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Novel convergence of solutions to 1D compressible Euler equations with spatiotemporal damping in critical case

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Abstract

This paper is concerned with the Cauchy problem for 1D compressible Euler equations with spatiotemporal damping in the critical case. We prove the existence of the solutions and their new convergence to the special diffusion waves by the technical time-weighted energy method, where the convergence rates are dependent on the spatial state of the spatiotemporal damping as $x \to \pm\infty$. These convergence results significantly improve and develop the previous studies of Geng et al. (2020) [10] and Matsumura and Nishihara (2024) [24].

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1. Introduction

We consider the Cauchy problem for the Euler equations with damping depending on the temporal and spatial variables

$$\begin{cases} v_t - u_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+, \\ u_t + p(v)_x = -\alpha(x,t)u, \\ (v,u)(x,0) = (v_0,u_0)(x) \to (v_\pm, u_\pm), \text{ as } x \to \pm\infty \ (v_\pm > 0), \end{cases} \tag{1.1}$$

which describes the one-dimensional compressible flow through porous media in Lagrangian coordinates, where u represents the flow velocity, $v > 0$ denotes the specific volume, and $p(v) > 0$ represents the pressure, with $p'(v) < 0$. The term $-\alpha(x,t)u$ represents the fractional external affection in the porous media flow, which is given by

$$\alpha(x,t) = \frac{\alpha_1(x)}{1+t}, \quad \text{with } \alpha_1(x) \to \underline{\alpha}_1 \ \text{ as } x \to \pm\infty, \tag{1.2}$$

where $\underline{\alpha}_1 > 2$ is a positive state constant.

When $\alpha(x,t) = 0$, the system (1.1) reduces to the standard compressible Euler equations, which have attracted considerable attention from a large number of analysts, triggering a great many crucial advancements [3–6,33,34]. It is well known that smooth solutions to (1.1) generally break down in finite time.

When $\alpha(x,t) = \underline{\alpha}_1 > 0$, in the constant damping case, the system (1.1) becomes the compressible Euler equations with damping. The damping effect usually stops the shock formation and guarantees the existence of the global solutions, once the initial data are smooth enough. In 1992, Hsiao-Liu [13] first showed that the solution of system (1.1) with the constant damping globally exists and converges time-asymptotically to nonlinear diffusion waves $(\bar{v},\bar{u})(\frac{x}{1+t})$, the self-similar solutions to its corresponding nonlinear porous media equations, in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$. Such convergence rates of $(v-\bar{v}, u-\bar{u})$ were improved by Nishihara [26] to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O)(1)(t^{-\frac{3}{4}}, t^{-\frac{5}{4}})$. Subsequently, Nishihara-Wang-Yang [28] further refined the result to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-\frac{3}{2}})$ when the initial perturbations are in $L^1(\mathbf{R})$. However, as Mei [25] pointed out, the self-similar solutions are not the best asymptotic profiles because $\int_{-\infty}^{\infty}(u-\bar{u}-\hat{u}) \neq 0$. By using twice anti-derivatives technique, Mei [25] recognized that the optimal asymptotic profiles are the specific solutions to the nonlinear porous media equation with selected initial conditions, and proved the better convergence rates: $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{2}}\log(2+t), t^{-2}\log(2+t))$. For the other interesting studies for the constant damping, we refer to [14,21–23,27,28,32,36–38,42,43].

When $\alpha(x,t) = \frac{\alpha_1}{(1+t)^\lambda}$ with $\lambda \neq 0$, the system (1.1) reduces to the compressible Euler equations with time-dependent damping. For $\lambda > 0$, the time-dependent damping becomes asymptotically weak in time, the so-called under-damping case [15]. For $\lambda < 0$, the damping becomes asymptotically strong in time, the so-called over-damping case [16]. In the weak damping case ($\lambda > 0$), as showed in [10], the damping effect $\alpha(x,t) = \frac{\alpha_1}{(1+t)^\lambda}$ is getting weaker as λ is increasing. In particular, when $\lambda = 1$, it is the critical damping case, because the hyperbolic effect dominates for the damped Euler equations as $\lambda > 1$, and the solutions must blow up and the shocks usually form; while, the parabolic effect coming from damping dominates for the damped Euler equations as $0 < \lambda < 1$, and the solutions globally exist in this case.

In the regular under-damping case ($0 < \lambda < 1$), Yin and his research group [11,12,29–31] first gave the criteria for the existence and non-existence of the solutions, see also the significant developments in [7,15,34,35], in particular, see the optimal convergence rates to the constant states: $\|\partial^\alpha(v-1,u)(t)\|_{L^2(R^n)} = O(1)(t^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)}, t^{-\frac{1+\lambda}{2}(\frac{n}{2}+|\alpha|)-\frac{1-\lambda}{2}})$ obtained by Ji-Mei in [15]. Regarding the convergence of the solutions to the corresponding diffusion waves, we refer to the interesting studies in [8,9,19,20].

In the critical under-damping case: $\alpha(x,t)=\frac{\alpha_1}{1+t}$, namely, $\lambda=1$, by variable scaling approach, Geng-Li-Mei [10] observed that, the parabolic mechanism and the hyperbolic mechanism both are important for the system, and cannot be ignored. The asymptotic profiles are not the ordinary diffusion waves (the self-similar solutions to the porous media equations), but the particular solutions with hyperbolic-parabolic structure (still denoted by $(\bar{v},\bar{u})(x,t)$). Furthermore, they [10] showed the convergence of the original solutions to the specified asymptotic profile $(\bar{v},\bar{u})(x,t)$ in the form of $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)t^{-\frac{1}{2}-\frac{\underline{\alpha}_1}{2}}$ for $2<\underline{\alpha}_1\le 4$, and $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)t^{-\frac{3}{2}}$ for $\underline{\alpha}_1>4$. Here, the convergence rates depend on the value of $\underline{\alpha}_1$ in two intervals. $\underline{\alpha}_1=2$ is the critical value above which the system admits global existence of solutions; otherwise, the damping effect $\frac{\alpha_1}{1+t}$ is too weak such that the system behaves like the pure Euler equations and the shock solutions must form [7,12,29,34].

For the over-damping case $\alpha(x,t)=\frac{\alpha_1}{(1+t)^\lambda}$ with $-1\le\lambda<0$, different from the common sense, the strong damping for the compressible Euler equations does not yield a better decay. In fact, as showed by Ji-Mei [16] for the over-damping case, the decay rates around the constant states are $\|(v-1,u)(t)\|_{L^2(R^n)} = O(1)(t^{-\frac{1+\lambda}{4}n}, t^{-\frac{1+\lambda}{4}n-\frac{1-\lambda}{2}})$ for $\lambda\in(-1,0)$, and $\|(v-1,u)(t)\|_{L^2(R^n)} = O(1)(|\ln(1+t)|^{-\frac{n}{4}}, t^{-1}|\ln(1+t)|^{-\frac{n}{4}-\frac{1}{2}})$ for $\lambda=-1$, which are much weaker than the regular under-damping case with $0<\lambda<1$. On the other hand, when the states $(v_\pm,u_\pm)$ are different, the solutions are expected to converge to the corresponding diffusion waves time-asymptotically with algebraic rates for $-1<\lambda<0$ and the algebraic decay involving $\log(1+t)$ for $\lambda=-1$, see [17,18,39] for the details.

Regarding the damping effect involving the space only: $\alpha(x,t)=\alpha_1(x)\to\underline{\alpha}_1>0$ as $x\to\pm\infty$, Matsumura-Nishihara [24] first studied this case and proved the convergence of the solutions to the corresponding diffusion waves by the energy method in form of $\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-\frac{3}{4}},t^{-\frac{3}{4}})$.

When the damping effect is dependent on both the space x and the time t, recently we [1,2] first investigated how the spatiotemporal damping $\alpha(x,t)=\frac{\alpha_1(x)}{(1+t)^\lambda}$ with $0\le\lambda<1$ to affect the structure of the solutions, and showed that: when $v_+=v_-$, then

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = \begin{cases} O(1)\Big((1+t)^{-\frac{3(\lambda+1)}{4}},(1+t)^{-\frac{\lambda+5}{4}}\Big), & 0\leqslant\lambda<\frac{3}{5},\\ O(1)\Big(\frac{\ln^{\frac{1}{2}}(2+t)}{(1+t)^{\frac{6}{5}}},\frac{\ln^{\frac{1}{2}}(2+t)}{(1+t)^{\frac{7}{5}}}\Big), & \lambda=\frac{3}{5},\\ O(1)\Big((1+t)^{\frac{\lambda-3}{2}},(1+t)^{\lambda-2}\Big), & \frac{3}{5}<\lambda<1, \end{cases}$$

whereas when $v_+\neq v_-$, then

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty} = \begin{cases} O(1)\Big((1+t)^{-\frac{3+3\lambda}{8}},(1+t)^{-\frac{7-\lambda}{8}}\Big), & 0\le\lambda\le\frac{1}{3},\\ O(1)\Big((1+t)^{-\frac{5-3\lambda}{8}},(1+t)^{-\frac{9-7\lambda}{8}}\Big), & \frac{1}{3}<\lambda<\frac{3}{5},\\ O(1)\Big((1+t)^{\lambda-1},(1+t)^{-\frac{3-3\lambda}{2}}\Big), & \frac{3}{5}\le\lambda<1. \end{cases}$$

Subsequently to our previous studies [1,2], in this paper we are interested in the critical case of spatiotemporal damping $\alpha(x,t)=\frac{\alpha_1(x)}{1+t}$ given in (1.2). Here, as demonstrated in [10], the parabolic structure from the damping effect and the hyperbolic structure from the system itself both play the crucial roles for the system, and cannot be ignored, see (2.4) below for the expected asymptotic profile by variable scaling technique. By the time-weighted energy method, in particular, by refining the decay estimates on the asymptotic profiles $(\bar{v},\bar{u})(x,t)$ which closely depends on the value of $\underline{\alpha}_1$, we obtain the following new convergence of the original solutions to the corresponding profiles as:

Case 1: when $v_+=v_-=:\underline{v}$, $u_+\neq u_-$, and $\int_{-\infty}^{\infty}[v_0(x)-\underline{v}]dx\neq\frac{u_+-u_-}{-\underline{\alpha}_1+1}$, then

- for $3\le\underline{\alpha}_1<4$, it holds

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{\underline{\alpha}_1}{4}},$$
$$\sum_{i+j=1}\|\partial_x^i\partial_t^j(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{\underline{\alpha}_1-1}{2}}.$$

- for $4\le\underline{\alpha}_1<6$, it holds

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-1},$$
$$\sum_{i+j=1}\|\partial_x^i\partial_t^j(u-\bar{u})(t)\|_{L^\infty}+\|(v-\bar{v})_x(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{\underline{\alpha}_1+2}{4}},$$
$$\|(v-\bar{v})_t(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{3}{2}}.$$

- for $\underline{\alpha}_1\ge 6$, it holds

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-1},$$
$$\sum_{i+j=1}\|\partial_x^i\partial_t^j(u-\bar{u})(t)\|_{L^\infty}+\|(v-\bar{v})_x(t)\|_{L^\infty}=O(1)(1+t)^{-2},$$
$$\|(v-\bar{v})_t(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{3}{2}}.$$

Case 2: when $v_+=v_-=:\underline{v}$, $u_+\neq u_-$, but $\int_{-\infty}^{\infty}[v_0(x)-\underline{v}]dx=\frac{u_+-u_-}{-\underline{\alpha}_1+1}$, then

- for $\frac{20}{9}\le\underline{\alpha}_1<\frac{12}{5}$, it holds

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{7\underline{\alpha}_1}{8}+1},$$
$$\sum_{i+j=1}\|\partial_x^i\partial_t^j(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)(1+t)^{-\frac{9\underline{\alpha}_1}{8}+\frac{3}{2}}.$$

- for $\frac{12}{5}\le\underline{\alpha}_1\le 4$, it holds

$$\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}=O(1)\ln^{\frac{1}{2}}(2+t)(1+t)^{\frac{\underline{\alpha}_1+2}{4}},$$

$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)\ln(2+t)(1+t)^{-\frac{\alpha_1}{2}}.$$

- for $4 < \underline{\alpha}_1 < 6$, it holds

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} &= O(1)(1+t)^{-1-\frac{\alpha_1}{4}}, \\ \|(v-\bar{v})_x(t)\|_{L^\infty} &= O(1)(1+t)^{-2}. \end{aligned}$$

- for $\underline{\alpha}_1 \geq 6$, it holds

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{5}{2}}, \\ \|(v-\bar{v})_x(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{9}{4}}. \end{aligned}$$

Case 3: when $v_+ = v_- =: \underline{v}$, $u_+ = u_- =: \underline{u}$, and $\int_{-\infty}^{\infty} [v_0(x) = \underline{v}]dx = 0$, then

- for $2 < \underline{\alpha}_1 < 4$, it holds

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\alpha_1+2}{4}}, \\ \sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\alpha_1}{2}}. \end{aligned}$$

- for $4 \leq \underline{\alpha}_1 < 6$, it holds

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{\alpha_1}{4}-1}, \\ \|(v-\bar{v})_x(t)\|_{L^\infty} &= O(1)(1+t)^{-2}. \end{aligned}$$

- for $\underline{\alpha}_1 \geq 6$, it holds

$$\begin{aligned} \|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{3}{2}}, \\ \sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{5}{2}}, \\ \|(v-\bar{v})_x(t)\|_{L^\infty} &= O(1)(1+t)^{-\frac{9}{4}}. \end{aligned}$$

These convergence rates are dependent on the value of $\underline{\alpha}_1$ in different intervals, and significantly improve and develop the previous studies [1,2,10,24]. However, due to the technical issue, there is still a gap for $2 < \underline{\alpha}_1 \le 3$. This would be our next goal.

The paper is organized as follows: In Section 2, for three cases, through analysis, we respectively select different asymptotic profiles and restate the problems. Meanwhile, we present the decay estimates of the asymptotic profiles, which plays the crucial roles in the subsequent proofs. For Sections 3, 4, and 5, we conduct decay estimates for the convergence of the original solutions to the corresponding asymptotic profiles in three cases, respectively. Before ending this section, let us introduce some notations.

Notations: $L^p = L^p(\mathbf{R})$ denotes the usual Lebesgue space with the norm

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbf{R}} |f(x)|^p dx \right)^{1/p} \quad (1 \le p < \infty),$$
$$|f|_\infty = \|f\|_\infty = \|f\|_{L^\infty} = \operatorname*{ess.sup}_{\mathbf{R}} |f(x)|.$$

The integral domain $\mathbf{R}$ is often abbreviated when it is clear. For any integer $k \ge 0$, $H^k = H^k(\mathbf{R})$ denotes the usual k-th order Sobolev space with the norm

$$\|f\|_{H^k} = \left(\sum_{j=0}^{k} \|\partial_x^k f\|_2^2 \right)^{1/2}.$$

When $k = 0$ and $p = 2$, we often use the notation $\|f\| = \|f\|_0$. For brevity, we write

$$\|f, g, \dots\|_{H^k \times H^l \times \dots} = \|f\|_{H^k} + \|g\|_{H^l} + \dots.$$

The set of k-times continuously differentiable functions in $\mathbf{R}$ with compact support is denoted by $C_0^k(\mathbf{R})$. The space $C^k([0, T]; X)$ consists of k-times differentiable functions from $[0, T]$ to the Hilbert space X. Additionally, by C or c, we denote a generic positive constant independent of the time t, whose value may change from line to line. For two quantities a and b, $a \sim b$ means $\frac{1}{C}|b| \le |a| \le C|b|$ for a given constant C.

2. Selection of asymptotic profiles and their properties

In this section, we reformulate problem as a quasi-linear wave equation with damping, based on heuristic reasoning. For convenience, we denote $\alpha(x, t)$ by α, then we present the results for the reformulated problem, which in turn yield the main results for the original problem (1.1). Rewrite (1.1) as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x + \alpha u = 0, \\ (v, u)(0, x) = (v_0, u_0)(x) \to (v_\pm, u_\pm), x \to \pm\infty, \end{cases} \tag{2.1}$$

under the condition (1.2). With the aim of searching for suitable asymptotic-state equations of (2.1), we carry out the following variable substitutions. For an arbitrarily ε, let

$$t \to (1+\bar{t})/\varepsilon, \quad x \to \bar{x}/\varepsilon, \quad v \to \bar{v}, \quad u \to \bar{u}, \quad \alpha_1 \to \bar{\alpha}_1$$

then, it follows from (2.1) that

$$\bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0,$$

and

$$\varepsilon \bar{u}_{\bar{t}} + \varepsilon p(\bar{v})_{\bar{x}} = -\frac{\varepsilon \bar{\alpha}_1 \bar{u}}{(\varepsilon + 1 + \bar{t})}. \tag{2.2}$$

Neglecting the small term with ε, we get the asymptotic-state equations

$$\begin{cases} \bar{v}_{\bar{t}} - \bar{u}_{\bar{x}} = 0, \\ \bar{u}_{\bar{t}} + p(\bar{v})_{\bar{x}} = -\frac{\bar{\alpha}_1}{1+\bar{t}} \bar{u}. \end{cases} \tag{2.3}$$

Therefore, by restricting ourselves to $v_+ = v_- =: \underline{v}$, but u_+ and u_- may not be equal, we consider the following linear system of equations:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{u}_t + p'(\underline{v})\bar{v}_x = -\frac{\alpha_1}{1+t}\bar{u}, \\ (\bar{v}, \bar{u})(x, 0) = (\bar{v}_0, \bar{u}_0)(x) \to (\underline{v}, u_\pm) \quad \text{as} \quad x \to \pm\infty. \end{cases} \tag{2.4}$$

This will be the expected asymptotic profile with particularly selected initial data $(\bar{v}_0, \bar{u}_0)(x)$ in their different cases.

Before we embark on our analysis, we assume

$$\begin{cases} p \in C^4(\mathbf{R}_+), \ \text{with} - p'(v) > 0 \ (v > 0), \\ \alpha_1 \in C^2(\mathbf{R}), \ \alpha_1 \geq \underline{\alpha}_1, \ \alpha_1 - \underline{\alpha}_1 \in L^2 \cap L^1 \ \textit{and} \ \alpha_{1x} \in L^2. \end{cases} \tag{2.5}$$

Case 1: $v_+ = v_- =: \underline{v}$, $u_+ \neq u_-$, and $\int_{-\infty}^{\infty} [v_0(x) - \underline{v}]dx \neq \frac{u_+ - u_-}{-\underline{\alpha}_1 + 1}$. In this case, we choose the initial data $\bar{v}_0(x)$ as

$$\bar{v}_0(x) = \underline{v} + \kappa \Phi_0(x), \tag{2.6}$$

where $\Phi_0(x)$ is a given smooth function such that

$$\Phi_0(x) \in L^1(\mathbf{R}) \quad \text{and} \quad \int_{-\infty}^{+\infty} \Phi_0(x)dx \neq 0,$$

and will be elaborated on in more detail in the subsequent discussion. Then, the constant κ is defined as

$$\kappa := \left(\int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx \right) / \int_{-\infty}^{+\infty} \Phi_0(x)dx, \tag{2.7}$$

so that

$$\int_{-\infty}^{+\infty}[v_0(x)-\underline{v}]dx-\kappa\int_{-\infty}^{+\infty}\Phi_0(x)dx=0.$$

Hence, the problem (2.4) can be reduced to

$$\begin{cases}\bar{v}_{tt}+p'(\underline{v})\bar{v}_{xx}+\frac{\alpha_1}{1+t}\bar{v}_t=0,\\(\bar{v},\bar{v}_t)(x,0)=(\bar{v}_0,\bar{u}_0')(x),\end{cases}\tag{2.8}$$

which is well studied by Wirth in [40,41]. With regard to the estimation of $\bar{v}$, we will conduct a discussion subsequently, and $\bar{u}(x,t)$ can be solved from the second equation of (2.4) as follows:

$$\bar{u}(x,t)=(1+t)^{-\underline{\alpha}_1}\bar{u}_0(x)-(1+t)^{-\underline{\alpha}_1}\int_0^t(1+s)^{\underline{\alpha}_1}p'(\underline{v})\bar{v}_x(x,s)ds.\tag{2.9}$$

Consequently, we obtain the anticipated asymptotic profile $(\bar{v},\bar{u})(x,t)$ for Case 1.

Case 2: $v_+=v_-=:\underline{v}$, $u_+\neq u_-$, and $\int_{-\infty}^{\infty}[v_0(x)-\underline{v}]dx=\frac{u_+-u_-}{-\underline{\alpha}_1+1}$. In this case, we can choose the initial data as

$$\begin{cases}\bar{v}_0(x)=v_0(x),\\\bar{u}_0(x)=u_-+(u_+-u_-)\int_{-\infty}^x m_1(y)dy,\end{cases}\tag{2.10}$$

where $\int_{-\infty}^{\infty}m_1(x)dx=1$ and $m_1(x)\in C_0^\infty(\mathbf{R})$, and the problem (2.4) can be reduced to

$$\begin{cases}\bar{v}_t-\bar{u}_x=0,\\\bar{u}_t+p'(\underline{v})\bar{v}_x=-\frac{\alpha_1}{1+t}\bar{u},\\(\bar{v}_0,\bar{u}_0)(x)=(v_0,\bar{u}_0)(x)\to(\underline{v},u_\pm),\quad\text{as}\quad x\to\pm\infty.\end{cases}\tag{2.11}$$

Based on the analysis of equation (2.11), applying the anti-derivatives technique can yield better decay estimates for the equation. However, due to $\int_{-\infty}^{\infty}\bar{u}_0dx\neq0$, when we transform the equation, its initial value condition fails to be L^2-bounded. Thus, to resolve this problem, we introduce the following modified equation:

$$\begin{cases}\hat{v}_t-\hat{u}_x=0,\\\hat{u}_t=-\frac{\underline{\alpha}_1}{1+t}\hat{u},\\(\hat{v}_0,\hat{u}_0)(x,0)\to(0,u_\pm),\quad\text{as}\quad x\to\pm\infty,\end{cases}\tag{2.12}$$

where $\hat{u}_0(x)=u_-+(u_+-u_-)\int_{-\infty}^x m_2(y)dy$, $\int_{-\infty}^{\infty}m_2(x)dx=1$ and $m_2(x)\in C_0^\infty(\mathbf{R})$. Hence, there exists an exact solution of (2.12) as follows:

$$\begin{cases}\hat{u}=(1+t)^{-\underline{\alpha}_1}\left(u_-+(u_+-u_-)\int_{-\infty}^x m_2(y)dy\right),\\\hat{v}=\frac{u_+-u_-}{-\underline{\alpha}_1+1}(1+t)^{-\underline{\alpha}_1+1}m_2(x),\end{cases}\tag{2.13}$$

which, together with (2.11), deduces

$$\begin{cases} \left(\bar{v}-\underline{v}-\hat{v}\right)_t-\left(\bar{u}-\hat{u}\right)_x=0, \\ \left(\bar{u}-\hat{u}\right)_t+p'(\underline{v})\left(\bar{v}-\underline{v}-\hat{v}\right)_x+\frac{\alpha_1}{1+t}\left(\bar{u}-\hat{u}\right)=-p'(\underline{v})\hat{v}_x. \end{cases} \tag{2.14}$$

Thus, we define $\phi=\int_{-\infty}^x\left(\bar{v}-\underline{v}-\hat{v}\right)dy$. Then, we have $\phi_t=\int_{-\infty}^x\left(\bar{v}-\underline{v}-\hat{v}\right)_t dy=\bar{u}-\hat{u}$, which follows from the first equation in (2.14), and the second equation in (2.14) can be written as

$$\phi_{tt}+p'(\underline{v})\phi_{xx}+\frac{\alpha_1}{1+t}\phi_t=-p'(\underline{v})\hat{v}_x. \tag{2.15}$$

As a result, we obtain the following reformulated problem:

$$\begin{cases} \phi_{tt}+p'(\underline{v})\phi_{xx}+\frac{\alpha_1}{1+t}\phi_t=-p'(\underline{v})\hat{v}_x, \\ \phi(0,x)=\phi_0(x):=\int_{-\infty}^x(v_0(y)-\underline{v}-\hat{v}(0,y))dy, \\ \phi_t(0,x)=\phi_1(x):=\bar{u}_0(x)-\hat{u}(0,x). \end{cases} \tag{2.16}$$

Case 3: $v_+=v_-=:\underline{v}$, $u_+=u_-=:\underline{u}$ and $\int_{-\infty}^{\infty}[v_0(x)-\underline{v}]dx=0$. By considering above-mentioned conditions, we look for a better convergence to a certain asymptotic profile and expect

$$\bar{v}(x,t)=\underline{v}, \tag{2.17}$$

which, together with (2.4), yields that

$$\bar{u}_x=0, \quad \bar{u}_t=-\frac{\alpha_1}{1+t}\bar{u},$$

and

$$\bar{u}(x,t)=\underline{u}(1+t)^{-\alpha_1}.$$

Hence, we obtain the anticipated asymptotic profile

$$(\bar{v},\bar{u})(x,t)=\left(\underline{v},\underline{u}(1+t)^{-\alpha_1}\right), \tag{2.18}$$

which has a better decay estimates compared with Case 1 in the subsequent discussions.

Theorem 2.1 *(Property of asymptotic profiles $(\bar{v},\bar{u})(x,t)$ in Case 1). Suppose $(\Phi_0,\bar{u}_0')\in H^3(\mathbf{R})\times H^2(\mathbf{R})$. Then, the Cauchy problem (2.4) admits a unique global smooth solution $\bar{v}(x,t)$, which satisfies, for $\underline{\alpha}_1>2$*

$$\sum_{0\le i+j\le\mu}(1+t)^{2(i+j)}\|\partial_x^i\partial_t^j(\bar{v}-\underline{v})(t)\|^2+\sum_{\mu<i+j\le 3}(1+t)^{\min\{\underline{\alpha}_1,6\}}\|\partial_x^i\partial_t^j(\bar{v}-\underline{v})(t)\|^2$$
$$\le C\left(\|\Phi_0\|_3^2+\|u_0'\|_2^2+|u_+|^2+|u_-|^2\right), \tag{2.19}$$

where $\mu := \min\{[\frac{\underline{\alpha}_1}{2}], 3\}$, *and for* $0 < \underline{\alpha}_1 < 2$,

$$(1+t)^{\underline{\alpha}_1-2}\|(\bar{v}-\underline{v})(t)\|^2 + \sum_{1\le i+j\le 3}(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j(\bar{v}-\underline{v})(t)\|^2 \le C\left(\|\Phi_0\|_3^2 + \|u_0'\|_2^2 + |u_+|^2 + |u_-|^2\right). \tag{2.20}$$

For the sake of our subsequent proof, we denote

$$E_0 := \|\Phi_0\|_{H^3} + \|u_0'\|_{H^2} + |u_+| + |u_-|.$$

Proof. Similar to the discussion in [10], we transform (2.8) into the following problem:

$$\begin{cases} W_{tt} + p'(\underline{v})W_{xx} + \frac{\underline{\alpha}_1}{1+t}W_t = 0, \\ (W, W_t)(x,0) = (W_0, W_1)(x), \end{cases} \tag{2.21}$$

where $W = \bar{v} - \underline{v}$ and

$$(W_0, W_1)(x) = (\kappa\Phi_0(x), \bar{u}_0'(x)).$$

When $0 < \underline{\alpha}_1 < 2$, by multiplying (2.21) by $(1+t)^{\underline{\alpha}_1}W_t$ and integrating it over $\mathbf{R}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int(1+t)^{\underline{\alpha}_1}(W_t^2 - p'(\underline{v})W_x^2)dx + \frac{\underline{\alpha}_1}{2}\int(1+t)^{\underline{\alpha}_1-1}W_t^2dx + \frac{\underline{\alpha}_1}{2}\int(1+t)^{\underline{\alpha}_1-1}p'(\underline{v})W_x^2dx = 0. \tag{2.22}$$

Also multiplying (2.21) by $(1+t)^{\underline{\alpha}_1-1}W$ and integrating it over $\mathbf{R}$ give

$$\frac{d}{dt}\int\left[(1+t)^{\underline{\alpha}_1-1}WW_t + \frac{1}{2}(1+t)^{\underline{\alpha}_1-2}W^2\right]dx - \int(1+t)^{\underline{\alpha}_1-1}p'(\underline{v})W_x^2dx - \int\left((1+t)^{\underline{\alpha}_1-1}W_t^2 + \frac{\underline{\alpha}_1-2}{2}(1+t)^{\underline{\alpha}_1-3}W^2\right)dx = 0. \tag{2.23}$$

Then, by calculating $\frac{2}{\underline{\alpha}_1}\cdot(2.22) + (2.23)$, one gets

$$\frac{d}{dt}\int\left[\frac{1}{\underline{\alpha}_1}(1+t)^{\underline{\alpha}_1}(W_t^2 - p'(\underline{v})W_x^2) + (1+t)^{\underline{\alpha}_1-1}WW_t + \frac{1}{2}(1+t)^{\underline{\alpha}_1-2}W^2\right]dx - \int\frac{\underline{\alpha}_1-2}{2}(1+t)^{\underline{\alpha}_1-3}W^2dx = 0. \tag{2.24}$$

Applying Cauchy's inequality, we can derive

$$\int(1+t)^{\underline{\alpha}_1-1}WW_tdx \le \int\frac{\varepsilon}{2}(1+t)^{\underline{\alpha}_1}W_t^2dx + \int\frac{1}{2\varepsilon}(1+t)^{\underline{\alpha}_1-2}W^2dx,$$

where $\varepsilon = \frac{\underline{\alpha}_1+2}{2\underline{\alpha}_1}$. Hence, by integrating (2.24) over $[0, t]$ yields

$$\frac{2-\underline{\alpha}_1}{4\underline{\alpha}_1}(1+t)^{\underline{\alpha}_1}\|W_t\|^2 + c_0(1+t)^{\underline{\alpha}_1}\|W_x\|^2 + \frac{2-\underline{\alpha}_1}{2(\underline{\alpha}_1+2)}(1+t)^{\underline{\alpha}_1-2}\|W\|^2 \le CE_0^2. \tag{2.25}$$

Next, differentiating (2.21) in t, we have

$$W_{ttt} + p'(\underline{v})W_{txx} + \frac{\underline{\alpha}_1}{1+t}W_{tt} - \frac{\underline{\alpha}_1}{(1+t)^2}W_t = 0. \tag{2.26}$$

By multiplying (2.26) by $(1+t)^{\underline{\alpha}_1}W_{tt}$ and integrating it over $\mathbf{R}$, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left[(1+t)^{\underline{\alpha}_1}(W_{tt}^2 - p'(\underline{v})W_{tx}^2) - \underline{\alpha}_1(1+t)^{\underline{\alpha}_1-2}W_t^2\right]dx\\ &+\frac{\underline{\alpha}_1}{2}\int(1+t)^{\underline{\alpha}_1-1}(W_{tt}^2 - p'(\underline{v})W_{tx}^2)dx\\ &+\frac{\underline{\alpha}_1(\underline{\alpha}_1-2)}{2}\int(1+t)^{\underline{\alpha}_1-3}W_t^2dx = 0.\end{aligned} \tag{2.27}$$

Then, by multiplying (2.26) by $(1+t)^{\underline{\alpha}_1-1}W_t$ and integrating it over $\mathbf{R}$, we have

$$\begin{aligned}&\frac{d}{dt}\int\left[(1+t)^{\underline{\alpha}_1-1}W_{tt}W_t + \frac{1}{2}(1+t)^{\underline{\alpha}_1-2}W_t^2\right]dx - \int(1+t)^{\underline{\alpha}_1-1}p'(\underline{v})W_{tx}^2dx\\ &-\int\left((1+t)^{\underline{\alpha}_1-1}W_{tt}^2 + \frac{3\underline{\alpha}_1-2}{2}(1+t)^{\underline{\alpha}_1-3}W_t^2\right)dx = 0.\end{aligned} \tag{2.28}$$

Hence, by calculating $\frac{2}{\underline{\alpha}_1}\cdot(2.27)+(2.28)$, integrating it over $[0, t]$, applying Cauchy's inequality and (2.25), one gets

$$(1+t)^{\underline{\alpha}_1}\|W_{tt}\|^2 + (1+t)^{\underline{\alpha}_1}\|W_{tx}\|^2 \le CE_0^2, \tag{2.29}$$

which, together with (2.21), yields

$$(1+t)^{\underline{\alpha}_1}\|W_{xx}\|^2 \le C(1+t)^{\underline{\alpha}_1}\left(\|W_{tt}\|^2 + (1+t)^{-2}\|W_t\|^2\right) \le CE_0^2. \tag{2.30}$$

Furthermore, by a similar calculation to (2.26)–(2.30), we can get the higher order estimate as follows

$$\sum_{i+j=3}(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^jW\|^2 \le CE_0^2. \tag{2.31}$$

When $2 < \underline{\alpha}_1 < 4$, we can obtain estimates for W and its derivatives from reference [10].

When $[\frac{\underline{\alpha}_1}{2}] = 2$, we can get the estimates of W, as well as its first - and second - order derivatives, from reference [10]. However, for its third - order derivatives, we have a better estimates compared to the results in the reference [10]. Taking the derivative of (2.21) with respect to t twice, we obtain

$$W_{tttt} + p'(\underline{v})W_{ttxx} + \frac{\underline{\alpha}_1}{1+t}W_{ttt} - 2\frac{\underline{\alpha}_1}{(1+t)^2}W_{tt} + 2\frac{\underline{\alpha}_1}{(1+t)^3}W_t = 0. \tag{2.32}$$

Multiply (2.32) by $(1+t)^{\underline{\alpha}_1}W_{ttt}$ and integrate with respect to x, and we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\Big[(1+t)^{\underline{\alpha}_1}(W_{ttt}^2 - p'(\underline{v})W_{ttx}^2) - 2\underline{\alpha}_1(1+t)^{\underline{\alpha}_1-2}W_{tt}^2\Big]dx \\
&+\frac{d}{dt}\int\Big(2\underline{\alpha}_1(1+t)^{\underline{\alpha}_1-3}W_{tt}W_t - \underline{\alpha}_1(\underline{\alpha}_1-3)(1+t)^{\underline{\alpha}_1-4}W_t^2\Big)dx \\
&+\frac{\underline{\alpha}_1}{2}\int(1+t)^{\underline{\alpha}_1-1}(W_{ttt}^2 + p'(\underline{v})W_{ttx}^2)dx + \underline{\alpha}_1(\underline{\alpha}_1-4)\int(1+t)^{\underline{\alpha}_1-3}W_{tt}^2dx \\
&+\underline{\alpha}_1(\underline{\alpha}_1-3)(\underline{\alpha}_1-4)\int(1+t)^{\underline{\alpha}_1-5}W_t^2dx = 0.
\end{aligned} \tag{2.33}$$

Then, by multiplying (2.32) by $(1+t)^{\underline{\alpha}_1-1}W_{tt}$ and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\Big[(1+t)^{\underline{\alpha}_1-1}W_{ttt}W_{tt} + \frac{1}{2}(1+t)^{\underline{\alpha}_1-2}W_{tt}^2 + \underline{\alpha}_1(1+t)^{\underline{\alpha}_1-4}W_t^2\Big]dx \\
&-\int\Big((1+t)^{\underline{\alpha}_1-1}\Big(W_{ttt}^2 + p'(\underline{v})W_{ttx}^2\Big) + \frac{2-5\underline{\alpha}_1}{2}(1+t)^{\underline{\alpha}_1-3}W_{tt}^2\Big)dx \\
&-\int\underline{\alpha}_1(\underline{\alpha}_1-4)(1+t)^{\underline{\alpha}_1-5}W_t^2dx = 0.
\end{aligned} \tag{2.34}$$

Hence, by calculating $\frac{2}{\underline{\alpha}_1}\cdot(2.33) + (2.34)$ and integrating it over $[0,t]$, one gets

$$(1+t)^{\underline{\alpha}_1}\|W_{ttt}\|^2 + (1+t)^{\underline{\alpha}_1}\|W_{ttx}\|^2 \le CE_0^2. \tag{2.35}$$

By differentiating (2.21) in x, we have

$$W_{ttx} + p'(\underline{v})W_{xxx} + \frac{\underline{\alpha}_1}{1+t}W_{tx} = 0. \tag{2.36}$$

From (2.26), (2.35) and (2.36), we can obtain

$$(1+t)^{\underline{\alpha}_1}\Big(\|W_{txx}\|^2 + \|W_{xxx}\|^2\Big) \le CE_0^2. \tag{2.37}$$

Referring to our previous proof, we obtain the estimates for ϕ and its derivatives in the case of $\left[\frac{\underline{\alpha}_1}{2}\right] \ge 3$, and thus omit the proof. □

Remark 2.1. If the initial values $(\Phi_0, \bar{u}_0')$ have higher-order derivatives and these derivatives belong to $L^2(\mathbf{R})$, then by using the method in proof of Theorem 2.1, we can conduct a more detailed partition of $\underline{\alpha}_1$ and obtain the corresponding decay estimates.

Corollary 2.1. *Under the assumptions of Theorem 2.1, for* $2 < \underline{\alpha}_1 < 4$*, we have*

$$\begin{aligned}
\|(\bar{v}-\underline{v})(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{1}{2}},\\
\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} &\le CE_0(1+t)^{-(\frac{1}{2}+\frac{\alpha_1}{4})},\\
\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{\alpha_1}{2}};
\end{aligned} \tag{2.38}$$

for $[\frac{\alpha_1}{2}]=2$*, we have*

$$\begin{aligned}
\|(\bar{v}-\underline{v})(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{1}{2}},\\
\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{3}{2}},\\
\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} &\le CE_0(1+t)^{-1-\frac{\alpha_1}{4}};
\end{aligned} \tag{2.39}$$

and for $[\frac{\alpha_1}{2}]\ge 3$*, we have*

$$\begin{aligned}
\|(\bar{v}-\underline{v})(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{1}{2}},\\
\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{3}{2}},\\
\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} &\le CE_0(1+t)^{-\frac{5}{2}}.
\end{aligned} \tag{2.40}$$

We define $\delta_1 := |u_+|+|u_-|$ and $I_0 := \|\phi_0\|_{H^4}+\|\phi_1\|_{H^3}+\delta_1 \ll 1$.

Theorem 2.2 *(Property of asymptotic profiles* $(\bar{v},\bar{u})(x,t)$ *in Case 2). Suppose* $\|\phi_0\|^2_{H^4}+\|\phi_1\|^2_{H^3}$ *and* $|u_+|+|u_-|$ *are sufficiently small. Thus, the Cauchy problem (2.11) has a unique global smooth solution* $\bar{v}(x,t)$*, which fulfills the following:* $(i\ge 0,\ j\ge 0)$

- *for* $2<\underline{\alpha}_1<\frac{8}{3}$*, then*

$$(1+t)^{\frac{3\alpha_1}{2}-2}\|(\bar{v}-\underline{v},\bar{u})(t)\|^2+\sum_{1\le i+j\le 3}(1+t)^{\underline{\alpha}_1}\|(\partial_x^i\partial_t^j\bar{v},\partial_x^i\partial_t^j\bar{u})(t)\|^2\le CI_0^2. \tag{2.41}$$

- *for* $\underline{\alpha}_1=\frac{8}{3}$*, then*

$$\frac{(1+t)^2}{\ln(2+t)}\|(\bar{v}-\underline{v},\bar{u})(t)\|^2+\sum_{1\le i+j\le 3}(1+t)^{\underline{\alpha}_1}\|(\partial_x^i\partial_t^j\bar{v},\partial_x^i\partial_t^j\bar{u})(t)\|^2\le CI_0^2. \tag{2.42}$$

- *for* $\frac{8}{3}<\underline{\alpha}_1<4$*, then*

$$(1+t)^2\|(\bar{v}-\underline{v},\bar{u})(t)\|^2+\sum_{1\le i+j\le 3}(1+t)^{\underline{\alpha}_1}\|(\partial_x^i\partial_t^j\bar{v},\partial_x^i\partial_t^j\bar{u})(t)\|^2\le CI_0^2. \tag{2.43}$$

- *for* $[\frac{\alpha_1}{2}]=2$, *then*

$$\sum_{0\le i+j\le 1}(1+t)^{2(i+j+1)}\|(\partial_x^i\partial_t^j(\bar v-\underline{v}),\partial_x^i\partial_t^j\bar u)(t)\|^2 + \sum_{2\le i+j\le 3}(1+t)^{\underline{\alpha}_1}\|(\partial_x^i\partial_t^j\bar v,\partial_x^i\partial_t^j\bar u)(t)\|^2\le CI_0^2. \tag{2.44}$$

- *for* $[\frac{\alpha_1}{2}]=3$, *then*

$$\sum_{0\le i+j\le 2}(1+t)^{2(i+j+1)}\|(\partial_x^i\partial_t^j(\bar v-\underline{v}),\partial_x^i\partial_t^j\bar u)(t)\|^2 + \sum_{i+j=3}(1+t)^{\underline{\alpha}_1}\|(\partial_x^i\partial_t^j\bar v,\partial_x^i\partial_t^j\bar u)(t)\|^2\le CI_0^2. \tag{2.45}$$

- *for* $[\frac{\alpha_1}{2}]\ge 4$, *then*

$$\sum_{0\le i+j\le 3}(1+t)^{2(i+j+1)}\|(\partial_x^i\partial_t^j(\bar v-\underline{v}),\partial_x^i\partial_t^j\bar u)(t)\|^2\le CI_0^2. \tag{2.46}$$

For the sake of our subsequent proof, we present the following a priori assumption:

- when $2<\underline{\alpha}_1<4$, then

$$N(T):=\sup_{0\le t\le T}\{(1+t)^{\frac{\underline{\alpha}_1}{2}}\|\phi_{tx}\|\}\le\delta,$$

- when $4\le\underline{\alpha}_1$, then

$$N(T):=\sup_{0\le t\le T}\{(1+t)^2\|\phi_{tx}\|\}\le\delta,$$

for some $\delta\ll 1$ and $0<T<\infty$.

Proof. When $2<\underline{\alpha}_1<4$, multiplying (2.16) by $(1+t)^2\phi_t$ and integrating it respect to x and t, we have

$$\begin{aligned}&\frac{1}{2}\int(1+t)^2(\phi_t^2-p'(\underline{v})\phi_x^2)dx+(\underline{\alpha}_1-1)\iint_0^t(1+\tau)\phi_t^2dxd\tau\\&+\iint_0^t(1+\tau)p'(\underline{v})W_x^2dxd\tau\\&\le C\left(\|\phi_0\|_{H^3}^2+\|\phi_1\|_{H^2}^2\right)-\iint_0^t(1+\tau)^2p'(\underline{v})\hat v_x\phi_tdxd\tau.\end{aligned} \tag{2.47}$$

Then, multiplying (2.16) by $(1+t)\phi$ and integrating it respect to x and t, we obtain

$$\frac{1}{2}\int\left[2(1+t)\phi\phi_t+(\underline{\alpha}_1-1)\phi^2\right]dx-\iint_0^t(1+\tau)p'(\underline{v})\phi_x^2dxd\tau-\iint_0^t(1+t)\phi_t^2dxd\tau$$

$$\leq C\left(\|\phi_0\|_{H^3}^2+\|\phi_1\|_{H^2}^2\right)-\iint_0^t(1+\tau)p'(\underline{v})\hat{v}_x\phi dxd\tau. \tag{2.48}$$

By calculating $\frac{\underline{\alpha}_1}{2(\underline{\alpha}_1-1)}\cdot(2.47)+(2.48)$ and through a discussion similar to [10], we can derive that

$$\frac{\underline{\alpha}_1-2}{8(\underline{\alpha}_1-1)}\int(1+t)^2\phi_t^2-\frac{\underline{\alpha}_1}{4(\underline{\alpha}_1-1)}\int(1+t)^2p'(\underline{v})\phi_x^2dx$$

$$+\frac{(\underline{\alpha}_1-1)(\underline{\alpha}_1-2)}{2(\underline{\alpha}_1+2)}\int\phi^2dx+\frac{\underline{\alpha}_1-2}{2}\iint_0^t(1+\tau)\phi_t^2dxd\tau$$

$$-\frac{\underline{\alpha}_1-2}{2(\underline{\alpha}_1-1)}\iint_0^t(1+\tau)p'(\underline{v})\phi_x^2dxd\tau$$

$$\leq-\iint_0^t\left(\frac{\underline{\alpha}_1}{2(\underline{\alpha}_1-1)}(1+\tau)^2p'(\underline{v})\hat{v}_x\phi_t+(1+\tau)p'(\underline{v})\hat{v}_x\phi\right)dxd\tau$$

$$+C\left(\|\phi_0\|_1^2+\|\phi_1\|^2\right). \tag{2.49}$$

Using Cauchy's inequality, we obtain

$$-\iint_0^t(1+\tau)p'(\underline{v})\hat{v}_x\phi dxd\tau=\iint_0^t(1+\tau)p'(\underline{v})\hat{v}\phi_xdxd\tau$$

$$\leq\delta_1\iint_0^t(1+\tau)\phi_x^2dxd\tau+\frac{C}{\delta_1}\iint_0^t(1+\tau)\hat{v}^2dxd\tau$$

$$\leq\delta_1\iint_0^t(1+\tau)\phi_x^2dxd\tau+CI_0. \tag{2.50}$$

Also, using Hölder's inequality and the a priori assumption gives

$$-\iint_0^t(1+\tau)^2p'(\underline{v})\hat{v}_x\phi_tdxd\tau=\iint_0^t(1+\tau)^2p'(\underline{v})\hat{v}\phi_{tx}dxd\tau$$

$$\le C\int_0^t (1+\tau)^2\|\phi_{tx}\|\|\hat{v}\|d\tau \le C\delta_1\int_0^t (1+\tau)^{3-\frac{3\underline{\alpha}_1}{2}}d\tau. \tag{2.51}$$

Hence, from (2.49)–(2.51), for $4-\frac{3\underline{\alpha}_1}{2}>0$,

$$\begin{aligned}&(1+t)^{\frac{3\underline{\alpha}_1}{2}-4}\|\phi\|^2+(1+t)^{\frac{3\underline{\alpha}_1}{2}-2}\left(\|\phi_t\|^2+\|\phi_x\|^2\right)\\&\quad+(1+t)^{\frac{3\underline{\alpha}_1}{2}-4}\int_0^t(1+\tau)\left(\|\phi_t\|^2+\|\phi_x\|^2\right)d\tau\\&\le CI_0^2,\end{aligned} \tag{2.52}$$

for $4-\frac{3\underline{\alpha}_1}{2}=0$,

$$\begin{aligned}&\|\phi\|^2+(1+t)^2\left(\|\phi_t\|^2+\|\phi_x\|^2\right)+\int_0^t(1+\tau)\left(\|\phi_t\|^2+\|\phi_x\|^2\right)d\tau\\&\le C\ln(2+t)I_0^2,\end{aligned} \tag{2.53}$$

and for $4-\frac{3\underline{\alpha}_1}{2}>0$,

$$\|\phi\|^2+(1+t)^2\left(\|\phi_t\|^2+\|\phi_x\|^2\right)+\int_0^t(1+\tau)\left(\|\phi_t\|^2+\|\phi_x\|^2\right)d\tau\le CI_0^2. \tag{2.54}$$

Next, we compute the decay estimates for ϕ_{xx}, ϕ_{tx} and ϕ_{tt}. By computing $\frac{2}{\underline{\alpha}_1}(1+t)^{\underline{\alpha}_1}\phi_{tt}\cdot\partial_t(2.16)+(1+t)^{\underline{\alpha}_1-1}\phi_t\cdot\partial_t(2.16)$, integrating it respect to x and t and applying Cauchy's inequality, we obtain

$$\begin{aligned}&(1+t)^{\underline{\alpha}_1}\left(\frac{1}{2\underline{\alpha}_1}\|\phi_{tt}\|^2-\frac{p'(\underline{v})}{\underline{\alpha}_1}\|\phi_{tx}\|^2\right)-\frac{\underline{\alpha}_1+1}{2}(1+t)^{\underline{\alpha}_1-2}\|\phi_t\|^2\\&\le-\iint_0^t\left(\frac{2}{\underline{\alpha}_1}(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{tx}\phi_{tt}+\underline{\alpha}_1(1+\tau)^{\underline{\alpha}_1-1}p'(\underline{v})\hat{v}_{tx}\phi_t\right)dxd\tau\\&\quad+C\left(\|\phi_0\|_3^2+\|\phi_1\|_2^2\right).\end{aligned} \tag{2.55}$$

From (2.51), (2.52) and (2.53), we have

$$-\iint_0^t\frac{2}{\underline{\alpha}_1}(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{tx}\phi_{tt}dxd\tau=-\iint_0^t\frac{d}{dt}\left(\frac{2}{\underline{\alpha}_1}(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{tx}\phi_t\right)dxd\tau$$

$$+\iint_0^t \left(\frac{2}{\underline{\alpha}_1}(1+\tau)^{\underline{\alpha}_1} p'(\underline{v})\hat{v}_{ttx}\phi_t + 2(1+\tau)^{\underline{\alpha}_1-1} p'(\underline{v})\hat{v}_{tx}\phi_t\right) dx d\tau$$

$$\leq C\int_0^t \left((1+t)^{\underline{\alpha}_1}\|\hat{v}_{ttx}\|\|\phi_t\| + (1+t)^{\underline{\alpha}_1-1}\|\hat{v}_{tx}\|\|\phi_t\|\right) d\tau$$

$$+ C\left(\|\phi_0\|_3^2 + \|\phi_1\|_2^2 + (1+t)^{\underline{\alpha}_1}|\|\hat{v}_{tx}\|\|\phi_t\|\right) \leq C I_0^2. \tag{2.56}$$

Similarly, we have

$$-\iint_0^t \underline{\alpha}_1(1+\tau)^{\underline{\alpha}_1-1} p'(\underline{v})\hat{v}_{tx}\phi_t dx d\tau \leq C I_0^2. \tag{2.57}$$

Hence, we have

$$(1+t)^{\underline{\alpha}_1}\left(\|\phi_{tt}\|^2 + \|\phi_{tx}\|^2\right) \leq C I_0^2. \tag{2.58}$$

Furthermore, from (2.16), we obtain

$$\phi_{xx} = -\frac{1}{p'(\underline{v})}\left(p'(\underline{v})\hat{v}_x + \phi_{tt} + \frac{\underline{\alpha}_1}{1+t}\phi_t\right), \tag{2.59}$$

or

$$\|\phi_{xx}\|^2 \leq C\left(\|\hat{v}_x\|^2 + \|\phi_{tt}\|^2 + (1+t)^{-2}\|\phi_t\|^2\right) \leq C I_0^2(1+t)^{-\underline{\alpha}_1}, \tag{2.60}$$

which, when combined with (2.58), shows the decay estimates for ϕ_{xx}, ϕ_{tx} and ϕ_{tt}.

Similar to the previous discussion, we can also obtain:

$$(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j\phi\| \leq C I_0^2,\ i+j = 3, 4.$$

When $[\frac{\underline{\alpha}_1}{2}] = 2$, similar to the previous discussion, the estimates for ϕ, its first derivatives, and its second derivatives are as follows:

$$(1+t)^{2(i+j)}\|\partial_x^i\partial_t^j\phi\| \leq C I_0^2,\ 0 \leq i+j \leq 2. \tag{2.61}$$

In order to obtain the decay estimate for the third derivative of ϕ, we will carry out the following steps. First, taking two derivatives of (2.16) with respect to t, we obtain

$$\phi_{tttt} + p'(\underline{v})\phi_{ttxx} + \frac{\underline{\alpha}_1}{1+t}\phi_{ttt} - 2\frac{\underline{\alpha}_1}{(1+t)^2}\phi_{tt} + 2\frac{\underline{\alpha}_1}{(1+t)^3}\phi_t = -p'(\underline{v})\hat{v}_{ttx}. \tag{2.62}$$

Then, multiplying (2.62) by $(1+t)^{\underline{\alpha}_1}\phi_{ttt}$ and integrating with respect to x and t, one obtains

$$\begin{aligned}
&\frac{1}{2}\int\left[(1+t)^{\underline{\alpha}_1}(\phi_{ttt}^2-p'(\underline{v})\phi_{ttx}^2)-2\underline{\alpha}_1(1+t)^{\underline{\alpha}_1-2}\phi_{tt}^2\right]dx\\
&\quad+\int\left(2\underline{\alpha}_1(1+t)^{\underline{\alpha}_1-3}\phi_{tt}\phi_t-\underline{\alpha}_1(\underline{\alpha}_1-3)(1+t)^{\underline{\alpha}_1-4}\phi_t^2\right)dx\\
&\quad+\frac{\underline{\alpha}_1}{2}\iint_0^t(1+\tau)^{\underline{\alpha}_1-1}(\phi_{ttt}^2-p'(\underline{v})\phi_{ttx}^2)dxd\tau\\
&\quad+\underline{\alpha}_1(\underline{\alpha}_1-4)\iint_0^t(1+\tau)^{\underline{\alpha}_1-3}\phi_{tt}^2dxd\tau\\
&\quad+\underline{\alpha}_1(\underline{\alpha}_1-3)(\underline{\alpha}_1-4)\iint_0^t(1+t)^{\underline{\alpha}_1-5}\phi_t^2dxd\tau\\
&\le C\left(\|\phi_0\|_3^2+\|\phi_1\|_2^2\right)-\iint_0^t(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{ttx}\phi_{ttt}dxd\tau\\
&\le C\left(\|\phi_0\|_3^2+\|\phi_1\|_2^2\right)-\int_0^t\frac{d}{dt}\int(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{ttx}\phi_{tt}dxd\tau\\
&\quad+\iint_0^t\left((1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{tttx}\phi_{tt}+\underline{\alpha}_1(1+\tau)^{\underline{\alpha}_1-1}p'(\underline{v})\hat{v}_{ttx}\phi_{tt}\right)dxd\tau. \qquad (2.63)
\end{aligned}$$

Also, multiplying (2.62) by $(1+t)^{\underline{\alpha}_1-1}\phi_{tt}$ and integrating with respect to x and t, one obtains

$$\begin{aligned}
&\int\left[(1+t)^{\underline{\alpha}_1-1}\phi_{ttt}\phi_{tt}+\frac{1}{2}(1+t)^{\underline{\alpha}_1-2}\phi_{tt}^2+\underline{\alpha}_1(1+t)^{\underline{\alpha}_1-4}\phi_t^2\right]dx\\
&\quad-\iint_0^t\left((1+\tau)^{\underline{\alpha}_1-1}\left(\phi_{ttt}^2+p'(\underline{v})\phi_{ttx}^2\right)+\frac{-5\underline{\alpha}_1}{2}(1+\tau)^{\underline{\alpha}_1-3}\phi_t^2\right)dx\\
&\quad-\iint_0^t\underline{\alpha}_1(\underline{\alpha}_1-4)(1+\tau)^{\underline{\alpha}_1-5}\phi_t^2dxd\tau\\
&\le C\left(\|\phi_0\|_3^2+\|\phi_1\|_2^2\right)-\iint_0^t(1+\tau)^{\underline{\alpha}_1}p'(\underline{v})\hat{v}_{tx}\phi_{tt}dxd\tau. \qquad (2.64)
\end{aligned}$$

By calculating $\frac{2}{\underline{\alpha}_1}\cdot(2.63)+(2.64)$ and through a discussion similar to the previous one, we can derive the following:

$$(1+t)^{\underline{\alpha}_1}\left(\|\phi_{ttt}\|^2+\|\phi_{ttx}\|^2\right)\le CI_0^2. \tag{2.65}$$

Differentiate (2.16) in t:

$$\phi_{ttt}+p'(\underline{v})\phi_{txx}+\frac{\underline{\alpha}_1}{1+t}\phi_{tt}-\frac{\underline{\alpha}_1}{(1+t)^2}\phi_t=-p'(\underline{v})\hat{v}_{tt}. \tag{2.66}$$

Differentiate (2.16) in x:

$$\phi_{ttx}+p'(\underline{v})\phi_{xxx}+\frac{\underline{\alpha}_1}{1+t}\phi_{tx}=-p'(\underline{v})\hat{v}_{tx}. \tag{2.67}$$

Therefore, from (2.65)–(2.67), we obtain

$$(1+t)^{\underline{\alpha}_1}\left(\|\phi_{txx}\|^2+\|\phi_{xxx}\|^2\right)\le CI_0^2. \tag{2.68}$$

By using a similar method, we can also obtain:

$$(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j\phi\|^2\le CI_0^2,$$

for $i+j=4$.

Referring to our previous proof, we can obtain the estimates for ϕ and its derivatives in the case of $[\frac{\underline{\alpha}_1}{2}]\ge 3$, and thus omit the proof.

Then, due to (2.13) and $\underline{\alpha}_1>2$, we obtain

$$\sum_{0\le i+j\le 3}\|(\partial_x^i\partial_t^j\hat{v},\partial_x^i\partial_t^j\hat{u})(t)\|^2\le CI_0^2(1+t)^{2\underline{\alpha}_1-2}\le CI_0^2(1+t)^{\underline{\alpha}_1}, \tag{2.69}$$

which, along with the estimates of ϕ and its derivatives of various orders, gives (2.41)–(2.46). □

Remark 2.2. If the initial values (ϕ_0,ϕ_1) have higher-order derivatives and these derivatives belong to $L^2(\mathbf{R})$, then by using the method in proof of Theorem 2.2, we can conduct a more detailed partition of $\underline{\alpha}_1$ and obtain the corresponding decay estimates.

Corollary 2.2. *Under the assumptions of Theorem 2.2, for $2<\underline{\alpha}_1<\frac{8}{3}$, we have*

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty}+\|\bar{u}(t)\|_{L^\infty}\le CI_0(1+t)^{\frac{1}{2}-\frac{5\underline{\alpha}_1}{8}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty}+\|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty}\le CI_0(1+t)^{-\frac{\underline{\alpha}_1}{2}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty}+\|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty}\le CI_0(1+t)^{-\frac{\underline{\alpha}_1}{2}};
\end{aligned} \tag{2.70}$$

for $\underline{\alpha}_1=\frac{8}{3}$, we have

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty}+\|\bar{u}(t)\|_{L^\infty}\le CI_0\ln^{\frac{1}{4}}(2+t)(1+t)^{-\frac{1}{2}-\frac{\underline{\alpha}_1}{4}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty}+\|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty}\le CI_0(1+t)^{-\frac{\underline{\alpha}_1}{2}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty}+\|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty}\le CI_0(1+t)^{-\frac{\underline{\alpha}_1}{2}};
\end{aligned} \tag{2.71}$$

for $\frac{8}{3} < \underline{\alpha}_1 < 4$, *we have*

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty} + \|\bar{u}(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{1}{2}-\frac{\alpha_1}{4}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} + \|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{\alpha_1}{2}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} + \|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{\alpha_1}{2}};
\end{aligned}\tag{2.72}$$

for $[\frac{\alpha_1}{2}] = 2$, *we have*

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty} + \|\bar{u}(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{3}{2}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} + \|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty} \le CI_0(1+t)^{-1-\frac{\alpha_1}{4}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} + \|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{\alpha_1}{2}};
\end{aligned}\tag{2.73}$$

for $[\frac{\alpha_1}{2}] = 3$, *we have*

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty} + \|\bar{u}(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{3}{2}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} + \|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{5}{2}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} + \|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{3}{2}-\frac{\alpha_1}{4}};
\end{aligned}\tag{2.74}$$

and for $[\frac{\alpha_1}{2}] \ge 4$, *we have*

$$\begin{aligned}
&\|(\bar{v}-\underline{v})(t)\|_{L^\infty} + \|\bar{u}(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{3}{2}},\\
&\|(\bar{v}_x,\bar{v}_t)(t)\|_{L^\infty} + \|(\bar{u}_x,\bar{u}_t)(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{5}{2}},\\
&\|(\bar{v}_{xx},\bar{v}_{tx})(t)\|_{L^\infty} + \|(\bar{u}_{tt},\bar{u}_{tx},\bar{u}_{xx})(t)\|_{L^\infty} \le CI_0(1+t)^{-\frac{7}{2}}.
\end{aligned}\tag{2.75}$$

3. Main results: convergence to asymptotic profiles

We are going to reformulate the problem (1.1) for Cases 1, 2, and 3, and present the corresponding main theorems for each.

Case 1: From (2.1) and (2.4)

$$\begin{cases}
(v-\bar{v})_t - (u-\bar{u})_x = 0,\\
(u-\bar{u})_t + (p(v)-p(\bar{v}))_x + \frac{\alpha_1}{1+t}(u-\bar{u})\\
= -\frac{\alpha_1-\underline{\alpha}_1}{1+t}\bar{u} - \big(p'(\bar{v}) - p'(\underline{v})\big)\bar{v}_x,
\end{cases}\tag{3.1}$$

where $(\bar{v},\bar{u})(x,t)$ is given by (2.8) and (2.9). We define

$$V(x,t) = \int_{-\infty}^{x} (v-\bar{v})(y,t)dy,$$

and from the first equation of (3.1), we have $V_t(x,t) = (u-\bar{u})(x,t)$. Since $u-\bar{u} \to 0$ as $x \to \pm\infty$ is expected, by integrating the first equation of (3.1) with respect to x and t, one gives

$$\int_{-\infty}^{+\infty} (v-\bar{v})(x,t)dx = \int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx - \kappa \int_{-\infty}^{+\infty} \Phi_0(x)dx = 0.$$

Thus, we may expect $V(\cdot,t) \in H^1$, and the second equation of (3.1) can be rewritten as

$$\begin{cases} V_{tt} + (p(\bar{v}+V_x) - p(\bar{v}))_x + \frac{\alpha_1}{1+t}V_t = F_1 + F_2 =: F, \\ V(x,0) = V_0(x), \\ V_t(x,0) = V_1(x), \end{cases} \tag{3.2}$$

where

$$F_1 := -\frac{\alpha_1 - \underline{\alpha}_1}{1+t}\bar{u}, \tag{3.3}$$

$$F_2 := -\left(p'(\bar{v}) - p'(\underline{v})\right)\bar{v}_x, \tag{3.4}$$

$$V_0(x) := \int_{-\infty}^{x} (v-\bar{v})(y,0)dy, \tag{3.5}$$

$$V_1(x) := (u-\bar{u})(x,0). \tag{3.6}$$

The following is the main results of case 1.

Lemma 3.1 *(Estimates of F). For $t \in \mathbf{R}_+$, the solution $\bar{v}(x,t)$ of (2.8), and $\bar{u}(x,t)$ defined by (2.9), it holds*

- *when $2 < \underline{\alpha}_1 < 4$, then*

$$\begin{cases} |F_1| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{4}-\frac{1}{2}}, \ \|F_2(t)\| \le CE_0(1+t)^{-\frac{3}{2}}, \\ |F_{1t}| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{4}-\frac{3}{2}}, \ \|F_{2t}(t)\| \le CE_0(1+t)^{-\frac{\underline{\alpha}_1+1}{2}}, \\ |F_{1tt}| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{2}-1}, \ \|F_{2tt}(t)\| \le CE_0(1+t)^{-\frac{\underline{\alpha}_1+1}{2}}. \end{cases} \tag{3.7}$$

- *when $[\frac{\underline{\alpha}_1}{2}] = 2$, then*

$$\begin{cases} |F_1| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{3}{2}}, \ \|F_2(t)\| \le CE_0(1+t)^{-\frac{3}{2}}, \\ |F_{1t}| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5}{2}}, \ \|F_{2t}(t)\| \le CE_0(1+t)^{-\frac{5}{2}}, \\ |F_{1tt}| \le CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{4}-2}, \ \|F_{2tt}(t)\| \le CE_0(1+t)^{-\frac{\underline{\alpha}_1+1}{2}}. \end{cases} \tag{3.8}$$

- *when* $[\frac{\alpha_1}{2}] \geq 3$, *then*

$$\begin{cases} |F_1| \leq CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{3}{2}},\ \|F_2(t)\| \leq CE_0(1+t)^{-\frac{3}{2}}, \\ |F_{1t}| \leq CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5}{2}},\ \|F_{2t}(t)\| \leq CE_0(1+t)^{-\frac{5}{2}}, \\ |F_{1tt}| \leq CE_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{7}{2}},\ \|F_{2tt}(t)\| \leq CE_0(1+t)^{-\frac{7}{2}}. \end{cases} \tag{3.9}$$

Proof. From (2.9), (3.3) and (3.4), we obtain

$$\begin{aligned} |F_1| &= \left|-\frac{\alpha_1 - \underline{\alpha}_1}{1+t}\bar{u}\right| \leq |\alpha_1 - \underline{\alpha}_1|(1+t)^{-1}|\bar{u}|_\infty \\ &\leq |\alpha_1 - \underline{\alpha}_1|(1+t)^{-1}\Bigg(|(1+t)^{-\underline{\alpha}_1}\bar{u}_0(x)| \\ &\quad + |(1+t)^{-\underline{\alpha}_1}\int_0^t (1+s)^{\underline{\alpha}_1} p'(\underline{v})\bar{v}_x(x,s)ds|\Bigg) \\ &\leq C|\alpha_1 - \underline{\alpha}_1|\left((1+t)^{-\underline{\alpha}_1-1}|\bar{u}_0(x)| + |\bar{v}_x|\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} |F_{1t}| &= \left|-\frac{\alpha_1 - \underline{\alpha}_1}{1+t}\bar{u}_t + (\alpha_1 - \underline{\alpha}_1)(1+t)^{-2}\bar{u}\right| \\ &\leq |\alpha_1 - \underline{\alpha}_1|\left((1+t)^{-1}|\bar{u}_t|_\infty + (1+t)^{-2}|\bar{u}|_\infty\right) \\ &\leq |\alpha_1 - \underline{\alpha}_1|\Bigg((1+t)^{-1}|\underline{\alpha}_1(1+t)^{-\underline{\alpha}_1-1}\bar{u}_0(x)| \\ &\quad + |\underline{\alpha}_1(1+t)^{-\underline{\alpha}_1-2}\int_0^t (1+s)^{\underline{\alpha}_1} p'(\underline{v})\bar{v}_x(x,s)ds| \\ &\quad + (1+t)^{-1}|p'(\underline{v})\bar{v}_x| + (1+t)^{-2}|\bar{u}|_\infty\Bigg), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|F_2\| &= \| - \left(p'(\bar{v}) - p'(\underline{v})\right)\bar{v}_x\| \leq C\||\bar{v} - \underline{v}|\bar{v}_x\| \\ &\leq C|\bar{v} - \underline{v}|_\infty\|\bar{v}_x\|, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \|F_{2t}\| &= \| - \left(p'(\bar{v}) - p'(\underline{v})\right)\bar{v}_{tx} - p''(\bar{v})\bar{v}_t\bar{v}_x\| \\ &\leq C\left(\||\bar{v} - \underline{v}|\bar{v}_{tx}\| + \|\bar{v}_t\bar{v}_x\|\right) \leq C\left(|\bar{v} - \underline{v}|_\infty\|\bar{v}_{tx}\| + |\bar{v}_t|_\infty\|\bar{v}_x\|\right). \end{aligned} \tag{3.13}$$

Consequently, based on Theorem 2.1, Corollary 2.1 and (3.10)–(3.13), the estimations of $|F_1|$, $|F_{1t}|$, $\|F_2(t)\|$ and $\|F_{2t}(t)\|$ can be derived. The estimation of $|F_{1tt}|$ and $\|F_{2tt}(t)\|$ can be obtained through a discussion similar to the above, so we omit it. Therefore, the proof is complete. □

Theorem 3.1 *(Convergence to asymptotic profiles $(\bar{v}, \bar{u})(x,t)$ in Case 1). Assume* (2.5)*, $i \geq 0$, and $j \geq 0$. When $\|V_0\|_3^2 + \|V_1\|_2^2$ and $\|\Phi_0\|_3 + \|u_0'\|_2 + |u_+| + |u_-|$ are sufficiently small, the Cauchy problem of* (3.11) *admits a unique global smooth solution (V_x, V_t) satisfying*

- *for $2 < \underline{\alpha}_1 < 3$, then*

$$(V_x, V_t) \in C(0, \infty; L^2(\mathbf{R})),$$

and

$$(1+t)\|\,(V_x, V_t)\,\|^2 \leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + \|\Phi_0\|_3 + \|u_0'\|_2 + |u_+| + |u_-|\right). \tag{3.14}$$

- *for $3 \leq \underline{\alpha}_1 < 4$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$\begin{aligned}&(1+t)\|\,(V_x, V_t)\,\|^2 + \sum_{2\leq i+j\leq 3}(1+t)^{\underline{\alpha}_1 - 1}\|\partial_x^i\partial_t^j V\|^2\\ &\leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + \|\Phi_0\|_3 + \|u_0'\|_2 + |u_+| + |u_-|\right).\end{aligned} \tag{3.15}$$

- *for $[\frac{\underline{\alpha}_1}{2}] = 2$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$\begin{aligned}&(1+t)\|\,(V_x, V_t)\,\|^2 + \sum_{i+j=2}(1+t)^3\|\partial_x^i\partial_t^j V\|^2\\ &\quad + \sum_{i+j=3, i\neq 3}(1+t)^{\underline{\alpha}_1 - 1}\|\partial_x^i\partial_t^j V\|^2 + (1+t)^3\|V_{xxx}\|^2\\ &\leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + \|\Phi_0\|_3 + \|u_0'\|_2 + |u_+| + |u_-|\right).\end{aligned} \tag{3.16}$$

- *for $[\frac{\underline{\alpha}_1}{2}] \geq 3$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$
\begin{aligned}
&(1+t)\|(V_x,V_t)\|^2+\sum_{i+j=2}(1+t)^3\|\partial_x^i\partial_t^j V\|^2\\
&+\sum_{i+j=3,i\neq 3}(1+t)^5\|\partial_x^i\partial_t^j V\|^2+(1+t)^3\|V_{xxx}\|^2\\
&\le C\left(\|V_0\|_3^2+\|V_1\|_2^2+\|\Phi_0\|_3+\|u_0'\|_2+|u_+|+|u_-|\right).
\end{aligned}
\tag{3.17}
$$

Corollary 3.1. *Let* $(\bar{v},\bar{u})$ *be the derived asymptotic profile in* (2.8) *and* (2.9)*. Under the assumptions of Theorem 3.1, one has*

- *for* $3\le \underline{\alpha}_1<4$*, then*

$$
\begin{aligned}
\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}&=O(1)(1+t)^{-\frac{\alpha_1}{4}},\\
\sum_{i+j=1}\|\partial_x^i\partial_t^j(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}&=O(1)(1+t)^{-\frac{\alpha_1-1}{2}}.
\end{aligned}
$$

- *for* $[\frac{\alpha_1}{2}]=2$*, then*

$$
\begin{aligned}
\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}&=O(1)(1+t)^{-1},\\
\sum_{i+j=1}\|\partial_x^i\partial_t^j(u-\bar{u})(t)\|_{L^\infty}+\|(v-\bar{v})_t(t)\|_{L^\infty}&=O(1)(1+t)^{-\frac{\alpha_1+2}{4}},\\
\|(v-\bar{v})_x(t)\|_{L^\infty}&=O(1)(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

- *for* $[\frac{\alpha_1}{2}]\ge 3$*, then*

$$
\begin{aligned}
\|(v-\bar{v},u-\bar{u})(t)\|_{L^\infty}&=O(1)(1+t)^{-1},\\
\sum_{i+j=1}\|\partial_x^i\partial_t^j(u-\bar{u})(t)\|_{L^\infty}+\|(v-\bar{v})_t(t)\|_{L^\infty}&=O(1)(1+t)^{-2},\\
\|(v-\bar{v})_x(t)\|_{L^\infty}&=O(1)(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

Case 2: From (2.1) and (2.4)

$$
\begin{cases}
(v-\bar{v})_t-(u-\bar{u})_x=0,\\
(u-\bar{u})_t+(p(v)-p(\bar{v}))_x+\frac{\alpha_1}{1+t}(u-\bar{u})\\
=-\frac{\alpha_1-\underline{\alpha}_1}{1+t}\bar{u}-\left(p'(\bar{v})-p'(\underline{v})\right)\bar{v}_x,
\end{cases}
\tag{3.18}
$$

where $(\bar{v},\bar{u})(x,t)$ is given by (2.8) and (2.9). We define

$$
V(x,t)=\int_{-\infty}^{x}(v-\bar{v})(y,t)dy,
$$

and from the first equation of (3.1), we have $V_t(x,t) = (u-\bar{u})(x,t)$. Since $u-\bar{u} \to 0$ as $x \to \pm\infty$ is expected, by integrating the first equation of (3.1) with respect to x and t, one gives

$$\int_{-\infty}^{+\infty} (v-\bar{v})(x,t)dx = \int_{-\infty}^{+\infty} [v_0(x) - \bar{v}_0(x)]dx = 0.$$

Thus, we may expect $V(\cdot,t) \in H^1$, and the second equation of (3.1) can be rewritten as

$$\begin{cases} V_{tt} + (p(\bar{v}+V_x) - p(\bar{v}))_x + \frac{\alpha_1}{1+t} V_t = G_1 + G_2, \\ V(x,0) = V_0(x), \\ V_t(x,0) = V_1(x), \end{cases} \tag{3.19}$$

where

$$G_1 := -\frac{\alpha_1 - \underline{\alpha}_1}{1+t}\bar{u}, \tag{3.20}$$

$$G_2 := -\left(p'(\bar{v}) - p'(\underline{v})\right)\bar{v}_x, \tag{3.21}$$

$$V_0(x) := \int_{-\infty}^{x} (v-\bar{v})(y,0)dy = 0, \tag{3.22}$$

$$V_1(x) := (u-\bar{u})(x,0) = 0. \tag{3.23}$$

The following is the main results of case 2.

Lemma 3.2 *(Estimates of G). For $t \in \mathbf{R}_+$, and the solution $(\bar{v},\bar{u})(x,t)$ of (2.11), it holds*

- *when $2 < \underline{\alpha}_1 < \frac{8}{3}$, then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5\underline{\alpha}_1}{8}-\frac{1}{2}}, \ \|G_2(t)\| \le CI_0(1+t)^{-\frac{9\underline{\alpha}_1}{8}+\frac{1}{2}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{2}-1}, \ \|G_{2t}(t)\| \le CI_0(1+t)^{-\frac{9\underline{\alpha}_1}{8}+\frac{1}{2}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\underline{\alpha}_1}{2}-1}, \ \|G_{2tt}(t)\| \le CI_0(1+t)^{-\frac{9\underline{\alpha}_1}{8}+\frac{1}{2}}. \end{cases} \tag{3.24}$$

- *when $\underline{\alpha}_1 = \frac{8}{3}$, then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|\frac{\ln^{\frac{1}{4}}(2+t)}{(1+t)^{\frac{13}{6}}}, \ \|G_2(t)\| \le CI_0\frac{\ln^{\frac{1}{4}}(2+t)}{(1+t)^{\frac{5}{2}}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{7}{3}}, \ \|G_{2t}(t)\| \le CI_0\frac{\ln^{\frac{1}{4}}(2+t)}{(1+t)^{\frac{5}{2}}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{7}{3}}, \ \|G_{2tt}(t)\| \le CI_0\frac{\ln^{\frac{1}{4}}(2+t)}{(1+t)^{\frac{5}{2}}}. \end{cases} \tag{3.25}$$

- *when* $\frac{8}{3} < \underline{\alpha}_1 < 4$, *then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{4}-\frac{3}{2}}, \ \|G_2(t)\| \le CI_0(1+t)^{-\frac{3\alpha_1}{4}-\frac{1}{2}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{2}-1}, \ \|G_{2t}(t)\| \le CI_0(1+t)^{-\frac{3\alpha_1}{4}-\frac{1}{2}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{2}-1}, \ \|G_{2tt}(t)\| \le CI_0(1+t)^{-\frac{3\alpha_1}{4}-\frac{1}{2}}. \end{cases} \tag{3.26}$$

- *when* $[\frac{\alpha_1}{2}] = 2$, *then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5}{2}}, \ \|G_2(t)\| \le CI_0(1+t)^{-\frac{7}{2}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{4}-2}, \ \|G_{2t}(t)\| \le CI_0(1+t)^{-\frac{\alpha_1}{2}-\frac{3}{2}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{2}-1}, \ \|G_{2tt}(t)\| \le CI_0(1+t)^{-\frac{\alpha_1}{2}-\frac{3}{2}}. \end{cases} \tag{3.27}$$

- *when* $[\frac{\alpha_1}{2}] = 3$, *then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5}{2}}, \ \|G_2(t)\| \le CI_0(1+t)^{-\frac{7}{2}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{7}{2}}, \ \|G_{2t}(t)\| \le CI_0(1+t)^{-\frac{9}{2}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{\alpha_1}{4}-\frac{5}{2}}, \ \|G_{2tt}(t)\| \le CI_0(1+t)^{-\frac{\alpha_1}{2}-\frac{3}{2}}. \end{cases} \tag{3.28}$$

- *when* $[\frac{\alpha_1}{2}] \ge 4$, *then*

$$\begin{cases} |G_1| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{5}{2}}, \ \|G_2(t)\| \le CI_0(1+t)^{-\frac{7}{2}}, \\ |G_{1t}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-\frac{7}{2}}, \ \|G_{2t}(t)\| \le CI_0(1+t)^{-\frac{9}{2}}, \\ |G_{1tt}| \le CI_0|\alpha_1 - \underline{\alpha}_1|(1+t)^{-5}, \ \|G_{2tt}(t)\| \le CI_0(1+t)^{-\frac{11}{2}}. \end{cases} \tag{3.29}$$

Using the method similar to that for proving Lemma 3.1, we can prove Lemma 3.2. Therefore, we omit the proof of Lemma 3.2.

Theorem 3.2 *(Convergence to asymptotic profiles $(\bar{v}, \bar{u})(x,t)$ in Case 2). Assume (2.5), $i \ge 0$, and $j \ge 0$. When $\|V_0\|_3^2 + \|V_1\|_2^2$ and I_0 are sufficiently small, the Cauchy problem of (3.19) admits a unique global smooth solution (V_x, V_t) satisfying*

- *when* $2 < \underline{\alpha}_1 < \frac{20}{9}$, *then*

$$(V_x, V_t) \in C(0, \infty; L^2(\mathbf{R})),$$

and

$$(1+t)^{\frac{4\alpha_1}{5}-1}\|(V_x, V_t)\|^2 \le C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + I_0\right). \tag{3.30}$$

- *when* $\frac{20}{9} \le \underline{\alpha}_1 < \frac{12}{5}$, *then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$(1+t)^{\frac{5\underline{\alpha}_1}{4}-1}\|(V_x,V_t)\|^2+\sum_{2\le i+j\le 3}(1+t)^{\frac{9\underline{\alpha}_1}{4}-3}\|\partial_x^i\partial_t^j V\|^2\le C\left(\|V_0\|_3^2+\|V_1\|_2^2+I_0\right). \tag{3.31}$$

- *when* $\frac{12}{5}\le\underline{\alpha}_1\le 4$*, then*

$$(V_x,V_t)\in C(0,\infty;H^2(\mathbf{R})),$$

and

$$(1+t)^2\|(V_x,V_t)\|^2+\sum_{2\le i+j\le 3}\frac{(1+t)^{\underline{\alpha}_1}}{\ln^2(2+t)}\|\partial_x^i\partial_t^j V\|^2\le C\left(\|V_0\|_3^2+\|V_1\|_2^2+I_0\right). \tag{3.32}$$

- *when* $4<\underline{\alpha}_1<6$*, then*

$$(V_x,V_t)\in C(0,\infty;H^2(\mathbf{R})),$$

and

$$\begin{aligned}&(1+t)^2\|(V_x,V_t)\|^2+\sum_{i+j=2}(1+t)^4\|\partial_x^i\partial_t^j V\|^2\\&+\sum_{i+j=3,i\neq 3}(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j V\|^2+(1+t)^4\|V_{xxx}\|^2\\&\le C\left(\|V_0\|_3^2+\|V_1\|_2^2+I_0\right).\end{aligned} \tag{3.33}$$

- *when* $[\frac{\underline{\alpha}_1}{2}]\ge 3$*, then*

$$(V_x,V_t)\in C(0,\infty;H^2(\mathbf{R})),$$

and

$$\begin{aligned}&(1+t)^2\|(V_x,V_t)\|^2+\sum_{i+j=2}(1+t)^4\|\partial_x^i\partial_t^j V\|^2\\&+\sum_{i+j=3,i\neq 3}(1+t)^6\|\partial_x^i\partial_t^j V\|^2+(1+t)^5\|V_{xxx}\|^2\\&\le C\left(\|V_0\|_3^2+\|V_1\|_2^2+I_0\right).\end{aligned} \tag{3.34}$$

Corollary 3.2. *Let* $(\bar{v},\bar{u})$ *be the derived asymptotic profile in* (2.12)*. Under the assumptions of Theorem* 3.2*, one has*

- *when* $\frac{20}{9} \leq \underline{\alpha}_1 < \frac{12}{5}$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{7\alpha_1}{8}+1},$$
$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{9\alpha_1}{8}+\frac{3}{2}}.$$

- *when* $\frac{12}{5} \leq \underline{\alpha}_1 \leq 4$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)\ln^{\frac{1}{2}}(2+t)(1+t)^{\frac{\alpha_1+2}{4}},$$
$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)\ln(2+t)(1+t)^{-\frac{\alpha_1}{2}}.$$

- *when* $4 < \underline{\alpha}_1 < 6$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}},$$
$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} = O(1)(1+t)^{-1-\frac{\alpha_1}{4}},$$
$$\|(v-\bar{v})_x(t)\|_{L^\infty} = O(1)(1+t)^{-2}.$$

- *when* $[\frac{\alpha_1}{2}] \geq 3$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}},$$
$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{5}{2}},$$
$$\|(v-\bar{v})_x(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{9}{4}}.$$

Remark 3.1. In fact, in the proof of Section 5, for $\frac{5}{12} < \underline{a}_1 \leq 4$, by modifying the prior assumption to:

$$N_2(T_2) := \sup_{0\leq t\leq T_2} \{\|V\| + (1+t)\|(V_t, V_x)\| + \sum_{2\leq i+j\leq 3} (1+t)^{\frac{\alpha_1}{2}} \|\partial_x^i \partial_t^j V\| \leq \delta$$

and through a similar discussion, we can actually obtain

$$\sum_{2\leq i+j\leq 3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V\|^2 \leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + I_0\right),$$
$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{\frac{\alpha_1+2}{4}}$$

and

$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{\alpha_1}{2}}.$$

Case 3: From (2.1), (2.4) and (2.18),

$$\begin{cases} (v-\bar{v})_t - (u-\bar{u})_x = 0, \\ (u-\bar{u})_t + p'(v)v_x + \frac{\alpha_1}{1+t}(u-\bar{u}) = -\frac{\alpha_1-\underline{\alpha}_1}{1+t}\bar{u}. \end{cases} \tag{3.35}$$

We define

$$V(x,t) = \int_{-\infty}^{x} (v-\bar{v})(y,t)dy = \int_{-\infty}^{x} \big(v(y,t) - \underline{v}\big)\, dy,$$

and from the first equation of (3.35), we have $V_t(x,t) = (u-\bar{u})(x,t)$. Since $u - \bar{u} \to 0$ as $x \to \pm\infty$ is expected, by integrating the first equation of (3.35) with respect to x and t, one gives

$$\int_{-\infty}^{+\infty} (v-\bar{v})(x,t)dx = \int_{-\infty}^{+\infty} [v_0(x) - \underline{v}]dx = 0.$$

Thus, we may expect $V(\cdot,t) \in H^1$, and the second equation of (3.35) can be rewritten as

$$\begin{cases} V_{tt} + \big(p(\underline{v}+V_x) - p(\underline{v})\big)_x + \frac{\alpha_1}{1+t}V_t = J, \\ V(x,0) = V_0(x), \\ V_t(x,0) = V_1(x), \end{cases} \tag{3.36}$$

where

$$J := -\frac{\alpha_1 - \underline{\alpha}_1}{1+t}\bar{u}, \tag{3.37}$$

$$V_0(x) := \int_{-\infty}^{x} (v(y,0) - \bar{v})\, dy, \tag{3.38}$$

$$V_1(x) := (u-\bar{u})(x,0). \tag{3.39}$$

The following is the main results of Case 3.

Lemma 3.3 *(Estimates of J). For $t \in \mathbf{R}_+$, when $(\bar{v},\bar{u})(x,t)$ satisfies (2.18), the following holds*

$$\begin{cases} \|J\| \le C|\underline{u}|(1+t)^{-1-\underline{\alpha}_1}, \ \|J_x\| \le C|\underline{u}|(1+t)^{-1-\underline{\alpha}_1}, \\ \|J_t\| \le C|\underline{u}|(1+t)^{-2-\underline{\alpha}_1}, \ \|J_{tt}\| \le C|\underline{u}|(1+t)^{-3-\underline{\alpha}_1}. \end{cases} \tag{3.40}$$

Theorem 3.3 *(Convergence to asymptotic profiles $(\bar{v}, \bar{u})(x,t)$ in Case 3). Assume (2.5), $i \geq 0$, and $j \geq 0$. When $\|V_0\|_3^2 + \|V_1\|_2^2$ and $|\underline{u}|$ are sufficiently small, the Cauchy problem of (3.36) admits a unique global smooth solution (V_x, V_t) satisfying*

- *for $2 < \underline{\alpha}_1 < 4$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$\begin{aligned}(1+t)^2\|(V_x, V_t)\|^2 + \sum_{2\leq i+j\leq 3}(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j V\|^2 \\ \leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + |\underline{u}|\right).\end{aligned} \tag{3.41}$$

- *for $[\frac{\underline{\alpha}_1}{2}] = 2$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$\begin{aligned}(1+t)^2\|(V_x, V_t)\|^2 + \sum_{i+j=2}(1+t)^4\|\partial_x^i\partial_t^j V\|^2 \\ + \sum_{i+j=3, i\neq 3}(1+t)^{\underline{\alpha}_1}\|\partial_x^i\partial_t^j V\|^2 + (1+t)^4\|V_{xxx}\|^2 \\ \leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + |\underline{u}|\right).\end{aligned} \tag{3.42}$$

- *for $[\frac{\underline{\alpha}_1}{2}] \geq 3$, then*

$$(V_x, V_t) \in C(0, \infty; H^2(\mathbf{R})),$$

and

$$\begin{aligned}(1+t)^2\|(V_x, V_t)\|^2 + \sum_{i+j=2}(1+t)^4\|\partial_x^i\partial_t^j V\|^2 \\ + \sum_{i+j=3, i\neq 3}(1+t)^6\|\partial_x^i\partial_t^j V\|^2 + (1+t)^5\|V_{xxx}\|^2 \\ \leq C\left(\|V_0\|_3^2 + \|V_1\|_2^2 + |\underline{u}|\right).\end{aligned} \tag{3.43}$$

Corollary 3.3. *Let $(\bar{v}, \bar{u})$ be the derived asymptotic profile in (2.18). Under the assumptions of Theorem 3.3, one has*

- *for* $2 < \underline{\alpha}_1 < 4$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{\underline{\alpha}_1+2}{4}},$$

$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{\underline{\alpha}_1}{2}}.$$

- *for* $[\frac{\underline{\alpha}_1}{2}] = 2$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}},$$

$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{\underline{\alpha}_1}{4}-1},$$

$$\|(v-\bar{v})_x(t)\|_{L^\infty} = O(1)(1+t)^{-2}.$$

- *for* $[\frac{\underline{\alpha}_1}{2}] \geq 3$, *then*

$$\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{3}{2}},$$

$$\sum_{i+j=1} \|\partial_x^i \partial_t^j (u-\bar{u})(t)\|_{L^\infty} + \|(v-\bar{v})_t(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{5}{2}},$$

$$\|(v-\bar{v})_x(t)\|_{L^\infty} = O(1)(1+t)^{-\frac{9}{4}}.$$

4. Decay estimates for Case 1

The main goal of this section is to obtain the decay rates of the solution of (3.2) and its derivatives, and thereby derive the decay estimate for the solution of the original problem. For some $\delta \ll 1$ and $0 < T_1 < \infty$, we are dedicated to estimating the solution of (3.2) under the following a priori assumptions:

- when $2 < \underline{\alpha}_1 < 3$,

$$N_1(T_1) := \sup_{0 \leq t \leq T_1} \{(1+t)^{-\frac{1}{2}} \|V\| + (1+t)^{\frac{1}{2}} \|(V_t, V_x)\|\} \leq \delta.$$

- when $3 \leq \underline{\alpha}_1 < 4$,

$$N_1(T_1) := \sup_{0 \leq t \leq T_1} \{(1+t)^{-\frac{1}{2}} \|V\| + (1+t)^{\frac{1}{2}} \|(V_t, V_x)\| + (1+t)^{\frac{\underline{\alpha}_1-1}{2}} \|(V_{tt}, V_{tx}, V_{tx})\|$$
$$+ (1+t)^{\frac{\underline{\alpha}_1-1}{2}} \|(V_{ttt}, V_{ttx}, V_{txx}, V_{xxx})\|\} \leq \delta.$$

- when $[\frac{\underline{\alpha}_1}{2}] = 2$,

$$N_1(T_1) := \sup_{0 \leq t \leq T_1} \{(1+t)^{-\frac{1}{2}} \|V\| + (1+t)^{\frac{1}{2}} \|(V_t, V_x)\| + (1+t)^{\frac{3}{2}} \|(V_{tt}, V_{tx}, V_{tx})\|$$

$$+(1+t)^{\frac{\alpha_1-1}{2}}\|(V_{ttt},V_{ttx},V_{txx})\|+(1+t)^{\frac{3}{2}}\|V_{xxx}\|\}\le\delta.$$

- when $[\frac{\alpha_1}{2}]\ge 3$,

$$N_1(T_1):=\sup_{0\le t\le T_1}\{(1+t)^{-\frac{1}{2}}\|V\|+(1+t)^{\frac{1}{2}}\|(V_t,V_x)\|+(1+t)^{\frac{3}{2}}\|(V_{tt},V_{tx},V_{tx})\|$$
$$+(1+t)^{\frac{5}{2}}\|(V_{ttt},V_{ttx},V_{txx})\|+(1+t)^{\frac{3}{2}}\|V_{xxx}\|\}\le\delta.$$

Before we start our proof, we present the following definition:

$$E_1:=\|V_0\|_3^2+\|V_1\|_2^2+E_0.$$

Lemma 4.1. *Under the assumptions of Theorem 3.1, it holds that*

$$\|V(t)\|^2+(1+t)^2\|(V_x,V_t)(t)\|^2+\int_0^t(1+s)\|(V_x,V_t)(s)\|^2ds$$
$$\le CE_1(1+t). \tag{4.1}$$

By multiplying equation (3.2) by $(1+t)^2V_t$, integrating it over $\mathbf{R}$, and applying integration by parts, we obtain:

$$\frac{d}{dt}\int(1+t)^2\frac{V_t^2}{2}dx+\int(1+t)^2\left(p(\bar v+V_x)-p(\bar v)\right)_xV_tdx$$
$$+\int\Big((1+t)\alpha_1(x)V_t^2-(1+t)V_t^2\Big)dx$$
$$\le\int(1+t)^2V_tFdx. \tag{4.2}$$

Through further calculations, we can obtain

$$\frac{d}{dt}\int\left((1+t)^2\frac{V_t^2}{2}+\int_0^{V_x}(1+t)^2\left(p(\bar v)-p(\bar v+s)\right)ds\right)dx$$
$$+\int(1+t)\left(\alpha_1(x)-1\right)V_t^2dx-\int C(1+t)^2|\bar v_t|_\infty V_x^2dx$$
$$-\iint_0^{V_x}2(1+t)\left(p(\bar v)-p(\bar v+s)\right)dsdx$$
$$\le\int(1+t)^2V_tFdx, \tag{4.3}$$

due to

$$
\begin{aligned}
&\int \left(p(\bar{v}+V_x)-p(\bar{v})\right)_x (1+t)^2\phi_t dx \\
&= \int \frac{d}{dt}\int_0^{V_x}(1+t)^2\left(p(\bar{v})-p(\bar{v}+s)\right)dsdx \\
&\quad + \int (1+t)^2\left(p(\bar{v}+V_x)-p(\bar{v})-p'(\bar{v})V_x\right)\bar{v}_t \\
&- \iint_0^{V_x} 2(1+t)\left(p(\bar{v})-p(\bar{v}+s)\right)dsdx \\
&\geq \int \frac{d}{dt}\int_0^{V_x}(1+t)^2\left(p(\bar{v})-p(\bar{v}+s)\right)dsdx \\
&\quad - \int C(1+t)^2|\bar{v}_t|_\infty V_x^2 dx \\
&- \iint_0^{V_x} 2(1+t)\left(p(\bar{v})-p(\bar{v}+s)\right)dsdx.
\end{aligned}
$$

Additionally, multiplying (3.2) by $(1+t)V$ and integrating it over $\mathbf{R}$, we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\int \left((1+t)V_tV+\alpha_1(x)\frac{V^2}{2}-\frac{1}{2}V^2\right)dx \\
&\quad + \int (1+t)\left(p(\bar{v})-p(\bar{v}+V_x)\right)V_x dx - \int (1+t)V^2 dx \\
&\leq \int (1+t)VFdx.
\end{aligned}
\tag{4.4}
$$

Notice that

$$
\begin{aligned}
&\int_0^{V_x}\left(p(\bar{v})-p(\bar{v}+s)\right)ds = p(\bar{v})V_x - \int_0^{V_x} p(\bar{v}+s)ds \\
&= p(\bar{v}+V_x)V_x - p'(\bar{v}+V_x)V_x^2 - \int_0^{V_x} p(\bar{v}+s)ds + \int_0^{V_x} p(\bar{v}+s)ds \\
&\quad - p(\bar{v}+V_x)V_x + p'(\bar{v}+V_x)\frac{V_x^2}{2} + o(V_x^2) \\
&= -p'(\bar{v}+V_x)\frac{V_x^2}{2} + o(V_x^2),
\end{aligned}
$$

and

$$(p(\bar{v})-p(\bar{v}+V_x))\,V_x=\frac{-p'(\bar{v}+V_x)}{2}V_x^2+o(V_x^2).$$

Hence, we see

$$\iint_0^{V_x} -2\,(p(\bar{v})-p(\bar{v}+s))\,dsdx \geq \frac{\underline{\alpha}_1+4}{6}\int p'(\bar{v}+V_x)V_x^2dx, \tag{4.5}$$

and

$$\int (p(\bar{v})-p(\bar{v}+V_x))\,V_xdx \geq \frac{\underline{\alpha}_1+2}{2\underline{\alpha}_1}\int -p'(\bar{v}+V_x)V_x^2dx. \tag{4.6}$$

From (4.5), (4.6) and Corollary 2.1, by calculating $k\cdot(4.3)+(4.4)$, the following is obtained:

$$\begin{aligned}
&\frac{d}{dt}\int k\left((1+t)^2\frac{V_t^2}{2}+\int_0^{V_x}(1+t)^2\,(p(\bar{v})-p(\bar{v}+s))\,ds\right)dx\\
&+\frac{d}{dt}\int\left((1+t)V_tV+\alpha_1(x)\frac{V^2}{2}-\frac{1}{2}V^2\right)dx\\
&\left(\frac{\underline{\alpha}_1+2}{2\underline{\alpha}_1}-k\frac{\underline{\alpha}_1+4}{6}\right)\int -p'(\bar{v}+V_x)(1+t)V_x^2dx\\
&-\int CE_0(1+t)V_x^2dx+\int (k\,(\alpha_1(x)-1)-1)\,(1+t)V_t^2dx\\
&\leq \int k(1+t)^2V_tFdx+\int(1+t)VFdx.
\end{aligned} \tag{4.7}$$

Due to Cauchy's inequality, we have

$$\int(1+t)V_tVdx \leq \frac{1}{2\varepsilon}\int(1+t)^2V_t^2dx+\frac{\varepsilon}{2}\int V^2dx. \tag{4.8}$$

Therefore, we select ε and k that satisfy the following conditions

$$\begin{cases}\frac{\underline{\alpha}_1+2}{2\underline{\alpha}_1}-k\frac{\underline{\alpha}_1+4}{6}>0,\ k\,(\alpha_1(x)-1)-1>k\left(\underline{\alpha}_1-1\right)-1>0,\ k>0,\\ \frac{1}{\varepsilon}<k,\ \varepsilon<\underline{\alpha}_1-1<\alpha(x)-1.\end{cases}$$

Additionally, the following specific choices for ε and k can be made:

$$k=\frac{2}{\underline{\alpha}_1},\ \varepsilon=\frac{3\underline{\alpha}_1-2}{4}. \tag{4.9}$$

Then, form (4.9), by integrating (4.7) with respect to t, we obtain

$$
\begin{aligned}
&\int \left(\frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1(3\underline{\alpha}_1 - 2)}(1+t)^2 V_t^2 + \frac{1}{2\underline{\alpha}_1}(1+t)^2 V_x^2 + \frac{\underline{\alpha}_1 - 2}{8} V^2 \right) dx \\
&+ \iint_0^t -\frac{\underline{\alpha}_1 - 2}{6\underline{\alpha}_1} p'(\bar{v} + V_x)(1+s) V_x^2 dxds - \iint_0^t CE_0(1+t)V_x^2 dxds \\
&+ \iint_0^t \frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1}(1+s)V_t^2 dxds \\
&\le CE_1 + \iint_0^t \frac{2}{\underline{\alpha}_1}(1+s)^2 V_t F dxds + \iint_0^t (1+s) V F dxds. \qquad (4.10)
\end{aligned}
$$

Next, let's estimate the last two terms of (4.10). With Corollary 2.1, the a priori assumption, and Hölder's inequality taken into consideration, we can get

$$
\begin{aligned}
&\iint_0^t (1+s) V F_1 dxds \le \iint_0^t |(1+s) V F_1| dxds \\
&\le CE_0 \iint_0^t |\alpha_1 - \underline{\alpha}_1| (1+s)^{-\frac{\underline{\alpha}_1}{4} + \frac{1}{2}} |V| dxds \\
&\le CE_0 \int_0^t \|\alpha_1 - \underline{\alpha}_1\|_{L^1} (1+s)^{-\frac{\underline{\alpha}_1}{4} + \frac{1}{2}} |V|_\infty ds \\
&\le CE_1 (1+t)^{-\frac{\underline{\alpha}_1}{4} + \frac{3}{2}}, \qquad (4.11) \\
&\iint_0^t (1+s) V F_2 dxds \le \int_0^t (1+s)\|V\| \|F_2\| ds \\
&\le CE_1(1+t), \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&\iint_0^t (1+s)^2 V_t F_1 dxds = \int_0^t \frac{d}{ds} \int (1+s)^2 V F_1 dxds \\
&- \iint_0^t 2(1+s) V F_1 dxds - \iint_0^t (1+s)^2 V F_{1t} dxds \\
&\le \int (1+t)^2 V F_1 dx + \iint_0^t 2(1+s)|V||F_1| dxds
\end{aligned}
$$

$$
\begin{aligned}
&+CE_1+\iint_0^t (1+s)^2|V||F_{1t}|dxds\\
&\le CE_0\int_0^t \|\alpha_1-\underline{\alpha}_1\|_{L^1}(1+s)^{-\frac{\alpha_1}{4}+\frac{1}{2}}|V|_\infty ds+CE_1\\
&\le CE_1(1+t)^{-\frac{\alpha_1}{4}+\frac{3}{2}}, \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint_0^t (1+s)^2 V_t F_2 dxds \le \int_0^t (1+s)^2\|V_t\|\|F_2\|ds\\
&\le CE_1(1+t). \qquad (4.14)
\end{aligned}
$$

Hence, from (4.10)–(4.14) and the smallness of E_1, we obtain

$$
\begin{aligned}
&\int \left((1+t)^2V_t^2+(1+t)^2V_x^2+V^2\right)dx\\
&\quad+\iint_0^t (1+s)V_x^2dxds+\iint_0^t (1+s)V_t^2dxds\\
&\le CE_1(1+t), \qquad (4.15)
\end{aligned}
$$

which implies (4.1). This completes the proof of Lemma 4.1.

Next, we will estimate the decay estimates of $\|V_{xx}\|$, $\|V_{tx}\|$, and $\|V_{tt}\|$.

Lemma 4.2. *Under the assumptions of Theorem 3.1, for* $0<\theta<1$*, it holds that*

- *when* $3\le\underline{\alpha}_1<4$*, then*

$$
\begin{aligned}
&(1+t)^{\underline{\alpha}_1-\theta}\|(V_{xx},V_{tx},V_{tt})(t)\|^2+\int_0^t (1+s)^{\underline{\alpha}_1-1-\theta}\|(V_{xx},V_{tx},V_{tt})(s)\|^2ds\\
&\le CE_1(1+t)^{1-\theta}. \qquad (4.16)
\end{aligned}
$$

- *when* $[\frac{\underline{\alpha}_1}{2}]\ge 2$*, then*

$$
\begin{aligned}
&(1+t)^{4-\theta}\|(V_{xx},V_{tx},V_{tt})(t)\|^2+\int_0^t (1+s)^{3-\theta}\|(V_{xx},V_{tx},V_{tt})(s)\|^2ds\\
&\le CE_1(1+t)^{1-\theta}. \qquad (4.17)
\end{aligned}
$$

Proof. Differentiate (3.2) in t,

$$V_{ttt} + (1+t)^{-1}\alpha_1(x)V_{tt} + \left(p'(\bar{v}+V_x)V_{tx} + \left(p'(\bar{v}+V_x) - p'(\bar{v})\right)\bar{v}_t\right)_x - (1+t)^{-2}\alpha_1(x)V_t = F_t. \tag{4.18}$$

Multiplying (4.18) by $(1+t)^\beta V_{tt}$, integrating over $\mathbf{R}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^\beta\frac{V_{tt}^2}{2} - \frac{1}{2}(1+t)^{\beta-2}\alpha_1(x)V_t^2\right)dx \\
&+\int\Big(-\frac{\beta}{2}(1+t)^{\beta-1}V_{tt}^2 + (1+t)^{\beta-1}\alpha_1(x)V_{tt}^2 \\
&+\frac{(\beta-2)}{2}(1+t)^{\beta-3}\alpha_1(x)V_t^2 - p'(\bar{v}+V_x)(1+t)^\beta V_{ttx}V_{tx} \\
&+\left((p'(\bar{v}+V_x)-p'(\bar{v}))\bar{v}_t\right)_x(1+t)^\beta V_{tt}\Big)dx \\
&=\int(1+t)^\beta V_{tt}F_t dx.
\end{aligned} \tag{4.19}$$

Through further calculations, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^\beta\frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2}(1+t)^\beta V_{tx}^2 - \frac{1}{2}(1+t)^{\beta-2}\alpha_1(x)V_t^2\right)dx \\
&+\int\Big(-\frac{\beta}{2}(1+t)^{\beta-1}V_{tt}^2 + (1+t)^{\beta-1}\alpha_1(x)V_{tt}^2 \\
&+\frac{p'(\bar{v}+V_x)}{2}(1+t)^\beta\,(V_{tx}+\bar{v}_t)\,V_{tx}^2 + \frac{\beta}{2}p'(\bar{v}+V_x)(1+t)^{\beta-1}V_{tx}^2 \\
&+p''(\bar{v}+V_x)(1+t)^\beta\bar{v}_tV_{xx}V_{tx} + (p'(\bar{v}+V_x)-p'(\bar{v}))\bar{v}_{tx}(1+t)^\beta V_{tt} \\
&+\left(p''(\bar{v}+V_x)-p''(\bar{v})\right)\bar{v}_t\bar{v}_x(1+t)^\beta V_{tt} + \frac{(\beta-2)}{2}(1+t)^{\beta-3}\alpha_1(x)V_t^2\Big)dx \\
&=\int(1+t)^\beta V_{tt}F_t dx.
\end{aligned} \tag{4.20}$$

Next, multiplying (4.18) by $(1+t)^{\beta-1}V_t$, integrating it over $\mathbf{R}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\beta-1}V_{tt}V_t + (\alpha_1(x)-\beta+1)\,(1+t)^{\beta-2}\frac{V_t^2}{2}\right)dx \\
&-\int\Big((1+t)^{\beta-1}V_{tt}^2 + \alpha_1(x)(1+t)^{\beta-3}V_t^2 + \frac{(\beta-2)\,(\alpha_1(x)-\beta+1)}{2}(1+t)^{\beta-3}V_t^2 \\
&+p'(\bar{v}+V_x)(1+t)^{\beta-1}V_{tx}^2 + \left(p'(\bar{v}+V_x)-p'(\bar{v})\right)\bar{v}_t(1+t)^{\beta-1}V_{tx}\Big)dx \\
&=\int(1+t)^{\beta-1}V_tF_t dx.
\end{aligned} \tag{4.21}$$

For $3 \le \underline{\alpha}_1 < 4$, applying the a priori assumption, Corollary 2.1 and Hölder's inequality, we can derive

$$\begin{aligned}
&\int \Big(\frac{p'(\bar{v}+V_x)}{2}(1+t)^{\beta}\,(V_{tx}+\bar{v}_t)\,V_{tx}^2 + (p'(\bar{v}+V_x)-p'(\bar{v}))\bar{v}_{tx}(1+t)^{\beta}V_{tt} \\
&\quad + p''(\bar{v}+V_x)(1+t)^{\beta}\bar{v}_t V_{xx}V_{tx} + \big(p''(\bar{v}+V_x)-p''(\bar{v})\big)\,\bar{v}_t\bar{v}_x(1+t)^{\beta}V_{tt} \\
&\quad - \big(p'(\bar{v}+V_x)-p'(\bar{v})\big)\,\bar{v}_t(1+t)^{\beta-1}V_{tx}\Big)dx \\
&\le C|V_{tx}+\bar{v}_t|_\infty\|V_{tx}\|^2(1+t)^{\beta} + C|V_x|_\infty\|\bar{v}_{tx}\|\|V_{tt}\|(1+t)^{\beta} \\
&\quad + C|\bar{v}_t|_\infty\|V_{tx}\|\|V_{xx}\|(1+t)^{\beta} + C|V_x|_\infty|\bar{v}_t|_\infty\|\bar{v}_x\|\|V_{tt}\|(1+t)^{\beta} \\
&\quad + C|V_x|_\infty\|\bar{v}_t\|\|V_{tt}\|(1+t)^{\beta-1} \\
&\le C(\delta+E_0)(1+t)^{\beta-\frac{\underline{\alpha}_1-1}{2}}\|V_{tx}\|^2 + CE_1(1+t)^{\beta-\frac{5\underline{\alpha}_1}{4}+\frac{1}{2}} \\
&\quad + CE_1(1+t)^{\beta-\underline{\alpha}-1} + CE_1(1+t)^{\beta-\frac{3\underline{\alpha}_1}{4}-\frac{3}{2}}.
\end{aligned} \tag{4.22}$$

By calculating (4.21) $+\, k_1 \cdot$ (4.20), and applying (4.22) and Lemma 4.1, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int k_1\left((1+t)^{\beta}\frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2}(1+t)^{\beta}V_{tx}^2 - \frac{1}{2}(1+t)^{\beta-2}\alpha_1(x)V_t^2\right)dx \\
&\quad + \frac{d}{dt}\int\left((1+t)^{\beta-1}V_{tt}V_t + (\alpha_1(x)-\beta+1)\,(1+t)^{\beta-2}\frac{V_t^2}{2}\right)dx \\
&\quad + \int\left(k_1\left(\alpha_1(x)-\frac{\beta}{2}\right)-1\right)(1+t)^{\beta-1}V_{tt}^2dx \\
&\quad - \int\left(1-k_1\frac{\beta}{2}\right)p'(\bar{v}+V_x)(1+t)^{\beta-1}V_{tx}^2dx \\
&\le C\int(1+t)^{\beta-3}V_t^2dx + CE_1(1+t)^{\beta-\frac{5\underline{\alpha}_1}{4}+\frac{1}{2}} \\
&\quad + \int k_1(1+t)^{\beta}V_{tt}F_tdx + \int(1+t)^{\beta-1}V_tF_tdx \\
&\quad + C(\delta+E_0)(1+t)^{\beta-\frac{\underline{\alpha}_1-1}{2}}\|V_{tx}\|^2 \\
&\le CE_1\left((1+t)^{\beta-4} + (1+t)^{\beta-\frac{5\underline{\alpha}_1}{4}+\frac{1}{2}}\right) + \int k_1(1+t)^{\beta}V_{tt}F_tdx \\
&\quad + \int(1+t)^{\beta-1}V_tF_tdx + C(\delta+E_0)(1+t)^{\beta-\frac{\underline{\alpha}_1-1}{2}}\|V_{tx}\|^2.
\end{aligned} \tag{4.23}$$

Subsequently, considering the a priori assumption, Lemma 3.1, and Hölder's inequality, it follows that

$$\begin{aligned}
&\int(1+t)^{\beta-1}V_tF_{1t}dxds \le \int(1+t)^{\beta-1}|V_t||F_{1t}|dx \\
&\le CE_0\|\alpha_1-\underline{\alpha}_1\|_{L^1}(1+t)^{\beta-\frac{\underline{\alpha}_1}{4}-\frac{5}{2}}|V_t|_\infty
\end{aligned}$$

$$
\leq CE_0(1+t)^{\beta-\frac{\alpha_1}{2}-\frac{5}{2}}, \tag{4.24}
$$

$$
\begin{aligned}
&\int (1+t)^{\beta-1} V_t F_{2t} dx \leq (1+t)^{\beta-1} \|V_t\| \|F_{2t}\| \\
&\leq CE_1(1+t)^{\beta-\frac{\underline{\alpha}_1}{2}-2},
\end{aligned} \tag{4.25}
$$

$$
\begin{aligned}
&\int (1+t)^{\beta} V_{tt} F_{1t} dx ds \leq \int (1+t)^{\beta} |V_{tt}||F_{1t}| dx \\
&\leq CE_0(1+t)^{\beta-\frac{\alpha_1}{4}-\frac{3}{2}} \|\alpha_1 - \underline{\alpha}_1\| \|V_{tt}\| \\
&\leq CE_0(1+t)^{\beta-\frac{3\alpha_1}{4}-1}
\end{aligned} \tag{4.26}
$$

and

$$
\begin{aligned}
&\int (1+t)^{\beta} V_{tt} F_{2t} dx ds \leq (1+t)^{\beta} |\|V_{tt}\| \|F_{2t}\| \\
&\leq CE_1(1+t)^{\beta-\underline{\alpha}_1}.
\end{aligned} \tag{4.27}
$$

By using Cauchy's inequality and Lemma 4.1, we have

$$
\begin{aligned}
&\int k_1 \left((1+t)^{\beta} \frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2} (1+t)^{\beta} V_{tx}^2 - \frac{1}{2}(1+t)^{\beta-2} \alpha_1(x) V_t^2 \right) dx \\
&\quad + \int \left((1+t)^{\beta-1} V_{tt} V_t + (\alpha_1(x) - \beta + 1)(1+t)^{\beta-2} \frac{V_t^2}{2} \right) dx \\
&\quad - \int \left(\alpha_1(x) - \beta + 1 - \frac{1}{2k_1} - \frac{k_1}{2} \alpha_1(x) \right) (1+t)^{\beta-2} \frac{V_t^2}{2} dx \\
&\geq \int \left((k_1 - \frac{k_1}{2})(1+t)^{\beta} \frac{V_{tt}^2}{2} - \frac{k_1 p'(\bar{v}+V_x)}{2} (1+t)^{\beta} V_{tx}^2 \right) dx \\
&\quad + \int \left(\alpha_1(x) - \beta + 1 - \frac{1}{2k_1} - \frac{k_1}{2} \alpha_1(x) \right) (1+t)^{\beta-2} \frac{V_t^2}{2} dx \\
&\quad - \int \left(\alpha_1(x) - \beta + 1 - \frac{1}{2k_1} - \frac{k_1}{2} \alpha_1(x) \right) (1+t)^{\beta-2} \frac{V_t^2}{2} dx \\
&\geq \int \left((k_1 - \frac{k_1}{2})(1+t)^{\beta} \frac{V_{tt}^2}{2} - \frac{k_1 p'(\bar{v}+V_x)}{2} (1+t)^{\beta} V_{tx}^2 \right) dx.
\end{aligned} \tag{4.28}
$$

From (4.23)–(4.27), we have

$$
\begin{aligned}
&\frac{d}{dt} \int k_1 \left((1+t)^{\beta} \frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2} (1+t)^{\beta} V_{tx}^2 - \frac{1}{2}(1+t)^{\beta-2} \alpha_1(x) V_t^2 \right) dx \\
&\quad + \frac{d}{dt} \int \left((1+t)^{\beta-1} V_{tt} V_t + (\alpha_1(x) - \beta + 1)(1+t)^{\beta-2} \frac{V_t^2}{2} \right) dx
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{d}{dt}\int\left(\alpha_1(x)-\beta+1-\frac{1}{2k_1}-\frac{k_1}{2}\alpha_1(x)\right)(1+t)^{\beta-2}\frac{V_t^2}{2}dx\\
&+\int\Big(k_1\Big(\alpha_1(x)-\frac{\beta}{2}\Big)-1\Big)(1+t)^{\beta-1}V_{tt}^2dx\\
&-\int\Big(1-k_1\frac{\beta}{2}\Big)p'(\bar v+V_x)(1+t)^{\beta-1}V_{tx}^2dx\\
\le{}& CE_1\left((1+t)^{\beta-4}+(1+t)^{\beta-\frac{5\alpha_1}{4}+\frac12}+(1+t)^{\beta-\underline{\alpha}_1}+1\right)\\
&-\frac{d}{dt}\int\left(\alpha_1(x)-\beta+1-\frac{1}{2k_1}-\frac{k_1}{2}\alpha_1(x)\right)(1+t)^{\beta-2}\frac{V_t^2}{2}dx\\
&+C(\delta+E_0)(1+t)^{\beta-\frac{\underline{\alpha}_1-1}{2}}\|V_{tx}\|^2. \qquad (4.29)
\end{aligned}
$$

Hence, choosing $\beta=\underline{\alpha}_1-\theta$ and $k_1=\frac{2}{\underline{\alpha}_1}$, integrating (4.29) with respect to t and applying $\frac{\underline{\alpha}_1-1}{2}\ge 1$, one has

$$
\begin{aligned}
&\int\left(\frac{k_1}{4}(1+t)^{\underline{\alpha}_1-\theta}V_{tt}^2-\frac{k_1p'(\bar v+V_x)}{2}(1+t)^{\underline{\alpha}_1-\theta}V_{tx}^2\right)dx\\
&\quad+\theta\iint_0^t(1+s)^{\underline{\alpha}_1-1-\theta}V_{tt}^2dxds+\theta\iint_0^t(1+s)^{\underline{\alpha}_1-1-\theta}V_{tx}^2dxds\\
&\le CE_1(1+t)^{1-\theta}+C(\delta+E_0)\int_0^t(1+s)^{1-\theta-\frac{\underline{\alpha}_1-1}{2}}(1+s)^{\underline{\alpha}_1-1}\|V_{tx}\|^2ds\\
&\le CE_1(1+t)^{1-\theta}+C(\delta+E_0)(1+t)^{1-\theta}\sup_{0\le t\le T}(1+t)^{\underline{\alpha}_1-1}\|V_{tx}\|^2, \qquad (4.30)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int-\frac{k_1p'(\bar v+V_x)}{2}(1+t)^{\underline{\alpha}_1-1}V_{tx}^2dx\\
&\le C(\delta+E_0)\sup_{0\le t\le T}(1+t)^{\underline{\alpha}_1-1}\|V_{tx}\|^2+CE_1, \qquad (4.31)
\end{aligned}
$$

for any $0\le t\le T$. From (4.31), using the smallness of δ and E_1, we have

$$
(1+t)^{\underline{\alpha}_1-1}\|V_{tx}(t)\|^2\le CE_1, \qquad (4.32)
$$

for any $0\le t\le T$, which, together with (4.31), implies

$$
\begin{aligned}
&(1+t)^{\underline{\alpha}_1-\theta}\|(V_{tt},V_{tx})(t)\|^2+\int_0^t(1+s)^{\underline{\alpha}_1-1-\theta}\|(V_{tt},V_{tx})(s)\|^2ds\\
&\le CE_1(1+t)^{1-\theta}. \qquad (4.33)
\end{aligned}
$$

Next, from (3.2), we have

$$V_{tt} + (p'(\bar{v} + V_x) - p'(\bar{v}))\bar{v}_x + (1+t)^{-1}\alpha_1 V_t - F = -p'(\bar{v} + V_x)V_{xx}, \tag{4.34}$$

which shows that

$$\begin{aligned} \|V_{xx}\|^2 \sim & \|V_{tt}\|^2 + (1+t)^{-1+\frac{\alpha_1}{2}}\|V_x\|^2 + (1+t)^{-2}\|V_t\|^2 \\ & + E_0(1+t)^{-\frac{\alpha_1}{2}-1} + E_0(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{4.35}$$

Combining (4.33) and (4.35), we can derive

$$\begin{aligned} & (1+t)^{\underline{\alpha}_1-\theta}\|V_{xx}(t)\|^2 + \theta\int_0^t (1+s)^{\underline{\alpha}_1-1-\theta}\|V_{xx}(s)\|^2 ds \\ & \leq CE_1(1+t)^{1-\theta}, \end{aligned} \tag{4.36}$$

which, together with (4.33), gives (4.16).

For $[\frac{\alpha_1}{2}] \geq 2$, choosing $\beta = 4-\theta$, multiplying (4.18) by $\left(\frac{1}{2}(1+t)^{4-\theta}V_{tt} + (1+t)^{3-\theta}V_t\right)$ and using a proof method similar to the one mentioned above, we can obtain

$$\begin{aligned} & (1+t)^{4-\theta}\|(V_{tx}, V_{tt})(t)\|^2 + \theta\int_0^t (1+s)^{3-\theta}\|(V_{tx}, V_{tt})(s)\|^2 ds \\ & \leq CE_1(1+t)^{1-\theta}. \end{aligned} \tag{4.37}$$

Similar to the proof of (4.36), from (4.34) and (4.37), we obtain

$$\begin{aligned} & (1+t)^{4-\theta}\|V_{xx}(t)\|^2 + \theta\int_0^t (1+s)^{3-\theta}\|V_{xx}(s)\|^2 ds \\ & \leq CE_1(1+t)^{1-\theta}, \end{aligned} \tag{4.38}$$

which, together with (4.37), gives (4.17). This completes the proof of Lemma 4.2. □

Similar to the proof of Lemma 4.2, for the following lemma, we only present the framework of its proof.

Lemma 4.3. *Under the assumptions of Theorem 3.1, for* $0 < \theta < 1$, $i \geq 0$ *and* $j \geq 0$, *it holds that*

- *when* $3 \leq \underline{\alpha}_1 < 4$, *then*

$$\sum_{i+j=3} (1+t)^{\underline{\alpha}_1-\theta}\|\partial_x^i\partial_t^j V(t)\|^2 + \theta\int_0^t \sum_{i+j=3} (1+s)^{\underline{\alpha}_1-1-\theta}\|\partial_x^i\partial_t^j V(s)\|^2 ds$$

$$\le CE_1(1+t)^{1-\theta}. \tag{4.39}$$

- *when* $[\frac{\alpha_1}{2}]=2$*, then*

$$\sum_{i+j=3,i\neq 3}(1+t)^{\underline{\alpha_1}-\theta}\|\partial_x^i\partial_t^j V(t)\|^2+\theta\int_0^t\sum_{i+j=3,i\neq 3}(1+s)^{\underline{\alpha_1}-1-\theta}\|\partial_x^i\partial_t^j V(s)\|^2 ds$$

$$+(1+t)^{4-\theta}\|\partial_x^3 V(t)\|^2+\theta\int_0^t(1+s)^{3-\theta}\|\partial_x^3 V(s)\|^2 ds\le CE_1(1+t)^{1-\theta}. \tag{4.40}$$

- *when* $[\frac{\alpha_1}{2}]\ge 3$*, then*

$$\sum_{i+j=3,i\neq 3}(1+t)^{6-\theta}\|\partial_x^i\partial_t^j V(t)\|^2+\theta\int_0^t\sum_{i+j=3,i\neq 3}(1+s)^{5-\theta}\|\partial_x^i\partial_t^j V(s)\|^2 ds$$

$$+(1+t)^{4-\theta}\|\partial_x^3 V(t)\|^2+\theta\int_0^t(1+s)^{3-\theta}\|\partial_x^3 V(s)\|^2 ds\le CE_1(1+t)^{1-\theta}. \tag{4.41}$$

Proof. Differentiate (3.2) in x,

$$V_{ttx}+(1+t)^{-1}\alpha_1(x)V_{tx}+\left(p'(\bar v+V_x)V_{xx}+\left(p'(\bar v+V_x)-p'(\bar v)\right)\bar v_x\right)_x$$
$$+(1+t)^{-1}\alpha_{1x}(x)V_t=F_x. \tag{4.42}$$

It can be obtained from (4.18) and (4.42) that

$$\|V_{txx}\|^2\sim\|V_{ttt}\|^2+(1+t)^{-2}\|V_{tt}\|^2+(|\bar v_x|_\infty^2+|V_{xx}|_\infty^2)\|V_{tx}\|^2+|\bar v_t|_\infty^2\|V_{xx}\|^2$$
$$+(1+t)^{-4}\|V_t\|^2+\left(|\bar v_x|_\infty^2|\bar v_t|_\infty^2+|\bar v_{tx}|_\infty^2\right)\|V_x\|^2+\|F_t\|^2, \tag{4.43}$$

and

$$\|V_{xxx}\|^2\sim\|V_{ttx}\|^2+(1+t)^{-2}\|V_{tx}\|^2+(|\bar v_x|_\infty^2+|V_{xx}|_\infty^2)\|V_{xx}\|^2$$
$$+\left(|\bar v_x|_\infty^4+|\bar v_{xx}|_\infty^2\right)\|V_x\|^2+(1+t)^{-2}|V_t|_\infty^2+\|F_x\|^2. \tag{4.44}$$

First, let's estimate the decay estimates of V_{ttt} and V_{ttx}. Differentiate (3.2) in t,

$$V_{tttt}+(1+t)^{-1}\alpha_1(x)V_{ttt}+\left(p'(\bar v+V_x)V_{tx}+\left(p'(\bar v+V_x)-p'(\bar v)\right)\bar v_t\right)_{tx}$$
$$-2(1+t)^{-2}\alpha_1(x)V_{tt}+2(1+t)^{-3}\alpha_1(x)V_t=F_{tt}. \tag{4.45}$$

By calculating $\left((1+t)^{\beta_2}V_{ttt}\cdot(4.45)+k_2(1+t)^{\beta_2-1}V_{tt}\cdot(4.45)\right)$ with β_2 will be determined later, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\Big((1+t)^{\beta_2}\frac{V_{ttt}^2}{2}-\frac{p'(\bar v+V_x)}{2}(1+t)^{\beta_2}V_{ttx}^2+k_2(1+t)^{\beta_2-1}V_{ttt}V_{tt}\\
&+\frac{k_2\alpha_1(x)}{2}(1+t)^{\beta_2-2}V_{tt}^2-(1+t)^{\beta_2-2}\alpha_1(x)V_{tt}^2-\frac{k_2(\beta_2-1)}{2}(1+t)^{\beta_2-2}V_{tt}^2\\
&+2(1+t)^{\beta_2-3}\alpha_1(x)V_tV_{tt}+k_2(1+t)^{\beta_2-4}\alpha_1(x)V_t^2\\
&-(\beta_2-3)(1+t)^{\beta_2-4}\alpha_1(x)V_t^2\Big)\,dx+\int\Big(\Big((1+t)^{\beta_2-1}\alpha_1(x)\\
&-k_2(1+t)^{\beta_2-1}-\frac{\beta_2}{2}(1+t)^{\beta_2-1}\Big)V_{ttt}^2\\
&+\Big(-k_2p'(\bar v+V_x)(1+t)^{\beta_2-1}+\frac{p''(\bar v+V_x)}{2}(\bar v_t+V_{tx})(1+t)^{\beta_2}\\
&+\frac{\beta_2p'(\bar v+V_x)}{2}(1+t)^{\beta_2-1}\Big)V_{ttx}^2+\Big(-2k_2(1+t)^{\beta_2-3}\alpha_1(x)\\
&-k_2\frac{\beta_2-2}{2}(1+t)^{\beta_2-3}\alpha_1(x)-2(1+t)^{\beta_2-3}\alpha_1(x)\\
&+\frac{k_2(\beta_2-1)(\beta_2-2)}{2}(1+t)^{\beta_2-3}+(\beta_2-2)(1+t)^{\beta_2-3}\alpha_1(x)\Big)V_{tt}^2\\
&+\Big(-k_2(\beta_2-4)(1+t)^{\beta_2-5}+(\beta_2-3)(\beta_2-4)(1+t)^{\beta_2-5}\Big)\alpha_1(x)V_t^2\Big)dx\\
&-C(1+t)^{\beta_2}A^{\frac12}\|V_{ttt}\|-C(1+t)^{\beta_2-1}B^{\frac12}\|V_{ttx}\|\\
&\le\int\Big((1+t)^{\beta_2}G_{tt}V_{ttt}+k_2(1+t)^{\beta_2-1}G_{tt}V_{tt}\Big)\,dx,
\end{aligned}
\tag{4.46}
$$

where

$$
\begin{aligned}
A=\int\Big(&p'''(\bar v+V_x)^2(\bar v_t+V_{tx})^2(\bar v_x+V_{xx})^2V_{tx}^2\\
&+p''(\bar v+V_x)^2(\bar v_{tx}+V_{txx})^2V_{tx}^2+p''(\bar v+V_x)^2(\bar v_t+V_{tx})^2V_{txx}^2\\
&+p''(\bar v+V_x)^2\bar v_{tt}^2V_{xx}^2+\big(p''(\bar v+V_x)-p''(\bar v)\big)^2\bar v_x^2\bar v_{tt}^2\\
&+\big(p'(\bar v+V_x)-p'(\bar v)\big)^2\bar v_{ttx}^2+p'''(\bar v+V_x)^2(\bar v_x+V_{xx})^2\bar v_t^2V_{tx}^2\\
&+p''(\bar v+V_x)^2\bar v_t^2V_{txx}^2+p''(\bar v+V_x)^2\bar v_{tx}^2V_{tx}^2\\
&+4\big(p''(\bar v+V_x)-p''(\bar v)\big)^2\bar v_t^2\bar v_{tx}^2+p'''(\bar v+V_x)^2\bar v_t^4V_{xx}^2\\
&+\big(p''(\bar v+V_x)-p''(\bar v)\big)^2\bar v_t^4\bar v_x^2\Big)dx,
\end{aligned}
\tag{4.47}
$$

and

$$
B=\int\Big(p''(\bar v+V_x)^2(\bar v_t+V_{tx})^2V_{tx}^2+p''(\bar v+V_x)^2\bar v_t^2V_{tx}^2
$$

$$+\left(p'(\bar{v}+V_x)-p'(\bar{v})\right)^2\bar{v}_{tt}^2+\left(p''(\bar{v}+V_x)-p''(\bar{v})\right)^2\bar{v}_t^4\Big)dx. \tag{4.48}$$

Next, for $3<\underline{\alpha}_1\le 4$, we choose $\beta_2=\underline{\alpha}_1-\theta$ and $k_2=\frac{\alpha_1}{2}$; for $[\frac{\alpha_1}{2}]=2$, we choose $\beta_2=\underline{\alpha}_1-\theta$ and $k_2=\frac{\alpha_1}{2}$; for $[\frac{\alpha_1}{2}]\ge 3$, we choose $\beta_2=6-\theta$ and $k_2=3$. And by referring to the previous proof, we can obtain the decay estimates of $\|V_{ttt}\|^2$ and $\|V_{ttx}\|^2$. Then, from (4.30), (4.44), the estimates of $\|(V_{ttt},V_{ttx})\|^2$, Lemma 3.1, Lemma 4.1 and Lemma 4.2, we can obtain the estimates of $\|(V_{txx},V_{xxx})\|^2$. This completes the proof of Lemma 4.3. □

From Lemma 4.1–Lemma 4.3, it simply follows that Theorem 3.1 is valid.

5. Decay estimates for Case 2

The primary objective of this section is to determine the decay rates of the solution of (3.18) and its derivatives, and subsequently deduce the decay estimate for the solution of the original problem. For some $\delta\ll 1$ and $0<T_2<\infty$, we focus on estimating the solution of (3.18) based on the following a priori assumptions:

- when $2<\underline{\alpha}_1<\frac{20}{9}$, then

$$N_2(T_2):=\sup_{0\le t\le T_2}\{(1+t)^{\frac{5\alpha_1}{8}-\frac{3}{2}}\|V\|+(1+t)^{\frac{5\alpha_1}{8}-\frac{1}{2}}\|(V_t,V_x)\|\}\le\delta.$$

- when $\frac{20}{9}\le\underline{\alpha}_1<\frac{12}{5}$, then

$$\begin{aligned}N_2(T_2):=\sup_{0\le t\le T_2}\{&(1+t)^{\frac{5\alpha_1}{8}-\frac{3}{2}}\|V\|+(1+t)^{\frac{5\alpha_1}{8}-\frac{1}{2}}\|(V_t,V_x)\|\\&+(1+t)^{\frac{9\alpha_1}{8}-\frac{3}{2}}\|(V_{tt},V_{tx},V_{xx})\|+(1+t)^{\frac{9\alpha_1}{8}-\frac{3}{2}}\|(V_{ttt},V_{ttx},V_{txx},V_{xxx})\|\}\le\delta.\end{aligned}$$

- when $\frac{12}{5}\le\underline{\alpha}_1\le 4$, then

$$\begin{aligned}N_2(T_2):=\sup_{0\le t\le T_2}\{&\|V\|+(1+t)\|(V_t,V_x)\|+\frac{(1+t)^{\frac{\alpha_1}{2}}}{\ln(2+t)}\|(V_{tt},V_{tx},V_{xx})\|\\&+\frac{(1+t)^{\frac{\alpha_1}{2}}}{\ln(2+t)}\|(V_{ttt},V_{ttx},V_{txx},V_{xxx})\|\}\le\delta.\end{aligned}$$

- when $4<\underline{\alpha}_1<6$, then

$$\begin{aligned}N_2(T_2):=\sup_{0\le t\le T_2}\{&\|V\|+(1+t)\|(V_t,V_x)\|+(1+t)^2\|(V_{tt},V_{tx},V_{xx})\|\\&+(1+t)^{\frac{\alpha_1}{2}}\|(V_{ttt},V_{ttx},V_{txx})\|+(1+t)^2\|V_{xxx}\|\}\le\delta.\end{aligned}$$

- when $[\frac{\alpha_1}{2}] \geq 3$, then

$$N_2(T_2) := \sup_{0\leq t\leq T_2} \{\|V\| + (1+t)\|(V_t, V_x)\| + \frac{(1+t)^2}{\ln(2+t)}\|(V_{tt}, V_{tx}, V_{xx})\| \\ + (1+t)^3\|(V_{ttt}, V_{ttx}, V_{txx})\| + (1+t)^{\frac{5}{2}}\|V_{xxx}\|\} \leq \delta.$$

Before we start our proof, we present the following definition:

$$I_1 := \|V_0\|_3^2 + \|V_1\|_2^2 + I_0.$$

Lemma 5.1. *Under the assumptions of Theorem 3.2, it holds that*

- *for* $2 < \underline{\alpha}_1 < \frac{12}{5}$, *then*

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \\ \leq CI_1(1+t)^{3-\frac{5\underline{\alpha}_1}{4}}. \tag{5.1}$$

- *for* $\frac{12}{5} \leq \underline{\alpha}_1$, *then*

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \\ \leq CI_1. \tag{5.2}$$

Proof. Similar to the discussion in Lemma 4.1, multiplying (3.26) by $\left(\frac{2}{\underline{\alpha}_1}(1+t)^2V_t + (1+t)V\right)$ and integrating it respect to x and t, we obtain

$$\int \left(\frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1(3\underline{\alpha}_1 - 2)}(1+t)^2V_t^2 + \frac{1}{2\underline{\alpha}_1}(1+t)^2V_x^2 + \frac{\underline{\alpha}_1 - 2}{8}V^2\right)dx \\ + \iint_0^t -\frac{\underline{\alpha}_1 - 2}{6\underline{\alpha}_1}p'(\bar{v} + V_x)(1+s)V_x^2 dxds - \iint_0^t CI_0(1+t)V_x^2 dxds \\ + \iint_0^t \frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1}(1+s)V_t^2 dxds \\ \leq \iint_0^t \frac{2}{\underline{\alpha}_1}(1+s)^2V_tGdxds + \iint_0^t (1+s)VGdxds. \tag{5.3}$$

Subsequently, we will proceed to estimate the final two terms of (5.3). By invoking Lemma 3.2, leveraging the a priori assumption, and applying Hölder's inequality, one gets

- when $2 < \underline{\alpha}_1 < \frac{12}{5}$, then

$$\iint_0^t (1+s)VG_1 dxds \le \iint_0^t |(1+s)VG_1| dxds$$

$$\le CI_0 \iint_0^t |\alpha_1 - \underline{\alpha}_1|(1+s)^{-\frac{5\underline{\alpha}_1}{8}+\frac{1}{2}}|V| dxds$$

$$\le CI_0 \int_0^t \|\alpha_1 - \underline{\alpha}_1\|_{L^1}(1+s)^{-\frac{5\underline{\alpha}_1}{8}+\frac{1}{2}}|V|_\infty ds$$

$$\le CI_0(1+t)^{-\frac{5\underline{\alpha}_1}{4}+3} + CI_0, \tag{5.4}$$

$$\iint_0^t (1+s)VG_2 dxds \le \int_0^t (1+s)\|V\|\|G_2\| ds$$

$$\le CI_0(1+t)^{-\frac{7\underline{\alpha}_1}{4}+4} + CI_0, \tag{5.5}$$

$$\iint_0^t (1+s)^2 V_t G_1 dxds \le \int_0^t (1+s)^2\|V_t\|\|G_1\| ds$$

$$\le CI_0 \int_0^t (1+s)^{2-\frac{5\underline{\alpha}_1}{8}-\frac{1}{2}-\frac{5\underline{\alpha}_1}{8}+\frac{1}{2}} ds$$

$$\le CI_0(1+t)^{3-\frac{5\underline{\alpha}_1}{4}} + CI_0, \tag{5.6}$$

and

$$\iint_0^t (1+s)^2 V_t G_2 dxds \le \int_0^t (1+s)^2\|V_t\|\|G_2\| ds$$

$$\le CI_0(1+t)^{-\frac{7\underline{\alpha}_1}{4}+4} + CI_0. \tag{5.7}$$

- when $\frac{12}{5} < \underline{\alpha}_1$, then

$$\iint_0^t (1+s)VG_1 dxds \le \iint_0^t |(1+s)VG_1| dxds$$

$$
\begin{aligned}
&\le CI_0 \iint_0^t |\alpha_1 - \underline{\alpha}_1|(1+s)^{-1}|V|dxds \\
&\le CI_0 \int_0^t \|\alpha_1 - \underline{\alpha}_1\|_{L^1}(1+s)^{-1}|V|_\infty ds \\
&\le CI_0, \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
&\iint_0^t (1+s)VG_2 dxds \le \int_0^t (1+s)\|V\|\|G_2\|ds \\
&\le CI_0, \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
&\iint_0^t (1+s)^2 V_t G_1 dxds \le \iint_0^t |(1+s)^2 V_t G_1|dxds \\
&\le CI_0 \iint_0^t |\alpha_1 - \underline{\alpha}_1||V_t|dxds \\
&\le CI_0 \int_0^t \|\alpha_1 - \underline{\alpha}_1\|_{L^1}|V_t|_\infty ds \\
&\le CI_0, \qquad (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint_0^t (1+s)^2 V_t G_2 dxds \le \int_0^t (1+s)^2\|V_t\|\|G_2\|ds \\
&\le CI_0. \qquad (5.11)
\end{aligned}
$$

Hence, from (5.3)–(5.7), we obtain

- for $2 < \underline{\alpha}_1 < \frac{12}{5}$, then

$$
\begin{aligned}
&\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \\
&\le CI_1(1+t)^{3-\frac{5\underline{\alpha}_1}{4}}. \qquad (5.12)
\end{aligned}
$$

- for $\frac{12}{5} < \underline{\alpha}_1$, then

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \\ \le CI_1. \tag{5.13}$$

When $\underline{\alpha}_1 = \frac{12}{5}$, for a suitably small $\bar{\theta} > 0$, multiply (3.26) by $\left(\frac{2}{\underline{\alpha}_1}(1+t)^{2+\bar{\theta}} V_t + (1+t)^{1+\bar{\theta}} V\right)$; through a discussion similar to the above proof, we can obtain

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \le CI_1.$$

This completes the proof of Lemma 5.1. □

Next, we will estimate the decay estimates of $\|V_{xx}\|$, $\|V_{tx}\|$, and $\|V_{tt}\|$.

Lemma 5.2. *Under the assumptions of Theorem 3.2, it holds that*

- *when* $\frac{20}{9} \le \underline{\alpha}_1 < \frac{12}{5}$, *for* $0 < \theta < \frac{-5\underline{\alpha}_1}{4} + 3$, *then*

$$(1+t)^{\underline{\alpha}_1 - \theta}\|(V_{xx}, V_{tx}, V_{tt})(t)\|^2 + \theta\int_0^t (1+s)^{\underline{\alpha}_1 - 1 - \theta}\|(V_{xx}, V_{tx}, V_{tt})(s)\|^2 ds \\ \le CI_1(1+t)^{\frac{-5\underline{\alpha}_1}{4} + 3 - \theta}. \tag{5.14}$$

- *when* $\frac{12}{5} \le \underline{\alpha}_1 \le 4$, *then*

$$(1+t)^{\underline{\alpha}_1}\|(V_{xx}, V_{tx}, V_{tt})(t)\|^2 \le CI_1 \ln^2(2+t). \tag{5.15}$$

- *when* $4 < \underline{\alpha}_1$, *then*

$$(1+t)^4\|(V_{xx}, V_{tx}, V_{tt})(t)\|^2 \le CI_1. \tag{5.16}$$

Proof. Following a similar approach to the discussion in Lemma 4.2, multiplying ∂_t(3.26) by $\left(k_3(1+t)^{\beta_3} V_{tt} + (1+t)^{\beta_3 - 1} V_t\right)$ and integrating it respect to x and t, we obtain

$$\int k_3\left((1+t)^{\beta_3}\frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2}(1+t)^{\beta} V_{tx}^2 - \frac{1}{2}(1+t)^{\beta_3 - 2}\alpha_1(x) V_t^2\right) dx \\ + \int \left((1+t)^{\beta_3 - 1} V_{tt} V_t + (\alpha_1(x) - \beta_3 + 1)(1+t)^{\beta_3 - 2}\frac{V_t^2}{2}\right) dx$$

$$
\begin{aligned}
&+\iint_0^t \Big(k_3\Big(\alpha_1(x)-\frac{\beta_3}{2}\Big)-1\Big)(1+s)^{\beta_3-1}V_{tt}^2dxds\\
&-\iint_0^t \Big(1-\frac{k_3\beta_3}{2}\Big)p'(\bar v+V_x)(1+s)^{\beta_3-1}V_{tx}^2dxds\\
&\le C\int_0^t |D|ds+\iint_0^t k_3(1+s)^{\beta_3}V_{tt}G_tdxds+\iint_0^t(1+s)^{\beta_3-1}V_tG_tdxds\\
&+C\iint_0^t(1+s)^{\beta_3-3}V_t^2dxds+CI_1,
\end{aligned}
\tag{5.17}
$$

where

$$
\begin{aligned}
D=&\int\Big(\frac{p'(\bar v+V_x)}{2}(1+t)^{\beta_3}\left(V_{tx}+\bar v_t\right)V_{tx}^2+(p'(\bar v+V_x)-p'(\bar v))\bar v_{tx}(1+t)^{\beta_3}V_{tt}\\
&+p''(\bar v+V_x)(1+t)^{\beta}\bar v_tV_{xx}V_{tx}+\big(p''(\bar v+V_x)-p''(\bar v)\big)\bar v_t\bar v_x(1+t)^{\beta_3}V_{tt}\\
&-\big(p'(\bar v+V_x)-p'(\bar v)\big)\bar v_t(1+t)^{\beta_3-1}V_{tx}\Big)dx\\
\le& C|V_{tx}+\bar v_t|_\infty\|V_{tx}\|^2(1+t)^{\beta_3}+C|V_x|_\infty\|\bar v_{tx}\|\|V_{tt}\|(1+t)^{\beta_3}\\
&+C|\bar v_t|_\infty\|V_{tx}\|\|V_{xx}\|(1+t)^{\beta_3}+C|V_x|_\infty|\bar v_t|_\infty\|\bar v_x\|\|V_{tt}\|(1+t)^{\beta_3}\\
&+C|V_x|_\infty\|\bar v_t\|\|V_{tt}\|(1+t)^{\beta_3-1}.
\end{aligned}
\tag{5.18}
$$

Applying Hölder's inequality, Lemma 3.2 and the a priori assumption, it holds that

- when $\frac{20}{9}\le\alpha_1<\frac{12}{5}$, by choosing $\beta_3=\underline{\alpha}_1-\theta,\ k_3=\frac{2}{\underline{\alpha}_1}$, then

$$
\begin{aligned}
|D|\le& C\left(\delta_1+I_0\right)(1+s)^{\underline{\alpha}_1-1-\theta}\|V_{tx}\|^2+CI_0(1+t)^{-\frac{3\underline{\alpha}_1}{2}+\frac{5}{2}-\theta}\\
&+CI_0(1+t)^{-\frac{7\underline{\alpha}_1}{4}+3-\theta}+CI_0(1+t)^{-2\underline{\alpha}_1+\frac{5}{2}-\theta}+CI_0(1+t)^{-\frac{3\underline{\alpha}_1}{2}+\frac{3}{2}-\theta},
\end{aligned}
\tag{5.19}
$$

and

$$
\begin{aligned}
&\iint_0^t\frac{2}{\underline{\alpha}_1}(1+s)^{\underline{\alpha}_1-\theta}V_{tt}G_tdxds+\iint_0^t(1+s)^{\underline{\alpha}_1-1-\theta}V_tG_tdxds\\
&\le C\int_0^t\Big((1+s)^{\underline{\alpha}_1-\theta}\|V_{tt}\|\|G_t\|+(1+s)^{\underline{\alpha}_1-1-\theta}\|V_t\|\|G_t\|\Big)ds\\
&\le CI_1(1+t)^{-\frac{5\underline{\alpha}_1}{4}+2-\theta}.
\end{aligned}
\tag{5.20}
$$

- when $\frac{12}{5} \le \alpha_1 \le 4$, by choosing $\beta_3 = \underline{\alpha}_1,\ k_3 = \frac{2}{\underline{\alpha}_1}$, then

$$|D| \le C(\delta_1 + I_0)(1+t)^{\underline{\alpha}_1 - 1}\|V_{tx}\|^2 + C I_0 \ln(2+t)(1+t)^{-\frac{11}{10}+\frac{1}{4}} \tag{5.21}$$

and

$$\begin{aligned}
&\iint_0^t \frac{2}{\underline{\alpha}_1}(1+s)^{\underline{\alpha}_1} V_{tt} G_t \,dxds + \iint_0^t (1+s)^{\underline{\alpha}_1 - 1} V_t G_t \,dxds \\
&\le C \int_0^t \Big((1+s)^{\underline{\alpha}_1}\|V_{tt}\|\|G_t\| + (1+s)^{\underline{\alpha}_1 - 1}\|V_t\|\|G_t\| \Big) ds \\
&\le C I_1 \ln^2(2+t).
\end{aligned} \tag{5.22}$$

- when $4 < \underline{\alpha}_1 < 6$, by choosing $\beta_3 = 4,\ k_3 = \frac{1}{2}$, then

$$|D| \le C(\delta_1 + I_0)(1+t)^3\|V_{tx}\|^2 + C I_0 (1+t)^{-\frac{3}{2}} \tag{5.23}$$

and

$$\begin{aligned}
&\iint_0^t \frac{1}{2}(1+s)^4 V_{tt} G_t \,dxds + \iint_0^t (1+s)^3 V_t G_t \,dxds \\
&\le C \int_0^t \Big((1+s)^4\|V_{tt}\|\|G_t\| + (1+s)^3\|V_t\|\|G_t\| \Big) ds \\
&\le C I_1.
\end{aligned} \tag{5.24}$$

Hence, relying on (5.17)–(5.24), applying Cauchy's inequality and making use of Lemma 5.1, and by using a discussion similar to that in Lemma 4.2, we obtain

- when $\frac{20}{9} \le \alpha_1 < \frac{12}{5}$, $\beta_3 = \underline{\alpha}_1 - \theta$ and $k_3 = \frac{2}{\underline{\alpha}_1}$, then

$$\begin{aligned}
&(1+t)^{\underline{\alpha}_1 - \theta}\|(V_{tx}, V_{tt})(t)\|^2 + \theta \int_0^t (1+s)^{\underline{\alpha}_1 - 1 - \theta}\|(V_{tx}, V_{tt})(s)\|^2 ds \\
&\le C I_1 (1+t)^{-\frac{5\underline{\alpha}_1}{4} + 3 - \theta}.
\end{aligned} \tag{5.25}$$

- when $\frac{12}{5} \le \alpha_1 \le 4$, $\beta_3 = \underline{\alpha}_1$ and $k_3 = \frac{2}{\underline{\alpha}_1}$, then

$$(1+t)^{\underline{\alpha}_1}\|(V_{tx}, V_{tt})(t)\|^2 \le C I_1 \ln^2(2+t). \tag{5.26}$$

- when $4 < \underline{\alpha}_1$, $\beta_3 = 4$ and $k_3 = \frac{1}{2}$, then

$$(1+t)^4 \|(V_{tx}, V_{tt})(t)\|^2 \le CI_1. \tag{5.27}$$

Referring to the proof of the estimates of V_{xx} in Lemma 4.2, we can also get the estimates of V_{xx} in Lemma 5.2. This completes the proof of Lemma 5.2. □

Similar calculations to Lemma 4.3 yield the following Lemma, the proof of which is omitted.

Lemma 5.3. *Let $i \ge 0$ and $j \ge 0$. Under the assumptions of Theorem 3.2, it holds that*

- *when $\frac{20}{9} \le \underline{\alpha}_1 < \frac{12}{5}$, for $0 < \theta < \frac{-5\underline{\alpha}_1}{4} + 3$, then*

$$\sum_{i+j=3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V(t)\|^2 \le CI_1 (1+t)^{-\frac{5\underline{\alpha}_1}{4}+3-\theta}. \tag{5.28}$$

- *when $\frac{12}{5} \le \underline{\alpha}_1 < 4$, then*

$$\sum_{i+j=3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V(t)\|^2 \le CI_1 \ln^2(2+t). \tag{5.29}$$

- *when $[\frac{\underline{\alpha}_1}{2}] = 2$, then*

$$\sum_{i+j=3, i\ne 3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V(t)\|^2 + (1+t)^4 \|\partial_x^3 V(t)\|^2 \le CI_1. \tag{5.30}$$

- *when $[\frac{\underline{\alpha}_1}{2}] \ge 3$, then*

$$\sum_{i+j=3, i\ne 3} (1+t)^6 \|\partial_x^i \partial_t^j V(t)\|^2 + (1+t)^4 \|\partial_x^3 V(t)\|^2 \le CI_1. \tag{5.31}$$

6. Decay estimates for Case 3

We now start to obtain the estimates of the solution of (3.36) and its derivative through calculations. After obtaining the estimates of the solution of (3.36) and its derivative, the proof of Theorem 3.3 holds naturally. For some $\delta \ll 1$ and $0 < T_3 < \infty$, we are dedicated to estimating the solution of (3.36) under the following a priori assumptions:

- when $2 < \underline{\alpha}_1 < 4$,

$$N_3(T_3) := \sup_{0\le t\le T_3} \{\|V\| + (1+t)\|(V_t, V_x)\| + (1+t)^{\frac{\underline{\alpha}_1}{2}} \|(V_{tt}, V_{tx}, V_{tx})\| + (1+t)^{\frac{\underline{\alpha}_1}{2}} \|(V_{ttt}, V_{ttx}, V_{txx}, V_{xxx})\|\} \le \delta.$$

- when $[\frac{\alpha_1}{2}] = 2$,

$$N_3(T_3) := \sup_{0\le t\le T_3}\{\|V\| + (1+t)\|(V_t, V_x)\| + (1+t)^2\|(V_{tt}, V_{tx}, V_{tx})\| \\ + (1+t)^{\frac{\alpha_1}{2}}\|(V_{ttt}, V_{ttx}, V_{txx})\| + (1+t)^2\|V_{xxx}\|\} \le \delta.$$

- when $[\frac{\alpha_1}{2}] \ge 3$,

$$N_3(T_3) := \sup_{0\le t\le T_3}\{\|V\| + (1+t)\|(V_t, V_x)\| + (1+t)^2\|(V_{tt}, V_{tx}, V_{tx})\| \\ + (1+t)^3\|(V_{ttt}, V_{ttx}, V_{txx})\| + (1+t)^{\frac{5}{2}}\|V_{xxx}\|\} \le \delta.$$

Before we start the proof, we present the following definition:

$$K_0 := \|V_0\|_3 + \|V_1\|_2 + |\underline{u}|.$$

Lemma 6.1. *Under the assumptions of Theorem 3.3, for $\underline{\alpha}_1 > 2$, it holds that*

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds \\ \le CK_0^2. \tag{6.1}$$

Proof. Multiplying (3.36) by $\left(\frac{2}{\underline{\alpha}_1}(1+t)^2V_t + (1+t)V\right)$, integrating it respect to x and t, and applying Hölder's inequality, one gets

$$\int \left(\frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1(3\underline{\alpha}_1 - 2)}(1+t)^2V_t^2 + \frac{1}{2\underline{\alpha}_1}(1+t)^2V_x^2 + \frac{\underline{\alpha}_1 - 2}{8}V^2\right)dx \\ + \iint_0^t -\frac{\underline{\alpha}_1 - 2}{6\underline{\alpha}_1}p'(\underline{v} + V_x)(1+s)V_x^2 dxds + \iint_0^t \frac{\underline{\alpha}_1 - 2}{\underline{\alpha}_1}(1+s)V_t^2 dxds \\ \le \iint_0^t \frac{2}{\underline{\alpha}_1}(1+s)^2V_tJdxds + \iint_0^t (1+s)VJdxds + CK_0^2 \\ \le C\int_0^t \left((1+s)^2\|V_t\|\|J\| + (1+s)\|V\|\|J\|\right)ds + CK_0^2. \tag{6.2}$$

Then, applying Lemma 3.3 and the a priori assumption to (6.2), we obtain

$$\|V(t)\|^2 + (1+t)^2\|(V_x, V_t)(t)\|^2 + \int_0^t (1+s)\|(V_x, V_t)(s)\|^2 ds$$
$$\leq CK_0^2. \tag{6.3}$$

The proof is complete. □

Lemma 6.2. *Under the assumptions of Theorem 3.3, it holds that*

- *when* $2 < \underline{\alpha}_1 < 4$, *then*

$$(1+t)^{\underline{\alpha}_1}\|(V_{xx}, V_{tx}, V_{tt})(t)\|^2 \leq CK_0^2. \tag{6.4}$$

- *when* $[\frac{\underline{\alpha}_1}{2}] \geq 2$, *then*

$$(1+t)^4\|(V_{xx}, V_{tx}, V_{tt})(t)\|^2 \leq CK_0^2. \tag{6.5}$$

Proof. Multiplying ∂_t(3.26) by $\Big(k_4(1+t)^{\beta_4}V_{tt} + (1+t)^{\beta_4-1}V_t\Big)$ and integrating it respect to x and t, we obtain

$$\begin{aligned}
&\int k_4\left((1+t)^{\beta_4}\frac{V_{tt}^2}{2} - \frac{p'(\bar{v}+V_x)}{2}(1+t)^{\beta_4}V_{tx}^2 - \frac{1}{2}(1+t)^{\beta_4-2}\alpha_1(x)V_t^2\right)dx \\
&+ \int\left((1+t)^{\beta_4-1}V_{tt}V_t + (\alpha_1(x) - \beta_4 + 1)(1+t)^{\beta_4-2}\frac{V_t^2}{2}\right)dx \\
&+ \iint_0^t \left(k_4\Big(\alpha_1(x) - \frac{\beta_4}{2}\Big) - 1\right)(1+s)^{\beta_4-1}V_{tt}^2 dxds \\
&- \iint_0^t \left(1 - \frac{k_4\beta_4}{2}\right)p'(\bar{v}+V_x)(1+s)^{\beta_4-1}V_{tx}^2 dxds \\
&\leq \iint_0^t k_4(1+s)^{\beta_4}V_{tt}J_t dxds + \iint_0^t (1+s)^{\beta_4-1}V_tJ_t dxds \\
&+ C\iint_0^t (1+s)^{\beta_4-3}V_t^2 dxds + C\iint_0^t (1+s)^{\beta_4}|V_{tx}|V_{tx}^2 dxds.
\end{aligned} \tag{6.6}$$

By applying Lemma 3.3, Lemma 6.1, the a priori assumption and Hölder's inequality and using a discussion similar to that in Lemma 4.2, we obtain

- for $2 < \underline{\alpha}_1 < 4$, by choosing $\beta_4 = \underline{\alpha}_1$, $k_4 = \frac{2}{\underline{\alpha}_1}$ then

$$
\begin{aligned}
&\iint_0^t \frac{2}{\underline{\alpha}_1}(1+s)^{\underline{\alpha}_1} V_{tt} G_t dxds + \iint_0^t (1+s)^{\underline{\alpha}_1 - 1} V_t G_t dxds \\
&\le C \int_0^t \Big((1+s)^{\underline{\alpha}_1} \|V_{tt}\| \|J_t\| + (1+s)^{\underline{\alpha}_1 - 1} \|V_t\| \|J_t\| \Big) \\
&\le C \int_0^t \Big((1+s)^{-\frac{\underline{\alpha}_1}{2} - 2} + (1+s)^{-4} \Big) \\
&\le C K_0^2,
\end{aligned}
\tag{6.7}
$$

and

$$
\begin{aligned}
&\iint_0^t (1+s)^{\underline{\alpha}_1 - 3} V_t^2 dxds + C \iint_0^t (1+t)^{\underline{\alpha}_1} |V_{tx}| V_{tx}^2 dxds \\
&\le C K_0^2 + C\delta \int_0^t (1+s)^{-\frac{\underline{\alpha}_1}{2}} (1+s)^{\underline{\alpha}_1} \|V_{tx}\|^2 ds \\
&\le C K_0^2 + C\delta \sup_{0 \le t \le T_3} (1+t)^{\underline{\alpha}_1} \|V_{tx}\|^2.
\end{aligned}
\tag{6.8}
$$

- for $[\frac{\underline{\alpha}_1}{2}] \ge 2$, by choosing $\beta_4 = 4$, $k_4 = \frac{1}{2}$ then

$$
\begin{aligned}
&\iint_0^t \frac{1}{2}(1+s)^4 V_{tt} G_t dxds + \iint_0^t (1+s)^3 V_t G_t dxds \\
&\le C \int_0^t \Big((1+s)^4 \|V_{tt}\| \|J_t\| + (1+s)^3 \|V_t\| \|J_t\| \Big) \\
&\le C \int_0^t (1+s)^{-\underline{\alpha}_1} ds \le C K_0^2,
\end{aligned}
\tag{6.9}
$$

and

$$
\iint_0^t (1+s) V_t^2 dxds + C \iint_0^t (1+t)^4 |V_{tx}| V_{tx}^2 dxds
$$

$$\leq CK_0^2 + +C\delta \int_0^t (1+s)^{-2}(1+s)^4 \|V_{tx}\|^2 ds$$
$$\leq CK_0^2 + C\delta \sup_{0\leq t\leq T_3} (1+t)^4 \|V_{tx}\|^2. \tag{6.10}$$

Hence, by applying Cauchy's inequality to (6.6), and then making use of (6.7)–(6.10) and the smallness of δ, one gets

- when $\left[\frac{\alpha_1}{2}\right] = 1$, $\beta_4 = \underline{\alpha}_1$ and $k_4 = \frac{2}{\underline{\alpha}_1}$, then

$$(1+t)^{\underline{\alpha}_1} \|(V_{tx}, V_{tt})(t)\|^2 \leq CK_0^2. \tag{6.11}$$

- when $\left[\frac{\alpha_1}{2}\right] \geq 2$, $\beta_4 = 4$ and $k_4 = \frac{1}{2}$, then

$$(1+t)^4 \|(V_{tx}, V_{tt})(t)\|^2 \leq CK_0^2. \tag{6.12}$$

In reference to the proof idea of the estimates of V_{xx} in Lemma 4.2, we can equally derive the estimates of V_{xx} in Lemma 6.2. Consequently, the proof of Lemma 6.2 is completed. □

Calculations similar to those in Lemma 4.3 lead to the following lemma. Therefore, its proof is omitted.

Lemma 6.3. *Under the assumptions of Theorem 3.3, for $i \geq 0$ and $j \geq 0$, it holds that*

- *when $2 < \underline{\alpha}_1 < 4$, then*

$$\sum_{i+j=3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V(t)\|^2 \leq CK_0^2. \tag{6.13}$$

- *when $[\frac{\alpha_1}{2}] = 2$, then*

$$\sum_{i+j=3, i\neq 3} (1+t)^{\underline{\alpha}_1} \|\partial_x^i \partial_t^j V(t)\|^2 + (1+t)^4 \|\partial_x^3 V(t)\|^2 \leq CK_0^2. \tag{6.14}$$

- *when $[\frac{\alpha_1}{2}] \geq 3$, then*

$$\sum_{i+j=3, i\neq 3} (1+t)^6 \|\partial_x^i \partial_t^j V(t)\|^2 + (1+t)^5 \|\partial_x^3 V(t)\|^2 \leq CK_0^2. \tag{6.15}$$

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Data availability

No data was used for the research described in the article.

References

[1] Y. Cai, C. Liu, M. Mei, Z. Wang, Asymptotic behavior of solutions to the Cauchy problem for 1D Euler equations with spatiotemporal damping: case 1. $v_+ = v_-$, J. Differ. Equ. 436 (2025) 113347, 37 pp.
[2] Y. Cai, C. Liu, M. Mei, Z. Wang, Asymptotic behavior of solutions to the Cauchy problem for 1D Euler equations with spatiotemporal damping: case 2. $v_+ \neq v_-$, preprint, 2025.
[3] H. Cai, G. Chen, T. Wang, Singularity formation for radially symmetric expanding wave of compressible Euler equations, SIAM J. Math. Anal. 55 (4) (2023) 2917–2947.
[4] G. Chen, Formation of singularity and smooth wave propagation for the non-isentropic compressible Euler equations, J. Hyperbolic Differ. Equ. 8 (4) (2011) 671–690.
[5] G.-Q. Chen, C. Dafermos, M. Slemrod, D. Wang, On two-dimensional sonic-subsonic flow, Commun. Math. Phys. 271 (2007) 635–647.
[6] G. Chen, R. Pan, S. Zhu, Singularity formation for the compressible Euler equations, SIAM J. Math. Anal. 49 (4) (2017) 2591–2614.
[7] S. Chen, H. Li, J. Li, M. Mei, K. Zhang, Global and blow-up solutions for compressible Euler equations with time-dependent damping, J. Differ. Equ. 268 (9) (2020) 5035–5077.
[8] H. Cui, H. Yin, J. Zhang, C. Zhu, Convergence to nonlinear diffusion waves for solutions of Euler equations with time-depending damping, J. Differ. Equ. 264 (7) (2018) 4564–4602.
[9] H. Cui, H. Yin, C. Zhu, Convergence rates to the Barenblatt solutions for the compressible Euler equations with time-dependent damping, J. Differ. Equ. 374 (2023) 761–788.
[10] S. Geng, Y. Lin, M. Mei, Asymptotic behavior of solutions to Euler equations with time-dependent damping in critical case, SIAM J. Math. Anal. 52 (2) (2020) 1463–1488.
[11] F. Hou, I. Witt, H.C. Yin, Global existence and blowup of smooth solutions of 3-D potential equations with time-dependent damping, Pac. J. Math. 292 (2) (2018) 389–426.
[12] F. Hou, H.C. Yin, On the global existence and blowup of smooth solutions to the multi-dimensional compressible Euler equations with time-depending damping, Nonlinearity 30 (2017) 2485–2517.
[13] L. Hsiao, T. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Commun. Math. Phys. 143 (3) (1992) 599–605.
[14] F. Huang, R. Pan, Z. Wang, L^1 convergence to the Barenblatt solution for compressible Euler equations with damping, Arch. Ration. Mech. Anal. 200 (2) (2011) 665–689.
[15] S. Ji, M. Mei, Optimal decay rates of the compressible Euler equations with time-dependent damping in $\mathbb{R}^n$: (I) under-damping case, J. Nonlinear Sci. 33 (1) (2023) 7, 47 pp.
[16] S. Ji, M. Mei, Optimal decay rates of the compressible Euler equations with time-dependent damping in $\mathbb{R}^n$: (II) overdamping case, SIAM J. Math. Anal. 55 (2) (2023) 1048–1099.
[17] H. Li, Optimal convergence rate to nonlinear diffusion waves for bipolar Euler-Poisson equations with critical overdamping, Appl. Anal. 102 (11) (2023) 2999–3019.
[18] H. Li, J. Li, M. Mei, K. Zhang, Optimal convergence rate to nonlinear diffusion waves for Euler equations with critical overdamping, Appl. Math. Lett. 113 (2021) 106882, 7 pp.
[19] H. Li, J. Li, M. Mei, K. Zhang, Convergence to nonlinear diffusion waves for solutions of p-system with time-dependent damping, J. Math. Anal. Appl. 456 (2) (2017) 849–871.
[20] L. Luan, M. Mei, B. Rubino, P. Zhu, Large-time behavior of solutions to Cauchy problem for bipolar Euler-Poisson system with time-dependent damping in critical case, Commun. Math. Sci. 19 (5) (2011) 1207–1231.
[21] T. Luo, H.H. Zeng, Global existence of smooth solutions and convergence to Barenblatt solutions for the physical vacuum free boundary problem of compressible Euler equations with damping, Commun. Pure Appl. Math. 69 (2016) 1354–1396.

[22] H. Ma, M. Mei, Best asymptotic profile for linear damped p-system with boundary effect, J. Differ. Equ. 249 (2) (2010) 446–484.
[23] P. Marcati, M. Mei, B. Rubino, Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping, J. Math. Fluid Mech. 7 (2002) S224–S240.
[24] A. Matsumura, K. Nishihara, Asymptotic behavior of solutions to the Cauchy problem for 1D p-system with space-dependent damping, SIAM J. Math. Anal. 56 (1) (2024) 993–1015.
[25] M. Mei, Best asymptotic profile for hyperbolic p-system with damping, SIAM J. Math. Anal. 42 (1) (2010) 1–23.
[26] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differ. Equ. 131 (2) (1996) 171–188.
[27] K. Nishihara, M. Nishikawa, Asymptotic behavior of solutions to the system of compressible adiabatic flow through porous media, SIAM J. Math. Anal. 33 (1) (2001) 216–239.
[28] K. Nishihara, W. Wang, T. Yang, L_p-convergence rate to nonlinear diffusion waves for p-system with damping, J. Differ. Equ. 161 (1) (2000) 191–218.
[29] X. Pan, Blow up of solutions to 1-d Euler equations with time-dependent damping, J. Math. Anal. Appl. 442 (2016) 435–445.
[30] X. Pan, Global existence of solutions to 1-d Euler equations with time-dependent damping, Nonlinear Anal. 132 (2016) 327–336.
[31] X. Pan, Global existence and asymptotic behavior of solutions to the Euler equations with time-dependent damping, Appl. Anal. 100 (16) (2021) 3546–3575.
[32] T. Sideris, B. Thomases, D. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, Commun. Partial Differ. Equ. 28 (2003) 795–816.
[33] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York-Berlin, 1983.
[34] Y. Sugiyama, Singularity formation for the 1D compressible Euler equations with variable damping coefficient, Nonlinear Anal. 170 (2018) 70–87.
[35] Y. Sugiyama, Global existence and blow-up for the 1D damped compressible Euler equations with time and space dependent perturbation, Nonlinear Anal. 250 (2025) 113658, 12 pp.
[36] Z. Tan, Y. Wang, Global solution and large-time behavior of the 3D compressible Euler equations with damping, J. Differ. Equ. 254 (2013) 1686–1704.
[37] Z. Tan, G. Wu, Large time behavior of solutions for compressible Euler equations with damping in $\mathbb{R}^3$, J. Differ. Equ. 252 (2012) 1546–1561.
[38] D. Wang, G.-Q. Chen, Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation, J. Differ. Equ. 144 (1998) 44–65.
[39] J. Xu, M. Mei, H. Li, Large-time behavior of solutions for unipolar Euler-Poisson equations with critical over-damping, Discrete Contin. Dyn. Syst. 43 (12) (2023) 4403–4428.
[40] J. Wirth, Wave equations with time-dependent dissipation. I. Non-effective dissipation, J. Differ. Equ. 222 (2) (2006) 487–514.
[41] J. Wirth, Wave equations with time-dependent dissipation. II. Effective dissipation, J. Differ. Equ. 232 (1) (2007) 74–103.
[42] N. Zhang, Optimal convergence rates to diffusion waves for solutions of p-system with damping on quadrant, J. Math. Anal. Appl. 512 (1) (2022) 126118, 10 pp.
[43] H. Zhao, Convergence to strong nonlinear diffusion waves for solutions of p-system with damping, J. Differ. Equ. 174 (1) (2001) 200–236.