(3.18)
$$\overline{\Xi}(w) < 0, \quad \overline{\Xi}'(w) > \frac{2}{\mathcal{J}}(1 + \tau(\overline{B} - \mathcal{J})) > 0 \quad \text{and} \quad \overline{\Xi}''(w) < 0 \quad \text{for} \quad w \in (-\mathcal{J}, 0).$$

Suppose that $F_2(0) = -h/2 < 0$ and it holds that $\frac{dF_2}{dw}(0) > 0$; then we consider the region $F \leq 0$ and $-\mathcal{J} < w \leq 0$. Furthermore, similar to the proof of Lemma 3.3, one see that

$$(F_2^2 - k^2\overline{\Xi}^2)' = 2\mathcal{J}\overline{\Xi} \cdot I(w; r_0, \overline{B}) + (F_2^2 - k^2\overline{\Xi}^2) \cdot \frac{2r_0\mathcal{J}}{\tau(w+\mathcal{J})^2(F_2 + k\overline{\Xi})}. \tag{3.19}$$

From the form of $I(w; r_0, \overline{B})$, we derive that the trajectory of I certainly changes sign on $(-\mathcal{J}, 0)$. Now we want to choose a suitable value of k to ensure

$$I(w) > 0 \quad \text{on} \quad \Big(-\frac{\mathcal{J}}{2}, 0\Big].$$

Setting $k = \frac{r_0}{8\tau}$ and $\tau < \min\{\frac{1}{\overline{B}}, \frac{r_0}{32}\}$, we obtain

$$\begin{aligned}
&I(w; r_0, \overline{B}) \\
&\quad \geq \frac{1}{\tau(w+\mathcal{J})^2} \cdot \left[(r_0 k - 4k^2\tau) - r_0 + \frac{k^2\tau^2}{\mathcal{J}^2}\left(2(w+\mathcal{J})^3 - \overline{B}(w+\mathcal{J})^2 - \mathcal{J}^2\overline{B}\right)\right] \\
&\quad \geq \frac{1}{\tau(w+\mathcal{J})^2} \cdot \left[(r_0 k - 4k^2\tau) - r_0 - 2\overline{B}k^2\tau^2\right] \\
&\quad = \frac{1}{\tau(w+\mathcal{J})^2} \cdot \left[k\left(r_0 - 4k\tau - \frac{r_0\overline{B}\tau}{4}\right) - r_0\right] \\
&\quad > \frac{r_0}{\tau(w+\mathcal{J})^2} \cdot \left(\frac{k}{4} - 1\right) > 0.
\end{aligned}$$

Then it follows that

$$F_2^2(w) \geq k^2\overline{\Xi}^2(w) \quad \text{for} \quad w \in \Big(-\frac{\mathcal{J}}{2}, 0\Big],$$

which implies by (3.18) that

$$F_2(w) < k\overline{\Xi}(w) < 0 \quad \text{for} \quad w \in \Big(-\frac{\mathcal{J}}{2}, 0\Big).$$

So we conclude from (3.17) that

$$\frac{dF_2}{dw}(w) > 0 \quad \text{for} \quad w \in \Big(-\frac{\mathcal{J}}{2}, 0\Big).$$

We next define $\hat{F} := F - F_2$; then for $r \in (r_0, x_0)$,

$$\frac{d\hat{F}}{dw} = r \cdot H(w, F; \overline{B}) - r_0 \cdot H(w, F_2; \overline{B}) + \frac{r}{F} \cdot K(w),$$

where K is given by Lemma 3.3. Thanks to $F(0) - F_2(0) = -h/2 < 0$, and as in Lemma 3.3, we apply the same argument to prove that

$$\frac{dF}{dw}(w) \geq \frac{dF_2}{dw}(w) > 0 \quad \text{for} \quad w \in \Big(-\frac{\mathcal{J}}{2}, 0\Big) \quad \text{and} \quad r \in (r_0, x_0).$$

By (3.3), one finds that

$$0 < \frac{dF(w)}{dw} \leq \frac{r\mathcal{J}}{\tau(w+\mathcal{J})^2} \leq \frac{4x_0}{\tau\mathcal{J}} \quad \text{and} \quad \frac{F(w)}{w} \leq \frac{4x_0}{\tau\mathcal{J}} \quad \text{for} \quad w \in \left(-\frac{\mathcal{J}}{2}, 0\right).$$

Thus for the region $F < 0$, it follows from (3.2) that

$$0 < w_r(r) \leq \frac{4x_0(w+\mathcal{J})^3}{\tau r_0(w+2\mathcal{J})} \leq \frac{2x_0\mathcal{J}^2}{r_0\tau} \quad \text{for} \quad r \in (r_0, x_0) \quad \text{and} \quad w \in \left(-\frac{\mathcal{J}}{2}, 0\right).$$

Then it implies that there exists an inverse function $r(w)$ such that $r_w(w) \geq C(\tau)$ on $(-\frac{\mathcal{J}}{2}, 0)$ where the constant C positively depends on τ. Suppose that τ is small enough and $x_0 - r_0 \leq C(\tau)$ with a small and positive constant C; then the trajectory cannot go back to the region $F \geq 0$ at the starting state, that is, $w(r_0) < 0$ and $F|_{w(r_0)} < 0$. Hence, the trajectories ending at $(0, -h)$ cannot start from the line $w = 0$. In addition, it's impossible that any trajectories end at $(0, h)$. The proof is finished. ∎

LEMMA 3.5. *There exists a constant τ_0 such that for any $0 < \tau < \tau_0$, system* (3.2) *has no transonic shock solution.*

Proof. See Theorem 5.13 in [19]. ∎

On the basis of Lemmas 3.3–3.5, and additionally according to Theorem 2.1 and Theorem 3.2 in [9], we can construct infinitely many continuous transonic solutions to system (3.2) and there is no transonic shock solution in the case of $\tau \ll 1$; then Theorem 3.1 is completely proved.

Naturally, we then discuss the interior regularity of the continuous transonic solutions referring to section 5 in [19], and the result is as follows.

THEOREM 3.6 (C^1-smooth transonic solutions). *Assume that $B \in C[r_0, r_1]$ and that there exists a constant $\tau_0 > 0$ such that $0 < \tau < \tau_0$; then system* (3.1) *has infinitely many C^1-smooth transonic solutions denoted in Theorem* 3.1 *satisfying*

$$m_{sub}(x_0) = m_{sup}(x_0), \quad \tilde{E}_{sub}(x_0) = \tilde{E}_{sup}(x_0),$$

and

$$m'_{sub}(x_0) = m'_{sup}(x_0) = \frac{\mathcal{J}}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - \frac{8}{\tau x_0}\big[\tau(B_0 - \mathcal{J}) + 1\big]}\right)$$

at the continuous points x_0.

Proof. From Theorem 3.1, it has been known that system (3.1) has a continuous transonic solution with the transonic transition point x_0. Now it suffices to prove that the continuous transonic solution is C^1-smooth on the neighborhood of x_0. Therefore, to achieve this purpose, we have to apply local analysis near the continuous point x_0. Here our adopted approach is really rigorous but tedious, so this proof is divided into three steps.

Step 1. First, one claims that all interior positive trajectories over (x_0, r_1) starting from the point $(0, 0)$ to system (3.2) are C^1-smooth on the neighborhood of $w = 0^+$. In fact, if $\frac{F(w)}{w} \leq 0$ near $w = 0^+$, then this implies from (3.3) and $\underline{B} > \mathcal{J}$ that $\frac{dF(w)}{dw} > 0$ on a neighborhood of $w = 0^+$, which is a contradiction to $F(0) = 0$. Thus, there is always a small constant ε_1 such that $\frac{F(w)}{w} > 0$ on $(0, \varepsilon_1)$. Because of $B \in C[r_0, r_1]$, we see that the function $\frac{dF}{dw}$ is smooth with respect to w. Hence the claim is verified.

Next we're going to prove that

$$\lim_{w\to 0^+} \frac{dF(w)}{dw} \text{ exists} = F'(0).$$

Here $F'(0)$ satisfies

$$F'(0) = \frac{2(\mathcal{J}-B_0)}{\mathcal{J}^2}\cdot\frac{x_0}{F'(0)} + \frac{x_0}{\tau\mathcal{J}} - \frac{2}{\tau\mathcal{J}^2}\cdot\frac{x_0}{F'(0)} \tag{3.20}$$

with $B_0 = B(x_0) > \mathcal{J}$. A direct computation indicates that $F'(0)$ has two solutions denoted by

$$\theta_1 = \frac{1}{2}\left(\frac{x_0}{\tau\mathcal{J}} + \sqrt{\left(\frac{x_0}{\tau\mathcal{J}}\right)^2 - \frac{8x_0}{\tau\mathcal{J}^2}\left[\tau(B_0-\mathcal{J})+1\right]}\right)$$

or

$$\theta_2 = \frac{1}{2}\left(\frac{x_0}{\tau\mathcal{J}} - \sqrt{\left(\frac{x_0}{\tau\mathcal{J}}\right)^2 - \frac{8x_0}{\tau\mathcal{J}^2}\left[\tau(B_0-\mathcal{J})+1\right]}\right).$$

Now we prepare to exclude the choice of θ_1. Actually, since $\theta_1 = O(\frac{1}{\tau})$ and $\theta_2 = C(\mathcal{J}) + O(\tau)$, one can see that $\frac{\theta_1}{2} > \theta_2 > 0$ when $\tau \ll 1$. After this, if

$$\lim_{w\to 0^+} \frac{dF(w)}{dw} \text{ exists} > \frac{\theta_1}{2},$$

then there exists a constant $\varepsilon_2 < \overline{B} - \mathcal{J}$ such that $\frac{F(w)}{w} \geq \frac{\theta_1}{2} > \theta_2$ on $(0,\varepsilon_2]$. We here set $\frac{F_1(\varepsilon_2)}{\varepsilon_2} := \frac{F(\varepsilon_2)}{\varepsilon_2} \geq \frac{\theta_1}{2}$. By Lemma 3.3, if $\tau \ll 1$, we see that for $w \in [\varepsilon_2, \overline{B}-\mathcal{J}]$,

$$I(w; x_0, \overline{B}) > 0.$$

Moreover, we can set $F_1(\varepsilon_2) \geq \frac{\theta_1\varepsilon_2}{2} > \frac{\overline{B}^2\Xi(\varepsilon_2)}{\overline{B}^2-\mathcal{J}^2} > 0$; then it follows from (3.8) and (3.11) that

$$F_1(w) \geq \frac{\overline{B}^2\Xi(w)}{\overline{B}^2-\mathcal{J}^2} > 0 \quad \text{on} \quad [\varepsilon_2, \overline{B}-\mathcal{J}].$$

Next as in Lemma 3.3, we also get that

$$\frac{dF_1}{dw}(w) > 0 \quad \text{for} \quad w \geq \overline{B}-\mathcal{J},$$

and further

$$\frac{dF}{dw}(w) \geq \frac{dF_1}{dw}(w) > 0 \quad \text{for} \quad w \geq \varepsilon_2.$$

Since $F(w)$ denotes an interior positive trajectory and goes back to the line $w = 0$, this is a contradiction.

Afterwards, it suffices to show that

$$\lim_{w\to 0^+} \frac{dF(w)}{dw} \text{ exists} = F'(0) = \theta_2.$$

Based on the above, we note that there exist constants ε_3 and θ_3 satisfying $\varepsilon_3 < \underline{B}-\mathcal{J}$ and $\theta_3 \leq \frac{\theta_1}{2}$, so that

$$\frac{F(w)}{w} \leq \frac{\theta_1}{2} \quad \text{on} \quad (0,\varepsilon_3] \quad \text{and} \quad \frac{F(\varepsilon_3)}{\varepsilon_3} = \theta_3.$$

In fact, one can take the variable r of (3.3) as a continuous function with respect to w over $[0, +\infty)$ and $r(0) = x_0$. Then, we discuss this issue into two cases, $\theta_2 < \theta_3 < \frac{\theta_1}{2}$ and $0 < \theta_3 < \theta_2$, and create a locally iterative approximation as follows.

In the case of $\theta_3 > \theta_2$, because of $B \in C[r_0.r_1]$ and $r \in C[0, \varepsilon_3]$, it then follows that

$$\begin{aligned}\frac{dF}{dw}(\varepsilon_3) &= r(\varepsilon_3)\left(\frac{(\varepsilon_3 + \mathcal{J} - B)(\varepsilon_3 + 2\mathcal{J})}{(\varepsilon_3 + \mathcal{J})^3} - \frac{\mathcal{J}(\varepsilon_3 + 2\mathcal{J})}{\tau(\varepsilon_3 + \mathcal{J})^4}\right) \cdot \frac{1}{\theta_3} + \frac{r(\varepsilon_3)\mathcal{J}}{\tau(\varepsilon_3 + \mathcal{J})^2} \\ &\geq -\left(\frac{2(B_0 - \mathcal{J})}{\mathcal{J}^2} + \frac{2}{\tau\mathcal{J}^2}\right) \cdot \frac{x_0}{\theta_3} + \frac{x_0}{\tau\mathcal{J}} - C(r(\varepsilon_3) - r(0) + \varepsilon_3) \\ &\geq \theta_3 + f_1(\theta_3) - C_1\varepsilon_3,\end{aligned}$$

where $C_1 > 0$ is a fixed constant depending on $(r_0, r_1, \mathcal{J}, \tau, \overline{B})$ but independent of ε_3, and

$$f_1(s) := -\left(\frac{2(B_0 - \mathcal{J})}{\mathcal{J}^2} + \frac{2}{\tau\mathcal{J}^2}\right) \cdot \frac{x_0}{s} - s + \frac{x_0}{\tau\mathcal{J}} \quad \text{for} \quad s > 0.$$

From (3.20), it's easy to see that f_1 is a positive function on $(\theta_2, \theta_1/2)$ and $f_1(\theta_2) = 0$. Indeed, we can find a small constant δ such that $\theta_2 + \delta < \theta_3$ and

$$f_1(s) \geq f_1(\theta_2 + \delta) \quad \text{for any} \quad s \in (\theta_2 + \delta, \theta_1/2).$$

Here we redefine $\varepsilon_3 = \frac{f_1(\theta_2+\delta)}{2C_1}$ with a sufficiently small δ; then

$$\frac{dF}{dw}(\varepsilon_3) \geq \theta_3 + \frac{f_1(\theta_3)}{2} > \frac{F}{w}(\varepsilon_3).$$

The above inequality means that the trajectory $F(w)$ keeps close to the line $F = \theta_2 w$ if w tends to zero, and the details of the progress are described below. By the continuity of $\frac{dF}{dw}$, there must exist a constant $\varepsilon_4 < \varepsilon_3$ satisfying $\frac{F(\varepsilon_4)}{\varepsilon_4} := \theta_4 \geq \theta_2 + \delta$ and

$$\frac{dF(w)}{dw} \geq \theta_3 \quad \text{for} \quad w \in (\varepsilon_4, \varepsilon_3).$$

Obviously, the point $(\theta_4, F(\theta_4))$ is under the line $F(w) = \theta_3 w$ in the plane (w, F), which indicates that $\theta_4 < \theta_3$. Noting that

$$\varepsilon_4 < \varepsilon_3 \leq \frac{f_1(\theta_2 + \delta)}{2C_1} \leq \frac{f_1(\theta_4)}{2C_1},$$

we then estimate

$$\frac{dF}{dw}(\varepsilon_4) \geq \theta_4 + f_1(\theta_4) - C_1\varepsilon_4 \geq \theta_4 + \frac{f_1(\theta_4)}{2} > \frac{F}{w}(\varepsilon_4).$$

Repeating the above process, we notice that the trajectory crossing the point $(\varepsilon_3, \theta_3\varepsilon_3)$ eventually intersects with the line $F(w) = (\theta_2+\delta)w$ at a point $(\hat{\varepsilon}, (\theta_2+\delta)\hat{\varepsilon})$. In nature, as the trajectory is sufficiently close to the line $w = 0$, the constant $\hat{\varepsilon}$ can keep getting small, and we can find a smaller constant δ because $f_1'(s) > 0$ near $s = \theta_2^+$. Note that the trajectory $F(w)$ only starts from the point $(0, 0)$, which is proved in Theorem 3.1. Therefore, it holds that

$$\delta \searrow 0^+ \quad \text{as} \quad \hat{\varepsilon} \searrow 0^+.$$

In other words, the trajectory of (3.2) and the line $F(w) = \theta_2 w$ are tangent at the point $(0, 0)$, that is

$$\lim_{w \mapsto 0^+} \frac{dF(w)}{dw} = \theta_2.$$

In the case of $\theta_3 < \theta_2$, we can conclude that

$$\frac{dF}{dw}(\varepsilon_3) \le \theta_3 + f_1(\theta_3) + C_2\varepsilon_3 \le \theta_3 + \frac{f_1(\theta_3)}{2} < \frac{F}{w}(\varepsilon_3)$$

if ε_3 is small enough. Here f_1 is negative on $(0, \theta_2)$ and C_2 is a positive constant. As in to the case of $\theta_3 > \theta_2$, we also observe that the slope of the interior positive trajectory of system (3.2) can get close to the constant θ_2 as $\varepsilon_3 \searrow 0$. In the end, we also obtain

$$\frac{dF(w)}{dw} \to \theta_2 \quad \text{as} \quad w \to 0^+.$$

Of course, when $\theta_3 = \theta_2$, the conclusion is obvious by the first two cases. The step is complete.

Step 2. In this step, from Lemma 3.4, we know that all interior negative trajectories ending at the point $(0, 0)$ to system (3.2) are C^1-smooth on a neighborhood of $w = 0^-$ for $r \in (r_0, x_0)$, where $x_0 - r_0 \ll 1$. Then a claim is stated that the limit of $\frac{dF}{dw}$ exists, and

$$\lim_{w \to 0^-} \frac{dF(w)}{dw} = F'(0).$$

Here,

$$F'(0) = \frac{1}{2}\left(\frac{x_0}{\tau\mathcal{J}} \pm \sqrt{\left(\frac{x_0}{\tau\mathcal{J}}\right)^2 - \frac{8x_0}{\tau\mathcal{J}^2}\left[\tau(B_0 - \mathcal{J}) + 1\right]}\right) = \theta_{1,2} > 0 \quad \text{if} \quad \tau \ll 1.$$

Also, it follows from (3.3) that $\frac{F(w)}{w} > 0$ near $w = 0^-$. If $\lim\limits_{w \to 0^-} \frac{dF(w)}{dw}$ exists $> \frac{\theta_1}{2}$, then there exists a constant $\varepsilon_5 < 0$ such that $\frac{F(w)}{w} \ge \frac{\theta_1}{2}$ on $[\varepsilon_5, 0)$. Further, we assume that $\frac{F_2(\varepsilon_5)}{\varepsilon_5} := \frac{F(\varepsilon_5)}{\varepsilon_5} \ge \frac{\theta_1}{2}$ where the function F_2 is defined in Lemma 3.4. Due to $x_0 - r_0 \ll 1$, we get that $w \ge -\mathcal{J}/2$. By Lemma 3.4, taking $k = 2$ in (3.19), when $\tau < \min\{\frac{r_0}{64}, \sqrt{\frac{r_0}{32\overline{B}}}\}$, we see that for $w \in (-\frac{\mathcal{J}}{2}, \varepsilon_5]$,

$$\begin{aligned} I(w; r_0, \overline{B}) &\ge \frac{1}{\tau(w + \mathcal{J})^2} \cdot \left[(r_0 - 16\tau) + \frac{4\tau^2}{\mathcal{J}^2}(2(w + \mathcal{J})^3 - \overline{B}(w + \mathcal{J})^2 - \mathcal{J}^2\overline{B})\right] \\ &\ge \frac{1}{\tau(w + \mathcal{J})^2} \cdot \left[r_0 - 16\tau - 8\overline{B}\tau^2\right] \\ &> \frac{1}{\tau(w + \mathcal{J})^2} \cdot \frac{r_0}{2} > 0. \end{aligned}$$

Obviously, $F_2(x_5) \le \frac{\theta_1\varepsilon_5}{2} < 2\overline{\Xi}(x_5) < 0$ when $\tau \ll 1$. By (3.19), we have

$$F_2^2(w) \ge 4\overline{\Xi}^2(w) \quad \text{on} \quad \Big(-\frac{\mathcal{J}}{2}, \varepsilon_5\Big].$$

Thus, $F_2(w) \le \frac{\theta_1\varepsilon_5}{2} < 2\Xi(w) < 0$ on $(-\frac{\mathcal{J}}{2}, \varepsilon_5]$. Now as similar to Lemma 3.4, it's proved that

$$\frac{dF(w)}{dw} > \frac{dF_2(w)}{dw} > 0 \quad \text{for} \quad w \in \Big(-\frac{\mathcal{J}}{2}, \varepsilon_5\Big] \quad \text{and} \quad r \in (r_0, x_0).$$

Hence the trajectory $F(w)$ ending at the point $(0,0)$ cannot go back the line $w=0$ for $r\in(r_0,x_0)$, so that $F'(0)=\theta_2$. Next there exists a constant ε_6, and we also have two cases, $\theta_2<\frac{F(\varepsilon_6)}{\varepsilon_6}\le\frac{\theta_1}{2}$ or $0<\frac{F(\varepsilon_6)}{\varepsilon_6}<\theta_2$. As in Step 1, by a rigorous local analysis, we derive that

$$\lim_{w\to 0^-}\frac{dF(w)}{dw}=\theta_2.$$

Step 3. From Steps 1–2, it is concluded that every transonic trajectory is C^1-smooth over (r_0,r_1) and the corresponding solution satisfies, at a continuous point x_0,

$$F(x_0)=w(x_0)=0$$

and

$$w'(x_0)=\lim_{r\to x_0}\frac{(w+\mathcal{J})^3}{r(w+2\mathcal{J})}\cdot\frac{F}{w}=\frac{\mathcal{J}^2}{2x_0}\cdot\lim_{w\to 0}\frac{dF(w)}{dw}=\frac{\mathcal{J}}{4}\left(\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2}-\frac{8}{\tau x_0}\left[\tau(B_0-\mathcal{J})+1\right]}\right).$$

Afterwards, by (3.1) and (3.2), we have

$$\tilde{E}_{sub}(x_0)=\tilde{E}_{sup}(x_0)=\frac{1}{\tau}-\frac{1}{x_0},\quad m_{sub}(x_0)=m_{sup}(x_0)=\mathcal{J},$$

and

$$m'_{sub}(x_0)=m'_{sup}(x_0)=w'(x_0)=\frac{\mathcal{J}}{4}\left(\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2}-\frac{8}{\tau x_0}\left[\tau(B_0-\mathcal{J})+1\right]}\right).$$

Thus, we complete the proof of this theorem. ∎

4. Transonic solutions in three dimensions. The existence of transonic solutions to the three-dimensional system (1.10) will be stated, especially, including transonic shock solutions and transonic C^1-smooth solutions. Analogously to Theorems 2.2, 3.1, and 3.6, and in a similar manner, we can obtain some results about transonic solutions to the following system:

$$\begin{cases}\left(1-\dfrac{\mathcal{J}^2}{m^2}\right)m_r=m(\tilde{E}+\dfrac{2}{r})-\dfrac{\mathcal{J}}{\tau},\\(r^2\tilde{E})_r=m-B,\\m(r_0)=m(r_1)=\mathcal{J}.\end{cases}\tag{4.1}$$

Theorem 4.1 (transonic shock solutions). *Assume that* $\tau\gg 1$ *and* $\underline{B}-2>\mathcal{J}$ *and that system* (4.1) *has a transonic shock solution* $(m_{tran},\tilde{E}_{tran})(r)$ *over* $[r_0,r_1]$, *satisfying the entropy condition* (1.11) *and the Rankine–Hugoniot condition* (1.12) *at a point* $x_0\in(r_0,r_1)$. *Because of the arbitrary choices of* x_0, *the transonic solutions are infinitely many.*

Theorem 4.2 (C^1-smooth transonic solutions). *Assume that* $B\in C[r_0,r_1]$ *satisfies* $\underline{B}-2>\mathcal{J}$ *and there exists a constant* $\tau_0(r_0,r_1,\overline{B})$ *such that for* $0<\tau<\tau_0$, *there exist infinitely many* C^1*-smooth transonic solutions* $(m_{tran},\tilde{E}_{tran})$ *to system* (4.1) *in the form of*

$$(m_{tran},\tilde{E}_{tran})(r)=\begin{cases}(m_{sup},\tilde{E}_{sup})(r) & \textit{for}\quad r\in[r_0,x_0),\\(m_{sub},\tilde{E}_{sub})(r) & \textit{for}\quad r\in[x_0,r_1].\end{cases}$$

Indeed,

$$m_{sub}(x_0) = m_{sup}(x_0), \quad \tilde{E}_{sub}(x_0) = \tilde{E}_{sup}(x_0),$$

and

$$m'_{sub}(x_0) = m'_{sup}(x_0) = \frac{\mathcal{J}}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - \frac{8}{\tau x_0^2}\left[\tau(B_0 - 2 - \mathcal{J}) + 2x_0\right]}\right)$$

at some continuous points x_0. There is also no transonic shock solution to system (4.1).

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