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Asymptotic behavior of solutions to the Cauchy problem for 1D p-system with spatiotemporal damping: Case 1. $v_+ = v_-$

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Abstract

This paper investigates the Cauchy problem for the p-system with spatiotemporal damping, modeling one-dimensional compressible flow through porous media in Lagrangian coordinates. We focus on the large-time asymptotic behavior of the system's solutions when the state constants for the specific volume are the same: $v_+ = v_-$, but the state constants for the velocity are different: $u_+ \neq u_-$. We show the convergence of the solutions to their diffusion waves with the different algebraic time decay rates according to different exponent of time-damping: $0 \leq \lambda < \frac{3}{5}$, $\lambda = \frac{3}{5}$ and $\frac{3}{5} < \lambda < 1$, respectively. Our analysis employs an energy method to establish a series of a priori estimates, offering new insights and theoretical support for understanding the long-time dynamics of compressible flows in porous media with spatially heterogeneous damping.

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1. Introduction

We consider the Cauchy problem for the p-system with damping depending on the temporal and spatial variables

$$\begin{cases} v_t - u_x = 0, \ (t,x) \in \mathbf{R}_+ \times \mathbf{R}, \\ u_t + p(v)_x = -\alpha(t,x)u, \\ (v,u)(0,x) = (v_0,u_0)(x) \to (v_\pm, u_\pm), \ x \to \pm\infty \ (v_\pm > 0), \end{cases} \tag{1.1}$$

which describes the one-dimensional compressible flow through porous media in Lagrangian coordinates, where $u = u(t,x)$ represents the flow velocity at time t and position x, $v > 0$ denotes the specific volume, and $p(v) > 0$ represents the pressure, with $p'(v) < 0$. The term $-\alpha(t,x)u$ represents the fractional external affection in the porous media flow, which is given by

$$\alpha(t,x) = \alpha_1(x)(1+t)^{-\lambda}, \quad \text{with } \alpha_1(x) \to \underline{\alpha}_1 \ \text{ as } x \to \pm\infty, \quad \text{and } 0 \le \lambda < 1, \tag{1.2}$$

where $\underline{\alpha}_1$ is a positive constant.

Our focus is on the large-time behavior of the solution (v,u) to (1.1), which is expected to approach the so-called diffusion wave $(\overline{v}, \overline{u})$ to the following porous media diffusion equation by the Darcy's law:

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha(t,x)\bar{u}, \end{cases} \quad \text{equivalently,} \quad \begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{v}_t - \left(\frac{-p(\bar{v})_x}{\alpha(t,x)}\right)_x = 0. \end{cases} \tag{1.3}$$

Let us draw the background of studies. For $\alpha(t,x) = 0$, the system (1.1) simplifies to the standard compressible Euler equations, which have garnered significant interest from many analysts, leading to numerous important developments. It is well known that smooth solutions to (1.1) generally break down in finite time.

For $\alpha(t,x) = \underline{\alpha}_1 > 0$ with $\lambda = 0$, the constant damping case, the system (1.1) becomes the compressible Euler equations with damping, modeling compressible flow through porous media. A vast body of literature explores the global existence and long-term behavior of smooth solutions to the compressible Euler equations with damping. Hsiao and Liu [8] were the first to demonstrate that the solution of system (1.1) converges time-asymptotically to nonlinear diffusion waves for α_1 is a constant. The convergence rates of $(v - \bar{v}, u - \bar{u})$ were further improved by the Nishihara [15] in L^2-sense, and by Nishihara-Wang-Yang [16] in L^1-sense, respectively. However, the so-called self-similar solutions are not the best asymptotic profiles as pointed out by Mei [14]. By using twice anti-derivatives technique, Mei [14] recognized that the best asymptotic profiles are the certain solutions to the nonlinear porous media equation with selected initial data, and further showed the better convergence rates. For the other interesting studies, we refer to [12,16,20,21].

For $\alpha(t,x) = \alpha_1(x) \to \underline{\alpha}_1 > 0$ with $\lambda = 0$, the spatial damping case, recently Matsumura and Nishihara [13] showed the convergence of the solutions to the diffusion waves by the energy method with new setting.

For $\alpha(t,x)=\frac{\alpha_1}{(1+t)^\lambda}$ with $\lambda\neq 0$, the system (1.1) reduces to the compressible Euler equations with time-dependent damping, where the damping effect $\frac{\alpha_1}{(1+t)^\lambda}$ is called under damping for $\lambda>0$ and over damping for $\lambda<0$. Pan [17] and Hou-Yin [6,7] first investigated the well-posedness and convergence for the 1-D and n-D under damping, respectively. The optimal convergence rates were showed by Ji-Mei [9] by Fourier analysis method recently. For the over damping case with $\lambda<0$, Li-Li-Mei-Zhang [11] and Ji-Mei [10] first investigated the large-time behavior of the solutions in 1-D and n-D cases respectively. For the other studies related to diffusion waves in bounded / unbounded domains, we refer to the following significant works [3,4,9–11,19]. For the blow-up results, we refer to [2,5,18].

In this series of studies, we consider the p-system with the spatiotemporal damping effect $-\alpha(t,x)u$. When the damping effect involves in both time t and the space x, the structure of system becomes more complicated and difficult, of course, it is more practical. Here, we first consider the case

$$v_+=v_- \text{ (and } u_+\neq u_- \text{ in general)}, \tag{1.4}$$

and will leave the case

$$v_+\neq v_- \text{ (and } u_+\neq u_- \text{ in general)}, \tag{1.5}$$

in the second part [1]. We realize that, different from the previous studies for the case with time-dependent damping $-\frac{\alpha_1}{(1+t)^\lambda}u$ only and the case with space-dependent damping $-\alpha_1(x)u$ only, there is a strong interaction by the space x and the time t for the spatiotemporal damping case. To overcome such a difficulty, some new techniques are developed for establishing energy estimates based on different value of λ with different setting up in this paper.

For the system (1.1), noting the damping effect (1.2), namely, $\alpha(t,x)=\alpha_1(x)(1+t)^{-\lambda}\to\underline{\alpha}_1(1+t)^{-\lambda}$ as $x\to\pm\infty$, we expect the asymptotic profile of (1.1) as follows:

$$\begin{cases} V_t-U_x=0,\\ p'(\underline{v})V_x+\underline{\alpha}_1(1+t)^{-\lambda}U=0,\\ V(t,\pm\infty)=\underline{v}\,(:=v_+=v_-)\,, \end{cases}$$

with the explicit expression in the form of

$$\begin{cases} V(t,x)=v_++\dfrac{\delta_0}{(4\pi\int_{-1}^t\mu(s)ds)^{\frac12}}\mathrm{e}^{-\frac{x^2}{4\int_{-1}^t\mu(s)ds}},\\ U(t,x)=\dfrac{-\mu(t)\delta_0 x}{2(4\pi)^{\frac12}(\int_{-1}^t\mu(s)ds)^{\frac32}}\mathrm{e}^{-\frac{x^2}{4\int_{-1}^t\mu(s)ds}}, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^+, \end{cases}$$

where $\mu(t)=\frac{|p'(\underline{v})|}{\underline{\alpha}_1}(1+t)^\lambda$. For details, we refer to (2.3)-(2.5) below. Our main target in this paper is to show the convergence of the original solution $(v,u)(t,x)$ to the profile $(V,U)(t,x)$ as follows, based on different values of λ:

$$\|(v-V)(t,\cdot)\|_{L^\infty(\mathbf{R})}=\begin{cases}O(1)(1+t)^{-\frac{3(\lambda+1)}{4}}, & 0\leqslant\lambda<\frac{3}{5},\\ O(1)\ln^{\frac{1}{2}}(2+t)(1+t)^{-\frac{6}{5}}, & \lambda=\frac{3}{5},\\ O(1)(1+t)^{\frac{\lambda-3}{2}}, & \frac{3}{5}<\lambda<1,\end{cases}$$

and

$$\|(u-U)(t,\cdot)\|_{L^\infty(\mathbf{R})}=\begin{cases}O(1)(1+t)^{-\frac{\lambda+5}{4}}, & 0\leqslant\lambda<\frac{3}{5},\\ O(1)\ln^{\frac{1}{2}}(2+t)(1+t)^{-\frac{7}{5}}, & \lambda=\frac{3}{5},\\ O(1)(1+t)^{\lambda-2}, & \frac{3}{5}<\lambda<1.\end{cases}$$

The organization of this paper is as follows: In Section 2, we reformulate the problems for each case of (1.4) as Cauchy problems for a second-order quasilinear wave equation with damping. In this section, the reformulated theorem will be explicitly stated, allowing us to derive our goals for (1.1) as corollaries in each case. Since the coefficient α also depends on time t, our proof methods differ from those in [13]. For $\lambda=0$, we improve the decay estimate of (ϕ_x,ϕ_t) and their higher-order derivatives compared to [13]. Additionally, for $\lambda=\frac{3}{5}$, compared to the critical case $\lambda=\frac{1}{7}$ in [3], we refine the proof method to achieve a sharper estimate. A standard energy method will be applied in these proofs, which necessitates a series of a priori estimates. The case for (1.4) will be demonstrated in Section 3.

Notations: For the function spaces, $L^p=L^p(\mathbf{R})\,(1\le p\le\infty)$ denotes the usual Lebesgue space with the norm

$$\|f\|_p=\|f\|_{L^p}=\left(\int_{\mathbf{R}}|f(x)|^p dx\right)^{1/p}\quad(1\le p<\infty),$$
$$|f|_\infty=\|f\|_\infty=\|f\|_{L^\infty}=\operatorname*{ess.sup}_{\mathbf{R}}|f(x)|.$$

The integral domain $\mathbf{R}$ is often abbreviated when it is clear. For any integer $k\ge0$, $H^k=H^k(\mathbf{R})$ denotes the usual kth order Sobolev space with the norm

$$\|f\|_{H^k}=\left(\sum_{j=0}^{k}\|\partial_x^k f\|_2^2\right)^{1/2}.$$

When $k=0$ and $p=2$, we often use the notation $\|f\|=\|f\|_0$. For brevity, we write

$$\|f,g,\ldots\|_{H^k\times H^l\times\ldots}=\|f\|_{H^k}+\|g\|_{H^l}+\ldots.$$

The set of k-times continuously differentiable functions in $\mathbf{R}$ with compact support is denoted by $C_0^k(\mathbf{R})$. The space $C^k([0,T];X)$ consists of k-times differentiable functions from $[0,T]$ to the Hilbert space X. Additionally, by C or c, we denote a generic positive constant independent of the time t, whose value may change from line to line. For two quantities a and b, $a \sim b$ means $\frac{1}{C}|b| \le |a| \le C|b|$ for a given constant C.

2. Reformulation of the problem and main results

In this section, we reformulate problem as a quasi-linear wave equation with damping, based on heuristic reasoning. For convenience, we denote $\alpha(t,x)$ by α, then present the results for the reformulated problem, which in turn yield the main results for the original problem (1.1). Rewrite (1.1) as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x + \alpha u = 0, \\ (v,u)(0,x) = (v_0,u_0)(x) \to (v_\pm, u_\pm), \ x \to \pm\infty, \end{cases} \tag{2.1}$$

under the condition (1.2). According to Darcy's law, the asymptotic profile $(\hat{v}, \hat{u})$ is expected to be given by (1.3), as discussed in previous studies. However, when α depends on x, we encounter difficulties in directly constructing $(\hat{v}, \hat{u})$ using

$$\bar{v}_t - \left(\frac{-p(\bar{v})_x}{\alpha}\right)_x = 0 \quad \text{and} \quad \bar{u} = \frac{-p(\bar{v})_x}{\alpha}. \tag{2.2}$$

In this paper, to circumvent this difficulty, we will introduce a simpler profile for (1.4), as in [13].

Here, we are going to discuss the simpler case (1.4). Given that both (1.2) and (1.4) are assumed, we adopt an asymptotic profile (V, U) defined by

$$\begin{cases} V_t - U_x = 0, \\ p'(\underline{v})V_x + \underline{\alpha}_1(1+t)^{-\lambda}U = 0, \\ V(t,\pm\infty) = \underline{v} \ (:= v_+ = v_-), \end{cases} \tag{2.3}$$

or

$$\begin{cases} V_t - \mu V_{xx} = 0, \\ V(t,\pm\infty) = \underline{v}, \quad \mu(t) := \frac{|p'(\underline{v})|(1+t)^\lambda}{\underline{\alpha}_1}, \\ U = \frac{|p'(\underline{v})|(1+t)^\lambda}{\underline{\alpha}_1} V_x = \mu V_x, \end{cases} \tag{2.4}$$

so that, the solutions to the above equations can be expressed explicitly as:

$$\begin{cases} V(t,x) = v_+ + \dfrac{\delta_0}{(4\pi\int_{-1}^t \mu(s)ds)^{\frac{1}{2}}} e^{-\frac{x^2}{4\int_{-1}^t \mu(s)ds}}, \\ U(t,x) = \dfrac{-\mu\delta_0 x}{2(4\pi)^{\frac{1}{2}}(\int_{-1}^t \mu(s)ds)^{\frac{3}{2}}} e^{-\frac{x^2}{4\int_{-1}^t \mu(s)ds}}, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^+, \end{cases} \tag{2.5}$$

where

$$\int_{-1}^{t} \mu(s)ds = \frac{|p'(\underline{v})|(1+t)^{1+\lambda}}{\underline{\alpha}_1(1+\lambda)}$$

and δ_0 represents a constant that will be determined later. Then, (2.4) can be reformulated as

$$\begin{cases} V_t - U_x = 0, \\ U_t + p(V)_x + \alpha U = U_t + (\alpha_1 - \underline{\alpha}_1)(1+t)^{-\lambda}U + (p(V) - p'(\underline{v})V)_x. \end{cases} \tag{2.6}$$

It is noted that $(V, U) \to (\underline{v}, 0)$ as $x \to \pm\infty$.

Next, in order to impose the condition $u(0, x) = u_0(x)$ while ensuring $u(x) \to u_\pm$ as $x \to \pm\infty$, we introduce a correction function $(\hat{v}, \hat{u})$. Meanwhile, according to the second equation of (2.1), we expect that

$$u(t, x) \sim e^{-\underline{\alpha}_1 \frac{(1+t)^{1-\lambda}-1}{1-\lambda}} u_\pm, \text{ as } x \to \pm\infty.$$

Consequently, we determine the correction function $(\hat{v}, \hat{v})$ such that it satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t + \alpha\hat{u} = 0, \\ (\hat{v}, \hat{u})(t, x) \to \left(0, e^{-\underline{\alpha}_1 \frac{(1+t)^{1-\lambda}-1}{1-\lambda}} u_\pm\right), \ x \to \pm\infty. \end{cases} \tag{2.7}$$

Thus we construct $\hat{v}, \hat{u}$ by

$$\begin{cases} \hat{u}(t, x) = e^{-\alpha_1 \frac{(1+t)^{1-\lambda}-1}{1-\lambda}} (u_- + (u_+ - u_-) \int_{-\infty}^{x} m_0(y)dy) \\ =: e^{-\alpha_1 \frac{(1+t)^{1-\lambda}-1}{1-\lambda}} M_0(x), \\ \hat{v}(t, x) = \frac{\partial}{\partial x}\left(\int_{\infty}^{t} e^{-\alpha_1 \frac{(1+s)^{1-\lambda}-1}{1-\lambda}} M_0(x)ds\right), \end{cases} \tag{2.8}$$

where $m_0 \in C_0^\infty(\mathbf{R})$ with $\int_{-\infty}^{\infty} m_0(y)dy = 1$. Hence, (2.7) holds with

$$\int_{-\infty}^{\infty} \hat{v}(0, x)dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\int_{\infty}^{0} e^{-\alpha_1 \frac{(1+s)^{1-\lambda}-1}{1-\lambda}} M_0(x)ds\right) dx = S(u_+ - u_-), \tag{2.9}$$

where $S = \int_{\infty}^{0} e^{-\underline{\alpha}_1 \frac{(1+s)^{1-\lambda}-1}{1-\lambda}} ds.$

Combining (2.1), (2.6), and (2.7), we have

$$\begin{cases} (v-V-\hat{v})_t-(u-U-\hat{u})_x=0, \\ (u-U-\hat{u})_t+(p(v)-p(V))_x+\alpha(u-U-\hat{u}) \\ \quad =-\{U_t+(\alpha_1-\underline{\alpha}_1)(1+t)^{-\lambda}U+(p(V)-p'(\underline{v})V)_x\}. \end{cases} \tag{2.10}$$

It is anticipated that $u-U-\hat{u}\to 0$ as $x\to\pm\infty$. Therefore, we have

$$\begin{aligned} \int_{-\infty}^{\infty}(v-V-\hat{v})(t,x)dx &= \int_{-\infty}^{\infty}\big(v_0(x)-V(0,x)-\hat{v}(0,x)\big)\,dx \\ &= \int_{-\infty}^{\infty}\big\{(v_0(x)-\underline{v})-(V(0,x)-\underline{v})\big\}\,dx+S(u_+-u_-) \\ &= \int_{-\infty}^{\infty}(v_0(x)-\underline{v})dx-\delta_0+S(u_+-u_-), \end{aligned} \tag{2.11}$$

and we can choose δ_0 by

$$\delta_0=\int_{-\infty}^{\infty}\big(v_0(x)-\underline{v}\big)\,dx+S(u_+-u_-). \tag{2.12}$$

Therefore, we have $\int_{-\infty}^{\infty}(v-V-\hat{v})(t,x)\,dx=0$ for any $t\geq 0$. Consequently, when we define ϕ by

$$\phi(t,x)=\int_{-\infty}^{x}(v-V-\hat{v})(t,y)dy, \tag{2.13}$$

which holds that $\phi_x=v-V-\hat{v}$ and $\phi_t=u-U-\hat{u}$, we may heuristically expect $\phi(t,\cdot)\in H^1$. Furthermore, the second equation of (2.10) can be written as

$$\begin{aligned} &\phi_{tt}+(p(V+\hat{v}+\phi_x)-p(V))_x+\alpha\phi_t \\ &=-\{U_t+(\alpha_1-\underline{\alpha}_1)(1+t)^{-\lambda}U+(p(V)-p'(\underline{v})V)_x\}, \end{aligned} \tag{2.14}$$

which can be derived into the following reformulated problem:

$$\begin{cases} \phi_{tt}+(p(V+\hat{v}+\phi_x)-p(V+\hat{v}))_x+\alpha\phi_t=F, \\ \phi(0,x)=\phi_0(x), \\ \phi_t(0,x)=\phi_1(x), \end{cases} \tag{2.15}$$

where

$$F := F_1 + F_2 + F_3 + F_4, \tag{2.16}$$

$$F_1 := -U_t, \tag{2.17}$$

$$F_2 := -(\alpha_1(x) - \underline{\alpha}_1)(1+t)^{-\lambda}U, \tag{2.18}$$

$$F_3 := -(p(V) - p'(\underline{v})V)_x, \tag{2.19}$$

$$F_4 := -(p(V+\hat{v}) - p(V))_x, \tag{2.20}$$

$$\phi_0(x) := \int_{-\infty}^{x} (v_0(y) - V(0,y) - \hat{v}(0,y))dy, \tag{2.21}$$

$$\phi_1(x) := u_0(x) - U(0,x) - \hat{u}(0,x). \tag{2.22}$$

Before we begin our next job, which is to demonstrate the existence and asymptotic behavior of the unique global-in-time solution ϕ to (2.15), we assume

$$\begin{cases} p \in C^4(\mathbf{R}_+), \quad \text{with } -p'(v) > 4c_0 > 0 \ (v > 0), \ (c_0 : \text{constant}) \\ \alpha_1 \in C^2(\mathbf{R}), \quad \text{with } \alpha_1 - \underline{\alpha}_1 \in L^2, \ |x|^{1/2}(\alpha_1 - \underline{\alpha}_1) \in L^2, \\ (1+|x|)|\alpha_{1x}| \in L^2, \ |\alpha_{1xx}| \in L^2, \ \text{and } \alpha_1 \geq \alpha_0 > 0 \ (\alpha_0 : \text{constant}), \end{cases} \tag{2.23}$$

and

$$v_0 - \underline{v} \in L^1 \cap H^2, \int_{-\infty}^{x} (v_0 - \underline{v})(y)dy \in L^2 \text{ and } u_0 - u_\pm \in L^2(\mathbf{R}_\pm), \ u_{0x} \in H^1, \tag{2.24}$$

which show that $(\phi_0, \phi_1) \in H^3 \times H^2$.

Before presenting our main results, we define

$$\delta_1 := \|v_0 - \underline{v}\|_{L^1} + |u_+| + |u_-|.$$

The following are the main results.

Lemma 2.1 *(Decay estimates of F). For $t \in \mathbf{R}_+$ and $\delta_1 = \|v_0 - \underline{v}\|_{L^1} + |u_+| + |u_-|$, it holds*

$$\begin{cases} \|\partial_x^k (V - \underline{v})\|_p \leq C\delta_0 (1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{k}{2}\right)(1+\lambda)}, \\ \|\partial_t^l (V - \underline{v})\|_p \leq C\delta_0 (1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+l\right)-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)\right)\lambda}, \\ \|\partial_x^k \partial_t^l (V - \underline{v})\|_p \leq C\delta_0 (1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{k}{2}+l\right)-\left(\frac{k}{2}+\frac{1}{2}\left(1-\frac{1}{p}\right)\right)\lambda}, \\ \|\partial_x^k \hat{v}\|_p + \|\partial_t^l \hat{v}\|_p \leq C(|u_+| + |u_-|)e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}} \leq C\delta_1 e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}}, \end{cases} \tag{2.25}$$

and

$$\begin{cases} \|F(t)\|^2 \le C\delta_1^2(1+t)^{-\frac{5(1+\lambda)}{2}} + C\delta_1^2(1+t)^{-\frac{7-\lambda}{2}}, \\ \|F_x(t)\|^2 \le C\delta_1^2(1+t)^{-3(1+\lambda)} + C\delta_1^2(1+t)^{-\frac{9+\lambda}{2}}, \\ \|F_t(t)\|^2 \le C\delta_1^2(1+t)^{-\frac{9+5\lambda}{2}} + C\delta_1^2(1+t)^{-\frac{11-\lambda}{2}}, \\ \|F_{tt}(t)\|^2 \le C\delta_1^2(1+t)^{-\frac{13+5\lambda}{2}} + C\delta_1^2(1+t)^{-\frac{15-\lambda}{2}}, \end{cases} \tag{2.26}$$

where $0 < c_1 < \alpha_0$, k and l are non-negative integers.

Proof. Due to (2.5), we can derive that

$$\begin{cases} \|\partial_x^k (V - \underline{v})\|_p \le C\delta_0(1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{k}{2}\right)(1+\lambda)}, \\ \|\partial_t^l (V - \underline{v})\|_p \le C\delta_0(1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+l\right)-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)\right)\lambda}, \\ \|\partial_x^k\partial_t^l (V - \underline{v})\|_p \le C\delta_0(1+t)^{-\left(\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{k}{2}+l\right)-\left(\frac{k}{2}+\frac{1}{2}\left(1-\frac{1}{p}\right)\right)\lambda}. \end{cases} \tag{2.27}$$

From (2.8), we have

$$\begin{aligned} \hat{v} =& \frac{\partial}{\partial x}\left(\int_\infty^t e^{-\alpha_1\frac{(1+s)^{1-\lambda}-1}{1-\lambda}} M_0(x) ds\right) \\ \le& |(u_+ - u_-)m_0(x)| \left|\int_\infty^t e^{-\alpha_1\frac{(1+s)^{1-\lambda}-1}{1-\lambda}} ds\right| \\ &+ |\alpha_1' M_0(x)| \left|\int_\infty^t -\frac{(1+s)^{1-\lambda}-1}{1-\lambda} e^{-\alpha_1\frac{(1+s)^{1-\lambda}-1}{1-\lambda}} ds\right| \\ \le& C|(u_+ - u_-)m_0(x)| \left|\int_\infty^t e^{-(\alpha_1-c_1)\frac{(1+s)^{1-\lambda}-1}{1-\lambda}} ds\right| e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}} \\ &+ C|\alpha_1' M_0(x)| \left|\int_\infty^t -\frac{(1+s)^{1-\lambda}-1}{1-\lambda} e^{-(\alpha_1-c_1)\frac{(1+s)^{1-\lambda}-1}{1-\lambda}} ds\right| e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}} \\ \le& C\left(|(u_+ - u_-)m_0(x)| + |\alpha_1' M_0(x)|\right) e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}}, \end{aligned}$$

therefore, we can derive that

$$\|\partial_x^k \hat{v}\|_p + \|\partial_t^l \hat{v}\|_p \le C(|u_+| + |u_-|)e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}} \le C\delta_1 e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}}. \tag{2.28}$$

Because of the first equation of (2.15) and (2.27), we obtain

$$\begin{aligned} &\|F_1(t)\|^2 \le \|U_t\|^2 = \|(\mu(t)V_x)_t\|^2 \le C\delta_0^2(1+t)^{-\frac{7-\lambda}{2}}, \\ &\|F_3(t)\|^2 \le \int |p'(V) - p'(\underline{v})|^2 |V_x|^2 dx \le C\|V_x\|_\infty^2 \|V - \underline{v}\|^2 \le C\delta_0^2(1+t)^{-\frac{5(1+\lambda)}{2}}, \end{aligned}$$

$$\|F_4(t)\|^2 \le C\delta_1^2 e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}}.$$

For F_2, from (2.5), (2.23) and (2.27), we have

$$\begin{aligned}
\|F_2(t)\|^2 &\le C\int |\alpha_1-\underline{\alpha}_1|^2|V_x|^2dx\\
&\le C\int |\alpha_1-\underline{\alpha}_1|^2|V_x|dx\cdot\|V_x\|_\infty\\
&\le C\int \frac{\delta_0|\alpha_1-\underline{\alpha}_1|^2|x|}{\left|\int_{-1}^t\mu(s)ds\right|}\cdot \mathrm{e}^{-\frac{x^2}{4\int_{-1}^t\mu(s)ds}}dx\cdot\|V_x\|_\infty\\
&\le C\delta_0\int \frac{|\alpha_1-\underline{\alpha}_1|^2|x|}{(1+t)^{\frac{3(1+\lambda)}{2}}}dx\cdot\|V_x\|_\infty\\
&\le C\delta_0(1+t)^{-\frac{3(1+\lambda)}{2}}\||\alpha_1-\underline{\alpha}_1|\cdot|x|^{\frac12}\|^2\|V_x\|_\infty\\
&\le C\delta_0^2(1+t)^{-\frac{5(1+\lambda)}{2}}.
\end{aligned}$$

Therefore, we have proved $\|F(t)\|^2\le C\delta_1^2(1+t)^{-\frac{5(1+\lambda)}{2}}+C\delta_1^2(1+t)^{-\frac{7-\lambda}{2}}$.

Next, we shall now establish the estimates for the higher-order derivatives of the function F. From (2.23), (2.27) and (2.28), one has

$$\begin{aligned}
\|F_{1x}(t)\|^2 &= \|U_{tx}\|^2=\|(\mu(t)V_{xx})_t\|^2\le C\delta_0^2(1+t)^{-\frac{9+\lambda}{2}},\\
\|F_{2x}(t)\|^2 &\le C\int(|\alpha_{1x}V_x|^2+|\alpha_1-\underline{\alpha}_1|^2V_{xx}^2)dx\\
&\le C(\|x^{-1}\cdot V_x\|_\infty^2\|x\cdot\alpha_{1x}\|^2+\|V_{xx}\|_\infty^2\|\alpha_1-\underline{\alpha}_1\|^2)\\
&\le C\delta_0^2(1+t)^{-3(1+\lambda)}(\|x\cdot\alpha_{1x}\|^2+\|\alpha_1-\underline{\alpha}_1\|^2)\\
&\le C\delta_0^2(1+t)^{-3(1+\lambda)},\\
\|F_{3x}(t)\|^2 &\le \int(p''(V)V_x^2+(p'(V)-p'(\underline{v}))V_{xx})^2dx\\
&\le C\left(\|V_x\|_\infty^2\|V_x\|^2+\|V-\underline{v}\|_\infty^2\|V_{xx}\|^2\right)\\
&\le C\delta_0^2(1+t)^{-\frac{7(1+\lambda)}{2}},\\
\|F_{4x}(t)\|^2 &\le C\delta_1^2e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}},
\end{aligned}$$

and

$$
\begin{aligned}
\|F_{1t}(t)\|^2 &= \|U_{tt}\|^2 = \|(\mu(t)V_x)_{tt}\|^2 \le C\delta_0(1+t)^{-\frac{11-\lambda}{2}},\\
\|F_{2t}(t)\|^2 &= C\|(\alpha_1-\underline{\alpha}_1)V_{tx}\|^2 \le C\|(\alpha_1-\underline{\alpha}_1)(1+t)^{\lambda}V_{xxx}\|^2\\
&\le C\delta_0^2(1+t)^{-\frac{9+5\lambda}{2}}\|x^{1/2}\cdot(\alpha-\underline{\alpha})\|^2 \le C\delta_0^2(1+t)^{-\frac{9+5\lambda}{2}},\\
\|F_{3t}(t)\|^2 &= \|(p'(V)-p'(\underline{v}))V_{tx}+p''(V)V_xV_t\|^2\\
&\le C\left(\|V-\underline{v}\|_\infty^2\|V_{tx}\|^2+\|V_x\|_\infty^2\|V_t\|^2\right)\\
&\le C\delta_0^2(1+t)^{-\frac{9+5\lambda}{2}},\\
\|F_{4t}(t)\|^2 &\le C\delta_1^2 e^{-\frac{c_1}{1-\lambda}t^{1-\lambda}}.
\end{aligned}
$$

Hence, we obtained $\|F_x(t)\|^2 \le C\delta_1^2(1+t)^{-3(1+\lambda)} + C\delta_1^2(1+t)^{-\frac{9+\lambda}{2}}$ and $\|F_t(t)\|^2 \le C\delta_1^2(1+t)^{-\frac{9+5\lambda}{2}} + C\delta_1^2(1+t)^{-\frac{11-\lambda}{2}}$. The estimation of $\|F_{tt}(t)\|^2$ follows a similar process as described above and is therefore omitted. □

Theorem 2.1 *(The case of $0\le\lambda<\frac{3}{5}$). Assume (2.23) and (2.24). When $\delta_1 = \|v_0-\underline{v}\|_{L^1}+|u_+|+|u_-|$ and $\|\phi_0\|_{H^3}+\|\phi_1\|_{H^2}$ are sufficiently small, then there exists a unique time-global solution satisfying*

$$
\phi \in C^k((0,\infty), H^{3-k}(\mathbf{R})), \quad k=0,1,2,3, \tag{2.29}
$$

and

$$
\phi_t \in C^k((0,\infty), H^{2-k}(\mathbf{R})), \quad k=0,1,2, \tag{2.30}
$$

furthermore, we have

$$
\begin{aligned}
&\sum_{k=0}^{2}(1+t)^{(1+\lambda)k}\|\partial_x^k\phi(t,\cdot)\|^2 + (1+t)^{\frac{5(1+\lambda)}{2}}\|\partial_x^3\phi(t,\cdot)\|^2\\
&+\sum_{k=0}^{2}(1+t)^{(1+\lambda)k+2}\|\partial_x^k\phi_t(t,\cdot)\|^2 + \int_0^t\left[\sum_{j=1}^{2}(1+s)^{(1+\lambda)j-1}\|\partial_x^j\phi(s,\cdot)\|^2\right.\\
&\left.+\sum_{j=0}^{2}(1+s)^{(1+\lambda)j+1}\|\partial_x^j\phi_t(s,\cdot)\|^2\right]ds + \frac{1}{1+t}\int_0^t(1+s)^{\frac{5(1+\lambda)}{2}}\|\partial_x^3\phi(s,\cdot)\|^2ds\\
\le& C(\|\phi_0\|_{H^3}^2+\|\phi_1\|_{H^2}^2+\delta_1).
\end{aligned} \tag{2.31}
$$

Theorem 2.2 *(The case of $\frac{3}{5}<\lambda<1$). Assume (2.23) and (2.24). When $\delta_1 = \|v_0-\underline{v}\|_{L^1}+|u_+|+|u_-|$ and $\|\phi_0\|_{H^3}+\|\phi_1\|_{H^2}$ are sufficiently small, then there exists a unique time-global solution satisfying*

$$
\phi \in C^k((0,\infty), H^{3-k}(\mathbf{R})), \quad k=0,1,2,3, \tag{2.32}
$$

and

$$\phi_t \in C^k((0,\infty), H^{2-k}(\mathbf{R})), \quad k = 0, 1, 2, \tag{2.33}$$

furthermore, we have

$$\begin{aligned}
&\sum_{k=0}^{2}(1+t)^{(1+\lambda)k+\frac{3}{2}-\frac{5\lambda}{2}}\|\partial_x^k\phi(t,\cdot)\|^2 + (1+t)^4\|\partial_x^3\phi(t,\cdot)\|^2\\
&\quad+\sum_{k=0}^{2}(1+t)^{(1+\lambda)k+\frac{7}{2}-\frac{5\lambda}{2}}\|\partial_x^k\phi_t(t,\cdot)\|^2 + \frac{1}{1+t}\int_0^t(1+s)^4\|\partial_x^3\phi(s,\cdot)\|^2 ds\\
&\le C(\|\phi_0\|_{H^3}^2 + \|\phi_1\|_{H^2}^2 + \delta_1),
\end{aligned} \tag{2.34}$$

and for any $\varepsilon \in (\frac{3}{2}-\frac{3\lambda}{2}, \lambda)$*, we have*

$$\begin{aligned}
&\int_0^t\left[\sum_{j=0}^{2}(1+s)^{(1+\lambda)(j-1)+\varepsilon}\|\partial_x^j\phi(s,\cdot)\|^2 + \sum_{j=0}^{2}(1+s)^{(1+\lambda)j+\varepsilon-\lambda+1}\|\partial_x^j\phi_t(s,\cdot)\|^2\right]ds\\
&\le C(\|\phi_0\|_{H^3}^2 + \|\phi_1\|_{H^2}^2 + \delta_1)(1+t)^{\varepsilon+\frac{3\lambda}{2}-\frac{3}{2}}.
\end{aligned} \tag{2.35}$$

Theorem 2.3 *(The case of* $\lambda = \frac{3}{5}$*). Assume* (2.23) *and* (2.24)*. When* $\delta_1 = \|v_0 - \underline{v}\|_{L^1} + |u_+| + |u_-|$ *and* $\|\phi_0\|_{H^3} + \|\phi_1\|_{H^2}$ *are sufficiently small, then there exists a unique time-global solution satisfying*

$$\phi \in C^k((0,\infty), H^{3-k}(\mathbf{R})), \quad k = 0, 1, 2, 3, \tag{2.36}$$

and

$$\phi_t \in C^k((0,\infty), H^{2-k}(\mathbf{R})), \quad k = 0, 1, 2, \tag{2.37}$$

furthermore, we have

$$\begin{aligned}
&\sum_{k=0}^{2}(1+t)^{\frac{8}{5}k}\|\partial_x^k\phi(t,\cdot)\|^2 + (1+t)^4\|\partial_x^3\phi(t,\cdot)\|^2\\
&\quad+\sum_{k=0}^{2}(1+t)^{\frac{8}{5}k+2}\|\partial_x^k\phi_t(t,\cdot)\|^2 + \int_0^t\left[\sum_{j=1}^{2}(1+s)^{\frac{8}{5}j-1}\|\partial_x^j\phi(s,\cdot)\|^2\right.\\
&\quad\left.+\sum_{j=0}^{2}(1+s)^{\frac{8}{5}j+1}\|\partial_x^j\phi_t(s,\cdot)\|^2\right]ds + \frac{1}{1+t}\int_0^t(1+s)^4\|\partial_x^3\phi(s,\cdot)\|^2 ds\\
&\le C(\|\phi_0\|_{H^3}^2 + \|\phi_1\|_{H^2}^2 + \delta_1)\ln(2+t).
\end{aligned} \tag{2.38}$$

Notice that $\phi_x = v - V - \hat{v},\ \phi_t = u - U - \hat{u}$, we immediately obtain the following convergence rates.

Corollary 2.1. *Under the assumptions of Theorem 2.1, system (2.1) possesses a uniquely global solution $(v,u)(t,x)$ satisfying*

$$\|(v-V)(t,\cdot)\|_{L^\infty(\mathbf{R})} \le \begin{cases} C(1+t)^{-\frac{3(\lambda+1)}{4}}, & 0 \leqslant \lambda < \frac{3}{5}, \\ C\ln^{\frac{1}{2}}(2+t)(1+t)^{-\frac{6}{5}}, & \lambda = \frac{3}{5}, \\ C(1+t)^{\frac{\lambda-3}{2}}, & \frac{3}{5} < \lambda < 1, \end{cases} \tag{2.39}$$

and

$$\|(u-U)(t,\cdot)\|_{L^\infty(\mathbf{R})} \le \begin{cases} C(1+t)^{-\frac{\lambda+5}{4}}, & 0 \leqslant \lambda < \frac{3}{5}, \\ C\ln^{\frac{1}{2}}(2+t)(1+t)^{-\frac{7}{5}}, & \lambda = \frac{3}{5}, \\ C(1+t)^{\lambda-2}, & \frac{3}{5} < \lambda < 1. \end{cases} \tag{2.40}$$

Remark 2.1. Based on the decay estimates of F, to match the decay rates of the linear parts for the energy estimates, we realize that $\frac{3}{5}$ is determined as the turning point of the decay rates of F for the nonlinear parts, see (3.18) and (3.41) below. Therefore, in our work, $\lambda = \frac{3}{5}$ is the critical case. When considering the critical case, it is often inevitable that some decay estimates will be lost compared with the expected results. Just as in the critical case where $\lambda = \frac{1}{7}$ discussed by Cui-Yin-Zhang-Zhu [3], the obtained decay estimate is reduced by $(1+t)^\beta$ compared with the expected results (where β is an arbitrary number greater than 0). In order to reduce such a loss of estimates, we have employed a new estimation method in our critical case $\lambda = \frac{3}{5}$. Eventually, our decay estimate only loses $\ln^{\frac{1}{2}}(2+t)$ compared with the expected results, which is better than the loss in the exponential form, also better than the previous studies.

Remark 2.2. When α_1 is a constant, conducting energy estimates is relatively straightforward. However, when α_1 depends on the spatial variable x, difficulties will arise from the decay energy estimates. In particular, the estimates of F are less than the ideal one compared to the constant case, which may cause our results to fall far short of expectations. To address these potential issues and ensure the success of our energy estimates and subsequent proofs, we impose specific conditions on α_1. Considering that our subsequent proofs require favorable decay estimates for F and its derivatives, the solutions of equation (1.1) are expected to possess a certain degree of smoothness. Therefore, in the description of α_1 in Hypothesis (2.23), we emphasize that α_1 has a certain degree of smoothness, and the product of α_1 by the weight function related to x still belongs to the function space $L^2(\mathbb{R})$.

3. Decay estimates to reformulated problem

The main goal of this section is to obtain the decay rates of the solution (ϕ_x, ϕ_t). We devote ourselves to the estimates of the solution $(\phi_x, \phi_t)(t,x)$ under the a priori assumption

$$N_1(T_1) := \sup_{0\le t\le T_1} \{\|\phi, \phi_t\|_{H^3\times H^2} + (1+t)\|\phi_{tx}\| + (1+t)\|\phi_{txx}\|\} \le \delta,$$

for some $\delta \ll 1$ and $0 < T_1 < \infty$. Then we will prove theorem in a series of several steps. To do that, we rewrite (2.15) as

$$\phi_{tt} + (p(\hat{V} + \phi_x) - p(\hat{V}))_x + \alpha\phi_t = F, \tag{3.1}$$

where $\hat{V} = V + \hat{v}$ and

$$F = -\{U_t + (\alpha_1 - \underline{\alpha}_1)(1+t)^{-\lambda}U + (p(V) - p'(\underline{v})V)_x + (p(V+\hat{v}) - p(V))_x\}. \tag{3.2}$$

We can derive that

$$\delta_1 = \|v_0 - \underline{v}\|_{L^1} + |u_+| + |u_-| \ge \min\{1, \underline{\alpha}\}\delta_0,$$

where δ_0 is defined as in (2.12), then we define that

$$I_0 := \|\phi_0, \phi_1\|_{H^3\times H^2} + \delta_1.$$

Lemma 3.1. *Under the assumptions of Theorem 2.1, when $\delta + \delta_1 \ll 1$, it holds that:*

- *For $0 \le \lambda < \frac{3}{5}$, then*

$$\|\phi\|^2 + (1+t)^{1+\lambda}(\|\phi_t\|^2 + \|\phi_x\|^2) + \int_0^t\int \left((1+s)\phi_t^2 + (1+s)^{\lambda}\phi_x^2\right) dxds$$
$$\le CI_0. \tag{3.3}$$

- *For $\frac{3}{5} < \lambda < 1$, then*

$$(1+t)^{\frac{3-5\lambda}{2}}\|\phi\|^2 + (1+t)^{\frac{5-3\lambda}{2}}\|\phi_x\|^2 + (1+t)^{\frac{5-3\lambda}{2}}\|\phi_t\|^2 \le CI_0, \tag{3.4}$$

and

$$\int_0^t\int \left((1+s)^{\varepsilon+1-\lambda}\phi_t^2 + (1+s)^{\varepsilon}\phi_x^2 + (1+s)^{\varepsilon-\lambda-1}\phi^2\right) dxds$$
$$\le CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}, \qquad \textit{for any} \quad \frac{3(1-\lambda)}{2} < \varepsilon < \lambda. \tag{3.5}$$

Proof. Let $\varepsilon \ge 0$ be a number which will be determined later. Multiplying (3.1) by $(1+t)^{\varepsilon}\phi$ and integrating the resulting equality with respect to x over $\mathbf{R}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\varepsilon}\phi_t\phi+(1+t)^{\varepsilon-\lambda}\alpha_1(x)\frac{\phi^2}{2}-\frac{\varepsilon}{2}(1+t)^{\varepsilon-1}\phi^2\right)dx\\
&\quad+\int(1+t)^{\varepsilon}\left(p(\hat{V})-p(\hat{V}+\phi_x)\right)\phi_x dx+\int\Big(-(1+t)^{\varepsilon}\phi_t^2\\
&\quad+\frac{\lambda-\varepsilon}{2}(1+t)^{\varepsilon-\lambda-1}\alpha_1(x)\phi^2+\frac{\varepsilon(\varepsilon-1)}{2}(1+t)^{\varepsilon-2}\phi^2\Big)dx\\
&\le\int(1+t)^{\varepsilon}\phi F dx.
\end{aligned}\tag{3.6}
$$

Next, multiplying (3.1) by $(1+t)^{\varepsilon+\lambda}\phi_t$ and integrating it over $\mathbf{R}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int(1+t)^{\varepsilon+\lambda}\frac{\phi_t^2}{2}dx+\int\left(p(\hat{V}+\phi_x)-p(\hat{V})\right)_x(1+t)^{\varepsilon+\lambda}\phi_t dx\\
&\quad+\int\left((1+t)^{\varepsilon}\alpha_1(x)\phi_t^2-\frac{\lambda+\varepsilon}{2}(1+t)^{\varepsilon+\lambda-1}\phi_t^2\right)dx\\
&\le\int(1+t)^{\varepsilon+\lambda}\phi_t F dx.
\end{aligned}\tag{3.7}
$$

The second term can be estimated as

$$
\begin{aligned}
&\int\left(p(\hat{V}+\phi_x)-p(\hat{V})\right)_x(1+t)^{\varepsilon+\lambda}\phi_t dx\\
&=\int\left(p(\hat{V})-p(\hat{V}+\phi_x)\right)(1+t)^{\varepsilon+\lambda}\phi_{tx}dx\\
&=\int\frac{d}{dt}\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}\left(p(\hat{V})-p(\hat{V}+s)\right)ds dx\\
&\quad+\int(1+t)^{\varepsilon+\lambda}\left(p(\hat{V}+\phi_x)-p(\hat{V})-p'(\hat{V})\phi_x\right)\hat{V}_t dx\\
&\quad-\int\int_0^{\phi_x}(\varepsilon+\lambda)(1+t)^{\varepsilon+\lambda-1}\left(p(\hat{V})-p(\hat{V}+s)\right)ds dx\\
&\ge\int\frac{d}{dt}\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}\left(p(\hat{V})-p(\hat{V}+s)\right)ds dx\\
&\quad-\int\int_0^{\phi_x}(\varepsilon+\lambda)(1+t)^{\varepsilon+\lambda-1}\left(p(\hat{V})-p(\hat{V}+s)\right)ds dx\\
&\quad-\int C(|V_t|_\infty+|\hat{v}_t|_\infty)(1+t)^{\varepsilon+\lambda}\phi_x^2 dx.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\int_0^{\phi_x}\Big(p(\hat V)-p(\hat V+s)\Big)ds = p(\hat V)\phi_x-\int_0^{\phi_x}p(\hat V+s)ds\\
&= p(\hat V+\phi_x)\phi_x - p'(\hat V+\phi_x)\phi_x^2-\int_0^{\phi_x}p(\hat V+s)ds+\int_0^{\phi_x}p(\hat V+s)ds\\
&\quad - p(\hat V+\phi_x)\phi_x + p'(\hat V+\phi_x)\frac{\phi_x^2}{2}+o(\phi_x^2)\\
&= -p'(\hat V+\phi_x)\frac{\phi_x^2}{2}+o(\phi_x^2),
\end{aligned}
$$

we can easily derive

$$
\begin{aligned}
\int -p'(\hat V+\phi_x)\phi_x^2dx &\ge \int\int_0^{\phi_x}\Big(p(\hat V)-p(\hat V+s)\Big)dsdx\\
&\ge \int -\frac14 p'(\hat V+\phi_x)\phi_x^2dx. \qquad (3.8)
\end{aligned}
$$

Hence, (3.7) can be derived as

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\varepsilon+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}\Big(p(\hat V)-p(\hat V+s)\Big)ds\right)dx\\
&\quad+\int\left((1+t)^{\varepsilon}\alpha_1(x)\phi_t^2-\frac{\lambda+\varepsilon}{2}(1+t)^{\varepsilon+\lambda-1}\Big(\phi_t^2-2p'(\hat V+\phi_x)\phi_x^2\Big)\right)dx\\
&\quad-\int C(1+t)^{\varepsilon+\lambda}(|V_t|_\infty+|\hat v_t|_\infty)\phi_x^2dx\\
&\le \int(1+t)^{\varepsilon+\lambda}\phi_t F dx. \qquad (3.9)
\end{aligned}
$$

Choosing $k=\frac{\alpha_0}{4}$, by calculating $k\cdot(3.6)+(3.9)$ and applying Cauchy's inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\Bigg((1+t)^{\varepsilon+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}(p(\hat V)-p(\hat V+s))ds\\
&\quad+k\left((1+t)^{\varepsilon}\phi_t\phi+(1+t)^{\varepsilon-\lambda}\alpha_1(x)\frac{\phi^2}{2}-\frac{\varepsilon}{2}(1+t)^{\varepsilon-1}\phi^2\right)\Bigg)dx\\
&\quad+\int\left((\alpha_1(x)-k)\,(1+t)^{\varepsilon}-\frac{\lambda+\varepsilon}{2}(1+t)^{\varepsilon+\lambda-1}\right)\phi_t^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int\Big(-k(1+t)^{\varepsilon}p'(\hat{V}+\phi_x)+(\lambda+\varepsilon)(1+t)^{\varepsilon+\lambda-1}p'(\hat{V}+\phi_x)\\
&-C(1+t)^{\varepsilon+\lambda}(|V_t|_\infty+|\hat{v}_t|_\infty)\Big)\phi_x^2dx\\
&+\int k\left(\frac{\lambda-\varepsilon}{2}(1+t)^{\varepsilon-\lambda-1}\alpha_1(x)+\frac{\varepsilon(\varepsilon-1)}{2}(1+t)^{\varepsilon-2}\right)\phi^2dx\\
\le&\int\Big((1+t)^{\varepsilon+\lambda}\phi_tF+k(1+t)^{\varepsilon}\phi F\Big)dx\\
=&\int\left(\delta_1^{\frac{1}{2}}(1+t)^{\frac{\varepsilon}{2}}\phi_t\cdot\frac{1}{\delta_1^{\frac{1}{2}}}(1+t)^{\lambda+\frac{\varepsilon}{2}}F+\delta_1^{\frac{1}{2}}(1+t)^{-\frac{\theta}{2}}\phi\cdot\frac{k}{\delta_1^{\frac{1}{2}}}(1+t)^{\varepsilon+\frac{\theta}{2}}F\right)dx\\
\le&\int\delta_1\Big((1+t)^{\varepsilon}\phi_t^2+(1+t)^{-\theta}\phi^2\Big)dx\\
&+\int\frac{1}{\delta_1}\Big((1+t)^{2\lambda+\varepsilon}+k^2(1+t)^{2\varepsilon+\theta}\Big)F^2dx, \qquad (3.10)
\end{aligned}
$$

for some constant $\theta > 0$ which will be determined below.

To obtain the estimates of (ϕ_x, ϕ_t) over finite time intervals, we define the functional as follows:

$$
\begin{aligned}
\mathcal{E}(t):=&\frac{(1+t)^{\lambda}}{2}\|\phi_t\|^2+\int\int_0^{\phi_x}(1+t)^{\lambda}\Big(p(\hat{V})-p(\hat{V}+s)\Big)dsdx\\
&+k\int\phi\phi_tdx+\frac{k(1+t)^{-\lambda}}{2}\|\alpha_1(x)^{\frac{1}{2}}\phi\|^2.
\end{aligned}
$$

By Cauchy's inequality, we obtain

$$
\begin{aligned}
k\int\phi\phi_tdx=&\int 2^{\frac{1}{2}}k(1+t)^{-\frac{\lambda}{2}}\phi\cdot(1+t)^{\frac{\lambda}{2}}\frac{\phi_t}{2^{\frac{1}{2}}}dx\\
\le&\frac{1}{4}(1+t)^{\lambda}\|\phi_t\|^2+k^2(1+t)^{-\lambda}\|\phi\|^2.
\end{aligned}
$$

Due to $0 < 4k = \alpha_0 \le \alpha_1(x)$, (3.8) and the above inequality, we can derive

$$
\mathcal{E}(t)\ge\frac{(1+t)^{\lambda}}{4}\|\phi_t\|^2+c_0(1+t)^{\lambda}\|\phi_x\|^2+k^2(1+t)^{-\lambda}\|\phi\|^2.
$$

By taking $\varepsilon = 0$ together with $\theta = 1+\lambda$ in (3.10), and using Lemma 2.1, we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t)+\int\left((\alpha_1(x)-k)\phi_t^2-kp'(\hat{V}+\phi_x)\phi_x^2+\frac{k\lambda}{2}(1+t)^{-\lambda-1}\phi^2\right)dx\\
\le&\int\left(\delta_1+\frac{\lambda}{2}(1+t)^{\lambda-1}\right)\phi_t^2dx+\int\delta_1(1+t)^{-\lambda-1}\phi^2dx+C\delta_1(1+t)^{\frac{-3+\lambda}{2}}
\end{aligned}
$$

$$+\int\Big(C(1+t)^{\frac{-3+\lambda}{2}}-\lambda(1+t)^{\lambda-1}p'(\hat V+\phi_x)\Big)\phi_x^2dx. \tag{3.11}$$

Using Gronwall's inequality to (3.11) on $[0,t]$, we obtain

$$\begin{aligned}&\int\Big((1+t)^\lambda\phi_t^2+(1+t)^\lambda\phi_x^2+(1+t)^{-\lambda}\phi^2\Big)dx\\&+\int_0^t\int\Big(\phi_t^2+\phi_x^2+(1+s)^{-\lambda-1}\phi^2\Big)dxds\\&\le C(t)(\mathcal{E}(0)+I_0)\le C(t)I_0.\end{aligned} \tag{3.12}$$

According to Lemma 2.1, we have $C(1+t)^{\varepsilon+\lambda}(|V_t|_\infty+|\hat v_t|_\infty)\le C(1+t)^{\varepsilon+\frac{\lambda-3}{2}}$. Let $T\gg1$ such that

$$\begin{cases}(\alpha_1(x)-k-\delta_1)(1+T)^\varepsilon-\frac{\lambda+\varepsilon}{2}(1+T)^{\varepsilon+\lambda-1}\ge k(1+T)^\varepsilon,\\ k^2(1+T)^{\varepsilon-\lambda}-\frac{k\varepsilon}{2}(1+T)^{\varepsilon-1}\ge\frac{k^2}{2}(1+T)^{\varepsilon-\lambda},\\ -k(1+T)^\varepsilon p'(\hat V+\phi_x)+(\lambda+\varepsilon)(1+T)^{\varepsilon+\lambda-1}p'(\hat V+\phi_x)\\ -C\delta_1(1+T)^{\varepsilon+\frac{\lambda-3}{2}}\ge kc_0(1+T)^\varepsilon,\end{cases} \tag{3.13}$$

for $\lambda<1$ and $\varepsilon\le1$. Hence, by setting $\lambda<1$, $\varepsilon\le1$ and $t\ge T\gg1$ in (3.10), and applying Lemma 2.1 along with (3.13) (where $k=\frac{\alpha_0}{4}$), we obtain

$$\begin{aligned}&\frac{d}{dt}\int\Bigg((1+t)^{\varepsilon+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}(p(\hat V)-p(\hat V+s))ds\\&+k\Big((1+t)^\varepsilon\phi_t\phi+(1+t)^{\varepsilon-\lambda}\alpha_1(x)\frac{\phi^2}{2}-\frac{\varepsilon}{2}(1+t)^{\varepsilon-1}\phi^2\Big)\Bigg)dx\\&+\int\Big(k(1+t)^\varepsilon\phi_t^2+kc_0(1+t)^\varepsilon\phi_x^2+\frac{\lambda-\varepsilon}{2}(1+t)^{\varepsilon-\lambda-1}\alpha_1(x)\phi^2\Big)dx\\&\le\int\Big(C(1+t)^{\varepsilon-2}\phi^2+\delta_1(1+t)^{-\theta}\phi^2\Big)dx+C\delta_1\Big((1+t)^{-\frac72+\frac{5\lambda}{2}+\varepsilon}\\&+(1+t)^{-\frac52-\frac{\lambda}{2}+\varepsilon}+(1+t)^{-\frac72+\frac{\lambda}{2}+2\varepsilon+\theta}+(1+t)^{-\frac52-\frac{5\lambda}{2}+2\varepsilon+\theta}\Big).\end{aligned} \tag{3.14}$$

Case1. $0\le\lambda<\frac35$

In this case, it is straightforward to see that $1<\min\left\{\frac52-\frac{5\lambda}{2},\frac32+\frac{\lambda}{2}\right\}$. Therefore, we can let $t\ge T\gg1$ and take $\varepsilon=\lambda$ in (3.14). Then, there exists a constant θ such that $1<\theta<\min\left\{\frac52-\frac{5\lambda}{2},\frac32+\frac{\lambda}{2}\right\}$, from which we obtain

$$\frac{d}{dt}\int\Bigg((1+t)^{2\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{2\lambda}(p(\hat V)-p(\hat V+s))ds+k\Big((1+t)^\lambda\phi_t\phi$$

$$
\begin{aligned}
&+\alpha_1(x)\frac{\phi^2}{2}-\frac{\lambda}{2}(1+t)^{\lambda-1}\phi^2\Big)\Big)dx+\int\Big(k(1+t)^{\lambda}\phi_t^2+kc_0(1+t)^{\lambda}\phi_x^2\Big)dx\\
\le&\int C\Big((1+t)^{\lambda-2}\alpha_1(x)\phi^2+(1+t)^{-\theta}\phi^2\Big)dx+C\delta_1\Big((1+t)^{-\frac{7}{2}+\frac{7\lambda}{2}}+(1+t)^{-\frac{5}{2}+\frac{\lambda}{2}}\\
&+(1+t)^{-\frac{7}{2}+\frac{5\lambda}{2}+\theta}+(1+t)^{-\frac{5}{2}-\frac{\lambda}{2}+\theta}\Big).
\end{aligned}
\tag{3.15}
$$

Due to $k=\frac{\alpha_0}{4}$ and (3.13), we define that

$$
\begin{aligned}
\mathcal{E}_1(t):=&\frac{(1+t)^{2\lambda}}{2}\|\phi_t\|^2+\int\int_0^{\phi_x}(1+t)^{2\lambda}\Big(p(\hat V)-p(\hat V+s)\Big)dsdx\\
&+k\Big((1+t)^{\lambda}\int\phi_t\phi dx+\frac{1}{2}\|\alpha_1(x)^{\frac{1}{2}}\phi\|^2-\frac{\lambda}{2}(1+t)^{\lambda-1}\|\phi\|^2\Big)\\
\ge&\frac{(1+t)^{2\lambda}}{4}\|\phi_t\|^2+c_0(1+t)^{2\lambda}\|\phi_x\|^2+k^2\|\phi\|^2-\frac{k\lambda}{2}(1+t)^{\lambda-1}\|\phi\|^2\\
\ge&\frac{(1+t)^{2\lambda}}{4}\|\phi_t\|^2+c_0(1+t)^{2\lambda}\|\phi_x\|^2+\frac{k^2}{2}\cdot\|\phi\|^2.
\end{aligned}
$$

Hence, from (3.15), we can derive

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_1(t)+\int\Big(k(1+t)^{\lambda}\phi_t^2+kc_0(1+t)^{\lambda}\phi_x^2\Big)dx\le C\Big((1+t)^{\lambda-2}+(1+t)^{-\theta}\Big)\mathcal{E}_1(t)\\
&+C\delta_1\Big((1+t)^{-\frac{7}{2}+\frac{5\lambda}{2}+\theta}+(1+t)^{-\frac{5}{2}-\frac{\lambda}{2}+\theta}\Big),
\end{aligned}
\tag{3.16}
$$

Using the Gronwall's inequality to (3.16) on $[T,t]$ and combining with (3.12), we have

$$
\mathcal{E}_1(t)+\int_0^t\int\Big((1+s)^{\lambda}\phi_t^2+(1+s)^{\lambda}\phi_x^2\Big)dxds\le C\left(C(T)I_0+\delta_1\right)\le CI_0. \tag{3.17}
$$

Furthermore, by setting $\varepsilon=1$ and $t\ge T\gg 1$ in (3.9), and applying (3.13) along with Lemma 2.1, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{1+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{1+\lambda}\Big(p(\hat V)-p(\hat V+s)\Big)ds\right)dx\\
&+\int\Big(k(1+t)\phi_t^2+(1+\lambda)(1+t)^{\lambda}p'(\hat V+\phi_x)\phi_x^2\Big)dx\\
&-\int C\delta_1(1+t)^{\frac{\lambda-1}{2}}\phi_x^2dx+\int\delta_1(1+t)\phi_t^2dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{d}{dt}\int\left((1+t)^{1+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{1+\lambda}\left(p(\hat{V})-p(\hat{V}+s)\right)ds\right)dx\\
&\quad+\int\left((1+t)\left(\alpha_1(x)-\delta_1\right)\phi_t^2-\frac{\lambda+1}{2}(1+t)^{\lambda}\left(\phi_t^2-2p'(\hat{V}+\phi_x)\phi_x^2\right)\right)dx\\
&\quad-\int C\delta_1(1+t)^{\frac{\lambda-1}{2}}\phi_x^2dx+\int\delta_1(1+t)\phi_t^2dx\\
&\leq\int(1+t)^{1+\lambda}\phi_tFdx\leq\int\left(\delta_1(1+t)\phi_t^2+\frac{1}{\delta_1}(1+t)^{1+2\lambda}F^2\right)dx\\
&\leq\int\delta_1(1+t)\phi_t^2dx+\delta_1\left((1+t)^{\frac{5\lambda-5}{2}}+(1+t)^{\frac{-\lambda-3}{2}}\right).
\end{aligned}
$$

Then, by integrating the above inequality over $[0,t]$ and using (3.17), we obtain the improved decay rates for the functions ϕ_t and ϕ_x as follows:

$$
(1+t)^{1+\lambda}(\|\phi_t\|^2+\|\phi_x\|^2)+\int_0^t\int(1+s)\phi_t^2dxds\leq CI_0, \tag{3.18}
$$

which, together with (3.17), deduces (3.3).

Case2. $\frac{3}{5}<\lambda<1$

In this case, due to $\frac{3(1-\lambda)}{2}<\lambda$, we can take $t\geq T\gg 1$, $\frac{3(1-\lambda)}{2}<\varepsilon<\lambda$ and $\theta=\lambda-\varepsilon+1$. Therefore, from (3.14), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\varepsilon+\lambda}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}(p(\hat{V})-p(\hat{V}+s))ds\right.\\
&\quad\left.+k\Big((1+t)^{\varepsilon}\phi_t\phi+(1+t)^{\varepsilon-\lambda}\alpha_1(x)\frac{\phi^2}{2}-\frac{\varepsilon}{2}(1+t)^{\varepsilon-1}\phi^2\Big)\right)dx\\
&\quad+\int\left(k(1+t)^{\varepsilon}\phi_t^2+kc_0(1+t)^{\varepsilon}\phi_x^2+\Big(\frac{\lambda-\varepsilon}{2}-\delta_1\Big)(1+t)^{\varepsilon-\lambda-1}\alpha_1(x)\phi^2\right)dx\\
&\leq\int C(1+t)^{\varepsilon-2}\alpha_1(x)\phi^2dx+C\delta_1\left((1+t)^{-\frac{7}{2}+\frac{5\lambda}{2}+\varepsilon}+(1+t)^{-\frac{5}{2}-\frac{\lambda}{2}+\varepsilon}\right.\\
&\quad\left.+(1+t)^{-\frac{5}{2}+\frac{3\lambda}{2}+\varepsilon}+(1+t)^{-\frac{3}{2}-\frac{3\lambda}{2}+\varepsilon}\right)\\
&\leq\int C(1+t)^{\varepsilon-2}\alpha_1(x)\phi^2dx+C\delta_1(1+t)^{-\frac{5}{2}+\frac{3\lambda}{2}+\varepsilon}.
\end{aligned} \tag{3.19}
$$

Here, based on $k=\frac{\alpha_0}{4}$ and (3.13), we define

$$
\mathcal{E}_2(t):=\frac{(1+t)^{\varepsilon+\lambda}}{2}\|\phi_t\|^2+k\Big((1+t)^{\varepsilon}\|\phi_t\phi\|_{L^1}+\frac{(1+t)^{\varepsilon-\lambda}}{2}\|\alpha_1(x)^{\frac{1}{2}}\phi\|^2\Big)
$$

$$
\begin{aligned}
&+\int\int_0^{\phi_x}(1+t)^{\varepsilon+\lambda}(p(\hat{V})-p(\hat{V}+s))dsdx-\frac{\varepsilon(1+t)^{\varepsilon-1}}{2}\|\phi\|^2\\
&\geq \frac{(1+t)^{\varepsilon+\lambda}}{4}\|\phi_t\|^2+c_0(1+t)^{\varepsilon+\lambda}\|\phi_x\|^2+k^2(1+t)^{\varepsilon-\lambda}\|\phi\|^2-\frac{\varepsilon}{2}(1+t)^{\varepsilon-1}\|\phi\|^2\\
&\geq \frac{(1+t)^{\varepsilon+\lambda}}{4}\|\phi_t\|^2+c_0(1+t)^{\varepsilon+\lambda}\|\phi_x\|^2+\frac{k^2}{2}\cdot(1+t)^{\varepsilon-\lambda}\|\phi\|^2.
\end{aligned}
$$

Therefore, we can derive from (3.19) that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_2(t)+\int\Big(k(1+t)^{\varepsilon}\phi_t^2+kc_0(1+t)^{\varepsilon}\phi_x^2+\Big(\frac{\lambda-\varepsilon}{2}-\delta_1\Big)(1+t)^{\varepsilon-\lambda-1}\phi^2\Big)dx\\
&\leq C(1+t)^{\lambda-2}\mathcal{E}_2(t)+C\delta_1(1+t)^{-\frac{5}{2}+\frac{3\lambda}{2}+\varepsilon}.
\end{aligned}
\tag{3.20}
$$

Using the Gronwall's inequality to (3.20) on $[T,t]$ and combining with (3.12), we obtain

$$
\begin{aligned}
&\mathcal{E}_2(t)+\int_0^t\int\Big((1+s)^{\varepsilon}\phi_t^2+(1+s)^{\varepsilon}\phi_x^2+(1+s)^{\varepsilon-\lambda-1}\phi^2\Big)dxds\\
&\leq C\delta_1(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}+C(T)I_0\leq CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}.
\end{aligned}
\tag{3.21}
$$

Furthermore, let $t\geq T\gg 1$, and substitute ε with $\varepsilon+1-\lambda$ in (3.9). By integrating over $[0,t]$ and applying (3.13) together with (3.21), we obtain an improved decay rates for the functions ϕ_t and ϕ_x as follows:

$$
\begin{aligned}
&(1+t)^{1+\varepsilon}(\|\phi_t\|^2+\|\phi_x\|^2)+\int_0^t\int(1+s)^{\varepsilon+1-\lambda}\phi_t^2dxds\\
&\leq CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon},
\end{aligned}
\tag{3.22}
$$

which, together with (3.21), deduces (3.4). This completes the proof of Lemma 3.1. □

In the subsequent lemma, we will deduce the decay rates for the higher-order derivatives of the global solution $\phi(t,x)$.

Lemma 3.2. *Under the assumptions of Theorem 2.1, when $\delta+\delta_1\ll 1$, it holds that:*

- *For $0\leq\lambda<\frac{3}{5}$, then*

$$
\begin{aligned}
&(1+t)^2\|\phi_t\|^2+(1+t)^{3+\lambda}(\|\phi_{tt}\|^2+\|\phi_{tx}\|^2)+(1+t)^{2+2\lambda}\|\phi_{xx}\|^2\\
&+\int_0^t\int\Big((1+s)^3\phi_{tt}^2+(1+s)^{2+\lambda}\phi_{tx}^2+(1+s)^{1+2\lambda}\phi_{xx}^2\Big)dxds
\end{aligned}
$$

$$\leq CI_0. \tag{3.23}$$

- *For* $\frac{3}{5} < \lambda < 1$*, then*

$$\begin{cases} (1+t)^{\frac{7-5\lambda}{2}}\|\phi_t\|^2 + (1+t)^{\frac{7-1\lambda}{2}}\|\phi_{xx}\|^2 + (1+t)^{\frac{9-3\lambda}{2}}\|\phi_{tx}\|^2 + (1+t)^{\frac{9-3\lambda}{2}}\|\phi_{tt}\|^2 \\ \leq CI_0, \\ \int_0^t \int \Big((1+s)^{3+\varepsilon-\lambda}\phi_{tt}^2 + (1+s)^{2+\varepsilon}\phi_{tx}^2 + (1+s)^{1+\varepsilon+\lambda}\phi_{xx}^2\Big)dxds \\ \leq CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}, \qquad \textit{for any} \quad \frac{3(1-\lambda)}{2} < \varepsilon < \lambda. \end{cases} \tag{3.24}$$

Proof. Differentiate (3.1) in t,

$$\begin{aligned} &\phi_{ttt} + (1+t)^{-\lambda}\alpha_1(x)\phi_{tt} + \Big(p'(\hat{V}+\phi_x)\phi_{tx} + \Big(p'(\hat{V}+\phi_x) - p'(\hat{V})\Big)\hat{V}_t\Big)_x \\ &-\lambda(1+t)^{-\lambda-1}\alpha_1(x)\phi_t = F_t. \end{aligned} \tag{3.25}$$

Multiplying (3.25) by $(1+t)^{\varepsilon_1+\lambda}\phi_{tt}$, integrating over $\mathbf{R}$, and applying Cauchy's inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}\int\Big((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2} - \frac{\lambda}{2}(1+t)^{\varepsilon_1-1}\alpha_1(x)\phi_t^2\Big)dx \\ &\quad + \int\Big(-\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tt}^2 + (1+t)^{\varepsilon_1}\alpha_1(x)\phi_{tt}^2 \\ &\quad + \frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2 - p'(\hat{V}+\phi_x)(1+t)^{\varepsilon_1+\lambda}\phi_{ttx}\phi_{tx} \\ &\quad + \Big((p'(\hat{V}+\phi_x) - p'(\hat{V}))\hat{V}_t\Big)_x(1+t)^{\varepsilon_1+\lambda}\phi_{tt}\Big)dx \\ &= \int(1+t)^{\varepsilon_1+\lambda}\phi_{tt}F_t dx \\ &\leq \int\frac{1}{\delta_1}(1+t)^{\varepsilon_1+2\lambda}F_t^2 dx + \int\delta_1(1+t)^{\varepsilon_1}\phi_{tt}^2 dx. \end{aligned} \tag{3.26}$$

Then, we have

$$\begin{aligned} &\frac{d}{dt}\int\Big((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2} - \frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2 \\ &\quad - \frac{\lambda}{2}(1+t)^{\varepsilon_1-1}\alpha_1(x)\phi_t^2\Big)dx + \int\Big((1+t)^{\varepsilon_1}\alpha_1(x)\phi_{tt}^2 \\ &\quad - \frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tt}^2 + \frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2\Big)dx \\ &\leq C\int\Big(\delta_1(1+t)^{\varepsilon_1+\frac{-3+\lambda}{2}} + (\delta+1)(1+t)^{\varepsilon_1+\lambda-1}\Big)\phi_{tx}^2 dx \end{aligned}$$

$$
\begin{aligned}
&+\int C\delta_1\left((1+t)^{-4+\varepsilon_1}\phi_x^2+(1+t)^{-3+\lambda+\varepsilon_1}\phi_{xx}^2\right)dx\\
&+C\delta_1\left((1+t)^{\frac{-11+5\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-9-\lambda}{2}+\varepsilon_1}\right)+\int\delta_1(1+t)^{\varepsilon_1}\phi_{tt}^2dx,
\end{aligned}\tag{3.27}
$$

because of

$$
\begin{aligned}
&\int -p'(\hat V+\phi_x)(1+t)^{\varepsilon_1+\lambda}\phi_{ttx}\phi_{tx}dx\\
&\quad+\int\left((p'(\hat V+\phi_x)-p'(\hat V))\hat V_t\right)_x(1+t)^{\varepsilon_1+\lambda}\phi_{tt}dx\\
&=-\frac{d}{dt}\int\frac{p'(\hat V+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2dx\\
&\quad+\int\frac{p''(\hat V+\phi_x)}{2}(\hat V_t+\phi_{tx})(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2dx\\
&\quad+\int\frac{(\varepsilon_1+\lambda)p'(\hat V+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tx}^2dx\\
&\quad+\int\Big(\left(p'(\hat V+\phi_x)-p'(\hat V)\right)\hat V_{tx}+p''(\hat V+\phi_x)\hat V_t\phi_{xx}\\
&\quad+\left(p''(\hat V+\phi_x)-p''(\hat V)\right)\hat V_t\hat V_x\Big)\phi_{tt}(1+t)^{\varepsilon_1+\lambda}dx\\
&\geq-\frac{d}{dt}\int\frac{p'(\hat V+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2dx\\
&\quad+\int\frac{p''(\hat V+\phi_x)}{2}(\hat V_t+\phi_{tx})(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2dx\\
&\quad+\int\Big(\frac{(\varepsilon_1+\lambda)p'(\hat V+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tx}^2-\delta_1(1+t)^{\varepsilon_1}\phi_{tt}^2\\
&\quad-\frac{C}{\delta_1}(1+t)^{\varepsilon_1+2\lambda}\left(\hat V_{tx}^2\phi_x^2+\hat V_t^2\hat V_x^2\phi_x^2+\hat V_t^2\phi_{xx}^2\right)\Big)dx,
\end{aligned}\tag{3.28}
$$

and $|\phi_{tx}|_\infty^2\leq C\|\phi_{tx}\|\|\phi_{txx}\|$, as well as the a priori assumption and Lemma 2.1.

Next, multiplying (3.25) by $(1+t)^{\varepsilon_1}\phi_t$, integrating it over $\mathbf{R}$, and applying Cauchy's inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\varepsilon_1}\phi_{tt}\phi_t+(1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2}-\frac{\varepsilon_1}{2}(1+t)^{\varepsilon_1-1}\phi_t^2\right)dx\\
&\quad-\int\Big((1+t)^{\varepsilon_1}\phi_{tt}^2+\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1-\lambda-1}\alpha_1(x)\phi_t^2-\frac{\varepsilon_1(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\phi_t^2\\
&\quad+\left(p'(\hat V+\phi_x)\phi_{tx}+\left(p'(\hat V+\phi_x)-p'(\hat V)\right)\hat V_t\right)(1+t)^{\varepsilon_1}\phi_{tx}\Big)dx\\
&=\int(1+t)^{\varepsilon_1}\phi_tF_tdx\leq C\frac{1}{\delta_1}\int(1+t)^{2\varepsilon_1-\varepsilon_1'}F_t^2dx+\delta_1\int(1+t)^{\varepsilon_1'}\phi_t^2dx.
\end{aligned}\tag{3.29}
$$

Then, applying Lemma 2.1, one gets

$$
\begin{aligned}
&\frac{d}{dt}\int\left((1+t)^{\varepsilon_1}\phi_{tt}\phi_t+(1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2}-\frac{\varepsilon_1}{2}(1+t)^{\varepsilon_1-1}\phi_t^2\right)dx\\
&\quad-\int\Big((1+t)^{\varepsilon_1}\phi_{tt}^2+p'(\hat{V}+\phi_x)(1+t)^{\varepsilon_1}\phi_{tx}^2\\
&\quad+\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1-\lambda-1}\alpha_1(x)\phi_t^2-\frac{\varepsilon_1(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\phi_t^2\Big)dx\\
&\le\int\delta_1(1+t)^{\varepsilon_1}\phi_{tx}^2dx+\int\left(C\delta_1(1+t)^{-3-\lambda+\varepsilon_1}\phi_x^2+C\delta_1(1+t)^{\varepsilon_1'}\phi_t^2\right)dx\\
&\quad+C\delta_1(1+t)^{\frac{-11+\lambda}{2}+2\varepsilon_1-\varepsilon_1'}+C\delta_1(1+t)^{\frac{-9-5\lambda}{2}+2\varepsilon_1-\varepsilon_1'}.
\end{aligned}
\tag{3.30}
$$

By calculating (3.27)+$k\cdot$(3.30), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\Bigg((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2-\frac{\lambda}{2}(1+t)^{\varepsilon_1-1}\alpha_1(x)\phi_t^2\\
&\quad+k\Big((1+t)^{\varepsilon_1}\phi_{tt}\phi_t+(1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2}-\frac{\varepsilon_1}{2}(1+t)^{\varepsilon_1-1}\phi_t^2\Big)\Bigg)dx\\
&\quad+\int\Bigg((1+t)^{\varepsilon_1}\left(\alpha_1(x)-k-\delta_1\right)\phi_{tt}^2-\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tt}^2\\
&\quad-k\left(p'(\hat{V}+\phi_x)+\delta_1\right)(1+t)^{\varepsilon_1}\phi_{tx}^2-C\delta_1(1+t)^{\varepsilon_1+\frac{-3+\lambda}{2}}\phi_{tx}^2\\
&\quad-C(\delta+1)(1+t)^{\varepsilon_1+\lambda-1}\phi_{tx}^2+\frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2\\
&\quad-k\Big(\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1-\lambda-1}\alpha_1(x)-\frac{\varepsilon_1(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\Big)\phi_t^2\Bigg)dx\\
&\le\int\left(C\delta_1(1+t)^{-3+\lambda+\varepsilon_1}\phi_{xx}^2+C\delta_1(1+t)^{-3-\lambda+\varepsilon_1}\phi_x^2+C\delta_1(1+t)^{\varepsilon_1'}\phi_t^2\right)dx\\
&\quad+C\delta_1\Big((1+t)^{\frac{-11+5\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-9-\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-11+\lambda}{2}+2\varepsilon_1-\varepsilon_1'}\\
&\quad+(1+t)^{\frac{-9-5\lambda}{2}+2\varepsilon_1-\varepsilon_1'}\Big).
\end{aligned}
\tag{3.31}
$$

By Cauchy's inequality, it is easy to see that ($k=\frac{\alpha_0}{4}$)

$$
\begin{aligned}
&\int\left((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2\right)dx\\
&\quad+\int k\left((1+t)^{\varepsilon_1}\phi_{tt}\phi_t+(1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2}\right)dx\\
&\ge\int\Bigg((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{4}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2
\end{aligned}
$$

$$
\begin{aligned}
&+k(1+t)^{\varepsilon_1-\lambda}\left(\frac{\alpha_1(x)}{2}-k\right)\phi_t^2\bigg)dx \\
&\geq \frac{(1+t)^{\varepsilon_1+\lambda}}{4}\|\phi_{tt}\|^2+c_0(1+t)^{\varepsilon_1+\lambda}\|\phi_{tx}\|^2 \\
&\quad +k^2(1+t)^{\varepsilon_1-\lambda}\|\phi_t\|^2. \qquad (3.32)
\end{aligned}
$$

Therefore, to obtain the estimates of (ϕ_{tx}, ϕ_{tt}) over finite time intervals, we apply (3.12) to (3.31) and use Lemma 2.1 along with the a priori assumption, one has

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_3(t)+\int\bigg((1+t)^{\varepsilon_1}\left(\alpha_1(x)-k-\delta_1\right)\phi_{tt}^2 \\
&\quad -k\left(p'(\hat{V}+\phi_x)+\delta_1\right)(1+t)^{\varepsilon_1}\phi_{tx}^2\bigg)dx \\
&\leq \int C\bigg(\left(\delta_1(1+t)^{\varepsilon_1+\frac{-3+\lambda}{2}}+(\delta+1)(1+t)^{\varepsilon_1+\lambda-1}\right)\phi_{tx}^2 \\
&\quad +\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tt}^2+\frac{k(\varepsilon_1+\lambda)}{2}(1+t)^{\varepsilon_1-\lambda-1}\alpha_1(x)\phi_t^2 \\
&\quad -\frac{k\varepsilon_1(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\phi_t^2-\frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2 \\
&\quad +C\delta_1(1+t)^{\varepsilon_1'}\phi_t^2+C\delta_1(1+t)^{-3-\lambda+\varepsilon_1}\phi_x^2\bigg)dx \\
&\quad +\int C\delta_1(1+t)^{-3+\lambda+\varepsilon_1}\phi_{xx}^2dx+\frac{d}{dt}\int\frac{k\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1-1}\phi_t^2dx \\
&\quad +C\delta_1\left((1+t)^{\frac{-11+5\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-9-\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-11+\lambda}{2}+2\varepsilon_1-\varepsilon_1'}\right. \\
&\quad \left.+(1+t)^{\frac{-9-5\lambda}{2}+2\varepsilon_1-\varepsilon_1'}\right) \\
&\leq C(1+t)^{-1}\mathcal{E}_3(t)+C(t)I_0+\frac{d}{dt}\int C(1+t)^{\varepsilon_1-1}\phi_t^2, \qquad (3.33)
\end{aligned}
$$

where ($k=\frac{\alpha_0}{4}$)

$$
\begin{aligned}
\mathcal{E}_3(t) :=&\int\bigg((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2 \\
&+k\Big((1+t)^{\varepsilon_1}\phi_{tt}\phi_t+(1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2}\Big)\bigg)dx \\
\geq&\frac{(1+t)^{\varepsilon_1+\lambda}}{4}\|\phi_{tt}\|^2+c_0(1+t)^{\varepsilon_1+\lambda}\|\phi_{tx}\|^2+k^2(1+t)^{\varepsilon_1-\lambda}\|\phi_t\|^2.
\end{aligned}
$$

Using the Gronwall's inequality on $[0, t]$ to (3.33), we obtain

$$
\begin{aligned}
&\mathcal{E}_3(t) + \int_0^t \int \left((1+s)^{\varepsilon_1} \phi_{tt}^2 + (1+s)^{\varepsilon_1} \phi_{tx}^2 \right) dx ds \\
&\le C(t)(\mathcal{E}_3(0) + C(t) I_0) \le C(t) I_0.
\end{aligned} \tag{3.34}
$$

Next, from (3.1), we have

$$
\phi_{tt} + (p'(\hat{V} + \phi_x) - p'(\hat{V}))\hat{V}_x + \alpha \phi_t - F = -p'(\hat{V} + \phi_x)\phi_{xx}, \tag{3.35}
$$

which shows that

$$
\begin{aligned}
\|\phi_{xx}\|^2 \sim &\|\phi_{tt}\|^2 + (1+t)^{-2\lambda-2}\|\phi_x\|^2 + (1+t)^{-2\lambda}\|\phi_t\|^2 \\
&+ \delta_1 (1+t)^{\frac{-7+\lambda}{2}} + \delta_1 (1+t)^{\frac{-5-5\lambda}{2}}.
\end{aligned} \tag{3.36}
$$

Combining (3.34) and (3.36), we can derive

$$
\|\phi_{xx}\|^2 \le C(t) I_0.
$$

Let $T_0 \gg 1$ such that

$$
(\alpha_1(x) - k - C\delta_1)(1+T_0)^{\varepsilon_1} - \frac{\lambda + \varepsilon_1}{2}(1+T_0)^{\varepsilon_1+\lambda-1} \ge k(1+T_0)^{\varepsilon_1}, \tag{3.37}
$$

and

$$
\begin{aligned}
&- k(p'(\hat{V} + \phi_x) + \delta_1)(1+T_0)^{\varepsilon_1} - C\delta_1 (1+T_0)^{\varepsilon_1 + \frac{-3+\lambda}{2}} - C(\delta+1)(1+T_0)^{\varepsilon_1+\lambda-1} \\
&\ge k c_0 (1+T)^{\varepsilon_1},
\end{aligned} \tag{3.38}
$$

for $k = \frac{\alpha_0}{4}$, $\lambda < 1$ and $\varepsilon_1 \le 3$. Let $t \ge T_0 \gg 1$ in (3.31) and use (3.36) together with (3.37), we obtain ($k = \frac{\alpha_0}{4}$)

$$
\begin{aligned}
&\frac{d}{dt}\int \Bigg((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2} - \frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2 - \frac{\lambda}{2}(1+t)^{\varepsilon_1-1}\alpha_1(x)\phi_t^2 \\
&\quad + k\left((1+t)^{\varepsilon_1}\phi_{tt}\phi_t + (1+t)^{\varepsilon_1-\lambda}\alpha_1(x)\frac{\phi_t^2}{2} - \frac{\varepsilon_1}{2}(1+t)^{\varepsilon_1-1}\phi_t^2 \right)\Bigg) dx \\
&\quad + \int \Bigg(k(1+t)^{\varepsilon_1}\phi_{tt}^2 + kc_0(1+t)^{\varepsilon_1}\phi_{tx}^2 + \frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2 \\
&\quad - k\left((\frac{\varepsilon_1+\lambda}{2})(1+t)^{\varepsilon_1-\lambda-1}\alpha_1(x) - \frac{\varepsilon_1(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2} \right)\phi_t^2 \Bigg) dx \\
&\le \int \left(C\delta_1(1+t)^{-3-\lambda+\varepsilon_1}\phi_t^2 + C\delta_1(1+t)^{-3-\lambda+\varepsilon_1}\phi_x^2 + C\delta_1(1+t)^{\varepsilon_1'}\phi_t^2 \right) dx
\end{aligned}
$$

$$
\begin{aligned}
&+ C\delta_1 \Big((1+t)^{\frac{-11+5\lambda}{2}+\varepsilon_1} + (1+t)^{\frac{-9-\lambda}{2}+\varepsilon_1} + (1+t)^{\frac{-11+\lambda}{2}+2\varepsilon_1-\varepsilon_1'} \\
&+(1+t)^{\frac{-9-5\lambda}{2}+2\varepsilon_1-\varepsilon_1'} \Big). \qquad (3.39)
\end{aligned}
$$

Case 1. $0 \le \lambda < \frac{3}{5}$

In this case, we can take $t \ge T_0 \gg 1$, $\varepsilon_1 = 2+\lambda$ and $\varepsilon_1' = 1$. Then, by integrating (3.39) over $[0,t]$ and using (3.18), we obtain

$$
\begin{aligned}
&\int \Big((1+t)^{2+2\lambda}\phi_{tt}^2 + (1+t)^{2+2\lambda}\phi_{tx}^2 + (1+t)^2\phi_t^2 \Big)\,dx \\
&+ \int_0^t \int \Big((1+s)^{2+\lambda}\phi_{tt}^2 + (1+s)^{2+\lambda}\phi_{tx}^2 \Big)\,dxds \\
&\le \int C \Bigg((1+t)^{2+2\lambda}\frac{\phi_{tt}^2}{2} - \frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{2+2\lambda}\phi_{tx}^2 - \frac{\lambda}{2}(1+t)^{1+\lambda}\alpha_1(x)\phi_t^2 \\
&+ k\Big((1+t)^{2+\lambda}\phi_{tt}\phi_t + (1+t)^2\alpha_1(x)\frac{\phi_t^2}{2} - \frac{2+\lambda}{2}(1+t)^{1+\lambda}\phi_t^2 \Big) \Bigg)\,dx \\
&+ \int_0^t \int \Big((1+s)^{2+\lambda}\phi_{tt}^2 + (1+s)^{2+\lambda}\phi_{tx}^2 \Big)\,dxds \\
&\le \int_0^t \int \Big(C\delta_1(1+s)^{-1}\phi_x^2 + (C\delta_1+1)(1+s)\phi_t^2 \Big)\,dxds \\
&+ C\delta_1 \Big((1+t)^{\frac{-3+5\lambda}{2}} + (1+t)^{\frac{-1-\lambda}{2}} \Big) + CI_0 \le CI_0. \qquad (3.40)
\end{aligned}
$$

Furthermore, setting $\varepsilon_1 = 3$ and assuming $t \ge T_0 \gg 1$ in (3.27), and integrating it over $[0,t]$. Then by applying (3.36), (3.37) and (3.40), we obtain an improved decay rates for the functions as follows:

$$
\begin{aligned}
&\int \Big((1+t)^{3+\lambda}\phi_{tt}^2 + (1+t)^{3+\lambda}\phi_{tx}^2 \Big)\,dx + \int_0^t \int \Big((1+s)^3\phi_{tt}^2 + (1+s)\phi_t^2 \Big)\,dxds \\
&\le \int C(1+t)^2\phi_t^2 dx + \int_0^t \int C\Big((1+s)^{2+\lambda}\phi_{tx}^2 + (1+s)^{-1}\phi_x^2 + (1+s)^{-\lambda}\phi_t^2 \Big)\,dxds \\
&+ C\delta_1 \Big((1+t)^{\frac{-3+5\lambda}{2}} + (1+t)^{\frac{-1-\lambda}{2}} \Big) + CI_0 \le CI_0. \qquad (3.41)
\end{aligned}
$$

Then, from the previous results and (3.36), we can derive the estimates of ϕ_{xx} as follows:

$$\begin{cases} (1+t)^{2+2\lambda}\|\phi_{xx}\|^2 \le CI_0, \\ \int_0^t \int (1+s)^{1+2\lambda}\phi_{xx}^2 dxdt \le CI_0, \end{cases} \tag{3.42}$$

which, together with (3.40) and (3.41), gives (3.23).

Case 2. $\frac{3}{5} < \lambda < 1$

In this case, we can take $t \ge T_0 \gg 1$, $\varepsilon_1 = 2+\varepsilon$ and $\varepsilon_1' = \varepsilon + 1 - \lambda$. Then, by integrating (3.39) over $[0,t]$ and using (3.22), we obtain

$$\begin{aligned}
&\int \left((1+t)^{2+\lambda+\varepsilon}\phi_{tt}^2 + (1+t)^{2+\lambda+\varepsilon}\phi_{tx}^2 + (1+t)^{2+\varepsilon-\lambda}\phi_t^2\right)dx \\
&\quad + \int_0^t \int \left((1+s)^{2+\varepsilon}\phi_{tt}^2 + (1+s)^{2+\varepsilon}\phi_{tx}^2\right)dxds \\
&\le \int C\Bigg((1+t)^{2+\lambda+\varepsilon}\frac{\phi_{tt}^2}{2} - \frac{p'(\hat V+\phi_x)}{2}(1+t)^{2+\lambda+\varepsilon}\phi_{tx}^2 - \frac{\lambda}{2}(1+t)^{1+\varepsilon}\alpha_1(x)\phi_t^2 \\
&\quad + k\left((1+t)^{2+\varepsilon}\phi_{tt}\phi_t + (1+t)^{2+\varepsilon-\lambda}\alpha_1(x)\frac{\phi_t^2}{2} - \frac{2+\varepsilon}{2}(1+t)^{1+\varepsilon}\phi_t^2\right)\Bigg)dx \\
&\quad + \int_0^t \int \left((1+s)^{2+\varepsilon}\phi_{tt}^2 + (1+s)^{2+\varepsilon}\phi_{tx}^2\right)dxds \\
&\le \int_0^t \int \left(C\delta_1(1+s)^{-1+\varepsilon-\lambda}\phi_x^2 + (C\delta_1+1)\,(1+s)^{1+\varepsilon-\lambda}\phi_t^2\right)dxds \\
&\quad + C\delta_1(1+t)^{\frac{-3+3\lambda}{2}+\varepsilon} + CI_0 \\
&\le CI_0(1+t)^{\frac{-3+3\lambda}{2}+\varepsilon}.
\end{aligned} \tag{3.43}$$

Furthermore, taking $\varepsilon_1 = \varepsilon + 3 - \lambda$ and assuming $t \ge T_0 \gg 1$, we integrate (3.27) over $[0,t]$ and apply (3.36), (3.37), and (3.42) to achieve an enhanced decay rates for the functions as follows:

$$\begin{aligned}
&\int \left((1+t)^{3+\varepsilon}\phi_{tt}^2 + (1+t)^{3+\varepsilon}\phi_{tx}^2\right)dx \\
&\quad + \int_0^t \int \left((1+s)^{3+\varepsilon-\lambda}\phi_{tt}^2 + (1+t)^{1+\varepsilon-\lambda}\phi_t^2\right)dxds \\
&\le \int_0^t \int C\left((1+s)^{2+\varepsilon}\phi_{tx}^2 + (1+s)^{-1+\varepsilon-\lambda}\phi_x^2 + (1+s)^{\varepsilon-2\lambda}\phi_t^2\right)dxds
\end{aligned}$$

$$
\begin{aligned}
&+\int C(1+t)^{2+\varepsilon-\lambda}\phi_t^2 dx + C\delta_1(1+t)^{\frac{-3+3\lambda}{2}+\varepsilon} + CI_0 \\
&\le CI_0(1+t)^{\frac{-3+3\lambda}{2}+\varepsilon}. \qquad (3.44)
\end{aligned}
$$

Then, from the previous results and (3.36), we can derive the estimates of ϕ_{xx} as follows:

$$
\begin{cases}
(1+t)^{\frac{7-\lambda}{2}}\|\phi_{xx}\|^2 \le CI_0, \\
\int_0^t\int (1+s)^{1+\lambda+\varepsilon}\phi_{xx}^2 dxdt \le CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}.
\end{cases} \qquad (3.45)
$$

which, when combined with (3.43) and (3.44), leads to (3.24). This completes the proof of Lemma 3.2. $\square$

Similar calculations to Lemma 3.2 yield the following Lemma, the proof of which is omitted.

Lemma 3.3. *Under the assumptions of Theorem 2.1, when $\delta+\delta_1 \ll 1$, it holds that:*

- *For $0 \le \lambda < \frac{3}{5}$, then*

$$
\begin{aligned}
&(1+t)^4\|\phi_{tt}\|^2 + (1+t)^{5+\lambda}\left(\|\phi_{ttt}\|^2 + \|\phi_{ttx}\|^2\right) + (1+t)^{4+2\lambda}\|\phi_{txx}\|^2 \\
&\quad + (1+t)^{\frac{5+5\lambda}{2}}\|\phi_{xxx}\|^2 + \int_0^t\int \Big((1+s)^5\phi_{ttt}^2 + (1+s)^{4+\lambda}\phi_{ttx}^2 \\
&\quad + (1+s)^{3+2\lambda}\phi_{txx}^2\Big) dxds + \frac{1}{1+t}\int_0^t\int (1+s)^{\frac{5+5\lambda}{2}}\phi_{xxx}^2 dxds \\
&\le CI_0. \qquad (3.46)
\end{aligned}
$$

- *For $\frac{3}{5} < \lambda < 1$, then*

$$
\begin{cases}
(1+t)^{\frac{11-5\lambda}{2}}\|\phi_{tt}\|^2 + (1+t)^4\|\phi_{xxx}\|^2 + (1+t)^{\frac{11-\lambda}{2}}\|\phi_{txx}\|^2 \\
+(1+t)^{\frac{13-3\lambda}{2}}(\|\phi_{ttx}\|^2 + \|\phi_{ttt}\|^2) + \frac{1}{1+t}\int\int_0^t (1+s)^4\phi_{xxx}^2 dxds \\
\le CI_0, \\
\int_0^t\int \Big((1+s)^{5+\varepsilon-\lambda}\phi_{ttt}^2 + (1+s)^{4+\varepsilon}\phi_{ttx}^2 + (1+s)^{3+\varepsilon+\lambda}\phi_{ttx}^2\Big) dxds \\
\le CI_0(1+t)^{-\frac{3}{2}+\frac{3\lambda}{2}+\varepsilon}, \qquad \textit{for any} \quad \frac{3(1-\lambda)}{2} < \varepsilon < \lambda.
\end{cases} \qquad (3.47)
$$

Recalling Lemmas 3.1-3.3, we establish the decay estimates (2.31), (2.34), and (2.35).

The proof of Theorem 2.3 shares similarities with previous discussions; thus, we present only the essential steps in the argument. Before beginning our proof, we introduce a new a priori assumption, differing from the previous one

$$
\begin{aligned}
N_2^2(T_1) := \sup_{0\le t\le T_1} \ln^{-1}(2+t)\Bigg\{&\sum_{k=0}^{2}(1+t)^{\frac{8}{5}k}\|\partial_x^k\phi(t,\cdot)\|^2 \\
&+(1+t)^4\|\partial_x^3\phi(t,\cdot)\|^2+\sum_{k=0}^{2}(1+t)^{\frac{8}{5}k+2}\|\partial_x^k\phi_t(t,\cdot)\|^2\Bigg\}\le\delta^2, \qquad (3.48)
\end{aligned}
$$

for some $\delta \ll 1$ and $0 < T_1 < \infty$. For the case $\lambda = \frac{3}{5}$, to obtain the corresponding estimates for (ϕ_x, ϕ_t) and their higher-order derivatives, our approach will differ somewhat from that in Section 3, focusing primarily on the estimates of ϕ itself and its first derivatives.

Lemma 3.4. *Under the assumptions of Theorem 2.1, if $\delta + \delta_1 \ll 1$, it holds that*

$$
\begin{aligned}
&\|\phi\|^2+(1+t)^{\frac{8}{5}}(\|\phi_t\|^2+\|\phi_x\|^2)+\int_0^t\int\left((1+s)\phi_t^2+(1+s)^{\frac{3}{5}}\phi_x^2\right)dxds \\
&\le C\ln(2+t)I_0. \qquad (3.49)
\end{aligned}
$$

Proof. To facilitate our subsequent proof, we define

$$
\tilde F := \sum_{i=2}^{4} F_i = -\left\{(\alpha_1-\underline{\alpha}_1)(1+t)^{-\lambda}U+(p(V)-p'(\underline{v})V)_x+(p(V+\hat v)-p(V))_x\right\}.
$$

From the proof of Lemma 2.1 on the estimate of $\|F\|^2$, we obtain

$$
\|\tilde F\|^2 \le C\sum_{i=2}^{4}\|F_i\|^2 \le C\delta_1^2(1+t)^{-\frac{5(1+\lambda)}{2}}. \qquad (3.50)
$$

As in the proof of Lemma 3.1, we can also obtain (3.6)-(3.14). From (3.12), we can obtain the estimates of (ϕ_x, ϕ_t) over finite time intervals. Next let $t \ge T \gg 1$ and take $\lambda = \varepsilon = \frac{3}{5}$ in (3.10), by applying Hölder's inequality and Cauchy's inequality, and using Lemma 2.1 along with (3.13), we can derive

$$
\begin{aligned}
&\frac{d}{dt}\int\Bigg((1+t)^{\frac{6}{5}}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\frac{6}{5}}(p(\hat V)-p(\hat V+s))ds \\
&+k\left((1+t)^{\frac{3}{5}}\phi_t\phi+\alpha_1(x)\frac{\phi^2}{2}-\frac{3}{10}(1+t)^{-\frac{2}{5}}\phi^2\right)\Bigg)dx \\
&+\int\left(k(1+t)^{\frac{3}{5}}\phi_t^2+kc_0(1+t)^{\frac{3}{5}}\phi_x^2\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int \left((1+t)^{\frac{6}{5}}\phi_t F + k(1+t)^{\frac{3}{5}}\phi F\right)dx - \int \delta_1(1+t)^{\frac{3}{5}}\phi_t^2 dx \\
&\quad + \int C(1+t)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dx \\
&\leq \int C(1+t)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dx + C\frac{1}{\delta_1}\int (1+t)^{\frac{9}{5}}F^2 dx \\
&\quad + \int k(1+t)^{\frac{3}{5}}\left(\phi F_1 + |\phi||\tilde{F}|\right)dx \\
&\leq \int C(1+t)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dx + C\delta_1\left((1+t)^{-\frac{7}{5}} + (1+t)^{-\frac{11}{5}}\right) \\
&\quad + \int C(1+t)^{\frac{3}{5}}\phi_x\left(\mu(t)V\right)_t dx + C(1+t)^{\frac{3}{5}}\left(\int \phi^2 dx\right)^{\frac{1}{2}}\left(\int \tilde{F}^2 dx\right)^{\frac{1}{2}} \\
&\leq \int C(1+t)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dx + C\delta_1\left((1+t)^{-\frac{7}{5}} + (1+t)^{-\frac{11}{5}}\right) \\
&\quad + \int \delta_1(1+t)^{\frac{3}{5}}\phi_x^2 dx + \int \frac{C}{\delta_1}(1+t)^{\frac{3}{5}}\left(\left(\mu(t)V\right)_t\right)^2 dx \\
&\quad + C(1+t)^{\frac{3}{5}}\left(\int \phi^2 dx\right)^{\frac{1}{2}}\left(\int \tilde{F}^2 dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.51}
$$

Due to Lemma 2.1, we can derive

$$
\int (1+t)^{\frac{3}{5}}\left(\left(\mu(t)V\right)_t\right)^2 dx \leq \int C\left((1+t)^{-\frac{1}{5}}V^2 + (1+t)^{\frac{9}{5}}V_t^2\right)dx \leq C\delta_1^2(1+t)^{-1}.
$$

Hence, by integrating (3.51) over $[0,t]$, based on (3.48) and (3.50), we can obtain

$$
\begin{aligned}
&\int \left((1+t)^{\frac{6}{5}}\frac{\phi_t^2}{2} + \int_0^{\phi_x}(1+t)^{\frac{6}{5}}\left(p(\hat{V}) - p(\hat{V}+s)\right)ds\right. \\
&\quad \left. + k\left((1+t)^{\frac{3}{5}}\phi_t\phi + \alpha_1(x)\frac{\phi^2}{2} - \frac{3}{10}(1+t)^{-\frac{2}{5}}\phi^2\right)\right)dx \\
&\quad + \int_0^t \int \left((1+s)^{\frac{3}{5}}\phi_t^2 + (1+s)^{\frac{3}{5}}\phi_x^2\right)dxds \\
&\leq \int_0^t \int C(1+s)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dxds + C\delta_1(1+s)^{-\frac{2}{5}} + CI_0 \\
&\quad + \int_0^t \int \delta_1(1+s)^{\frac{3}{5}}\phi_x^2 dx + \int_0^t \int \frac{C}{\delta_1}(1+s)^{\frac{3}{5}}\left(\left(\mu(t)V\right)_t\right)^2 dx.
\end{aligned}
$$

$$
\leq \int_0^t \int C(1+s)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dxds + C\ln(2+t)I_0. \tag{3.52}
$$

In order to estimate $\int_0^t \int C(1+s)^{-\frac{7}{5}}\alpha_1(x)\phi^2 dxds$, we let $t \geq T \gg 1$, by taking $\lambda = \frac{3}{5}$ and $\varepsilon = \frac{1}{5}$ in (3.10), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int \Bigg((1+t)^{\frac{4}{5}}\frac{\phi_t^2}{2} + \int_0^{\phi_x}(1+t)^{\frac{4}{5}}(p(\hat{V})-p(\hat{V}+s))ds \\
&\quad + k\left((1+t)^{\frac{1}{5}}\phi_t\phi + \alpha_1(x)(1+t)^{-\frac{2}{5}}\frac{\phi^2}{2} - \frac{1}{10}(1+t)^{-\frac{4}{5}}\phi^2\right)\Bigg) dx \\
&\quad + \int \left((\alpha(x)-k-\delta_1)(1+t)^{\frac{1}{5}} - \frac{2}{5}(1+t)^{-\frac{1}{5}} \right)\phi_t^2 dx \\
&\quad + \int \Big(-kp'(\hat{V}+\phi_x)(1+t)^{\frac{1}{5}} + \frac{4}{5}p'(\hat{V}+\phi_x)(1+t)^{-\frac{1}{5}} \\
&\quad - C(1+t)^{\frac{4}{5}}(|V_t|_\infty + |\hat{v}_t|_\infty)\Big)\phi_x^2 dx \\
&\quad + \int \frac{4}{5}\left(\frac{\alpha_1(x)}{4}(1+t)^{-\frac{7}{5}} - \frac{1}{10}(1+t)^{-\frac{9}{5}}\right)\phi^2 dx \\
&\leq \int \delta_1(1+t)^{-\frac{7}{5}}\phi^2 dx + \int \frac{C}{\delta_1}\left((1+t)^{\frac{7}{5}} + (1+t)^{\frac{9}{5}}\right)F^2 dx.
\end{aligned}
$$

Then integrating it over $[0,t]$ and using Lemma 2.1 along with (3.13) to it, we have

$$
\begin{aligned}
&\int \Bigg((1+t)^{\frac{4}{5}}\frac{\phi_t^2}{2} + \int_0^{\phi_x}(1+t)^{\frac{4}{5}}(p(\hat{V})-p(\hat{V}+s))ds \\
&\quad + k\left((1+t)^{\frac{1}{5}}\phi_t\phi + \alpha_1(x)(1+t)^{-\frac{2}{5}}\frac{\phi^2}{2} - \frac{1}{10}(1+t)^{-\frac{4}{5}}\phi^2\right)\Bigg) dx \\
&\quad + \int_0^t \int \left(k(1+s)^{\frac{1}{5}}\phi_t^2 + kc_0(1+s)^{\frac{1}{5}}\phi_x^2 + \frac{k^2}{2}(1+s)^{-\frac{7}{5}}\phi^2 \right) dxds \\
&\leq \int_0^t \int \frac{C}{\delta_1}\left((1+s)^{\frac{7}{5}} + (1+s)^{\frac{9}{5}}\right)F^2 dxds \\
&\quad + \int_0^t \int \delta_1(1+s)^{-\frac{7}{5}}\phi^2 dxds + CI_0
\end{aligned}
$$

$$\leq CI_0+\int_0^t\int \delta_1(1+s)^{-\frac{7}{5}}\phi^2dxds, \tag{3.53}$$

which implies that $\int_0^t\int(1+s)^{-\frac{7}{5}}\phi^2dxds\leq CI_0$. Therefore, returning to (3.52), we obtain

$$\begin{aligned}&(1+t)^{\frac{6}{5}}\|\phi_t\|^2+(1+t)^{\frac{6}{5}}\|\phi_x\|^2+\|\phi\|^2+\int_0^t\int\left((1+s)^{\frac{3}{5}}\phi_t^2+(1+s)^{\frac{3}{5}}\phi_x^2\right)dxds\\&\leq C\ln(2+t)I_0.\end{aligned} \tag{3.54}$$

Furthermore, for $\varepsilon=1$ and $t\geq T\gg 1$, if we integrate (3.9) over $[0,t]$ and use Lemma 2.1, (3.13) and (3.54), we can get the better decay rates of the functions ϕ_t and ϕ_x as follows:

$$\begin{aligned}&\int\left((1+t)^{\frac{8}{5}}\frac{\phi_t^2}{2}+\int_0^{\phi_x}(1+t)^{\frac{8}{5}}\left(p(\hat{V})-p(\hat{V}+s)\right)ds\right)dx\\&\quad+\int_0^t\int(1+s)\phi_t^2dxds\\&\leq\int_0^t\int C(1+s)^{\frac{3}{5}}(\phi_t^2+\phi_x^2)dxds+\int_0^t\int C(1+s)^{\frac{11}{5}}F^2dxds\\&\leq\int_0^t\int C(1+s)^{\frac{3}{5}}(\phi_t^2+\phi_x^2)dx+C\ln(2+t)\delta_1\\&\leq C\ln(2+t)I_0,\end{aligned} \tag{3.55}$$

or

$$(1+t)^{\frac{8}{5}}(\|\phi_t\|^2+\|\phi_x\|^2)+\int_0^t\int(1+s)\phi_t^2dxds\leq C\ln(2+t)I_0, \tag{3.56}$$

which, when combined with (3.55), gives (3.49). This completes the proof of Lemma 3.4. □

Lemma 3.5. *Under the assumptions of Theorem 2.1, if $\delta+\delta_1\ll 1$ it holds that.*

$$\begin{aligned}&(1+t)^2\|\phi_t\|^2+(1+t)^{\frac{18}{5}}(\|\phi_{tt}\|^2+\|\phi_{tx}\|^2)+(1+t)^{\frac{16}{5}}\|\phi_{xx}\|^2\\&\quad+\int_0^t\int\left((1+s)^3\phi_{tt}^2+(1+s)^{\frac{13}{5}}\phi_{tx}^2+(1+s)^{\frac{11}{5}}\phi_{xx}^2\right)dxds\end{aligned}$$

$$\leq C\ln(2+t)I_0. \tag{3.57}$$

Proof. Due to $|\phi_{tx}|_\infty \leq C\|\phi_{tx}\|^{\frac{1}{2}}\|\phi_{txx}\|^{\frac{1}{2}} \leq C\delta(1+t)^{-\frac{23}{10}}\ln^{\frac{1}{2}}(2+t) \leq C\delta(1+t)^{-1}$ and (3.48), referring to (3.25), (3.26) and (3.28), we can derive

$$\begin{aligned}
&\frac{d}{dt}\int\Big((1+t)^{\varepsilon_1+\lambda}\frac{\phi_{tt}^2}{2}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\varepsilon_1+\lambda}\phi_{tx}^2\\
&\quad-\frac{\lambda}{2}(1+t)^{\varepsilon_1-1}\alpha_1(x)\phi_t^2\Big)dx+\int\Big((1+t)^{\varepsilon_1}\alpha_1(x)\phi_{tt}^2\\
&\quad-\frac{\varepsilon_1+\lambda}{2}(1+t)^{\varepsilon_1+\lambda-1}\phi_{tt}^2+\frac{\lambda(\varepsilon_1-1)}{2}(1+t)^{\varepsilon_1-2}\alpha_1(x)\phi_t^2\Big)dx\\
&\leq\int C\left(\delta_1(1+t)^{\varepsilon_1+\frac{-3+\lambda}{2}}+(\delta+1)(1+t)^{\varepsilon_1+\lambda-1}\right)\phi_{tx}^2dx\\
&\quad+\int C\delta_1\left((1+t)^{-4+\varepsilon_1}\phi_x^2+(1+t)^{-3+\lambda+\varepsilon_1}\phi_{xx}^2\right)dx\\
&\quad+C\delta_1\left((1+t)^{\frac{-11+5\lambda}{2}+\varepsilon_1}+(1+t)^{\frac{-9-\lambda}{2}+\varepsilon_1}\right)+\int\delta_1(1+t)^{\varepsilon_1}\phi_{tt}^2dx.
\end{aligned}\tag{3.58}$$

As in the proof of Lemma 3.2, we can also obtain (3.29)-(3.39). From (3.34), we can obtain the estimates of (ϕ_x,ϕ_t) over finite time intervals. Next set $t\geq T_0\gg 1$ and let $\lambda=\frac{3}{5}$, $\varepsilon_1=\frac{13}{5}$ and $\varepsilon_1'=1$ in (3.39), by integrating it over $[0,t]$ and using (3.56), we obtain

$$\begin{aligned}
&\int\left((1+t)^{\frac{16}{5}}\phi_{tt}^2+(1+t)^{\frac{16}{5}}\phi_{tx}^2+(1+t)^2\phi_t^2\right)dx\\
&\quad+\int_0^t\int\left((1+s)^{\frac{13}{5}}\phi_{tt}^2+(1+s)^{\frac{13}{5}}\phi_{tx}^2\right)dxds\\
&\leq\int C\Bigg((1+t)^{\frac{16}{5}}\frac{\phi_{tt}^2}{2}-\frac{p'(\hat{V}+\phi_x)}{2}(1+t)^{\frac{16}{5}}\phi_{tx}^2-\frac{3}{10}(1+t)^{\frac{8}{5}}\alpha_1(x)\phi_t^2\\
&\quad+k\left((1+t)^{\frac{13}{5}}\phi_{tt}\phi_t+(1+t)^2\alpha_1(x)\frac{\phi_t^2}{2}-\frac{13}{10}(1+t)^{\frac{8}{5}}\phi_t^2\right)\Bigg)dx\\
&\quad+\int_0^t\int\left((1+s)^{\frac{13}{5}}\phi_{tt}^2+(1+s)^{\frac{13}{5}}\phi_{tx}^2\right)dxds\\
&\leq\int_0^t\int\left(C\delta_1(1+s)^{-1}\phi_x^2+(C\delta_1+1)(1+s)\phi_t^2\right)dxds\\
&\quad+C\delta_1\ln(2+t)+CI_0\leq C\ln(2+t)I_0.
\end{aligned}\tag{3.59}$$

Furthermore, for $\varepsilon_1=3$ and $t\geq T_0\gg 1$, if we integrate (3.27) over $[0,t]$ and use (3.37) along with (3.56), we can get the better decay rates of the functions ϕ_t and ϕ_x as follows:

$$
\begin{aligned}
&\int \left((1+t)^{\frac{18}{5}} \frac{\phi_{tt}^2}{2} - p'(\hat{V}+\phi_x)(1+t)^{\frac{18}{5}} \frac{\phi_{tx}^2}{2} - \frac{3}{10}(1+t)^2 \alpha_1(x)\phi_t^2 \right) dx \\
&\quad + \int_0^t \int (1+s)^3 \alpha_1(x) \phi_{tt}^2 dxds \\
&\le \int_0^t \int C(1+s)^{\frac{13}{5}} (\phi_{tt}^2 + \phi_{tx}^2) dxds \\
&\quad + \int_0^t \int C\delta_1 \left((1+s)^{-1}\phi_x^2 + (1+s)^{\frac{3}{5}} \phi_{xx}^2 \right) dxds \\
&\le C\delta_1 \ln(2+t) + CI_0 + \int \delta_1 (1+t)^3 \phi_{tt}^2 dx.
\end{aligned} \tag{3.60}
$$

Due to (3.48), we have $\|\phi_{xx}\|^2 \le \delta^2 \ln(2+t)(1+t)^{-\frac{16}{5}} \le C\delta^2(1+t)^{-\frac{11}{5}}$. Therefore, combining (3.55) and (3.60), we can derive

$$
(1+t)^{\frac{18}{5}} (\|\phi_{tt}\|^2 + \|\phi_{tx}\|^2) + \int_0^t \int (1+s)^3 \phi_{tt}^2 dxds \le C \ln(2+t) I_0. \tag{3.61}
$$

Then, from the previous results and (3.36), we can derive the estimates of ϕ_{xx} as follows:

$$
\begin{cases}
(1+t)^{\frac{16}{5}} \|\phi_{xx}\|^2 \le CI_0 \ln(2+t), \\
\int_0^t \int (1+s)^{\frac{11}{5}} \phi_{xx}^2 dxdt \le CI_0 \ln(2+t),
\end{cases} \tag{3.62}
$$

which, by combining with (3.59) and (3.61), results in (3.57). This completes the proof of Lemma 3.5. □

The next theorem is analogous to the discussion in Lemma 3.5, hence, we omit its proof.

Lemma 3.6. *Under the assumptions of Theorem 2.1, if $\delta+\delta_1 \ll 1$ it holds that.*

$$
\begin{aligned}
&(1+t)^4 \|\phi_{tt}\|^2 + (1+t)^{\frac{28}{5}} \left(\|\phi_{ttt}\|^2 + \|\phi_{ttx}\|^2 \right) + (1+t)^{\frac{26}{5}} \|\phi_{txx}\|^2 \\
&\quad + (1+t)^4 \|\phi_{xxx}\|^2 + \int_0^t \int \left((1+s)^5 \phi_{ttt}^2 + (1+s)^{\frac{23}{5}} \phi_{ttx}^2 \right. \\
&\quad \left. + (1+s)^{\frac{21}{5}} \phi_{txx}^2 \right) dxds + \frac{1}{1+t} \int_0^t \int (1+s)^4 \phi_{xxx}^2 dxds
\end{aligned}
$$

$$\leq C\ln(2+t)I_0. \tag{3.63}$$

Invoking Lemmas 3.4-3.6, we derive the decay estimates (2.38).

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Data availability

No data was used for the research described in the article.

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