§7. The sketch-specification of monoidal categories

The difficulty in sketch-specifying the concept of monoidal category is with arranging that the morphisms of sketches become in the doctrine the *not necessarily strict* monoidal functors. To achieve this effect, we modify the concept of monoidal category to what we call "anamonoidal category". The resulting concept can be directly treated by the approach of the present paper. Also, the ana-version has some other virtues pointed out below.

The idea is that all structure on a category should be determined *exactly* up to isomorphism; in our case, the unit object and the tensor products should be determined exactly up to isomorphism. This has two aspects. One is that, although (e.g.) the tensor of objects A and B is not unique, any two values of it are isomorphic. The other is that if C is isomorphic to (a value of) $A \otimes B$, then C itself is a possible value of $A \otimes B$. The right way of doing this involves "specifications", a new part of the structure.

I note that some of the ideas of this section appear, in a greater generality, in [31].

An *anamonoidal category* $\mathbf{C}$ is given by data (i) to (vii), subject to conditions (viii) to (xvi).

(i) An *underlying category*, also denoted by $\mathbf{C}$.

(ii) Abstract sets (classes) $\mathrm{Un}[\mathbf{C}]$, $\mathrm{Ten}[\mathbf{C}]$ of *unit specifications* and *tensor-specifications*, respectively.

(iii) Each unit specification $u \in \mathrm{Un}[\mathbf{C}]$ picks out an object I_u, a *unit object*; and any two $u, v \in \mathrm{Un}[\mathbf{C}]$ pick out an isomorphism $\mathrm{i}_{u,v} : \mathrm{I}_u \xrightarrow{\cong} \mathrm{I}_v$.

(iv) Each tensor specification s picks out a triple $[s] = (A, B, C) = (\mathrm{A}_s, \mathrm{B}_s, \mathrm{C}_s)$ of three specific objects; we write $A \otimes_s B$ for C although A, B are also determined by s (that is, it is not the case that we have an operation $\otimes_s$ defined on all pairs (A, B)). It is possible that two *different* $s, t \in \mathrm{Ten}[\mathbf{C}]$ pick out the *same* triple $(\mathrm{A}_s, \mathrm{B}_s, \mathrm{C}_s) = (\mathrm{A}_t, \mathrm{B}_t, \mathrm{C}_t)$.

(v) Given two tensor specifications s, t, with corresponding objects $A, B, A \otimes_s B$

and C, D, $C \otimes_t D$, together with arrows $f : A \to C$, $g : B \to D$, we have the arrow $f \otimes_{s,\, t} g : A \otimes_s B \longrightarrow C \otimes_t D$ determined by the pair (s, t).

(vi) $\lambda_{u,\, s} : I_u \otimes_s A \xrightarrow{\cong} A$, $\rho_{u,\, t} : A \otimes_t I_u \xrightarrow{\cong} A$ for any appropriate specifications $u \in \mathrm{Un}[\boldsymbol{C}]$, $s, t \in \mathrm{Ten}[\boldsymbol{C}]$ ("appropriate" means, for s, that $[s] = (I_u, A, I \otimes_s A)$).

(vii) With any four appropriate specifications s_0, s_1, s_2, s_3, giving rise to $A \otimes_{s_0} B$, $B \otimes_{s_1} C$, $A \otimes_{s_2} (B \otimes_{s_1} C)$, $(A \otimes_{s_0} B) \otimes_{s_3} C$, we have a corresponding *associativity isomorphism*

$$\alpha_{s_0,\, s_1,\, s_2,\, s_3} : A \otimes_{s_2} (B \otimes_{s_1} C) \xrightarrow{\cong} (A \otimes_{s_0} B) \otimes_{s_3} C)$$

determined by (s_0, s_1, s_2, s_3).

(viii) (Existence of unit and of tensor) $\mathrm{Un}[\boldsymbol{C}]$ is inhabited: it is not empty; for any $A, B \in \mathrm{Ob}(\boldsymbol{C})$, there is at least one $s \in \mathrm{Ten}[\boldsymbol{C}]$ such that $\mathrm{A}_s = A$, $\mathrm{B}_s = B$.

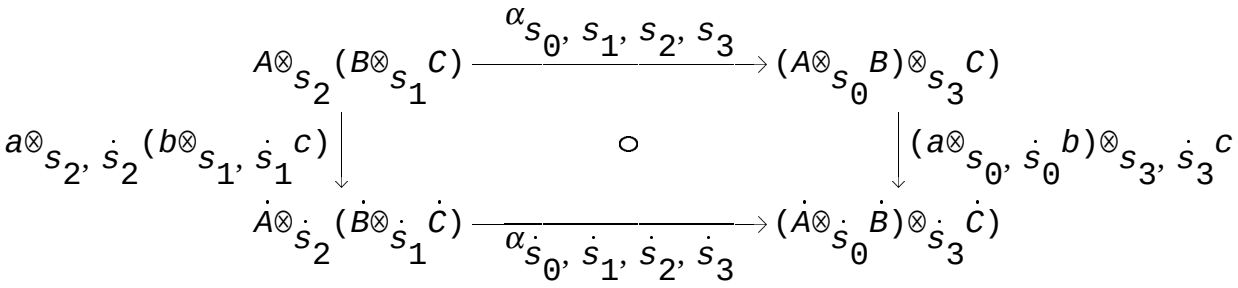
(ix) $\mathrm{i}_{v,\, w} \circ \mathrm{i}_{u,\, v} = \mathrm{i}_{u,\, w}$ ($u, v, w \in \mathrm{Un}[\boldsymbol{C}]$).

(x) (Functoriality of the tensor) $1_A \otimes_{s,\, s} 1_B = 1_{A \otimes_s B}$ ($A, B \in Ob(\boldsymbol{C})$, s giving rise to $A \otimes_s B$); for any $a : A \longrightarrow A'$, $a' : A' \longrightarrow A"$, $b : B \longrightarrow B'$, $b' : B' \longrightarrow B"$ and specifications s, s', $s"$ giving rise to $A \otimes_s B$, $A' \otimes_{s'} B'$, $A" \otimes_{s"} B"$, resp., we have $(a' \otimes_{s',\, s"} b') \circ (a \otimes_{s,\, s'} b) = (a' \circ a) \otimes_{s,\, s"} (b' \circ b)$.

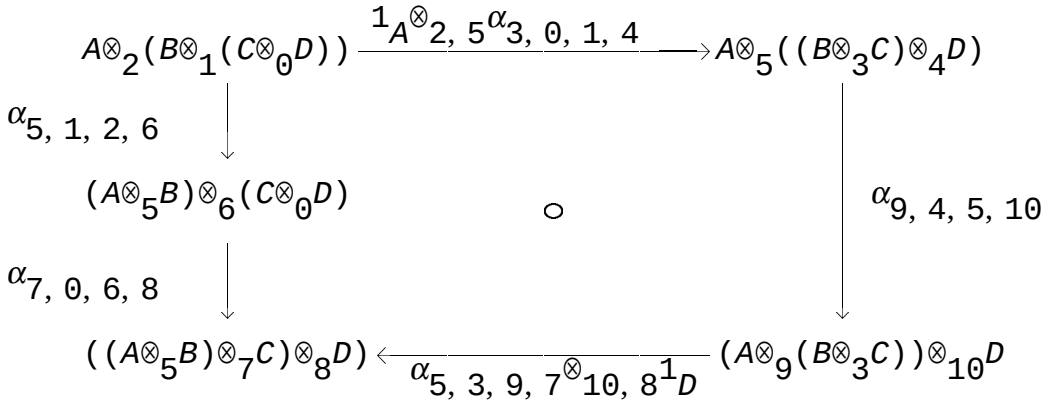
(xi) (Naturality of λ and ρ)

$$\begin{array}{ccc} I_u \otimes_s A & \xrightarrow{\lambda_{u,\, s}} & A \\ {\scriptstyle i_{u,\, v} \otimes_{s,\, t} f} \downarrow & \circ & \downarrow f \\ I_v \otimes_t B & \xrightarrow[\lambda_{v,\, t}]{} & B \end{array} \qquad \begin{array}{ccc} A \otimes_p I_u & \xrightarrow{\rho_{u,\, p}} & A \\ {\scriptstyle f \otimes_{p,\, q} i_{u,\, v}} \downarrow & \circ & \downarrow f \\ B \otimes_q I_v & \xrightarrow[\rho_{v,\, q}]{} & B \end{array}$$

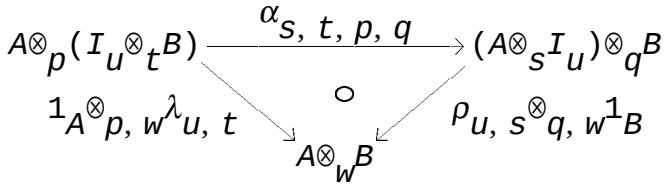
(xii) (Naturality of α)



(xiii) (Associativity coherence)



(xiv) (Triangular coherence)



(xv) (*Unique transfer under isomorphism of unit specification*) For any $u \in \mathrm{Un}[\boldsymbol{C}]$ and isomorphism $\theta : I_u \xrightarrow{\cong} A$, there is a unique $v \in \mathrm{Un}[\boldsymbol{C}]$ such that $\mathrm{I}_v = A$ and $\mathrm{i}_{u,\ v} = \theta$.

(xvi) (*Unique transfer under isomorphism of tensor specification*) Given $s \in \mathrm{Ten}[\boldsymbol{C}]$ giving rise to $A \otimes_s B$, and any isomorphism $i : A \otimes_s B \xrightarrow{\cong} C$, there is a unique $t \in \mathrm{Ten}[\boldsymbol{C}]$ such that $[t] = (A, B, C)$ and $i = 1_A \otimes_{s,\ t} 1_B$.

This says that if we have specified a tensor, and we have an isomorphism of the tensor to another object, we are free to re-specify the tensor as the new object, with marking the given

isomorphism as the corresponding tensor of the identity arrows involved; moreover, the notion of tensor-specification is "tight" in the sense that there is a *unique* new specification with the described properties.

This definition is rather complicated; what is its *raison d'etre*?

(A) We have the classical notion of monoidal category as described in [23], [8]. Given any monoidal category $(\boldsymbol{C}, I, \otimes, \alpha, \lambda, \rho)$, we can define an anamonoidal one, $\boldsymbol{C}_{(a)}$, as follows. We put $\mathrm{Ten}[\boldsymbol{C}_{(a)}]$ to consist of all entities of the form $s = (A, B, i : A \otimes B \xrightarrow{\cong} C) = (A_s, B_s, i_s : A_s \otimes B_s \xrightarrow{\cong} C_s)$; of course, $A \otimes_s B \;\overline{\mathrm{def}}\; C$. Given $s, t \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, and $a : A_s \to A_t$, $b : B_s \to B_t$, we define $a \otimes_{s,t} b$ by the following commutative diagram:

$$\begin{array}{ccc} A_s \otimes B_s & \xrightarrow{a \otimes b} & A_t \otimes B_t \\ {\scriptstyle i_s}\downarrow & \circ & \downarrow{\scriptstyle i_t} \\ C_s & \xrightarrow[a \otimes_{s,t} b]{} & C_t \end{array} \quad .$$

In particular, we have $s[A, B] \;\overline{\mathrm{def}}\; (A, B, 1_{A \otimes B}) \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, and we obtain that for any $s \in \mathrm{Ten}[\boldsymbol{C}_{(a)}]$, $i_s = (1_{A_s} \otimes_s, s[A_s, B_s] 1_{B_s})^{-1}$. The associativity isomorphisms are defined, very predictably, to make

$$\begin{array}{ccc} A \otimes_{s_2} (B \otimes_{s_1} C) & \xrightarrow{\alpha_{s_0, s_1, s_2, s_3}} & (A \otimes_{s_0} B) \otimes_{s_3} C) \\ \uparrow{\scriptstyle i_{s_2}} & & {\scriptstyle i_{s_3}}\uparrow \\ A \otimes (B \otimes_{s_1} C) & \circ & (A \otimes_{s_0} B) \otimes C) \\ \uparrow{\scriptstyle 1_A \otimes i_{s_1}} & & {\scriptstyle i_{s_0} \otimes 1_C}\uparrow \\ A \otimes (B \otimes C) & \xrightarrow[\alpha_{A, B, C}]{} & (A \otimes B) \otimes C) \end{array}$$

commute. One easily sees that the unique transfer property for tensor specifications holds. I leave the definition of the rest of the structure of $\boldsymbol{C}_{(a)}$ and the verification of the axioms to the reader.

Conversely, given any anamonoidal category $\mathbf{C}$, we pick (axiom of choice), to any pair (A, B) of objects, a particular tensor specification $s = s_{A,B}$ with $A_s = A$, $B_s = B$, and define $A \otimes B = A \otimes_s B$. The rest of the data for a monoidal category $\mathbf{C}_{(s)}$ are now supplied easily. Note also that this does not use the unique transfer properties. Finally, one notes that, for $\mathbf{C}$ an anamonoidal category, $\mathbf{C}_{(s)(a)}$ is *isomorphic* to $\mathbf{C}$; for this, we do have to use the unique transfer properties.

(B) The main argument in favor of the new concept that it naturally gives the right notion of morphism of monoidal categories; this is the direct notion of structure-preserving map. A morphism $F : \mathbf{C} \longrightarrow \mathbf{D}$ of anamonoidal categories, an "anamonoidal functor", is the same as a functor between the underlying categories, together with maps $F : \mathrm{Un}[\mathbf{C}] \longrightarrow \mathrm{Un}[\mathbf{D}]$, $F : \mathrm{Ten}[\mathbf{C}] \longrightarrow \mathrm{Ten}[\mathbf{D}]$ which are compatible, in the straightforward sense, with the data. That is,

$$F(I_u) = I_{F(u)},$$

$$F(i_{u,v}) = i_{Fu,Fv},$$

$$F(A \otimes_s B) = (FA) \otimes_{Fs} FB,$$

$$F(a \otimes_{s,t} b) = (Fa) \otimes_{Fs,Ft} Fb,$$

$$F(\lambda_{u,s}) = \lambda_{F(u),F(s)}, \qquad F(\rho_{u,s}) = \rho_{F(u),F(s)},$$

$$F(\alpha_{s_0,s_1,s_2,s_3}) = \mathrm{a}_{Fs_0,Fs_1,Fs_2,Fs_3}.$$

Recall ([8]) the notion of a (not necessarily strict) monoidal functor $F : \mathbf{C} \longrightarrow \mathbf{X}$. Consider the correspondence between the two types of monoidal category discussed above. If $\mathbf{C}$, $\mathbf{D}$ are monoidal, then we have a *bijection*

$$\hom(\mathbf{C},\mathbf{D}) \longrightarrow \hom(\mathbf{C}_{(a)},\mathbf{D}_{(a)}); \qquad (1)$$

(of course, the first hom-set is that of the monoidal functors, the second is that of the morphisms of anamonoidal categories): given $F \in \hom(\mathbf{C},\mathbf{D})$, the corresponding $F_{(a)} \in \hom(\mathbf{C}_{(a)},\mathbf{D}_{(a)})$ has its functor-part the same as F, and for

$s = (A, B, i : A \otimes B \xrightarrow{\cong} C) \in \mathrm{Ten}[\mathbf{C}_{(a)}]$, $F_{(a)}(s) = (FA, FB, j : FA \otimes FB \xrightarrow{\cong} FC)$ has $j = F(i) \circ \mathrm{i}_{A, B}$, the transition isomorphism $\mathrm{i}_{A, B} : FA \otimes FB \xrightarrow{\cong} F(A \otimes B)$ being given as part of the data for F . It is an instructive calculation to see that this mapping is indeed a bijection. In other words, the new definition of "anamonoidal category" *automatically* gives the right notion of morphism, which in the traditional framework is somewhat complicated in its appearance (and *e.g.* is not given, presumably for that reason, in [23]).

(C) The concept of a morphism of parallel anamonoidal functors is also more natural than the usual formulation. If $\mathbf{C} \underset{G}{\overset{F}{\longrightarrow}} \mathbf{D} \in \mathrm{hom}(\mathbf{C}, \mathbf{D})$, then $h : F \longrightarrow G$ is a natural transformation between the functor-parts of F and G such that

(C.i) for any $u \in \mathrm{Un}[\mathbf{C}]$, $h_{\mathrm{I}_u} = \mathrm{i}_{Fu, Gu}$; and

(C.ii) for any $s \in \mathrm{Ten}[\mathbf{C}]$, $h_{A \otimes_s B} : F(A \otimes_s B) \longrightarrow G(A \otimes_s B)$, i.e. $h_{A \otimes_s B} : FA \otimes_{Fs} FB \longrightarrow GA \otimes_{Gs} GB$, is equal to $h_A \otimes_{Fs, Gs} h_B$.

Then, keeping in mind the usual notion for the monoidal case, we see that the mapping (1) is in fact an *isomorphism* of categories.

One feature of the non-strict notion of morphism of monoidal categories (in either the usual or the new sense) is the (*unique*) *transfer property for functors*: "any functor isomorphic to a given one is just as good as the given one". More precisely, given anamonoidal $\mathbf{C}$, $\mathbf{D}$ and $F \in \mathrm{hom}(\mathbf{C}, \mathbf{D})$, a functor $G : \mathbf{C} \longrightarrow \mathbf{D}$ between the underlying categories, and $h : F \xrightarrow{\cong} G$, a natural isomorphism of functors, we can endow G with the structure of a anamonoidal functor $\mathbf{C} \longrightarrow \mathbf{D}$, actually in a unique way that makes h into an arrow between parallel anamonoidal functors. Furthermore, if $\mathbf{C}$ is an anamonoidal category, $\mathbf{D}$ is a category, $F : \mathbf{C} \longrightarrow \mathbf{D}$ is an equivalence-functor between categories, there is a way of endowing $\mathbf{D}$ with the structure of a anamonoidal category and F with the structure of a anamonoidal functor (actually, equivalence).

(D) Perhaps the best thing for showing that the present approach to the definition of monoidal category is natural is to consider concrete examples. On p. 159 of [23], we find the description of the monoidal category $\langle \mathbf{Ab}, \otimes, \ldots \rangle$ of Abelian groups. A few lines above from this description, we find the parenthetical phrase "(any chosen)" in connection with Cartesian

product, for the purposes of an example of monoidal category. The fact that a tensor product has to be *chosen* for each pair of Abelian groups in order to define $\langle \mathbf{Ab}, \otimes, \ldots \rangle$ is not mentioned, but it is clearly intended. The point of our approach is the avoidance of a choice. We define the anamonoidal category $\langle \mathbf{Ab}, \mathtt{Ten[Ab]}, \ldots \rangle$ by putting $\mathtt{Ten[Ab]}$ to be the set of all entities $S=(A, B, f:A\times B \longrightarrow C)$ such that f is a bilinear function *universal* among bilinear functions from $A\times B$ to Abelian groups; we put $\mathtt{A}_S=A$, $\mathtt{B}_S=B$ and $A\otimes_S B=C$. The rest of the definition should be clear. The definition satisfies the unique transfer properties, in particular the one for tensor-specifications; this is a consequence of the fact that the universal bilinear arrow $f:A\times B \longrightarrow C$ is determined up to a unique isomorphism. This fact is of course crucial to ensure that the new sense of $\mathbf{Ab}$ being a *ana*monoidal category is related to its being a monoidal category in the usual sense in the tight way described above.

On the basis of the above discussion, I adopt the position that anamonoidal categories are a good (or even better) substitute for monoidal categories, and proceed to exhibit their sketch-specification.

I remind the reader that in section 1, we have introduced the construction $\mathbf{G}\|\mathcal{G}$, which is an "indexed" version of the simpler one, $\mathbf{G}\,|\,\mathcal{G}$, and even pointed out that the category $\mathbf{G}\|\mathcal{G}$ is "better"; it is a (finite, presheaf) topos, while $\mathbf{G}\,|\,\mathcal{G}$ is not (in general). Now, the construction $\mathbf{G}\|\mathcal{G}$ becomes essential.

For the notion of (*ana*)*monoidal sketch*, we start by taking $\mathbf{G} = \mathtt{cSk}$. Let $\mathtt{Un}$ be the c-sketch whose underlying graph has a single object, say $\mathtt{3}$, and no arrows; the specification sets $\mathtt{I[Un]}, \mathtt{CT[Un]}$ are empty. Let $\mathtt{Ten}$ be the c-sketch whose underlying graph has the three objects $\mathtt{0}, \mathtt{1}$ and $\mathtt{2}$, with no arrows; the specification sets $\mathtt{I[Ten]}, \mathtt{CT[Ten]}$ are empty. We will write $\mathtt{0\otimes 1}$ for $\mathtt{2}$, because of the use of $\mathtt{Ten}$ (to be clarified step-by-step).

Consider $\mathtt{cSk}\|\mathtt{\{Un,Ten\}}$. An object S of $\mathtt{cSk}\|\mathtt{\{Un,Ten\}}$ is a c-sketch $|S|$, together with abstract sets $\mathtt{Un}[S]$, $\mathtt{Ten}[S]$, and assignments $u\mapsto \bar{u}$, $s\mapsto \bar{s}$ of a map $\bar{u}:\mathtt{Un} \longrightarrow |\mathtt{S}|$, $\bar{s}:\mathtt{Ten} \longrightarrow |S|$ to each $u\in\mathtt{Un}[S]$, $s\in\mathtt{Ten}[S]$. Intuitively, an object of $\mathtt{cSk}\|\mathtt{\{Un,Ten\}}$ is a c-sketch, together with specifications of instances of tensor-products and units, with the specifications having "individualizing" tags; thus, the very same map $\bar{s} = \bar{t}$ may correspond to two different $s, t\in\mathtt{Ten}[S]$.

Next, we perform a construction of the type $\mathbf{G}|\mathcal{G}$, with $\mathbf{G} = \mathtt{cSk}\|\{\mathtt{Un},\mathtt{Ten}\}$. We will define objects $\hat{\mathtt{U}}\mathtt{nArr}$, $\hat{\mathtt{L}}\mathtt{eftUn}$, $\hat{\mathtt{R}}\mathtt{ightUn}$, $\hat{\mathtt{T}}\mathtt{enArr}$ and $\hat{\mathtt{A}}\mathtt{ssoc}$, of $\boldsymbol{G}$. We will put

$$\mathtt{MonSk} \underset{\mathtt{def}}{=} \boldsymbol{G}|\{\hat{\mathtt{U}}\mathtt{nArr},\hat{\mathtt{L}}\mathtt{eftUn},\hat{\mathtt{R}}\mathtt{ightUn},\hat{\mathtt{T}}\mathtt{enArr},\hat{\mathtt{A}}\mathtt{ssoc}\}, \qquad (2)$$

the *category of monoidal sketches*. We will give sketch-entailments (arrows) $\mathcal{R}[\mathtt{Mon}]$ in $\mathtt{MonSk}$ so that the specification $(\mathtt{MonSk},\mathcal{R}[\mathtt{Mon}])$ gives, as $\mathtt{MonSk}:\mathcal{R}[\mathtt{Mon}]$ (see Section 2), the category of anamonoidal categories (up to isomorphism). For the exposition, it is better to mix the descriptions of the specification-types (in the brackets in the definition of $\mathtt{MonSk}$) and the axioms. I will number the axioms in the style (Axn) .

(Ax1) $\quad \mathtt{Ex[Un]} : \left|\begin{array}{c}\_\_\_\_\\ \_\_\_\_\end{array}\right| \longrightarrow \left|\begin{array}{c}\_\_\_\_\\ 3\\ \_\_\_\_\end{array}\right|$

$\mathtt{Un[]}=\{\mathtt{id}_{\mathtt{Un}}\}$

$\bar{\mathtt{i}}\mathtt{d}_{\mathtt{Un}} = \mathtt{id}_{\mathtt{Un}}$

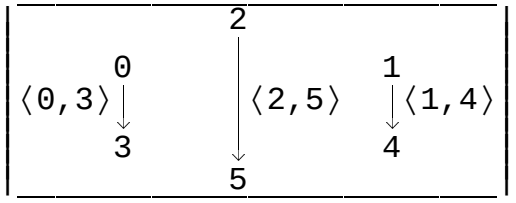
What is the effect of $\mathtt{Ex[Un]}$? A sketch S satisfies it iff there is a map $\varphi:\hat{\mathtt{E}}\mathtt{x[Un]} \longrightarrow S$ (we continue the custom of using $\hat{}$ for the codomain of a sketch-map), that is, iff there is $s \in \mathtt{Un}[S]$; note that φ necessarily maps 3 to the "value" $\bar{s}(3)$ of s in S. That is, $\mathtt{Ex[Un]}$ expresses the *existence* of (at least one) unit object.

(Ax2) $\quad \mathtt{Ex[Ten]}: \left|\begin{array}{cc}\hline 0 & 1\\ \hline\end{array}\right| \longrightarrow \left|\begin{array}{ccc}\hline 0 & 1 & 0\otimes 1\\ \hline\end{array}\right|$

$\mathtt{Ten[]}=\{\mathtt{id}_{\mathtt{Ten}}\}$

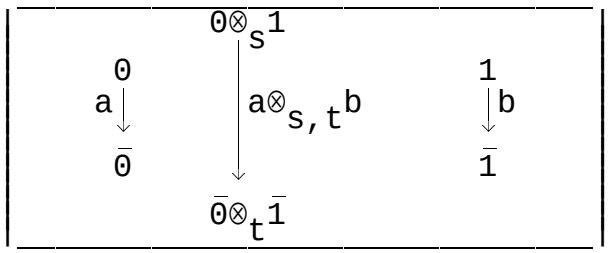
$\bar{\mathtt{i}}\mathtt{d}_{\mathtt{Ten}} = \mathtt{id}_{\mathtt{Ten}}$

From the specification-types in (2), let me start with $\hat{\mathtt{T}}\mathtt{enArr}$; it has the following underlying c-sketch $|\hat{\mathtt{T}}\mathtt{enArr}|$:

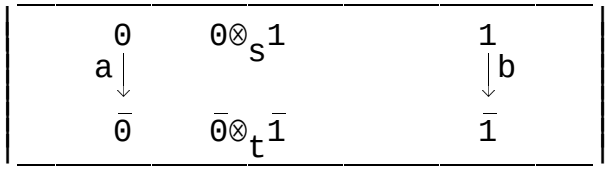


$$\left|\begin{array}{ccccc} & & 2 & & \\ & 0 & \downarrow & 1 & \\ \langle 0,3\rangle & \downarrow & \langle 2,5\rangle & \downarrow & \langle 1,4\rangle \\ & 3 & \downarrow & 4 & \\ & & 5 & & \end{array}\right|$$

with empty $\mathtt{I[]}$ and $\mathtt{CT[]}$. In addition, $\hat{\mathtt{T}}\mathtt{enArr[Ten]} = \{\langle 0,1,2\rangle, \langle 3,4,5\rangle\}$; $|\langle 3,4,5\rangle| : \mathtt{Ten} \longrightarrow |\mathtt{TenArr}|$ is the map for which $0 \mapsto 3$, $1 \mapsto 4$, $2 \mapsto 5$; $|\langle 0,1,2\rangle|$ acts as the identity. We will write $\mathtt{s}$, $\mathtt{t}$ for $\langle 0,1,2\rangle$ and $\langle 3,4,5\rangle$, resp., and



for $\hat{\mathtt{T}}\mathtt{enArr}$. Let $\mathtt{TenArr}^{+}$ be the $\mathtt{MonSk}$-sketch with underlying $\boldsymbol{G}$-sketch $\hat{\mathtt{T}}\mathtt{enArr}$, and only additional specification $\mathtt{id} \in \hat{\mathtt{T}}\mathtt{enArr[]}$; and let $\ulcorner \mathtt{TenArr} \urcorner$ be



,

that is, $\hat{\mathtt{T}}\mathtt{enArr}$ without the middle arrow.

We adopt the axiom

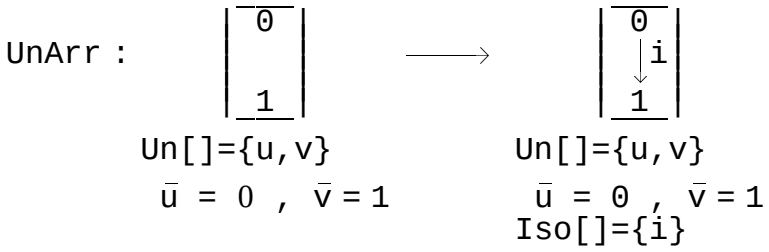
(Ax3) $\qquad \mathtt{Ex[TenArr]} : \ulcorner \mathtt{TenArr} \urcorner \longrightarrow \mathtt{TenArr}^{+}$.

(as usual, all arrows in specified sketch-axioms are inclusions as far as possible).
$\mathtt{Ex[TenArr]}$ expresses the existence of the tensor-product of arrows.

The uniqueness of tensors of arrows is expressed by the codiagonal

(Ax4) $\qquad \mathtt{Uni[TenArr]} : \mathtt{TenArr}^{+} \underset{\ulcorner \mathtt{TenArr} \urcorner}{\sqcup} \mathtt{TenArr}^{+} \longrightarrow \mathtt{TenArr}^{+}$.

Based on the arrow $\mathtt{UnArr}:\ulcorner\mathtt{UnArr}\urcorner\longrightarrow\hat{\mathtt{U}}\mathtt{nArr}$, in detail



(here, for convenience, we used isomorphism specifications, adjoined to $\mathtt{cSk}$ as a first step; see Section 5; of course, its use here may be considered an abbreviation), the axioms

(Ax5) $\mathtt{Ex[UnArr]}$,

(Ax6) $\mathtt{Uni[UnArr]}$

are formed similarly to the previous two.

Before describing $\hat{\mathtt{A}}\mathtt{ssoc}$, note that $\hat{\mathtt{A}}\mathtt{ssoc}$ will have an underlying c-sketch $|\hat{\mathtt{A}}\mathtt{ssoc}|$, and the latter has an underlying graph $||\hat{\mathtt{A}}\mathtt{ssoc}||$.

$||\hat{\mathtt{A}}\mathtt{ssoc}||$ has six objects, $\mathtt{0}$, $\mathtt{1}$, $\mathtt{2}$, $\mathtt{0\otimes1}$, $\mathtt{1\otimes2}$, $\mathtt{0\otimes(1\otimes2)}$ and $\mathtt{(0\otimes1)\otimes2}$; it has four arrows: $\mathtt{a:0\otimes(1\otimes2)}\longrightarrow\mathtt{(0\otimes1)\otimes2}$, $\mathtt{a}^{-1}\mathtt{:(0\otimes1)\otimes2}\longrightarrow\mathtt{0\otimes(1\otimes2)}$, $1_{\mathtt{a}}$ and $1_{\mathtt{a}^{-1}}$. $\mathtt{CT[}|\hat{\mathtt{A}}\mathtt{ssoc}|\mathtt{]}$ and $\mathtt{I[}|\hat{\mathtt{A}}\mathtt{ssoc}|\mathtt{]}$ has specifications expressing that $\mathtt{a}$, $\mathtt{a}^{-1}$ are inverses. $\mathtt{Ten[}\hat{\mathtt{A}}\mathtt{ssoc]}$ has four elements, $\mathtt{s}_i$ for $i<4$, to specify the four tensors in question; e.g., $\bar{\mathtt{s}}_3$ maps $\mathtt{0}\mapsto\mathtt{0}$, $\mathtt{1}\mapsto\mathtt{1\otimes2}$, $\mathtt{2}\mapsto\mathtt{0\otimes(1\otimes2)}$.

$\ulcorner\mathtt{Assoc}\urcorner$ is the same as $\hat{\mathtt{A}}\mathtt{ssoc}$ but without $\mathtt{a}$ and $\mathtt{a}^{-1}$. The axioms

(Ax7) $\mathtt{Ex[Assoc]}$,

(Ax8) $\mathtt{Uni[Assoc]}$

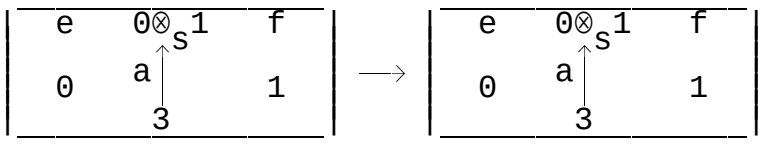
are then formed, based on $\ulcorner$Assoc$\urcorner \longrightarrow \hat{}$Assoc , as before. The axioms

(Ax9)-(Ax12): Ex[LeftUn] , Un[LeftUn] , Ex[RightUn] , Un[RightUn]

are similar.

The axioms so far express clauses (i) to (viii) of the definition of anamonoidal category. Clauses (ix) to (xiv) are expressed by corresponding axioms (Ax13) to (Ax18) in a straightforward way, involving diagrams similar to those in the clauses themselves; I leave these to the reader. Finally, we have axioms corresponding to the transfer clauses (xv), (xvi).

(Ax19) IsoTransEx[Ten] :



Ten[]={s}
Iso[]={a}
I[]={e,f}

t∈Ten[] :
t̄ :0↦0,1↦1,2↦3
σ∈TenArr : σ̄:⟨0,3⟩↦e,
σ̄:⟨1,4⟩↦f,σ̄:⟨2,5⟩↦a

(Ax20) IsoTransUni[Ten] : $T \underset{S}{\sqcup} T \longrightarrow T$,

with S = $\ulcorner$IsoTransEx[Ten]$\urcorner$, T = $\hat{}$IsoTransEx[Ten] .

The preceding two axioms express the existence and uniqueness parts, respectively, in clause (xvi). Finally, we have axioms

(Ax21) IsoTransEx[Un] ,

(Ax22) IsoTransUni[Un]

expressing (xv).

Any anamonoidal category gives rise to a MonSk-sketch, in the obvious way; the resulting

sketch allows the unique reconstruction of the anamonoidal category; we may identify the two objects.

Inspection shows that we have:

an object of `MonSk` *is isomorphic to a anamonoidal category iff it satisfies all* sketch-*axioms of category, and all of the* sketch-*axioms of anamonoidal category* (Ax1) *to* (Ax22). *Moreover, under this identification, the morphisms of anamonoidal categories are bijectively mapped onto maps of sketches.*

Variants and extensions of the concepts of monoidal category can be treated in a similar way. Symmetric and braided monoidal categories will have additional specifications and axioms corresponding to the symmetry and the braiding, respectively. E.g., the specification of the symmetry is by the sketch $\hat{\mathtt{Symm}}$ =

$$\left|\begin{array}{c} \overline{A\otimes_s B \xrightarrow[A \quad B]{\sigma_{s,t}} B\otimes_t A} \\ \mathtt{Ten[]=\{s,t\}} \end{array}\right| \quad ;$$

symmetric monoidal categories will be specified in the sketch-category $\mathtt{MonSk}\,|\,\{\hat{\mathtt{Symm}}\}$. Two of the axioms are `Ex[Symm]` and `Uni[Symm]` , expressing the existence and the uniqueness of the symmetry; note that the symmetry depends on tensor-specifications.

The treatment of closed monoidal categories is more understandable if it is based explicitly on [31]; I will not go into it here.