Champlain College – St.-Lambert

MATH 201-203: Calculus II

Review Questions for Final Exam

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1. Find integrals.

$$\text{(a)} \int \frac{x\ln(x^2+1)}{x^2+1}dx, \qquad \text{(b)} \int e^x \cos x dx,$$
$$\text{(c)} \int x\sin^2 x dx, \qquad \text{(d)} \int \frac{1}{x^3-2x^2+x}dx.$$

2. Evaluate each integral and test if it is convergent or divergent.

$$\text{(a)} \int_{-\infty}^{0} xe^x dx, \qquad \text{(b)} \int_0^2 \frac{1}{x^2-2x}dx.$$

3. Let A be a region bounded by $y = x^2$ and $y = x$, and V be a solid obtained by rotating A about the x-axis.

   (a) Find the area of A.

   (b) Find the volume of V.

4. Find the solution to the differential equation:

$$y' = xye^{-x^2}, \qquad y(0) = \sqrt{e}.$$

5. Test convergence or divergence of the sequences:

$$\text{(a) } a_n = \frac{2n+1}{4n-5}, \qquad \text{(b) } a_n = \frac{(-1)^n 5^{n+1}}{7^n+2}.$$

6. Test convergence or divergence of the series:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n+2}{n^3+n+1}, \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{5^n+1}{7^n+9}.$$

7. Find the interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{nx^n}{n^3+1}.$$

8. Find Maclaurin series of the function:

$$f(x) = \frac{x}{1+x^3}.$$

Solutions to Review Questions

1(a). **Method 1**.

$$\begin{aligned}
&\int \frac{x\ln(x^2+1)}{x^2+1}dx && [\text{ substitute: } u = x^2+1,\ du = 2xdx\ ]\\
&= \int \frac{\ln u}{u}\frac{du}{2} = \frac{1}{2}\int \frac{\ln u}{u}du && [\text{ substitute: } v = \ln u,\ dv = \frac{1}{u}du\ ]\\
&= \frac{1}{2}\int vdv = \frac{1}{4}v^2 + C = \frac{1}{4}(\ln u)^2 + C\\
&= \frac{1}{4}\ln^2(x^2+1) + C.
\end{aligned}$$

Method 2. Substitute $u = \ln(x^2+1)$, then $du = \frac{2x}{x^2+1}dx$. So,

$$\int \frac{x\ln(x^2+1)}{x^2+1}dx = \frac{1}{2}\int udu = \frac{u^2}{4} + C = \frac{1}{4}\ln^2(x^2+1) + C.$$

1(b).

$$\begin{aligned}
&\int e^x \cos x dx\\
&\qquad [\text{integration by parts: } f(x) = e^x,\ g'(x) = \cos x,\\
&\qquad\qquad \Rightarrow f'(x) = e^x,\ g(x) = \sin x]\\
&= e^x \sin x - \int e^x \sin x dx\\
&\qquad [\text{again, integration by parts: } f_2(x) = e^x,\ g_2'(x) = \sin x,\\
&\qquad\qquad \Rightarrow f_2'(x) = e^x,\ g_2(x) = -\cos x]\\
&= e^x \sin x - \Big(- e^x \cos x - \int [-e^x \cos x] dx \Big)\\
&= e^x \sin x + e^x \cos x - \int e^x \cos x dx
\end{aligned}$$

which implies

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C.$$

So,

$$\int e^x \cos x dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.$$

1(c). Since $\sin^2 x = \frac{1-\cos 2x}{2}$, then

$$\begin{aligned}
\int x\sin^2 x dx &= \int x \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{2}\int x dx - \frac{1}{2}\int x\cos 2x dx \\
&= \frac{x^2}{4} - \frac{1}{2}\int x\cos 2x dx \qquad [\text{ substitute: } u = 2x, du = 2dx] \\
&= \frac{x^2}{4} - \frac{1}{8}\int u\cos u \ du \\
&\qquad [\text{ integration by parts: } f(u) = u, \ g'(u) = \cos u, \\
&\qquad\qquad \Rightarrow f'(u) = 1, g(u) = \sin u] \\
&= \frac{x^2}{4} - \frac{1}{8}\left[u\sin u - \int \sin u \ du\right] \\
&= \frac{x^2}{4} - \frac{1}{8}\left[u\sin u + \cos u\right] + C \\
&= \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C.
\end{aligned}$$

1(d). Since $x^3 - 2x^2 + x = x(x-1)^2$, we then try the following partial fractions

$$\frac{1}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

for some constants A, B and C. Comparing the numerators, we have

$$A(x-1)^2 + Bx(x-1) + Cx = 1.$$

Thus, let $x = 0$, we get $A = 1$, and $x = 1$ we have $C = 1$. Furthermore, let $x = 2$, and use $A = C = 1$, we obtain $B = -1$. So, we can integrate

$$\int \frac{1}{x^3 - 2x^2 + x} dx = \int \left[\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right] dx = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C.$$

2(a).

$$\begin{aligned}
\int_{-\infty}^{0} xe^x dx &= \lim_{t\to-\infty}\int_t^0 xe^x dx \\
&\qquad [\text{integration by parts: } f(x)=x,\ g'(x)=e^x, \Rightarrow f'=1,\ g=e^x] \\
&= \lim_{t\to-\infty}\Big(xe^x - \int_t^0 e^x dx\Big) \\
&= \lim_{t\to-\infty}(xe^x - e^x)\Big|_t^0 \\
&= \lim_{t\to-\infty}[-1-(te^t - e^t)] \\
&= [-1-(0-0)] = -1,
\end{aligned}$$

where $\lim_{t\to-\infty} e^t = 0$, and by the l'Hospital law,

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{(t)'}{(e^{-t})'} = \lim_{t\to-\infty}\frac{1}{-e^{-t})} = \lim_{t\to-\infty} e^t = 0.$$

So, this improper integral is convergent.

2(b). Since $x^2-2x = x(x-2)$, so $x=0$ and $x=2$ both are singular points of the integrand, and the integral is improper at both the upper-limit 2 and the lower-limit 0. On the other hand, the integrand can be reduced to the partial fractions

$$\frac{1}{x^2-2x} = \frac{1}{2}\Big(\frac{1}{x-2}-\frac{1}{x}\Big),$$

thus, it holds

$$\begin{aligned}
\int_0^2 \frac{1}{x^2-2x}dx &= \int_0^1 \frac{1}{x^2-2x}dx + \int_1^2 \frac{1}{x^2-2x}dx \\
&= \lim_{t\to0^+}\int_t^1 \frac{1}{x^2-2x}dx + \lim_{s\to2^-}\int_1^s \frac{1}{x^2-2x}dx \\
&= \lim_{t\to0^+}\int_t^1 \frac{1}{2}\Big(\frac{1}{x-2}-\frac{1}{x}\Big)dx + \lim_{s\to2^-}\int_1^s \frac{1}{2}\Big(\frac{1}{x-2}-\frac{1}{x}\Big)dx \\
&= \lim_{t\to0^+}\frac{1}{2}(\ln|x-2|-\ln|x|)\Big|_t^1 + \lim_{s\to2^-}\frac{1}{2}(\ln|x-2|-\ln|x|)\Big|_1^s \\
&= \lim_{t\to0^+}\frac{1}{2}[(\ln|1|-\ln|1|)-(\ln|t-2|-\ln|t|)] \\
&\quad + \lim_{s\to2^-}\frac{1}{2}[(\ln|s-2|-\ln|s|)-(\ln|1|-\ln|1|)] \\
&= \frac{1}{2}[0-(\ln 2-\ln 0^+)] + \frac{1}{2}[(\ln 0^+ - \ln 2)-0] \\
&= -\infty, \qquad [\text{ because } \ln 0^+ = -\infty].
\end{aligned}$$

So, it is divergent.

3(a). The intersection points of $y = x^2$ and $y = x$ are $(0, 0)$ and $(1, 1)$. For $0 \le x \le 1$, the top curve is $y = x$ and the bottom curve is $y = x^2$. So, the area bounded by these two curves for $0 \le x \le 1$ is

$$A = \int_a^b [Y_{top} - Y_{bottom}]dx = \int_0^1 [x - x^2]dx = \Big(\frac{x^2}{2} - \frac{x^3}{3}\Big)\Big|_0^1 = \frac{1}{6}.$$

3(b).

$$V = V_{outer} - V_{inner} = \pi \int_0^1 (x)^2 dx - \pi \int_0^1 (x^2)^2 dx = \frac{2}{15}\pi.$$

4. Separate the variables to the equation to have

$$\frac{dy}{y} = xe^{-x^2} dx.$$

Then integrate it to yield

$$\int \frac{dy}{y} = \int xe^{-x^2} dx.$$

By substituting $u = -x^2$, we have

$$\int xe^{-x^2} dx = \int e^u(-\frac{1}{2})du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

So, we then have

$$\ln |y| = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C,$$

namely,

$$y = \pm e^{-\frac{1}{2}e^{-x^2} + C} = \pm e^C e^{-\frac{1}{2}e^{-x^2}} =: C_1 e^{-\frac{1}{2}e^{-x^2}},$$

where C_1 is an arbitrary constant. Notice that $y(0) = \sqrt{e}$, we have

$$\sqrt{e} = C_1 e^{-\frac{1}{2}},$$

i.e., $C_1 = e$. So, the particular solution is

$$y = e^{1-\frac{1}{2}e^{-x^2}}.$$

5(a).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{4n-5} = \lim_{n\to\infty} \frac{(2n+1)/n}{(4n-5)/n} = \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{4 - \frac{5}{n}} = \frac{1}{2}.$$

So, it is convergent.

5(b). Since

$$-\frac{5^{n+1}}{7^n+2} \le a_n = \frac{(-1)^n 5^{n+1}}{7^n+2} \le \frac{5^{n+1}}{7^n+2},$$

and

$$\lim_{n\to\infty} \frac{5^{n+1}}{7^n+2} = \lim_{n\to\infty} \frac{5^{n+1}/7^n}{(7^n+2)/7^n} = \lim_{n\to\infty} \frac{5(\frac{5}{7})^n}{1+\frac{2}{7^n}} = 0.$$

By the squeeze theorem, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 5^{n+1}}{7^n+2} = 0.$$

So, it is convergent.

6(a). Let $a_n = \frac{n+2}{n^3+n+1}$ and $b_n = \frac{n}{n^3} = \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2}{n^3+n+1} \Big/ \frac{1}{n^2} = 1,$$

by the limit comparison test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2}{n^3+n+1}$ and the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ both have the same convergence or divergence. Notice that, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, because it is a p-series with $p = 2 > 1$, so the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+2}{n^3+n+1}$ is also convergent.

6(b). Let $a_n = \frac{5^n+1}{7^n+9}$ and $b_n = \frac{5^n}{7^n} = (\frac{5}{7})^n$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5^n+1}{7^n+9} \Big/ (\frac{5}{7})^n = 1,$$

by the limit comparison test, the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5^n+1}{7^n+9}$ and the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (\frac{5}{7})^n$ both have the same convergence or divergence. Notice that, $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (\frac{5}{7})^n$ is convergent, because it is a geometric-series with $r = \frac{5}{7} < 1$, so the series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{5^n+1}{7^n+9}$ is also convergent.

7. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n}{n^3+1} \Big/ \frac{n+1}{(n+1)^3+1} = \lim_{n\to\infty} \frac{n((n+1)^3+1)}{(n^3+1)(n+1)} = 1.$$

So, the series $\sum_{n=1}^{\infty} \dfrac{n(x+1)^n}{n^3+1}$ is convergent for x in $(a-R, a+R) = (0-1, 0+1) = (-1, 1)$. Furthermore, at the endpoint $x = 1$, the series becomes $\sum_{n=1}^{\infty} \dfrac{n}{n^3+1}$, which is convergent.

In fact, let $b_n = \frac{n}{n^3} = \frac{1}{n^2}$, since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3+1} \Big/ \frac{1}{n^2} = 1 \neq 0,$$

by the limit comparison test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ and the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ both have the same convergence or divergence. Notice that, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, because it is a p-series with $p = 2$ (> 1), then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is also convergent. While, at the other endpoint $x = -1$, the series becomes $\sum_{n=1}^{\infty} \dfrac{(-1)^n n}{n^3+1}$, which is absolutely convergent, because $\left|\dfrac{(-1)^n n}{n^3+1}\right| = \dfrac{n}{n^3+1}$, and $\sum_{n=1}^{\infty} \dfrac{n}{n^3+1}$ is convergent as showed before. Therefore, the interval of convergence for $\sum_{n=1}^{\infty} \dfrac{x^n}{n^2}$ is $[-1, 1]$.

8.

$$f(x) = \frac{x}{1+x^3} = x\frac{1}{1-(-x^3)} = x\sum_{n=0}^{\infty}(-x^3)^n = x\sum_{n=0}^{\infty}(-1)^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n x^{3n+1},$$

for $x \in (-1, 1)$.