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Steady hydrodynamic model of semiconductors with sonic boundary and transonic doping profile

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Abstract

As shown in [17,18], for the hydrodynamic model of semiconductors represented by Euler-Poisson equations with sonic boundary and subsonic/supersonic doping profile, the structure of stationary solutions are very complicated. It may possess various solutions like subsonic/supersonic/transonic flows. In this paper, we consider a more challenging case where the doping profile is transonic, which is categorized into two types: subsonic-dominated and supersonic-dominated. In the subsonic-dominated case, we show that the system has a unique interior subsonic solution, at least one interior supersonic solution and infinitely many transonic solutions under the suitable assumptions. However, the difference with the case of subsonic doping profile is that the interior subsonic solution and interior supersonic solution may not exist in special cases when the relaxation time is small. In the supersonic-dominated case, the non-existence and existence of all types of solutions are also obtained. The approach adopted is the technical compactness analysis combining the Green's function method. Here, the results obtained perfectly develop the existing studies.

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Keywords: Euler-Poisson equations; Hydrodynamic model of semiconductors; Sonic boundary; Transonic doping profile; Subsonic-dominated; Supersonic-dominated

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1. Introduction

For the charged fluid particles such as electrons and holes in semiconductor devices, the presented model is the hydrodynamic system of semiconductors, the so-called Euler-Poisson equations [6]:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = \rho \Phi_x - \dfrac{\rho u}{\tau}, \qquad t > 0, \;\; x \in \Omega. \\ \Phi_{xx} = \rho - b(x). \end{cases} \tag{1.1}$$

Here $\rho(t,x)$, $u(t,x)$ and $\Phi(t,x)$ represent the electron density, the velocity and the electrostatic potential, respectively. $P(\rho)$ is the pressure-density relation, physically represented by

$$P(\rho) = T\rho^{\gamma}.$$

Here $T > 0$ is the constant temperature and $\gamma \geq 1$ represents the adiabatic exponent. In the paper, we mainly consider the isothermal case, i.e. $\gamma = 1$. The function $b(x) > 0$ is the doping profile standing for the density of impurities in semiconductor device. The constant $\tau > 0$ denotes the momentum relaxation time. Ω is the bounded domain. Without loss of generality, we take $\Omega = [0, 1]$ in the 1-dimensional case.

In order to study the behavior of solutions to (1.1), it is significant and necessary to classify the steady-state solutions in different cases.

Let $J := \rho u$, the current density of the electrons, and $E := \Phi_x$, the electric field, then the corresponding stationary system to (1.1) is reduced to

$$\begin{cases} J = \text{constant}, \\ \left(\dfrac{J^2}{\rho} + P(\rho)\right)_x = \rho E - \dfrac{J}{\tau}, \qquad x \in (0,1), \\ E_x = \rho - b(x), \end{cases} \tag{1.2}$$

From gas dynamics, we call $c := \sqrt{P'(\rho)} = \sqrt{T}$ the sound speed for $P(\rho) = T\rho$. So the stationary flow is supersonic/sonic/subsonic if the fluid velocity satisfies

$$\text{fluid velocity}: \quad u = \frac{J}{\rho} \gtreqqless c = \sqrt{T}: \quad \text{sound speed.} \tag{1.3}$$

In the following, we consider only the current-controlled flow; thus we assume that the current density J is prescribed. Note that if the couple $(\rho, E)(x)$ is a solution to (1.2) for a given constant J, then the couple $(\rho, -E)(1-x)$ is a solution to (1.2) with respect to $-J$ and $b(1-x)$. Hence, it is sufficient to consider only the case $J \geq 0$. Without loss of generality, we assume in the paper that

$$J = T = 1.$$

Then, (1.2) is reduced to

$$\begin{cases} \left(1-\dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x). \end{cases} \tag{1.4}$$

From (1.3), it can be checked that $\rho > 1$ denotes the subsonic flow, $\rho = 1$ means the sonic flow and $0 < \rho < 1$ stands for the supersonic flow. Thus, the sonic boundary condition to the system (1.4) is subjected by:

$$\rho(0) = \rho(1) = 1. \tag{1.5}$$

Dividing the first equation of (1.4) by ρ and differentiating the resulting equation with respect to x, and using the second equation of (1.4), we obtain

$$\begin{cases} \left[\left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x\right]_x + \dfrac{1}{\tau}\left(\dfrac{1}{\rho}\right)_x - [\rho - b(x)] = 0, \quad x \in (0,1). \\ \rho(0) = \rho(1) = 1 \qquad \text{(sonic boundary)}. \end{cases} \tag{1.6}$$

When the doping profile is the subsonic case, namely, $b(x) > 1$, we [17] first proved that, there exist a unique subsonic steady-state solution, at least one supersonic steady-state solutions, infinitely many transonic-shock solutions when the relaxation time is large, and infinitely many C^1-smooth transonic solutions when the relaxation time is small. Furthermore, we [18] studied the case of supersonic doping profile with $0 < b(x) < 1$, and classified the existence and non-existence of subsonic/supersonic/transonic steady-state solutions. However, when the doping profile is transonic, namely $b(x) > 1$ for some x in the domain and $b(x) < 1$ for the other part of the domain, the study remains open and is also more challenging. To treat such a unknown case is the main target of the present paper.

Throughout this paper, we denote

$$\underline{b} := \operatorname*{essinf}_{x\in(0,1)} b(x) \quad \text{and} \quad \bar{b} := \operatorname*{esssup}_{x\in(0,1)} b(x),$$

and assume that the doping profile $b(x) \in L^\infty(0,1)$ is transonic, which satisfies

$$\underline{b} \le b(x) \le \overline{b} \quad \text{and} \quad 0 < \underline{b} < 1 < \bar{b}, \quad x \in [0,1].$$

Then we define a set:

$$\mathcal{M} := \left\{x \in [0,1] : \underline{b} \le b(x) \le 1 + \mu\right\}, \tag{1.7}$$

where μ is a strictly positive constant such that $1 + \mu < \bar{b}$. Indeed, the doping profile $b(x)$ is named *subsonic-dominated* if $|\mathcal{M}| \ll 1$. Here, $|\cdot|$ denotes the measure of a set. In the same way, defining

$$\mathcal{N} := \left\{x \in [0,1] : 1 \le b(x) \le \bar{b}\right\}, \tag{1.8}$$

we call $b(x)$ *supersonic-dominated* if $|\mathcal{N}| \ll 1$. Recalling from [17,18], we define the interior subsonic/transonic/supersonic solutions as follows.

Definition 1.1 *(Subsonic/supersonic solutions).* $\rho(x)$ is called an interior subsonic (correspondingly, interior supersonic) solution of (1.6) if $\rho(0) = \rho(1) = 1$ but $\rho > 1$ (correspondingly, $0 < \rho(x) < 1$) for $x \in (0, 1)$, and $(\rho(x) - 1)^2 \in H_0^1(0, 1)$, and it holds that for any $\varphi \in H_0^1(0, 1)$

$$\int_0^1 \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right) \rho_x \varphi_x dx + \frac{1}{\tau} \int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0,$$

which is equivalent to

$$\frac{1}{2} \int_0^1 \frac{\rho + 1}{\rho^3} \left((\rho - 1)^2\right)_x \varphi_x dx + \frac{1}{\tau} \int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0.$$

Once $\rho = \rho(x)$ is determined by (1.6), in view of the first equation of (1.4), the electric field $E(x)$ can be solved by

$$E(x) = \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right) \rho_x + \frac{1}{\tau\rho} = \frac{(\rho + 1)[(\rho - 1)^2]_x}{2\rho^3} + \frac{1}{\tau\rho}.$$

In this way, we could obtain the interior subsonic/supersonic solutions to system (1.4)-(1.5).

Definition 1.2 *(Shock transonic solutions).* $\rho(x) > 0$ is called a transonic shock solution of system (1.4)-(1.5) if $\rho(0) = \rho(1) = 1$ and it is separated by a point $x_0 \in (0, 1)$ in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in [0, x_0], \\ \rho_{sub}(x), & x \in (x_0, 1], \end{cases}$$

where $0 < \rho_{sup}(x) < 1$ and $\rho_{sub}(x) > 1$ satisfy the entropy condition at x_0

$$0 < \rho_{sup}(x_0^-) < 1 < \rho_{sub}(x_0^+), \tag{1.9}$$

and the Rankine-Hugoniot condition

$$\begin{aligned} &\rho_{sup}(x_0^-) + \frac{1}{\rho_{sup}(x_0^-)} = \rho_{sub}(x_0^+) + \frac{1}{\rho_{sub}(x_0^+)}, \\ &E_{sup}(x_0^-) = E_{sub}(x_0^+). \end{aligned} \tag{1.10}$$

Set $\rho_l = \rho_{sup}(x_0^-)$ and $\rho_r = \rho_{sub}(x_0^+)$, it follows from (1.10) that

$$\rho_l \rho_r = 1.$$

Definition 1.3 *(C^1-smooth transonic solutions).* $\rho(x) > 0$ is called a C^1-smooth transonic solution of system (1.4)-(1.5) if $\rho(0) = \rho(1) = 1$ and it is separated by a point $x_0 \in (0, 1)$ in the form

$$\rho(x)=\begin{cases}\rho_{sup}(x), & x\in[0,x_0],\\ \rho_{sub}(x), & x\in(x_0,1],\end{cases}$$

where $0<\rho_{sup}(x)<1$ and $\rho_{sub}(x)>1$, and satisfy the smoothness conditions at x_0

$$\rho_{sup}(x_0^-)=1=\rho_{sub}(x_0^+),\quad \rho'_{sup}(x_0^-)=\rho'_{sub}(x_0^+). \tag{1.11}$$

The hydrodynamic model of semiconductors, introduced by Bløtekjær [6], has been extensively concerned in recent years because of its ability of modeling hot electron effects which are not accounted for the classical drift-diffusion model. For more mathematical derivations on the hydrodynamic model, we refer to the books [15,21,22]. Regarding the existence/non-existence of steady-states to the stationary hydrodynamic model with different physical backgrounds and different boundary conditions, it has been intensively studied. In 1990 and 1993, Degond and Markowich [8,9] first investigated the existence and uniqueness of subsonic solutions with the strong subsonic background (namely the current $J\ll 1$) in one dimension, and for potential flow in three dimension, respectively. Then, with the higher dimensions case and different boundary conditions, more discussion on the subsonic flows can be found in [2–5,13,14,16,23,24]; see also the references therein. For the case of the supersonic flows, Peng and Violet [25] proved the existence and uniqueness of supersonic solutions with a strongly supersonic background (i.e. $J\gg 1$), which was extended to the two dimensions case by M.Bae [5]. In addition, the corresponding investigations on the transonic flows have been drawn more interests and attention. For one dimension case, Ascher et al. [1] and Rosini [26] showed the existence of transonic solutions by phase plane analysis. Under the assumption of a non-flat doping profile $b(x)$, Gamba [10] and Gamba and Morawetz [11] constructed a transonic solution in 1-D and 2-D via artificial viscosity, respectively. Luo and Xin [20] then considered the Euler-Poisson system without the semiconductor effect $\frac{1}{\tau}$ where the structure of transonic solutions were studied thoroughly by phase plane analysis. Moreover, they [19] proved the existence of transonic shock solutions with a small perturbation of a constant doping profile.

When the system (1.2) is with sonic boundary, a critical boundary case, we [17,18] first technically classified the structure of all types of solutions to (1.2) when the doping profile is either subsonic or supersonic (including sonic case), respectively, and remarkably characterized the well-posedness and regularity of the solutions under the effect of the doping profile, semiconductor and the sonic boundary. In the case of the subsonic doping profile, there exist a unique interior subsonic, at least one interior supersonic solution, infinitely many transonic shock solutions when the relaxation time is sufficiently large (i.e. $\tau\gg 1$), and infinitely many C^1-smooth transonic solutions when the relaxation time is sufficiently small (i.e. $\tau\ll 1$). The primary proof of the results consists of the technical compactness analysis, phase plane analysis and the energy method. When $b(x)$ is supersonic or sonic, the non-existence of all type of solutions was discussed in the case of a small doping profile and a small relaxation time, and they proved the existence of supersonic and transonic shock solutions if the doping profile is close to the sonic line and the semiconductor effect is almost ignored ($\tau\gg 1$).

Subsequently to the first two cases with subsonic/supersonic doping profile studied in [17,18], it is inevitable for us to consider the more complicated case of transonic doping profile $b(x)$ for the hydrodynamic model when the boundaries are on sonic line. Since it is much overcomplicated to focus on the case of the general transonic doping profile, we divide this case into two types: subsonic-dominated and supersonic-dominated. When the doping profile $b(x)$ is subsonic-dominated (i.e. $|\mathcal{M}|\ll 1$) and the semiconductor effect vanishes (i.e. $\frac{1}{\tau}=0$), we show the existence of a unique interior subsonic solution and at least one interior supersonic solution by the technical compactness analysis and the Green function method. Here, the Green's function method is mainly applied to estimate the low bound of the iterative solutions in the proof of Schauder fixed point Theorem. Then, with the case of $\tau < +\infty$, there exist a unique interior subsonic solution, at least one interior supersonic solution and infinitely many transonic shock solution when $\tau \gg 1$ via phase plane analysis. One new phenomenon is that there may be no subsonic solution nor supersonic solution with some special cases when the semiconductor effect is strong (i.e. $\tau \ll 1$). Moreover, with the strong semiconductor effect, it is proved that there exist infinitely many C^1-smooth transonic solutions instead of transonic shock solutions when the doping profile is a step function. On the other hand, when the doping profile $b(x)$ is supersonic-dominated (i.e. $|\mathcal{N}| \ll 1$), the results are partially extended from the case of a supersonic doping profile. There is no interior subsonic solution, interior supersonic solution or transonic shock solution when the doping profile is small in integral form or the relaxation time is small. Furthermore, the existence of interior supersonic solutions and transonic shock solutions is proved again in a similar way.

The main results of the paper are summarized as follows.

Theorem 1.1 *(Subsonic-dominated doping profile). Let the doping profile be subsonic-dominated, satisfying $b(x) \in L^\infty(0,1)$. There exists a constant $\varepsilon_0 > 0$ such that $0 < |\mathcal{M}| < \varepsilon_0$.*

1. *Assume that $\mathcal{M} \subset [\alpha_1, 1]$ with a constant $\alpha_1 > 0$, then system (1.4)-(1.5) admits a unique pair of interior subsonic solution $(\rho, E)(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ and*

$$1 + C\sin(\pi x) \le \rho(x) \le \bar{b} \ \text{ on } \ [0,1]$$

where $C(\mu, \tau) < \bar{b} - 1$ is a small and positive constant.

2. *Assume that $\mathcal{M} \subset [0, \alpha_2]$ with a constant $\alpha_2 < 1$, then system (1.4)-(1.5) has at least one pair of interior supersonic solution $(\rho, E)(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ and*

$$0 < \rho(x) \le 1.$$

3. *Particularly, assume $\alpha_1 = 0$ and $\alpha_2 = 1$, and if the relaxation time is infinite, i.e. $\tau = \infty$, the results of part 1 and 2 also hold.*
4. *If τ is large, (1.4)-(1.5) has infinitely many transonic shock solutions $(\rho_{trans}, E_{trans})(x)$, satisfying the entropy condition (1.9) and the Rankine-Hugoniot condition (1.10) at the jump location x_0.*
5. *When τ is small enough and the doping profile $b(x)$ is a step function, (1.4)-(1.5) has infinitely many C^1-smooth transonic solutions.*

Theorem 1.2 *(Supersonic-dominated doping profile). Let the doping profile be supersonic-dominated, satisfying $b(x) \in L^\infty(0,1)$ and $|\mathcal{N}| \ll 1$. There exists a constant $\varepsilon_0 > 0$ such that $0 < |\mathcal{N}| \le \varepsilon_0$, then:*

1. *There is no interior subsonic solution to (1.4)-(1.5);*

2. *There is no interior supersonic solution nor transonic shock solution to* (1.4)-(1.5) *if*

$$\hat{b} := \int_{[0,1]\backslash\mathcal{N}} b(x)dx \ll 1$$

or the relaxation time τ *is small and* $\hat{b} \le 1-\epsilon$ *for a parameter* ϵ.
3. *There exist at least one interior supersonic solution and infinitely many transonic solutions to* (1.4)-(1.5) *when the doping profile* $b(x)$ *satisfies*

$$1-\epsilon \le b(x) \le \bar{b} \quad \text{with a small parameter } \epsilon,$$

and the relaxation time τ *is large.*

Remark 1.1.

1. In Parts 1 and 2 of Theorem 1.1, $C^{\frac{1}{2}}[0,1]$ is the optimal Hölder space for the global regularity of the interior subsonic/supersonic solutions, which also matches the case of subsonic doping profile shown in [17].
2. In Parts 1 and 2 of Theorem 1.1, when $\alpha_1 = 0$ and $\alpha_2 = 1$, two spacial cases in Appendix imply that there may be no interior subsonic solution or supersonic solution to (1.4)-(1.5) when $\tau \ll 1$. However, if τ is large enough, the results could be obtained with the general subsonic-dominated doping profile $b(x)$.
3. In Part 5 of Theorem 1.1, there is no transonic shock solution to (1.4)-(1.5) when τ is small enough, just as showed in [17]. In the case of $\tau \ll 1$, (1.4)-(1.5) has only infinity many C^1-smooth transonic solution.

The remaining part of this paper is arranged as follows. In Section 2, we mainly prove the existence of interior subsonic/supersonic/transonic solutions to (1.4)-(1.5) when the doping profile $b(x)$ is subsonic-dominated. When $\tau = +\infty$, there exist a unique interior subsonic solution and at least one interior supersonic solution. In the case $0 < \tau < +\infty$, we have the same results when $\alpha_1 > 0$ and $\alpha_2 < 1$. Also, (1.4)-(1.5) has infinitely many transonic shock solutions if $\tau \gg 1$. In addition, Section 3 is devoted to proving the existence/non-existence of stationary solutions with a supersonic-dominated doping profile. We list and prove the non-existence of interior subsonic/supersonic/transonic solutions when $\hat{b} \ll 1$ or $\tau \ll 1$ and $\hat{b} \le 1-\epsilon$. Furthermore, there exist at least one interior supersonic solution and infinitely many transonic shock solutions to (1.4)-(1.5) when $1-\epsilon \le b \le \bar{b}$ and $\tau \gg 1$.

2. Subsonic-dominated doping profile

Let the doping profile $b(x)$ be subsonic-dominated. The purpose of this section is to prove Theorem 1.1, which will be divided in steps for each case.

2.1. Interior subsonic solution

The subsection is devoted to investigating the interior subsonic solutions of system (1.4)-(1.5). When the semiconductor effect vanishes, we only assume that $b(x) \in L^\infty(0,1)$ and $|\mathcal{M}| \ll 1$.

Here the set $\mathcal{M}$ is defined by (1.7). With the strong semiconductor effect, we need to give an additional condition that $\mathcal{M} \subset [\alpha_1, 1]$ where $\alpha_1 > 0$.

1. The case of $\tau = \infty$. In this case, we prove that there exists a unique interior solution to system (1.4)-(1.5). The main adopted approach is the technical compactness method from [17], inspired by an artificial viscosity approximation. However, due to the effect of transonic doping profile, we apply the Green's function method to guarantee the lower bound of solution in the proof of Schauder fixed point theorem. This method has the advantage that the degree of subsonic dominance is intuitively observed. Then, system (1.4)-(1.5) without the term of $\frac{1}{\tau}$ becomes

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E, \\ E_x = \rho - b(x), \\ \rho(0) = \rho(1) = 1. \end{cases} \tag{2.1}$$

Theorem 2.1. *Assume that* $b(x) \in L^\infty(0,1)$ *and* $0 < \underline{b} < 1 < \bar{b}$, *there exists a constant* $\varepsilon_0 > 0$ *such that* $0 < |\mathcal{M}| < \varepsilon_0$, *then* (2.1) *admits a unique interior solution* $\rho(x)$ *satisfying* $\rho \in C^{\frac{1}{2}}[0,1]$ *and*

$$1 + C\sin(\pi x) \le \rho(x) \le \bar{b} \ \ on \ \ [0,1],$$

where $C = C(\mu) < \bar{b} - 1$ *is a positive constant.*

Owing to the boundary degeneration of (2.1), we first solve the following approximate system as similar to that of Theorem 2.1 [17]:

$$\begin{cases} \left[\left(\dfrac{1}{\rho_j} - \dfrac{j^2}{\rho_j^3}\right)(\rho_j)_x\right]_x - [\rho_j - b(x)] = 0, \quad x \in (0,1), \\ \rho_j(0) = \rho_j(1) = 1, \end{cases} \tag{2.2}$$

where the parameter j is a constant and $0 < j < 1$. Thus, (2.2) is uniformly elliptic on $[0,1]$, because $\frac{1}{\rho_j} - \frac{j^2}{\rho_j^3} > 0$ for $\rho_j \ge 1$. Then, we continue to transform equation (2.2) as follows. Define a function w with respect of ρ_j, and let

$$w'(\rho_j) := \frac{1}{\rho_j} - \frac{j^2}{\rho_j^3} \quad \text{and} \quad w(1) := 0, \quad \rho_j \in [1, +\infty).$$

Thus, we get

$$w(\rho_j) = \ln \rho_j + \frac{j^2}{2\rho_j^2} - \frac{j^2}{2}, \quad \rho_j \in [1, +\infty). \tag{2.3}$$

Since w is increasing over $[1, +\infty)$, an inverse function $\rho_j(w)$ can be denoted by

$$\rho_j(w) := w^{-1}(\rho_j), \quad w \in [0, +\infty],$$

then we extend $\rho_j(w)$ to the whole space $\mathbb{R}$, still labeled by $\rho_j(w)$, in the form

$$\rho_j(w) := \begin{cases} \rho_j(w), & w \geq 0 \\ \rho_j(-w), & w < 0. \end{cases}$$

From (2.2) and (2.3), one can see that w is a function of x and $w(0) = w(1) = 0$. Thus, (2.2) can be transformed to the following system:

$$\begin{cases} w_{xx} = \rho_j(w) - b(x), \\ w(0) = w(1) = 0. \end{cases} \tag{2.4}$$

Clearly, a positive solution of (2.4) always corresponds to an interior subsonic solution of (2.2). Hence, let us then prove the well-posedness of (2.4).

Lemma 2.1. *Assume that $b(x) \in L^\infty(0,1)$ and $0 < \underline{b} < 1 < \bar{b}$, there exists a constant $\varepsilon_0 > 0$ such that $0 < |\mathcal{M}| < \varepsilon_0$, then (2.4) admits a weak solution $w(x)$ satisfying $w \in H_0^1(0,1)$ and*

$$C \sin(\pi x) \leq w(x) \leq M \quad on \quad [0,1],$$

where $C = C(\mu) < M = \ln \bar{b}$ is a small and positive constant.

Proof. Let $M = \ln \bar{b}$, we set

$$S_0 := \left\{ \omega(x) : \omega \in C^0[0,1] \,|\, 0 \leq \omega \leq M \quad \text{and} \quad \omega(0) = \omega(1) = 0 \right\}.$$

First, subtract κw from both sides of (2.4), that is

$$\begin{cases} w_{xx} - \kappa w = -[(b-1) + \beta(w)], \\ w(0) = w(1) = 0, \end{cases} \tag{2.5}$$

where κ is a constant to be determined later and

$$\beta(w) := \kappa w - (\rho_j(w) - 1).$$

Then we define an operator $\Phi : \bar{w} \to w$ by solving the following linearized equation of (2.5)

$$\begin{cases} w_{xx} - \kappa w = -[(b-1) + \beta(\bar{w})], \\ w(0) = w(1) = 0. \end{cases} \tag{2.6}$$

By L^2 theory of elliptic equation and Sobolev imbedding theorem [12], one can see that (2.6) has a unique solution $w \in C^0[0,1]$ for every $\bar{w} \in S_0$, then we need to prove that $0 \leq w(x) \leq M$ for $x \in [0,1]$. Multiplying (2.6) by $\hat{w} := \max\{0, w - M\}$, we obtain

$$\int_0^1 \left(|\hat{w}_x|^2 + \kappa w \hat{w} \right) dx = \int_0^1 [(b-1) + \beta(\bar{w})] \, \hat{w} dx.$$

Because $b \leq \bar{b}$ and $\hat{w} \geq 0$, we get

$$0 \leq \int_0^1 |\hat{w}_x|^2 dx \leq \int_0^1 \left[(\bar{b}-1) + \beta(\bar{w}) - \kappa w\right] \hat{w} dx. \tag{2.7}$$

Suppose that κ is large enough such that $\beta'(M) > 0$ for any $j \in (0,1)$, it is easy to see that

$$\max_{\bar{w} \in S_0} \beta(\bar{w}) \leq \max \{\beta(0), \beta(M)\} \leq \max \left\{0, \kappa M - (\bar{b}-1)\right\}$$

where we have used

$$\rho_j(M) \geq \rho_j(M_j) = \bar{b}, \quad M_j = \ln \bar{b} + \frac{j^2}{2\bar{b}^2} - \frac{j^2}{2}.$$

Then, choose κ sufficiently large to satisfy $\kappa \geq \dfrac{\bar{b}-1}{M}$, and it follows that

$$\max_{\bar{w} \in S_0} \frac{(\bar{b}-1) + \beta(\bar{w})}{\kappa} \leq M,$$

further and

$$(\bar{b}-1) + \beta(\bar{w}) - \kappa w \leq \kappa(M - w).$$

Thus, it follows from (2.7) that

$$w(x) \leq M, \quad x \in [0,1].$$

Next, we claim that $w(x) > 0$ for $x \in (0,1)$ and $w(0) = w(1) = 0$ when $0 < |\mathcal{M}| < \varepsilon_0$. First of all, by the construction method of Green's function in [7], Green's function of (2.6) is

$$\Gamma(x,\xi) := \begin{cases} C_0 w_0(x) w_1(\xi) & \text{for} \quad x \leq \xi, \\ C_0 w_0(\xi) w_1(x) & \text{for} \quad x > \xi, \end{cases} \tag{2.8}$$

where $C_0 := \dfrac{e^{\sqrt{\kappa}} - e^{-\sqrt{\kappa}}}{2\sqrt{\kappa}}$, $w_0(x) := \dfrac{e^{\sqrt{\kappa}x} - e^{-\sqrt{\kappa}x}}{e^{\sqrt{\kappa}} - e^{-\sqrt{\kappa}}}$ and $w_1(x) := \dfrac{e^{\sqrt{\kappa}(1-x)} - e^{\sqrt{\kappa}(x-1)}}{e^{\sqrt{\kappa}} - e^{-\sqrt{\kappa}}}$. Hence, the solution of (2.6) is described as follow:

$$w(x) = \int_0^1 \Gamma(x,\xi) \left[(b(\xi) - 1) + \beta(\bar{w})(\xi)\right] d\xi \quad x \in [0,1]. \tag{2.9}$$

Then we divide the proof of the claim into two steps.

Step 1. Assume that $x \in [s_0, 1 - s_0]$ where s_0 is a number such that $0 < s_0 < \frac{1}{2}$. Suppose that κ is large enough such that

$$\min_{x\in[0,1],\bar{w}\in S_0} \beta(\bar{w})(x) \geq -\frac{\mu}{2}.$$

Hereafter, Select a constant κ that meets all above criteria and fix it. Thanks to the monotonicity of w_0 and w_1, it then follows from (1.7), (2.8) and (2.9) that for $x \in [s_0, 1-s_0]$

$$\begin{aligned}
w(x) &\geq C_0 w_1(x)\int_0^x w_0(\xi)\Big[(b(\xi)-1)-\frac{\mu}{2}\Big]d\xi + C_0 w_0(x)\int_x^1 w_1(\xi)\Big[(b(\xi)-1)-\frac{\mu}{2}\Big]d\xi \\
&\geq C_0 w_1(x)\left[\frac{\mu}{2}\int_0^{x-\varepsilon_0} w_0(\xi)d\xi - (1-\underline{b}+\frac{\mu}{2})\int_{x-\varepsilon_0}^{x} w_0(\xi)d\xi\right] \\
&\quad + C_0 w_0(x)\left[\frac{\mu}{2}\int_{x+\varepsilon_0}^{1} w_1(\xi)d\xi - (1-\underline{b}+\frac{\mu}{2})\int_{x}^{x+\varepsilon_0} w_1(\xi)d\xi\right] \\
&\geq \frac{\mu}{2}\cdot C_0 w_1(x)\left[\big(g_1(x-\varepsilon_0)-g_1(0)\big) - \left(1+\frac{2(1-\underline{b})}{\mu}\right)\varepsilon_0 g_1'(x)\right] \\
&\quad + \frac{\mu}{2}\cdot C_0 w_0(x)\left[\big(g_2(x+\varepsilon_0)-g_2(1)\big) + \left(1+\frac{2(1-\underline{b})}{\mu}\right)\varepsilon_0 g_2'(x)\right],
\end{aligned}$$

where $g_1(x) := \dfrac{e^{\sqrt{\kappa}x}+e^{-\sqrt{\kappa}x}}{\sqrt{\kappa}(e^{\sqrt{\kappa}}-e^{-\sqrt{\kappa}})}$ and $g_2(x) := \dfrac{e^{\sqrt{\kappa}(1-x)}+e^{\sqrt{\kappa}(x-1)}}{\sqrt{\kappa}(e^{\sqrt{\kappa}}-e^{-\sqrt{\kappa}})}$. Thus, suppose that $\varepsilon_0 < \frac{s_0}{2}$ is small enough, then we obtain that $w(x) \geq C(s_0) > 0$ on $[s_0, 1-s_0]$.

Step 2. Assume that $x \in [0, s_0)$, a direct calculation conclude that

$$\begin{aligned}
w(x) &\geq -(1-\underline{b}+\frac{\mu}{2})\cdot C_0 w_1(x)\int_0^x w_0(\xi)d\xi \\
&\quad + C_0 w_0(x)\left[\frac{\mu}{2}\int_x^1 w_1(\xi)d\xi - (1-\underline{b}+\mu)\int_x^{x+\varepsilon_0} w_1(\xi)d\xi\right]. \\
&\geq -(1-\underline{b}+\mu)\cdot\left[C_0 w_1(x)\int_0^x w_0(\xi)d\xi + C_0 w_0(x)\int_x^{x+\varepsilon_0} w_1(\xi)d\xi\right] \\
&\quad + \frac{\mu}{2}C_0 w_0(x)\int_x^1 w_1(\xi)d\xi.
\end{aligned} \tag{2.10}$$

Denote

$$F_1(x) := C_0 w_1(x)\int_0^x w_0(\xi)d\xi, \quad F_2(x) := C_0 w_0(x)\int_x^1 w_1(\xi)d\xi,$$

hence we directly compute that there exists a small number $l > 0$ such that

$$F_2'(x) > F_1'(x) \geq 0, \;\; F_1''(x) > 0 \text{ and } F_2''(x) < 0, \quad x \in [0, l]. \tag{2.11}$$

Set $0 < 2\varepsilon_0 < s_0 < l$. First, we could suppose $\varepsilon_0 = \varepsilon_0(\mu, \underline{b}, s_0)$ sufficiently small to satisfy

$$(1 - \underline{b} + \mu) \int_x^{x+\varepsilon_0} w_1(\xi) d\xi \leq \frac{\mu}{4} \int_x^1 w_1(\xi) d\xi, \quad x \in [0, s_0].$$

From (2.10) and (2.11), we get for $x \in [0, s_0)$

$$\begin{aligned} w(x) &\geq -(1 - \underline{b} + \mu) F_1(x) + \frac{\mu}{4} F_2(x) \\ &\geq \frac{F_1(x)}{4} \left[\mu \frac{F_2(x)}{F_1(x)} - 4(1 - \underline{b} + \mu) \right] \\ &\geq \frac{F_1(x)}{4} \left[\mu \frac{F_2'(s_0)}{F_1'(s_0)} - 4(1 - \underline{b} + \mu) \right]. \end{aligned}$$

Then, it follows from (2.11) that $\dfrac{F_2'(s_0)}{F_1'(s_0)}$ is strictly decreasing for $s_0 \in (0, l]$ and $F_1'(0) = 0$. After that, we choose $s_0 > 0$ sufficiently small and derive that $w(x) > 0$ for $x \in (0, s_0)$ and $w(0) = 0$. Similarly, because $F_1(x)$ and $F_2(x)$ are mutually symmetric with respect to the line $x = \dfrac{1}{2}$, we also get $w(x) > 0$ on $(1 - s_0, 1)$ and $w(1) = 0$. Combining with Steps 1-2, the claim is complete.

It is easy to see that $w \in S_0$. By Arzelà-Ascoli theorem, the operator Φ is a compact map. In addition, Φ is continuous by a standard continuity argument. Thus, we conclude that there exists a fixed point of Φ by Schauder fixed point Theorem. So (2.4) has a unique positive solution $w \in C^0[0, 1]$. From (2.4), it follows that $w_{xx} \in L^\infty(0, 1)$, then we get $w \in H_0^1(0, 1)$.

Finally, it remains to prove the lower boundedness estimate. Define $\Theta(x) := w - C \sin(\pi x)$ on $[0, 1]$, where w is the solution of (2.6) and C is a positive constant. Then we have the next linear system

$$\begin{cases} \Theta_{xx} - \kappa \Theta = -[(b - 1) + \beta(\bar{w}) - C(\pi^2 + \kappa) \sin(\pi x)], \\ \Theta(0) = \Theta(1) = 0. \end{cases} \tag{2.12}$$

Thus, suppose that C is small enough such that $C(\pi^2 + \kappa) \leq \frac{\mu}{4}$. By Green function method, we also see that (2.12) has a positive solution $\Theta \in S_0$ for every $\bar{w} \in S_0$ when ε_0 is small enough. Obviously, the solution $w(x)$ of (2.4) satisfies

$$w(x) \geq C \sin(\pi x), \quad x \in [0, 1].$$

The proof is complete. $\square$

Proof of Theorem 2.1. From Lemma 2.1, we know that there is an approximation solution $\rho_j(x)$ of (2.1) when $0 < j < 1$. Hereafter, it follows from (2.3) that

$$\rho_j(x) \geq 1 + w(x) \geq 1 + C\sin(\pi x), \quad x \in [0,1]$$

where C is a small positive constant, independent of j. From (2.2), and by phase-plane analysis, we have

$$\rho_j(x) \leq \bar{b}.$$

Then, recalled the proof of Theorem 2.1 [17], as $j \to 1^-$, we can obtain that (2.1) has a unique interior subsonic solution $\rho \in C^{\frac{1}{2}}[0,1]$. The proof is complete.

2. The case of $0 < \tau < \infty$. In this case, we also want to get the existence of the interior subsonic solutions by referring to the proof of Theorem 2.1. Nevertheless, for the linearized system (2.6), its occurrence of a first order term would be an obstacle for proving the existence theorem. Therefore, we have to apply phase-plane analysis to prove the existence of interior subsonic solutions. However, there may be no interior subsonic solution with the general supersonic-dominated doping profile when $\tau \ll 1$. Here define a step doping profile $b(x)$, that is

$$b(x) := \begin{cases} \bar{b}, & x \in [0, \alpha_1], \\ \underline{b}, & x \in (\alpha_1, \alpha_2), \\ \bar{b}, & x \in [\alpha_2, 1], \end{cases} \qquad 0 \leq \alpha_1 < \alpha_2 \leq 1. \tag{2.13}$$

When $b(x)$ is denoted by (2.13) and $\tau < \frac{1}{2\sqrt{\bar{b}^3+\bar{b}}}$, Proposition 4.1, shown in Appendix, indicates that no interior subsonic solution exists to (1.4)-(1.5) with the case of $\alpha_1 = 0$ and $\alpha_2 \ll 1$. Inspired by this, (1.4)-(1.5) is supplemented with an additional assumption: $\mathcal{M} \subset [\alpha_1, 1]$, $\alpha_1 > 0$. The main result is stated as follows.

Theorem 2.2. *Assume that* $b(x) \in L^\infty(0,1)$*, and there exist numbers* $\alpha_1 > 0$ *and* $\varepsilon_0 > 0$ *such that* $\mathcal{M} \subset [\alpha_1, 1]$ *and* $0 < |\mathcal{M}| < \varepsilon_0$*, then system* (1.4)-(1.5) *has a unique interior subsonic solution* $\rho(x)$ *satisfying* $\rho \in C^{\frac{1}{2}}[0,1]$ *and*

$$1 + C\sin(\pi x) \leq \rho(x) \leq \bar{b} \;\; on \;\; [0,1],$$

where $C(\tau, \mu) < \bar{b} - 1$ *is a positive constant.*

Proof. For clarity, we divide the proof into six steps.

Step 1. Let δ and η be small numbers to be determined later satisfying $0 < \delta < \eta \ll 1$. Define

$$\tilde{b}(x) := \begin{cases} 1 + \mu, & x \in \mathcal{M}, \\ b(x), & x \in [0,1] \backslash \mathcal{M}, \end{cases} \tag{2.14}$$

and $\tilde{b}(x)$ is extended periodically to $\mathbb{R}^+$. Now let us consider the following boundary value problem with subsonic boundary condition

$$\begin{cases} \left(1-\dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - \tilde{b}(x), \\ \rho(0) = \rho(L) = 1+\delta, \end{cases} \tag{2.15}$$

where $L \geq \frac{1}{4}$ is a positive constant and $1+\mu \leq \tilde{b}(x) \leq \bar{b}$ over $[0, L]$. From Theorem 2.1 [17], it's easy to see that there exists a unique interior subsonic solution $(\bar{\rho}, \bar{E})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ to (2.15), and $(\bar{\rho}, \bar{E})(x)$ satisfies

$$1 + m\sin(\frac{\pi x}{L}) \leq \bar{\rho}(x) \leq \bar{b} \quad \text{and} \quad \bar{E}(0) \geq \frac{1}{\tau}, \quad x \in [0, L], \tag{2.16}$$

where $m = m(\tau, \mu, L)$ is a positive and small constant, independent of δ. Obviously, by analyzing the monotonicity of the solution $(\bar{\rho}, \bar{E})(x)$, we know that there exists a number x_1 such that

$$\bar{\rho}(x_1) = \max_{0\leq x\leq L} \bar{\rho}(x) \triangleq \bar{\rho}_0, \quad \bar{E}(x_1) = \frac{1}{\bar{\rho}_0 \tau},$$

thus $\bar{\rho}_0(\tau, \mu)$ is a positive constant and

$$1 + m \leq \bar{\rho}_0 \leq \bar{b}.$$

Let $\tilde{E} := \bar{E} - \frac{1}{\tau\bar{\rho}}$, we first prove that

$$\tilde{E}(L) \leq -\Lambda,$$

where the constant Λ is independent of δ. In fact, that is

$$\frac{d\tilde{E}}{d\bar{\rho}} = \frac{(\bar{\rho}^2-1)(\bar{\rho}-\tilde{b})}{\bar{\rho}^3\tilde{E}} + \frac{1}{\tau\bar{\rho}^2} \geq \frac{(\bar{\rho}^2-1)(\bar{\rho}-\tilde{b})}{\bar{\rho}^3\tilde{E}}.$$

Integrating the above inequality over $[x_1, L]$, and $\tilde{b} < \bar{b}$, we get

$$\begin{aligned} \frac{\tilde{E}^2(L)}{2} &\geq \int_{x_1}^{L} \left[(1-\frac{1}{\bar{\rho}^2}) - \frac{\bar{b}(\bar{\rho}^2-1)}{\bar{\rho}^3}\right] dx \\ &= \frac{2+2\delta-\bar{b}}{2(1+\delta)^2} + 1 + \delta - \bar{b}\ln(1+\delta) - \left[\frac{2\bar{\rho}_0-\bar{b}}{2\bar{\rho}_0^2} + \bar{\rho}_0 - \bar{b}\ln\bar{\rho}_0\right] \\ &= \frac{\delta[2\bar{b}-2-(2-\bar{b})\delta]}{2(1+\delta)^2} + \delta - \bar{b}\ln(1+\delta) + h(\bar{\rho}_0), \end{aligned}$$

where

$$h(s) := 1 + \frac{2-\bar{b}}{2} - \frac{2s-\bar{b}}{2s^2} - s + \bar{b}\ln s, \quad s \in (1, \bar{b}).$$

Notice that $h(1) = 0$ and $h'(s) = \frac{(s-\bar{b})(1-s^2)}{s^3} > 0$ for $s \in (1, \bar{b})$. When δ is small such that $|\frac{\delta[2\bar{b}-2-(2-\bar{b})\delta]}{2(1+\delta)^2} + \delta - \bar{b}\ln(1+\delta)| \le \frac{h(1+m)}{2}$, because $\bar{\rho}_0 \ge 1+m$, we obtain

$$\tilde{E}^2(L) \ge h(1+m) > 0.$$

If $\tilde{E}(L) > 0$, we get $\bar{\rho}'(x) > 0$ in the neighborhood of L^-, which contradicts to $\bar{\rho}(L) = 1+\delta$ and $\bar{\rho}(x) > 1+\delta$ near the end point $x = L$. Thus

$$\tilde{E}(L) \le -\sqrt{h(1+m)} \triangleq -\Lambda(\tau, \mu, L).$$

Step 2. Here, let $L = \frac{1}{2}$ and denote the corresponding solution of (2.15) by $(\bar{\rho}_1, \bar{E}_1)(x)$. We assert that there exists a number $0 < x_2 < \frac{1}{2}$ such that

$$\left|\frac{1}{2} - x_2\right| \le C\eta^2, \quad \bar{\rho}_1(x_2) = 1+\eta \quad \text{and} \quad |\bar{E}_1(\frac{1}{2}) - \bar{E}_1(x_2)| \le C\eta^2, \tag{2.17}$$

where the constant C doesn't depend on η and δ. As in Step 1, note that

$$\bar{\rho}_1(\frac{1}{2})\bar{E}_1(\frac{1}{2}) - \frac{1}{\tau} \le -\Lambda_1 \quad \text{with a positive constant } \Lambda_1.$$

In the phase-plane (ρ, E), it follows from the first two equations of (2.15) that both of the functions $\bar{\rho}_1$ and $\bar{E}_1$ keep decreasing in the neighborhood of $\frac{1}{2}^-$. Clearly, there exists a last number $x_2 < \frac{1}{2}$ such that $\bar{\rho}_1$ attains $1+\eta$ and

$$\bar{\rho}_1(x_2)\bar{E}_1(x_2) < \frac{1}{\tau},$$

where $0 < 1+\eta < \min\left\{\max\limits_{0\le x\le 1/2} \bar{\rho}_1(x), 1+\mu\right\}$. Therefore, over $x \in [x_2, \frac{1}{2}]$, we have

$$\bar{E}_1(x) = \bar{E}_1(\frac{1}{2}) + \int_x^{\frac{1}{2}} (\tilde{b} - \bar{\rho}_1)dx \le -\frac{\Lambda_1}{1+\delta} + \frac{1}{\tau} + \bar{b}\left(\frac{1}{2} - x_2\right).$$

Note that $\bar{\rho}_1 \in [1+\delta, 1+\eta]$ for $x \in [x_2, \frac{1}{2}]$, then

$$\bar{\rho}_1(x)\bar{E}_1(x) - \frac{1}{\tau} \le \frac{\eta}{\tau} - \Lambda_1 + \frac{3\bar{b}}{2}\left(\frac{1}{2} - x_2\right) \le -\frac{3\Lambda_1}{4} + \frac{3\bar{b}}{2}\left(\frac{1}{2} - x_2\right) \le -\frac{\Lambda_1}{2},$$

if $\eta \le \min\left\{\frac{1}{2}, \frac{\tau\Lambda_1}{4}\right\}$ and

$$\frac{1}{2} - x_2 \le \frac{\Lambda_1}{6\bar{b}}. \tag{2.18}$$

Hence, from the first equation of (2.15), there exists a number $\zeta \in [x_2, \frac{1}{2}]$ such that

$$\frac{1}{2} - x_2 = \frac{\bar{\rho}_1(\frac{1}{2}) - \bar{\rho}_1(x_1)}{(\bar{\rho}_1)_x(\zeta)} = \frac{(\eta-\delta)(1-\bar{\rho}_1^2(\zeta))}{\bar{\rho}_1^2(\zeta)(\bar{\rho}_1(\zeta)\bar{E}_1(\zeta) - 1/\tau)} \le \frac{2\eta^2(\eta+2)}{(1+\eta)^2\Lambda_1} \le \frac{5\eta^2}{\Lambda_1} \text{ if } \eta \le \frac{1}{2}.$$

If $\eta \le \dfrac{\Lambda_1}{\sqrt{30\bar{b}}}$, then (2.18) holds. Thus, we choose $\eta = \eta(\tau, \bar{b}, \mu) \le \min\left\{\frac{1}{2}, \frac{\tau\Lambda_1}{4}, \frac{\Lambda_1}{\sqrt{30\bar{b}}}\right\}$. Afterwards,

$$|\bar{E}_1(\frac{1}{2}) - \bar{E}_1(x_2)| \le \left|\int_{x_2}^{\frac{1}{2}} (\bar{\rho}_1 - \tilde{b})dx\right| \le \bar{b} \cdot \left|\frac{1}{2} - x_2\right| \le C\eta^2,$$

where the constant C only depends on τ, μ and $\bar{b}$.

Step 3. Then we discuss the ODE system with the subsonic initial value:

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x), \\ (\rho(0), E(0)) = (1+\delta, \bar{E}_1(0)) \end{cases} \tag{2.19}$$

where the doping profile $b(x)$ have been extended periodically to $\mathbb{R}^+$. First, denote the subsonic solution of (2.19) by $(\rho_1, E_1)(x)$. Because $\mathcal{M} \in [\alpha_1, 1]$, one can see that $b(x) = \tilde{b}(x)$ over $[0, \alpha_1]$, thus $(\rho_1, E_1)(x) = (\bar{\rho}_1, \bar{E}_1)(x)$ for $x \in [0, \alpha_1]$. In this step, taking $\varepsilon_0 \ll 1$, we claim that there exists a number $x_3 > 0$ such that

$$\frac{1}{2} - C\eta \le x_3 \le \frac{1}{2} + C\eta,$$

and a subsonic solution of (2.19) satisfies

$$\rho_1(x_3) = 1 + \delta \quad \text{and} \quad |E_1(x_3) - \bar{E}_1(1/2)| \le C\eta.$$

Set $\phi := \bar{\rho}_1 - \rho_1$, $\psi := \bar{E}_1 - E_1$. Here as similar to Step 3 of the proof of Theorem 4.2 [17], by (2.15) and (2.19), (ϕ, ψ) satisfies

$$\begin{cases} \phi_x = \dfrac{\bar{\rho}_1^3\psi}{(\bar{\rho}_1+1)(\bar{\rho}_1-1)} + \dfrac{\bar{\rho}_1^2\rho_1^2 - \bar{\rho}_1^2 - \bar{\rho}_1\rho_1 - \rho_1^2)\phi E_1}{(\bar{\rho}_1+1)(\bar{\rho}_1-1)(\rho_1+1)(\rho_1-1)} \\ \qquad + \dfrac{(\bar{\rho}_1+\rho_1)\phi}{\tau(\bar{\rho}_1+1)(\bar{\rho}_1-1)(\rho_1+1)(\rho_1-1)}, \\ \psi_x = \phi + b - \tilde{b}, \\ (\phi(\alpha_1), \psi(\alpha_1)) = 0. \end{cases} \tag{2.20}$$

Define the solution space

$$X_T := \{(\phi, \psi) \in C[\alpha_1, T] | \phi(\alpha_1) = \psi(\alpha_1) = 0, \quad |\phi| \le \eta/2, |\psi| \le \eta/2\}.$$

Next, we would show *a priori* estimate

$$\phi^2(x) + \psi^2(x) \le \eta^2/4 \quad \text{on} \quad x \in [\alpha_1, x_2]. \tag{2.21}$$

Multiplying the first equation of (2.20) by ϕ and the second one by ψ, adding them together and integrating the final equation over $[\alpha_1, x_2]$, and noting that $|\bar{\rho}_1 - \rho_1| \le \eta/2$, by Young's inequality, we have

$$\int_{\alpha_1}^{x_2} (\phi^2 + \psi^2)_x dx \le \left(\frac{C}{\eta^2} + \frac{C}{\tau\eta^2}\right) \int_{\alpha_1}^{x_2} (\phi^2 + \psi^2) dx + C \int_{\alpha_1}^{x_2} (\tilde{b} - b) dx,$$

where C is a constant independent of τ, δ and η. Then by Gronwall's integral inequality and

$$\int_{\alpha_1}^{x_2} (\tilde{b} - b) dx \le C\varepsilon_0,$$

one can see that

$$\phi^2 + \psi^2 \le C\varepsilon_0 e^{\left(\frac{1}{\eta^2} + \frac{1}{\tau\eta^2}\right)}.$$

Hence, taking $\varepsilon_0 \ll 1$ such that $C\varepsilon_0 e^{(\frac{1}{\eta^2} + \frac{1}{\tau\eta^2})} \le \frac{1}{4}$, we get (2.21) and further

$$|\bar{\rho}_1 - \rho_1| \le \eta/2, \quad |\bar{E}_1 - E_1| \le \eta/2, \tag{2.22}$$

which leads to

$$1 + \eta/2 = \bar{\rho}_1(x_2) - \eta/2 \le \rho_1(x_2) \le \bar{\rho}_1(x_2) + \eta/2 \le 1 + 3\eta/2,$$

and

$$\bar{E}_1(\frac{1}{2}) - C\eta \le \bar{E}_1(x_2) - \eta/2 \le E_1(x_2) \le \bar{E}_1(x_2) + \eta/2 \le \bar{E}_1(\frac{1}{2}) + c\eta. \tag{2.23}$$

Now taking x_2 as the initial value, we extend the solution (ρ_1, E_1) to the state satisfying $\rho_1 = 1 + \delta$. Herewith we need to prove that there exists a number x_3 such that

$$x_2 < x_3 \le x_2 + C\eta^2,$$

and a subsonic solution of (2.19) satisfies

$$\rho_1(x_3) = 1 + \delta, \quad E_1(x_3) < \frac{1}{\tau} \quad \text{and} \quad |E_1(x_3) - \bar{E}_1(x_2)| \le C\eta^2.$$

In fact, it is easy to check that $\rho_1(x_2)E_1(x_2) - 1/\tau \le -\frac{\Lambda_1}{2}$ if $\eta \ll 1$, then we point out that ρ_1 is decreasing in the neighborhood of x_2^+. Here, near x_2^+, noticing that $1+\delta \le \rho_1 \le 1+\frac{3\eta}{2}$, we have

$$\begin{aligned} E_1(x) &= E_1(x_2) + \int\limits_{x_2}^{x} (\rho_1 - b)dx \\ &\le -\frac{\Lambda_1}{2(1+\frac{3\eta}{2})} + \frac{1}{\tau} + \bar{b}(x - x_2) \end{aligned}$$

and further

$$\begin{aligned} \rho_1(x)E_1(x) - 1/\tau &\le -\frac{\Lambda_1}{2(1+\frac{3\eta}{2})} + \frac{3\eta}{2\tau} + \bar{b}(1+\frac{3\eta}{2})(x - x_2) \\ &\le -\frac{\Lambda_1}{4} + \frac{3\eta}{2\tau} + 2\bar{b}(x - x_2) \\ &\le -\frac{\Lambda_1}{8} \end{aligned}$$

if $\eta \le \min\{\frac{\tau\Lambda_1}{24}, \frac{1}{2}\}$ and

$$x - x_2 \le \frac{\Lambda_1}{32\bar{b}}. \tag{2.24}$$

Therefore, as similar to Step 2, if ρ_1 keeps decreasing, there exists a number $x_3 > x_2$ such that the solution $\rho_1(x)$ satisfies $\rho_1(x_3) = 1+\delta$, and for a number $\bar{\zeta} \in [x_2, x_3]$,

$$x_3 - x_2 = \frac{\rho_1(x_3) - \rho_1(x_2)}{(\rho_1)_x(\bar{\zeta})} \le \frac{12\eta^2(\frac{3}{2}\eta + 2)}{\Lambda_1} \le \frac{36\eta^2}{\Lambda_1}.$$

Clearly, if $\eta \le \frac{\Lambda_1}{24\sqrt{2\bar{b}}}$, (2.24) holds. Then, from (2.17), (2.22) and (2.23),

$$\frac{1}{2} - C\eta \le x_2 \le x_3 \le x_2 + C\eta \le \frac{1}{2} + C\eta,$$

and

$$\bar{E}_1(\frac{1}{2}) - C\eta \le E_1(x_2) - C\eta \le E_1(x_3) \le E_1(x_2) + C\eta \le \bar{E}_1(\frac{1}{2}) + C\eta.$$

Step 4. Now, let us focus on the following system

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x), \\ \rho(0) = \rho(1) = 1+\delta. \end{cases} \tag{2.25}$$

Then, we need to prove that there exists a subsonic solution of (2.25) by a continuity method. Now take $L = \frac{3}{2}$ in Steps 2-3. Denote the solution of (2.15) by $(\bar{\rho}_2(x), \bar{E}_2(x))$, and the solution of (2.19) by $(\rho_2(x), E_2(x))$, there exists a number x_4 such that

$$\frac{3}{2} - C\eta \leq x_4 \leq \frac{3}{2} + C\eta,$$
$$\rho_2(x_4) = 1 + \delta \quad \text{and} \quad \bar{E}_2(\frac{3}{2}) - C\eta \leq E_2(x_4) \leq \bar{E}_2(\frac{3}{2}) + C\eta.$$

Note that the solution of (2.19) is continuously dependent on the length L with a small parameter ε_0. Since the solution $\bar{\rho}$ is unique, one can see that $\bar{E}_1(0) < \bar{E}_2(0)$. By a continuity method, there exists a constant $E_0 \in [\bar{E}_1(0), \bar{E}_2(0)]$ such that (2.25) has a subsonic solution $(\rho^\delta(x), E^\delta(x))$ with the conditions:

$$E^\delta(0) = E_0 \quad \text{and} \quad E^\delta(1) < \frac{1}{\tau}.$$

Step 5. Let us now prove the existence of subsonic solution of (1.4)-(1.5). Referring to Step 7 of the proof of Theorem 4.2 [17], multiplying the first equation of (2.25) by $\frac{1}{\rho^\delta}((\rho^\delta - 1 - \delta)^2)_x$ and using the second equation of (2.25), we have

$$\begin{aligned}
&\int_0^1 \frac{2\delta(\rho^\delta + 1)(\rho^\delta - 1 - \delta)((\rho^\delta)_x)^2}{(\rho^\delta)^3} + \frac{(\rho^\delta + 1)|((\rho^\delta - 1 - \delta)^2)_x|^2}{2(\rho^\delta)^3} dx \\
&\leq \int_0^1 (\bar{b} - \rho^\delta)(\rho^\delta - 1 - \delta)^2 dx \leq \int_0^1 \bar{b}(\rho^\delta - 1 - \delta)^2 dx \\
&\leq \frac{1}{4}\int_0^1 |((\rho^\delta - 1 - \delta)^2)_x|^2 dx + \bar{b}^2.
\end{aligned} \tag{2.26}$$

Hence, it then follows from (2.26) that

$$||(\rho^\delta - 1 - \delta)^2||_{H_1(0,1)} \leq C.$$

Moreover, by the compact imbedding $H^1(0,1) \hookrightarrow C^{\frac{1}{2}}[0,1]$, there exists a function ρ, as $\delta \to 0^+$, up to a subsequence,

$$\begin{aligned}
&(\rho^\delta - 1 - \delta)^2 \rightharpoonup (\rho - 1)^2, \text{ weakly in } H^1(0,1), \\
&(\rho^\delta - 1 - \delta)^2 \to (\rho - 1)^2, \text{ strongly in } C^{\frac{1}{2}}[0,1].
\end{aligned}$$

So we know that (1.4)-(1.5) has a weak interior subsonic solution $(\rho(x), E(x))$ for $x \in [0,1]$ that satisfies sonic boundary condition.

Step 6. In this step, we prove the uniqueness and the lower bound of the interior subsonic solution. First, the proof of uniqueness and regularity are directly given by that of Theorem 2.1

and Proposition 2.5 [17], respectively. Now, we only need to estimate the lower bound. We realize that there is an interior solution $\bar{\rho}(x)$ to (2.15) when $\delta = 0$, thereinto the corresponding initial value $\bar{E}(0)$ satisfies $\bar{E}(0) = E(0)$ and $\bar{\rho}(x) = \rho(x)$ over $[0, \alpha_1]$. Here (ρ, E) is derived from Step 5. As in Steps 2-3, there exists a number x_5 such that $x_5 \geq 1 - C\eta_0^2$ and the solution $(\rho, E)(x)$ satisfies

$$\rho(x_5) = 1 + \eta_0, \quad |\rho(x) - \bar{\rho}(x)| \leq \frac{\eta_0}{2}$$

over $[\alpha_1, x_5]$ with fixed constants $0 < \eta_0 \ll 1$ and $\varepsilon_0 \ll 1$. Here, we notice that ρ keeps decreasing on $[x_5, 1]$. Thus, combining with (2.16), we easily get $\rho(x) \geq \min\{1 + c_1 \sin(\pi x), 1 + \frac{\eta_0}{2}\}$ for $x \in (0, x_5)$ where $c_1 > 0$ is a small constant. Next, as similar to Step 3, there exists a positive constant c_2 such that

$$E(x) = E(x_5) + \int_{x_5}^{x} (\rho - b)dx \leq -c_2 + \frac{1}{\tau} + \bar{b}(1 - x_5) \leq -\frac{c_2}{2} + \frac{1}{\tau}$$

if $\eta_0 \ll 1$ and $1 - x_5 \leq \frac{c_2}{2\bar{b}}$. Then for $x \in [x_5, 1)$

$$\rho_x(x) = \frac{\rho^2(x)(\rho(x)E(x) - 1/\tau)}{\rho^2(x) - 1} \leq \frac{(1 + \eta_0)^2}{\eta_0(\eta_0 + 2)} \cdot \left(-\frac{c_2}{2} + \frac{\eta_0}{\tau}\right) \triangleq -c_3 < 0, \quad \text{if} \quad \eta_0 \leq \frac{c_2\tau}{4},$$

thus $\rho(x) \geq -c_3(x - 1)$ over $[x_5, 1)$ for ε_0 sufficiently small. We choose

$$C = \min\left\{c_1, \frac{\eta_0}{2}, c_3\right\},$$

such that

$$\rho(x) \geq 1 + C\sin(\pi x) \ \text{ on } \ [0, 1].$$

The proof is complete. □

2.2. Interior supersonic solution

1. The case of $\tau = \infty$. In this case, we show the existence of interior supersonic solutions of (1.4)-(1.5) when the relaxation time is infinite.

Theorem 2.3. *Assume that* $b(x) \in L^\infty(0, 1)$*, there exists a constant* $\varepsilon_0 > 0$ *such that* $0 < |\mathcal{M}| < \varepsilon_0$*, then* (2.1) *admits an interior supersonic solution* $\rho(x) \in C^{\frac{1}{2}}[0, 1]$ *satisfying* $\underline{\rho} \leq \rho(x) < 1$ *over* $(0, 1)$ *for some positive constant* $\underline{\rho}$*, and* $\rho(0) = \rho(1) = 1$.

Proof. As similar to that of Theorem 2.1, we first solve the approximate system, a uniformly elliptic equation,

$$\begin{cases} \left[\left(\frac{1}{\rho_k} - \frac{k^2}{\rho_k^3} \right) (\rho_k)_x \right]_x - [\rho_k - b(x)] = 0, & x \in (0,1), \\ \rho_k(0) = \rho_k(1) = 1, \end{cases} \tag{2.27}$$

where $1 < k < +\infty$. Set the electron velocity $u_k(x) = \dfrac{k}{\rho_k(x)}$, (2.27) becomes

$$\begin{cases} \left((u_k - \frac{1}{u_k})(u_k)_x \right)_x - \left(\frac{k}{u_k} - b \right) = 0, & x \in (0,1), \\ u_k(0) = u_k(1) = k. \end{cases} \tag{2.28}$$

Omit the subscripts k for simplicity and set a function v with respect to u

$$v'(u) := u - \frac{1}{u} \quad \text{and} \quad v(k) := 0, \quad u \in [k, +\infty),$$

so we get

$$v(u) = \left(\frac{1}{2}u^2 - \ln u \right) - \left(\frac{1}{2}k^2 - \ln k \right), \quad u \in [k, +\infty). \tag{2.29}$$

It is easy to see from (2.29) that v is an increasing and concave function of u over $[k, +\infty]$. Then we define an inverse function by

$$u(v) = v^{-1}(u), \quad v \in [0, +\infty).$$

Next, set

$$\tilde{u}(v) := \begin{cases} u(v), & v \geq 0, \\ u(-v), & v < 0. \end{cases}$$

Therefore, (2.28) is transformed into the following equation

$$\begin{cases} v_{xx} = -\left(b(x) - \frac{k}{\tilde{u}(v)} \right), & x \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

After that, we define the operator $\Psi : \bar{v} \to v$ by solving the linearized system

$$\begin{cases} v_{xx} = -\left(b(x) - \frac{k}{\tilde{u}(\bar{v})} \right), & x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{2.30}$$

where $\tilde{u}(\bar{v}) \geq k$. Set the solution space by

$$S^0 := \left\{ v(x) : v \in C^0[0,1] | 0 \leq v(x) \leq \overline{M} \text{ and } v(0) = v(1) = 0 \right\}$$

and let $\bar{v} \in S^0$. Here $\overline{M}$ is a constant to be determined later. We only need to show that $0 \le v(x) \le \overline{M}$ for $x \in [0,1]$. Multiplying (2.30) by v, and using $\frac{k}{\tilde{u}(\bar{v})} \le 1$, we have

$$\begin{aligned}
\int_0^1 {v_x}^2 dx &\le \int_0^1 \left(b - \frac{k}{\tilde{u}(\bar{v})}\right) v dx \\
&\le \frac{1}{2}\int_0^1 \left(b - \frac{k}{\tilde{u}(\bar{v})}\right)^2 dx + \frac{1}{2}\int_0^1 v^2 dx \\
&\le \frac{1}{2}\int_0^1 \left(b^2+1\right) dx + \frac{1}{2}\int_0^1 {v_x}^2 dx.
\end{aligned}$$

Then it follows that

$$\int_0^1 {v_x}^2 dx \le \int_0^1 \left(b^2+1\right) dx \triangleq \overline{M}.$$

Hence, one can show that

$$v \le \overline{M}.$$

Next, we claim the following result: assume that $b(x) \in L^\infty(0,1)$, there exists $\varepsilon_0 > 0$ such that $0 < |\mathcal{M}| < \varepsilon_0$, then $v(x) > 0$ for $x \in (0,1)$ and $v(0) = v(1) = 0$. We can obtain Green's function of (2.30) by

$$K(x,\xi) := \begin{cases} (1-\xi)x & \text{for} \quad x \le \xi, \\ (1-x)\xi & \text{for} \quad x > \xi. \end{cases}$$

Then the solution of (2.30) is given by

$$v(x) = \int_0^1 K(x,\xi)\left(b - \frac{k}{\tilde{u}(\bar{v})}\right) d\xi \triangleq \Pi_1(x) + \Pi_2(x).$$

Here,

$$\Pi_1(x) := (1-x)\int_0^x \xi(b-1)d\xi + x\int_x^1 (1-\xi)(b-1)d\xi,$$

$$\Pi_2(x) := (1-x)\int_0^x \xi\big(1 - \frac{k}{\tilde{u}(\bar{v})}\big)d\xi + x\int_x^1 (1-\xi)\big(1 - \frac{k}{\tilde{u}(\bar{v})}\big)d\xi.$$

Because $1 - \frac{k}{\bar{u}(\bar{v})} \geq 0$, one can see that $\Pi_2(x) > 0$ over $(0, 1)$. Thus, it suffices to prove $\Pi_1(x) > 0$ for $x \in (0, 1)$ when $0 < |\mathcal{M}| < \varepsilon_0$. It is easy to check that the corresponding proof is similar to that in Theorem 2.1. So we conclude that $v(x) > 0$ for $x \in (0, 1)$ and $v(0) = v(1) = 0$ when ε_0 is small.

Hence, the existence of approximate solution $\rho_k(x)$ is proved by using Schauder fixed point Theorem. Then, let $k \to 1^+$, we can get the uniform bound of the approximate solution and the lower bound as in the proof of Theorem 3.1 [17]. Also, we discover the global regularity $C^{1/2}$ of the interior supersonic solution. The proof is complete. $\square$

2. The case of $0 < \tau < +\infty$. In this case, $b(x)$ is given by (2.13). When $\alpha_2 - \alpha_1 \ll 1$, $\alpha_2 = 1$ and $\tau < \frac{1}{3\sqrt{\bar{b}}}$, there is no interior supersonic solution. Thereinto, the proof of the non-existence is shown by Proposition 4.2 in Appendix. Thus, we show the existence of interior supersonic solutions to (1.4)-(1.5) when $\mathcal{M} \subset [0, \alpha_2]$, $\alpha_2 < 1$. As analogous to Theorem 2.2, the main theorem is as follow.

Theorem 2.4. *Assume that* $b(x) \in L^\infty(0, 1)$*, and there exist numbers* $\alpha_2 < 1$ *and* $\varepsilon_0 > 0$ *such that* $\mathcal{M} \subset [0, \alpha_2]$ *and* $0 < |\mathcal{M}| < \varepsilon_0$*, then system* (1.4)-(1.5) *admits an interior supersonic solution* $\rho(x)$ *satisfying* $\rho \in C^{\frac{1}{2}}[0, 1]$.

Proof. *Step 1.* In this situation, different from the proof of Theorem 2.2, we want to solve the next ODE system (2.32) by tracing back to the initial point from the end point. To do this, we need to extend $b(x)$ periodically to $[-1, 1]$ and denote $b_{\bar{L}}(x) := b(1 - \bar{L} + x)$ over $[0, \bar{L}]$, then define

$$\tilde{b}_{\bar{L}}(x) := \begin{cases} b_{\bar{L}}(x), & b_{\bar{L}}(x) \geq 1 + \mu, \\ 1 + \mu, & b_{\bar{L}}(x) < 1 + \mu, \end{cases}$$

where $\bar{L} \leq 2$ is a positive constant, thus $b_1(x) = b(x)$. Here, set $\delta \ll 1$ and $\bar{L} \geq \frac{1}{4}$. Let's consider the following boundary value problem with supersonic boundary condition

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - \tilde{b}_{\bar{L}}(x), \\ \rho(0) = \rho(\bar{L}) = 1 - \delta. \end{cases} \tag{2.31}$$

We know that the existence of supersonic solutions of (2.31) is proved from [17], then still define the solution by $(\bar{\rho}, \bar{E})(x)$ such that

$$0 < \underline{\rho} \leq \bar{\rho}(x) \leq 1 - \delta \quad \text{for} \quad x \in [0, 1].$$

Hereinto there exists a minimum point y_1 such that

$$\bar{\rho}(y_1) = \min_{x \in [0, \bar{L}]} \bar{\rho}(x) = \underline{\rho} < 1, \quad \bar{E}(y_1) = \frac{1}{\underline{\rho}\tau}.$$

As similar to the proof of Lemma 4.1 [17] and Step 1 in the proof of Theorem 2.2 before, it follows that

$$C_1(\bar{L}) \leq \underline{\rho} \leq C_2(\bar{L}), \quad \bar{E}(0) - \frac{1}{\tau} \geq C_3(\bar{L}),$$

where C_1, C_2 and C_3 are positive constants, independent of δ. Let η be a small constant such that $\delta < \eta \ll 1$, following Step 2 of Theorem 2.2, one can see that there exists a number $0 < y_2 \leq C\eta^2$ such that

$$\bar{\rho}(y_2) = 1 - \eta \quad \text{and} \quad |\bar{E}(y_2) - \bar{E}(0)| \leq C\eta^2.$$

Step 2. Firstly, set $\bar{L} = \frac{1}{2}$ and denote the solution of (2.31) by $(\bar{\rho}_1, \bar{E}_1)(x)$. Let us focus on the ODE system:

$$\begin{cases} \left(1 - \dfrac{1}{\hat{\rho}^2}\right)\hat{\rho}_x = \hat{\rho}\hat{E} - \dfrac{1}{\tau}, \\ \hat{E}_x = \hat{\rho} - b_{\hat{L}}(x), \\ (\hat{\rho}(\hat{L}), \hat{E}(\hat{L})) = (1 - \delta, \bar{E}(\bar{L})), \end{cases} \tag{2.32}$$

where $\hat{L} \leq 2$ is a positive constant to be determined later. Denote $(\hat{\rho}_1, \hat{E}_1)(x)$ by the solution of (2.32). By the definition of $b_{\hat{L}}(x)$, $(\hat{\rho}_1(x), \hat{E}_1(x)) = (\bar{\rho}_1(\bar{L} - \hat{L} + x), \bar{E}_1(\bar{L} - \hat{L} + x))$ over $[\hat{L} - \alpha_2, \hat{L}]$. In the step, we actually try to seek the initial state of the system (2.32) when the end state is known. Referring to Steps 2-4 of Theorem 2.2, we know that there exists a number $\hat{L}_1 > 0$ such that $|\hat{L}_1 - \frac{1}{2}| \leq C\eta$ and the solution of (2.32) satisfies

$$\hat{\rho}_1(0) = \hat{\rho}_1(\hat{L}_1) = 1 - \delta, \quad |\hat{E}_1(0) - \bar{E}_1(y_2)| \leq C\eta$$

when $\varepsilon_0 \ll 1$. In addition, let $\bar{L} = \frac{3}{2}$, and denote the corresponding solution of (2.31) by $(\bar{\rho}_2, \bar{E}_2)(x)$ and the solution of (2.32) by $(\hat{\rho}_2, \hat{E}_2)(x)$. There exists a number $\hat{L}_2$ such that $|\hat{L}_2 - \frac{3}{2}| \leq C\eta$, and the solution of (2.32) satisfies

$$\hat{\rho}_2(0) = \hat{\rho}_2(\hat{L}_2) = 1 - \delta, \quad |\hat{E}_2(0) - \bar{E}_2(0)| \leq C\eta.$$

Observing that the solution of (2.31) is structurally stable with respect of the length $\bar{L}$, even if the solution may not be unique, we still could use a continuity method to obtain a supersonic solution $(\rho^\delta, E^\delta)(x)$ for system (1.4) with supersonic boundary condition $\rho(0) = \rho(1) = 1 - \delta$ when ε_0 is small enough.

Step 3. As in Step 5 of the proof of Theorem 2.4 [18] and Step 5 of Theorem 2.2, it follows that

$$\left\|(1 - \delta - \rho^\delta)^2\right\|_{H^1} \leq C.$$

Therefore, by the compact imbedding $H^1(0,1) \hookrightarrow C^{\frac{1}{2}}[0,1]$, there exists a function ρ^0, as $\delta \to 0^+$, up to a subsequence,

$$(1-\delta-\rho^{\delta})^2 \rightharpoonup (1-\rho^0)^2, \text{ weakly in } H^1(0,1),$$
$$(1-\delta-\rho^{\delta})^2 \to (1-\rho^0)^2, \text{ strongly in } C^{\frac{1}{2}}[0,1].$$

So one can show that system (1.4)-(1.5) has a weak interior supersonic solution $\rho^0(x)$ for $x \in [0,1]$. □

2.3. Transonic shock solution

For example, with a step doping profile, there is no transonic shock solution if $\tau \ll 1$ and $|\mathcal{M}| \ll 1$, whose proof is similar to Theorem 5.13 [17]. Now we mainly consider the interior transonic shock solutions of system (1.4)-(1.5) when $\tau \gg 1$.

Theorem 2.5. *Assume that $b(x) \in L^\infty(0,1)$, and there exists a number $\varepsilon_0 > 0$ such that $0 < |\mathcal{M}| < \varepsilon_0$. Moreover, when $\tau \gg 1$, system (1.4)-(1.5) has infinitely many transonic shock solutions $(\rho_{trans}, E_{trans})(x)$ for $x \in [0,1]$, satisfying the entropy condition (1.9) and the Rankine-Hugoniot condition (1.10) at the jump location x_0.*

Proof. *Step 1.* Let δ and η be small numbers to be defined later satisfying $0 < \delta < \eta \ll 1$. Firstly, let us concern the following boundary value problem:

$$\begin{cases} \left(1-\dfrac{1}{\rho^2}\right)\rho_x = \rho E, \\ E_x = \rho - \tilde{b}(x), \\ \rho(0) = \rho(L) = 1-\delta, \end{cases} \tag{2.33}$$

where $L \geq \frac{1}{4}$ is a positive constant and $\tilde{b}(x) \in L^\infty(\mathbb{R}^+)$ is define by (2.14). From [17], we know that (2.33) has a supersonic solution $(\tilde{\rho}, \tilde{E})(x)$ such that $\tilde{E}(L) \leq -C(L) < 0$. Here C is a constant independent of τ. Next, in the case of $\tau \gg 1$ and $\varepsilon_0 \ll 1$, we reconsider the ODE system,

$$\begin{cases} \left(1-\dfrac{1}{\rho^2}\right)\rho_x = \rho E - \frac{1}{\tau}, \\ E_x = \rho - b(x), \\ (\rho(0), E(0)) = (1-\delta, \tilde{E}(0)), \end{cases} \tag{2.34}$$

and the corresponding solution $(\rho, E)(x)$. Applying the proof of Theorem 4.2 [17] and Theorem 2.2, there exists a length $\mathcal{L}$ of the solution of (2.34) such that

$$|L-\mathcal{L}| \leq C\eta, \quad \rho(\mathcal{L}) = 1-\delta \quad \text{and} \quad |E(\mathcal{L}) - \tilde{E}(L)| \leq C\eta.$$

Step 2. Set $\rho_l = 1-\eta < 1-\delta$ and take a jump point $\hat{\mathcal{L}} \in (0,\mathcal{L})$ as the last number such that $\rho(\hat{\mathcal{L}}^-) = \rho_l$, and note that

$$|E_r - \tilde{E}(L)| \leq C\eta.$$

Then we claim that there exists a number z such that the system (2.34) has a subsonic solution $(\rho_{sub}, E_{sub})(x)$ on $[\hat{\mathcal{L}}, z]$ satisfying

$$\rho(\hat{\mathcal{L}}^+) = \rho_r = \frac{1}{1-\eta}, \quad \rho(z) = 1+\delta \quad \text{and} \quad z - \hat{\mathcal{L}} \leq C\eta.$$

In fact, when $C\eta \leq -\frac{\tilde{E}(L)}{2}$, we get

$$\rho_r E_r - \frac{1}{\tau} \leq \frac{1}{1-\eta}(\tilde{E}(L) + C\eta) \leq \frac{\tilde{E}(L)}{2} < 0.$$

Then, one can easily show that there exists a decreasing subsonic solution $\rho_{sub}(x)$ near the point $\hat{\mathcal{L}}$, and we have

$$\begin{aligned} E_{sub}(x) &= E_r + \int_{\hat{\mathcal{L}}}^{x} (\rho_{sub} - b(x))dx \\ &\leq \tilde{E}(L) + C\eta + (x - \hat{\mathcal{L}})\rho_r \\ &\leq \frac{\tilde{E}(L)}{4} \end{aligned}$$

if

$$|x - \hat{\mathcal{L}}| \leq -(1-\eta)\frac{\tilde{E}(L)}{4}. \tag{2.35}$$

Next, we prove that $z - \hat{\mathcal{L}} \leq C\eta$. First, from the first equation of (2.34), noting that

$$(\rho_{sub})_x = \frac{\rho_{sub}E_{sub} - \frac{1}{\tau}}{1 - \frac{1}{(\rho_{sub})^2}} \leq \frac{\tilde{E}(L)}{4\eta(2-\eta)} < \frac{\tilde{E}(L)}{8\eta},$$

we have

$$z - \hat{\mathcal{L}} = \frac{\rho_{sub}(z) - \rho_{sub}(\hat{\mathcal{L}})}{\int_0^1 (\rho_{sub})_x (sz + (1-s)\hat{\mathcal{L}})ds} \leq -\frac{\eta}{1-\eta} \cdot \frac{8\eta}{\tilde{E}(L)} \leq 8\eta \quad \text{if} \quad \eta < \min\{-\frac{\tilde{E}(L)}{2}, \frac{1}{2}\}.$$

So assuming that $\eta \leq -\frac{\tilde{E}(L)}{64}$, it is easy to see that (2.35) holds and $\rho_{sub}(z) = 1+\delta$. Then, we can construct a transonic shock solution to (2.34) on $[0, z]$ as follow

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \hat{\mathcal{L}}], \\ (\rho_{sub}, E_{sub})(x), & x \in (\hat{\mathcal{L}}, z], \end{cases}$$

satisfying

$$\rho(0) = 1 - \delta, \quad \rho(z) = 1 + \delta \quad \text{and} \quad 0 < \rho(\hat{\mathcal{L}}^-) < 1 - \eta < \rho(\hat{\mathcal{L}}^+).$$

Here, $\hat{\mathcal{L}}$ is a jump location and the R-H condition is satisfied at this point.

Step 3. Next, we argue by a continuity method. Then, we observe that the length L is continuous with respect to the initial value $\tilde{E}(0)$. First, set $L = \dfrac{1}{2}$ and denote the corresponding solution of (2.34) by $(\rho^{(1)}, E^{(1)})(x)$. From Step 2, we conclude that there exists a number z' such that

$$\frac{1}{2} - C\eta \le z' \le \frac{1}{2} + C\eta, \quad \rho^{(1)}(z') = 1 + \delta,$$

and

$$-E^{(1)}(\frac{1}{2}) - C\eta \le E(z') \le -E^{(1)}(\frac{1}{2}) + C\eta.$$

So we have a transonic solution to (2.34) on $[0, z']$ with $L = \frac{1}{2}$ in the form of

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \hat{\mathcal{L}}_1], \\ (\rho_{sub}, E_{sub})(x), & x \in (\hat{\mathcal{L}}_1, z']. \end{cases}$$

Here, $\hat{\mathcal{L}}_1$ is a jump location and the R-H condition is satisfied at this point. If $L = \dfrac{3}{2}$ (correspondingly, $(\rho^{(2)}, E^{(2)})(x)$), we know that there exists an interval $[0, z'']$ with

$$\frac{3}{2} - C\eta \le z'' \le \frac{3}{2} + C\eta, \quad \rho^{(2)}(0) = 1 - \delta, \quad \rho^{(2)}(z'') = 1 + \delta,$$

and

$$-E^{(2)}(\frac{3}{2}) - C\eta \le E(z'') \le -E^{(2)}(\frac{3}{2}) + C\eta.$$

For fixed ε_0, η, any $\tilde{E}_0 \in (E^{(1)}(0), E^{(2)}(0))$, there exists a length $z' < L < z''$ of transonic solutions, which is corresponding to the initial value $\tilde{E}_0$. Thus, we realize that (1.4) has a transonic shock solution $(\rho_{trans}, E_{trans})(x)$ for $x \in [0, 1]$ with $\rho(0) = 1 - \delta$, $\rho(1) = 1 + \delta$, satisfying the entropy condition and the Rankine-Hugoniot condition at a jump location x_0^δ in $[0, 1]$. As in step 7 of the proof of Theorem 4.2 [17], we have a transonic shock solution for (1.4)-(1.5) by letting $\delta \to 0^+$. Because of the arbitrary choices of η, transonic shock solutions are infinitely many. The proof is complete. □

2.4. C^1-smooth transonic solution

Theorem 2.6. *Assume that τ is small and $b(x) \in L^\infty(0, 1)$ is a step function, denoted by (2.13), there exists a constant $\varepsilon_0(\tau) > 0$ such that $\alpha_2 - \alpha_1 \le \varepsilon_0$, then (1.4)-(1.5) has infinitely many C^1-smooth transonic solutions.*

Proof. Assume that $b(x) \equiv \bar{b} > 1$ and $0 < \tau < \tau_0(\bar{b})$. Here τ_0 is a small constant. From Theorem 5.12 [17], we know that (1.4)-(1.5) has infinitely many C^1-smooth transonic solutions in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in [0, x_0], \\ \rho_{sub}(x), & x \in (x_0, 1], \end{cases}$$

where x_0 is a the location of transition, and the solutions satisfy the smoothness condition (1.11) at x_0. Moreover,

$$E_{sup}(x_0) = E_{sub}(x_0) = \frac{1}{\tau}, \quad \rho'_{sup}(x_0) = \rho'_{sub}(x_0) = \frac{1}{4}\left(\frac{1}{\tau} - \sqrt{\frac{1}{\tau^2} - 8(\bar{b} - 1)}\right). \tag{2.36}$$

Now, let $x_0 \notin [\alpha_1, \alpha_2]$ and $b(x)$ is denoted by (2.13). Then as in the proof of Theorem 2.2 and Theorem 2.4, we find that there exist a supersonic solution $\rho_{sup}(x)$ on $[0, x_0]$ and a subsonic solution $\rho_{sub}(x)$ on $[x_0, 1]$, which satisfy (1.11) and (2.36) at x_0. Because the choice of the location x_0 is arbitrary, C^1-smooth transonic solution are infinitely many. The proof is complete. $\square$

3. Supersonic-dominated doping profile

In this section, we will discuss the existence/non-existence of all types of the solutions during supersonic-dominated doping profile, satisfying $|\mathcal{N}| \ll 1$. Here, the set $\mathcal{N}$ is defined by (1.8).

3.1. Non-existence of interior subsonic/supersonic/transonic solutions

Theorem 3.1. *Assume that $b(x) \in L^\infty(0, 1)$ is supersonic-dominated and*

$$\int_{[0,1]\backslash\mathcal{N}} b(x)dx \le 1 - \epsilon \quad \textit{with a parameter } \epsilon.$$

Afterwards, there exists a constant $\varepsilon_0 < \epsilon/\bar{b}$ such that $0 < |\mathcal{N}| \le \varepsilon_0$, then (1.4)-(1.5) has no interior subsonic solution.

Proof. Let $\varepsilon_0 > 0$ be a small number, and suppose that there is an interior subsonic solution $\rho(x)$ of (1.4)-(1.5) when $|\mathcal{N}| = \varepsilon_0$. Moreover, the solution $\rho(x)$ satisfies

$$\rho(x) > 0, \quad x \in (0, 1) \quad \text{and} \quad \rho(0) = \rho(1) = 1.$$

Then set a function of ρ,

$$w'(\rho) := \frac{1}{\rho} - \frac{1}{\rho^3} \quad \text{for} \quad \rho \in [1, +\infty), \quad w(1) = 0,$$

and further

$$w(\rho) = \ln \rho + \frac{1}{2\rho^2} - \frac{1}{2} \quad \text{for} \quad \rho \ge 1.$$

Thus, (1.4) can be reduced to

$$\left(w_x + \frac{1}{\tau\rho}\right)_x = \rho - b(x), \quad x \in (0,1). \tag{3.1}$$

Then set $f(x) := w_x(x) + \frac{1}{\tau\rho(x)}$, and we get $f \in H^1(0,1)$ because (3.1) holds in the sense of distribution. By the compact imbedding $H^1(0,1) \hookrightarrow C^{1/2}[0,1]$, we have $f \in C^{1/2}[0,1]$. From Proposition 2.5 [17], one can see that $\rho \in C^{1/2}[0,1]$. Thus, notice that $w_x = f - \frac{1}{\tau\rho} \in C^{1/2}[0,1]$, then $w \in C^{1+1/2}[0,1]$.

Integrating (3.1) over $[0,1]$ with respect to x, we obtain

$$w_x(1) - w_x(0) = \int_0^1 (\rho - b)dx \tag{3.2}$$

where we have used

$$\int_0^1 \left(\frac{1}{\tau\rho}\right)_x dx = 0.$$

Noting that

$$w(x) > 0, \quad x \in (0,1) \quad \text{and} \quad w(0) = w(1) = 0,$$

we have

$$w_x(1) - w_x(0) \le 0,$$

and

$$\int_0^1 (\rho - b)dx \ge 1 - \int_0^1 b(x)dx > 1 - (1 - \epsilon + \varepsilon_0\bar{b}) > \epsilon - \varepsilon_0\bar{b}.$$

Herewith we get a contradiction when $0 < \varepsilon_0 < \epsilon/\bar{b}$. Then, for any $0 < |\mathcal{N}| \le \varepsilon_0 < \epsilon/\bar{b}$, there is no interior subsonic solution of (1.4)-(1.5). □

Next, we consider the non-existence of interior supersonic solutions and transonic shock solutions when $|\mathcal{N}| \ll 1$. The results are as follows.

Theorem 3.2. *Assume that $b(x) \in L^\infty(0,1)$ is supersonic-dominated, and there exists a constant $\varepsilon_0 > 0$ such that $0 < |\mathcal{N}| \le \varepsilon_0$, no interior supersonic solution to (1.4)-(1.5) exists when*

$$\int_{[0,1]\setminus\mathcal{N}} b(x)dx \ll 1$$

or the relaxation time τ *is small and*

$$\int_{[0,1]\backslash\mathcal{N}} b(x)dx \le 1-\epsilon \quad \textit{with a parameter } \epsilon.$$

Proof. First of all, when $|\mathcal{N}| = \varepsilon_0$, we assume that $\rho(x)$ is an interior supersonic solution of (1.4)-(1.5) satisfying

$$\rho(x) < 1, \quad x \in (0,1) \quad \text{and} \quad \rho(0) = \rho(1) = 1.$$

The velocity $u(x) = \dfrac{1}{\rho(x)}$ satisfies

$$\begin{cases} \left(u - \dfrac{1}{u}\right) u_x = E - \dfrac{u}{\tau}, \\ E_x = \dfrac{1}{u} - b(x). \end{cases} \tag{3.3}$$

Case 1: $\hat{b} \ll 1$, where $\hat{b} := \int_{[0,1]\backslash\mathcal{N}} b(x)dx$. Multiplying the first equation of (3.3) by $\big((u-1)^2\big)_x$, and integrating the resulting equation over $(0,1)$, we have

$$\int_0^1 \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx = \int_0^1 \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx \tag{3.4}$$

where we have used

$$u((u-1)^2)_x = \frac{1}{3}((u-1)^2(2u+1))_x.$$

Then, it follows from (3.4) that

$$\begin{aligned} &\int_0^1 \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\ &\le \int_0^1 \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx \\ &\le \frac{1}{4}\int_0^1 (u(x)-1)^4 dx + \int_0^1 b^2(x) dx \\ &\le \frac{1}{4}\int_0^1 \left|\left[(u(x)-1)^2\right]_x\right|^2 dx + (\hat{b} + \varepsilon_0 \bar{b}^2). \end{aligned} \tag{3.5}$$

Thus, we get from (3.5) that

$$\int_0^1 |[(u(x)-1)^2]_x|^2 dx \le 4(\hat{b}+\varepsilon_0 \bar{b}^2),$$

and further

$$u(x) \le 1 + [2(\hat{b}+\varepsilon_0 \bar{b}^2)]^{1/2} \quad \text{on} \quad [0,1],$$

which leads to

$$\frac{1}{1+[2(\hat{b}+\varepsilon_0 \bar{b}^2)]^{1/2}} \le \rho(x) \le 1 \quad \text{on} \quad [0,1].$$

Similarly, we set

$$w(\rho) = \ln \rho + \frac{1}{2\rho^2} - \frac{1}{2} \quad \text{for} \quad 0 < \rho \le 1.$$

Hence, (3.1) and (3.2) also hold. Because

$$w(x) > 0, \quad x \in (0,1) \quad \text{and} \quad w(0) = w(1) = 0,$$

we get

$$w_x(1) - w_x(0) \le 0.$$

In addition, when ε_0 is small enough and $\hat{b} \ll 1$, then

$$\int_0^1 (\rho - b) dx \ge \frac{1}{1+[2(\hat{b}+\varepsilon_0 \bar{b}^2)]^{1/2}} - (\hat{b}+\varepsilon_0 \bar{b}) > 0.$$

Herewith this is a contradiction.

Case 2: $\tau \ll 1$ and $\hat{b} \le 1-\epsilon$, where ϵ is a small constant. There exists a maximal point $\hat{x}$ such that $u(x) \le u(\hat{x})$ for $x \in [0,1]$ because $u \in C[0,1]$, and the first equation of (3.3) gives

$$E(\hat{x}) = \frac{u(\hat{x})}{\tau}. \tag{3.6}$$

As in (3.4), because of $u(\hat{x}) > 1$ and (3.6), we have

$$\begin{aligned} \int_{\hat{x}}^1 \frac{u(x)+1}{2u(x)} \left| \left[(u(x)-1)^2 \right]_x \right|^2 dx &= \int_{\hat{x}}^1 \left(b(x) - \frac{1}{u(x)} \right) (u(x)-1)^2 dx - \frac{(u(\hat{x})-1)^3}{3\tau} \\ &\le \bar{b}(u(\hat{x})-1)^2 - \frac{(u(\hat{x})-1)^3}{3\tau}. \end{aligned} \tag{3.7}$$

Then, if $u(\hat{x}) - 1 \geq \epsilon/2 > 0$, by choosing $\tau < \dfrac{\epsilon}{6\bar{b}}$, one can derive from (3.7) that

$$\int_{\hat{x}}^{1} \frac{u(x)+1}{2u(x)} \left| \left[(u(x)-1)^2 \right]_x \right|^2 dx < 0,$$

which is a contradiction. Otherwise, when $0 < u(\hat{x}) - 1 \leq \epsilon/2$, $\hat{b} \leq 1-\epsilon$ and $\varepsilon_0 < \dfrac{\epsilon}{2\bar{b}}$, as similar to the case of $\hat{b} \ll 1$,

$$\int_0^1 (\rho - b)dx \geq \frac{1}{1+\epsilon/2} - (\hat{b} + \bar{b}\varepsilon_0) \geq \frac{\epsilon}{2} - \bar{b}\varepsilon_0 > 0,$$

that is in contradiction with

$$w_x(1) - w_x(0) \leq 0.$$

The proof is complete. $\square$

Theorem 3.3. *Suppose all the assumptions remain the same as mentioned in Theorem 3.2, and there exists a small constant $\varepsilon_0 > 0$ such that $0 < |\mathcal{N}| \leq \varepsilon_0$, then no transonic shock solution to the system (1.4)-(1.5) exists.*

Proof. Let $(\rho, E)(x)$ be the pair of transonic solutions separated by a point $\hat{y}$ in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), x \in (0, \hat{y}), \\ \rho_{sub}(x), x \in (\hat{y}, 1), \end{cases}$$

and

$$\rho_l \rho_r = 1, \quad E_l = E_r \quad \text{with} \quad \rho_l < 1 \quad \text{and} \quad \rho_r > 1.$$

If $\hat{b} \ll 1$, we simply divide the proof into two cases, $0 < \hat{y} < \frac{1}{2}$ and $\frac{1}{2} \leq \hat{y} < 1$. In the case $0 < \hat{y} < \frac{1}{2}$: first of all, one can prove that

$$E_l < \frac{1}{\tau} - \frac{1}{8}$$

when $\hat{b} \ll 1$ and ε_0 is small enough. In fact, if $E_l \geq 1/\tau - \frac{1}{8}$ and $\hat{b} + \bar{b}\varepsilon_0 < \frac{1}{8}$,

$$E(y) = E_r + \int_{\hat{y}}^{y} (\rho_{sub} - b)dx \geq \frac{1}{\tau} - \frac{1}{8} + \frac{1}{4} - (\hat{b} + \bar{b}\varepsilon_0) > \frac{1}{\tau}$$

for all $y \in [\frac{3}{4}, 1)$. Thus, $(\rho_{sub})_x(y) > 0$ for $y \in [\frac{3}{4}, 1)$, that is in contradiction with $\rho_{sub}(1) = 1$ and $\rho_{sub}(y) > 1$ over $[\frac{3}{4}, 1)$. Then because $\hat{b} + \bar{b}\varepsilon_0 < \frac{1}{8}$, it follows that

$$E(0) = E_l - \int_0^{\hat{y}} (\rho_{sup} - b)dx < E_l + \int_0^{\hat{y}} b(x)dx < \frac{1}{\tau} - \frac{1}{8} + (\hat{b} + \bar{b}\varepsilon_0) < \frac{1}{\tau}.$$

Thus, $(\rho_{sup})_x(y) > 0$ because $E(y) < \frac{1}{\tau}$ near the end point $y = 1$, which contradicts to $\rho(x) = \rho_{sup}(x) < 1$ for $x \in (0, \hat{y}]$ and $\rho(0) = 1$.

Next, in the case of $\frac{1}{2} \leq \hat{y} < 1$, we will prove that

$$E_l < \frac{1}{\tau} + \bar{b}\varepsilon_0 \tag{3.8}$$

where ε_0 is small enough. If $E_l \geq 1/\tau + \bar{b}\varepsilon_0$, we have

$$E_{sub}(x) = E_l + \int_{\hat{y}}^{x} (\rho_{sub} - b)ds > E_l - \bar{b}\varepsilon_0 \geq \frac{1}{\tau}$$

and

$$\rho_{sub}(x)E_{sub}(x) - \frac{1}{\tau} > E_{sub}(x) - \frac{1}{\tau} > 0$$

for $x \in [\hat{y}, 1)$. Thus, $(\rho_{sub})_x(x) > 0$ on $[\hat{y}, 1)$, which in combination with $\rho_r > 1$ contradicts to $\rho_{sub}(1) = 1$. Hence, $E_l < 1/\tau + \bar{b}\varepsilon_0$.

Then multiplying the first equation of (3.3) by $\big((u-1)^2\big)_x$, and integrating the resulting equation over $(0, \hat{y})$, we get

$$\begin{aligned}
&\int_0^{\hat{y}} \frac{u(x)+1}{2u(x)} \left|\Big[(u(x)-1)^2\Big]_x\right|^2 dx \\
&= \int_0^{\hat{y}} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx + (u_l - 1)^2\left(E_l - \frac{2u_l+1}{3\tau}\right) \\
&= \int_0^{\hat{y}} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx + (u_l - 1)^2\left(E_l - \frac{1}{\tau}\right) - \frac{2(u_l-1)^3}{3\tau}.
\end{aligned} \tag{3.9}$$

Furthermore, it is easy to check from (3.8) and (3.9) that

$$\int_0^{\hat{y}} \frac{u(x)+1}{2u(x)} \left|\Big[(u(x)-1)^2\Big]_x\right|^2 dx < \int_0^{\hat{y}} b(x)(u(x)-1)^2 dx + \bar{b}\varepsilon_0(u_l-1)^2.$$

Then, by Hölder inequality and Sobolev imbedding theorem, it's verified that

$$(u(\check{x})-1)^4 \le C\int_0^{\hat{y}}\left|\left[(u(x)-1)^2\right]_x\right|^2 dx \le C\left(\int_0^{\hat{y}} b(x)^2 dx + \bar{b}\varepsilon_0 (u(\check{x})-1)^2\right)$$

where $u(\check{x}) = \max\limits_{x\in[0,\hat{y}]} u(x)$ and C is a positive constant, independent of ε_0. Therefore,

$$(u(\check{x})-1)^2 \le C\left(\bar{b}\varepsilon_0 + \sqrt{(\bar{b}\varepsilon_0)^2 + 4(\hat{b}+\bar{b}^2\varepsilon_0)}\right) =: \Upsilon,$$

where we have used

$$\int_0^{\hat{y}} b(x)^2 dx \le \hat{b} + \bar{b}^2\varepsilon_0.$$

Afterwards, for $\hat{b}$ and ε_0 sufficiently small, we get

$$E(\hat{y}) - E(0) = \int_0^{\hat{y}} (\rho_{sup} - b) dx \ge \frac{1}{2(1+\sqrt{\Upsilon})} - (\hat{b} + \bar{b}\varepsilon_0) > \frac{1}{4}.$$

Thus,

$$E(0) < E(\hat{y}) - \frac{1}{4} \le \frac{1}{\tau} + \bar{b}\varepsilon_0 - \frac{1}{4} \le \frac{1}{\tau},$$

that is a contradiction.

If $\tau \ll 1$ and $\hat{b} \le 1-\epsilon$, we first assume that there exists a minimal point $\check{y} \in (0,\hat{y})$ such that $(\rho_{sup})_x(\check{y}) = 0$, $0 < \rho_{sup}(\check{y}) < \rho_l$ and $E(\check{y}) = \dfrac{u(\check{y})}{\tau}$. Similar to (3.9), we have

$$\begin{aligned}
&\int_{\check{y}}^{\hat{y}} \frac{u(x)+1}{2u(x)}\left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\
&= \int_{\check{y}}^{\hat{y}} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx + (u_l-1)^2\left(E_l - \frac{2u_l+1}{3\tau}\right) \\
&\quad - (u(\check{y})-1)^2\left(E(\check{y}) - \frac{2u(\check{y})+1}{3\tau}\right) \\
&< \bar{b}(u(\check{y})-1)^2 + \bar{b}\varepsilon_0(u_l-1)^2 - \frac{1}{3\tau}((u_l-1)^3 + (u(\check{y})-1)^3) \\
&< 2\bar{b}(u(\check{y})-1)^2 - \frac{1}{3\tau}(u(\check{y})-1)^3
\end{aligned} \tag{3.10}$$

where we have used $E_l < 1/\tau + \bar{b}\varepsilon_0$ and $u_l < u(\breve{y})$. As showed in the case of $\tau \ll 1$ of Theorem 3.2, we discuss the value of $u(\breve{y})$: if $u(\breve{y}) - 1 \geq \delta > 0$, choosing $\tau < \dfrac{\delta}{6\bar{b}}$, one can obtain

$$\int_{\breve{y}}^{\hat{y}} \frac{u(x)+1}{2u(x)} \left|\Big[(u(x)-1)^2\Big]_x\right|^2 dx < 0.$$

This yields a contradiction. Then we only consider the case of $0 < u(\breve{y}) - 1 \leq \delta$. Herewith, by choosing $\bar{b}\varepsilon_0 < \frac{\epsilon}{2}$ such that $\int_0^1 b(x)dx = \hat{b} + \bar{b}\varepsilon_0 < 1 - \frac{\epsilon}{2}$, then there exists a number $c_0 > 0$ such that

$$\int_0^{c_0} (1 - b(x))dx = \int_{c_0}^{1} (1 - b(x))dx > \frac{\epsilon}{4}.$$

Case 1: $0 < \hat{y} < c_0$. We claim that

$$E_l < \frac{1}{\tau} - \frac{\epsilon}{16}$$

when ε_0 is small enough. Assume that $E_l \geq \dfrac{1}{\tau} - \dfrac{\epsilon}{16}$ and $\bar{b}\varepsilon_0 < \dfrac{\epsilon}{16}$,

$$\begin{aligned}
E(1) &= E_r + \int_{\hat{y}}^{1} (\rho_{sub} - b)dx \\
&\geq E_r + \int_{c_0}^{1} (1-b)dx - \bar{b}\varepsilon_0 \\
&> \frac{1}{\tau} - \frac{\epsilon}{16} + \frac{\epsilon}{4} - \bar{b}\varepsilon_0 \\
&> \frac{1}{\tau} + \frac{\epsilon}{4}.
\end{aligned}$$

Thus, this is a contradiction, and further $E_l < \frac{1}{\tau} - \frac{\epsilon}{16}$. Note that $\rho_{sup}(x) = 1/u(x) \geq \frac{1}{1+\delta}$ over $[0, \hat{y}]$, and choose δ and ε_0 small such that $\delta + \bar{b}\varepsilon_0 < \frac{\epsilon}{16}$, then it follows that

$$\begin{aligned}
E(0) &= E_l - \int_0^{\hat{y}} (\rho_{sup} - b)dx \\
&\leq E_l - \int_0^{\hat{y}} (\frac{1}{1+\delta} - 1)dx + \bar{b}\varepsilon_0
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{\tau} - \frac{\epsilon}{16} + \delta + \bar{b}\varepsilon_0 \\
&< \frac{1}{\tau},
\end{aligned}
$$

which contradicts to $\rho(x) = \rho_{sup}(x) < 1$ for $x \in (0, \hat{y}]$ and $\rho(0) = 1$.

Case 2: $c_0 \le \hat{y} < 1$. Noting that $E_l < 1/\tau + \bar{b}\varepsilon_0$, we have

$$
E(0) = E_l - \int_0^{\hat{y}} (\rho_{sup} - b)dx < \frac{1}{\tau} + 2\bar{b}\varepsilon_0 - \frac{\epsilon}{4} + \delta < \frac{1}{\tau} - \frac{\epsilon}{8},
$$

when $\delta + \bar{b}\varepsilon_0 < \frac{\epsilon}{16}$. This is a contradiction. Then, if $\hat{y}$ is the minimal point of $\rho_{sup}(x)$ on $[0, \hat{y}]$, as in (3.9), we obtain

$$
\int_0^{\hat{y}} \frac{u(x)+1}{2u(x)} \left| \left[(u(x)-1)^2 \right]_x \right|^2 dx < \bar{b}(1+\varepsilon_0)(u(\hat{y})-1)^2 - \frac{1}{3\tau}(u(\hat{y})-1)^3.
$$

As in (3.10), when $\tau < \dfrac{\delta}{6\bar{b}}$ and $\delta + \bar{b}\varepsilon_0 < \dfrac{\epsilon}{16}$, we have same results. The proof is complete. □

3.2. Existence of interior supersonic/transonic solutions

In the subsection, we expect to show the existence of solutions when the doping profile $b(x)$ is supersonic-dominated. Of course, in the case of the supersonic doping profile ($b < 1$), the existence of the supersonic/transonic solutions have been studied in [18] when $b(x)$ is close to the sonic line and the relaxation time is large. By the compactness technique and the continuity theorem, we have same results about the existence of interior supersonic/transonic solutions. Because the method used is similar to the former, by referring to the proof of Theorem 4.2 [17], Theorem 2.4 and Theorem 2.5 [18], and Theorem 2.2, we state the theorem and omit the proof.

Theorem 3.4. *There exist at least one interior supersonic solution and infinitely many transonic shock solutions to* (1.4)-(1.5) *for* $|\mathcal{N}| \ll 1$, *when* $\tau \gg 1$ *and the doping profile* $b(x) \in L^\infty(0, 1)$ *satisfies*

$$
1 - \epsilon \le b(x) \le \bar{b} \quad \textit{with a small parameter } \epsilon.
$$

4. Appendix. Non-existence of solutions

Now we give a special case that the system (1.4)-(1.5) has no subsonic solution with a subsonic-dominated doping profile. Referring to Section 5 of [17], we set

$$
F = E - \frac{1}{\tau\rho}, \quad n = \rho - 1.
$$

Then (1.4) is transformed to

$$\begin{cases} n_x = \dfrac{(n+1)^3 F}{(n+2)n}, \\ F_x = n+1-b+\dfrac{(n+1)F}{\tau(n+2)n}. \end{cases} \tag{4.1}$$

In the phase plane, we know that all trajectories of (4.1) satisfy

$$\frac{dF}{dn} = \frac{(n+1-b)(n+2)}{(n+1)^3} \cdot \frac{n}{F} + \frac{1}{\tau(n+1)^2}.$$

Here, the doping profile $b(x)$ is a step function, defined in (2.13) and $\alpha_1 = 0$, $\alpha_2 = \varepsilon_0$. A proposition is stated as follow.

Proposition 4.1. *When the parameter ε_0 is sufficiently small, then, no interior subsonic solution exists to* (1.4)-(1.5)*, no matter how small ε_0 is.*

Proof. First, we define a curve function

$$\Xi(n) := -\frac{\tau(n+1-\bar{b})(n+2)n}{n+1}$$

where the curve is determined by the equation

$$\frac{(n+1-\bar{b})(n+2)}{(n+1)^3} \cdot \frac{n}{F} + \frac{1}{\tau(n+1)^2} = 0.$$

From Lemma 5.3 [17], in the case of $b \equiv \bar{b}$, we know that all trajectories to system (4.1) satisfy

$$F(n) \le \frac{3}{2}\Xi(n) \quad \text{for} \quad n \ge 0,$$

when $0 < \tau < \frac{1}{3\sqrt{\bar{b}^3+\bar{b}}}$.

Assume that $n(\varepsilon_0) = n_1 > 0$ and choose ε_0 sufficiently small such that $F(n) \ge 0$ over $[0, n_1]$, then we want to prove $F(n) > \frac{3}{2}\Xi(n)$ over $[0, n_1]$. As in the proof of Lemma 5.2 [17], noting that

$$\Xi'(n) = -\tau\left(2 - \bar{b} + 2n - \frac{\bar{b}}{(n+1)^2}\right) \quad \text{for} \quad n \ge 0,$$

a direct calculation conclude that

$$\begin{aligned} (F^2 - \frac{9}{4}\Xi^2)' &= \frac{2(F - \frac{3}{2}\Xi)}{\tau(n+1)^2} + 2\Xi\left[\frac{1}{2\tau(n+1)^2} + \frac{9}{4}\tau\left(2 - \bar{b} + 2n - \frac{\bar{b}}{(n+1)^2}\right)\right] \\ &\quad + \frac{2(\bar{b} - b)(n+2)n}{(n+1)^3} \\ &= \left(F^2 - \frac{9}{4}\Xi^2\right) \cdot \frac{2}{\tau(n+1)^2(F + \frac{3}{2}\Xi)} + 2\Xi \cdot \Delta_1 + \Delta_2 \quad \text{on} \quad [0, n_1] \end{aligned} \tag{4.2}$$

where $\Delta_1 := \frac{1}{2\tau(n+1)^2} + \frac{9}{4}\tau\left(2-\bar{b}+2n-\frac{\bar{b}}{(n+1)^2}\right)$ and $\Delta_2 := \frac{2(\bar{b}-\underline{b})(n+2)n}{(n+1)^3}$. When $\tau^2 < \frac{1}{9(\bar{b}^3+\bar{b})}$, we have $\Delta_1 > 0$ as in the proof of Lemma 5.3 [17]. Then, one can see that $\Delta_2 > 0$ and $\Xi(n) > 0$ on $(0, n_1]$, which in combination with $\Xi(0) = 0$ and $F(0) \geq 0$ lead to $F(n) > \frac{3}{2}\Xi(n)$ on $[0, n_1]$. Thus, $F(n_1) > \frac{3}{2}\Xi(n_1)$. Indeed, over $(n_1, \bar{b}-1)$, we also get

$$(F^2 - \frac{9}{4}\Xi^2)' = (F^2 - \frac{9}{4}\Xi^2)\frac{2}{\tau(n+1)^2(F+\frac{3}{2}\Xi)} + 2\Xi\cdot\Delta_1 > 0, \tag{4.3}$$

which gives $F(n) > \frac{3}{2}\Xi(n)$ on $(n_1, \bar{b}-1)$. Because $(\bar{b}-1, 0)$ is a saddle point in phase plane (n, F) when $b \equiv \bar{b}$, the trajectory of the system (4.1) will go to infinity. This is a contradiction. Thus, no subsonic solution to (1.4)-(1.5) exists. □

As similar to Proposition 4.1, let us reconsider (1.4)-(1.5) with the step doping profile and $\alpha_1 = 1 - \varepsilon_0, \alpha_2 = 0$. Then, we will prove the non-existence of supersonic solution no matter how small ε_0 is if τ is small enough.

Proposition 4.2. *When the parameter ε_0 is sufficiently small, then, no interior supersonic solution exists to* (1.4)-(1.5).

Proof. We consider the interior supersonic solution $n(x) < 0$ over $(0, 1)$. Assume that $n(1-\varepsilon_0) = n_2 < 0$, thus $F(n_2) < 0$ when $\varepsilon_0 \ll 1$. Next we will focus on the region $F \leq 0$, and as in the proof of Lemma 5.7 [17], we only want to prove

$$F(n_2) < \frac{3}{2}\Xi(n_2)$$

on $[n_2, 0]$. Over $[n_2, 0]$, (4.2) also holds. When $\tau < \frac{1}{3\sqrt{\bar{b}}}$, we have $\Delta_1 > 0$ for $n \in (-1, 0)$. Moreover, $\Delta_2 < 0$, $\Xi(n) < 0$ and $F(n) < 0$ on $[n_2, 0)$. Thus, from (4.2), $F(n) < \frac{3}{2}\Xi(n)$ on $[n_2, 0]$, and further $F(n_2) < \frac{3}{2}\Xi(n_2)$. However, over $(-1, n_2)$, (4.3) holds, and we get

$$F(n) < \frac{3}{2}\Xi(n)$$

on $(-1, n_2]$ by lemma 5.8 [17]. Then, we have $\lim_{n\to -1} F(n) = -\infty$. No supersonic solution to (1.4)-(1.5) exists. □

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