§6. Further examples

In this section, we will see that the method of specifying a doctrine by sketches is not restricted to adjointness doctrines.

I will first consider the concept of **Grothendieck topology** [3]. I restrict attention (because of the interest in completeness; see the next section) to categories with finite limits; and also to *finitary topologies*, that is, ones generated by finite covering families. The latter restriction is a part of the general limitation in this paper to finitary logic.

The view of Grothendieck topology here is that it axiomatizes the notion of "surjective family of arrows". We start with $\mathtt{pbSk} = \mathtt{Graph}\,|\,\{\hat{\mathtt{Pb}}\}$, the category of sketches for categories with pullbacks; the axioms $\mathcal{R}_r[\mathtt{Pb}]$ will be assumed (although their use will be explicitly mentioned every time). I will use the symbols $\mathtt{A}$, $\mathtt{B}_i$ ($i \in \mathbb{N}$) for objects, and $\mathtt{u}_i : \mathtt{B}_i \longrightarrow \mathtt{A}$ for arrows, in certain specific sketches in $\mathtt{pbSk}$; they are fixed; e.g., $\mathtt{A}$ is 2; etc. For any $n \in \mathbb{N}$ (including $n=0$), the graph (in $\mathtt{Graph}$) whose objects are $\mathtt{A}$, and $\mathtt{B}_i$ for $i<n$, and whose arrows are $\mathtt{u}_i : \mathtt{B}_i \longrightarrow \mathtt{A}$ for $i<n$, is called $\mathtt{Cov}_n$. As usual, $\mathtt{Cov}_n$ is considered also a sketch in $\mathtt{pbSk}$, with empty specification sets. The sketch-category $\mathtt{GrtopfinSk}$, the category of sketches for finitary Grothendieck topologies (or sites), is, by definition, $\mathtt{pbSk}\,|\,\{\mathtt{Cov}_n : n \in \mathbb{N}\}$. In other words, a site-sketch is a graph, with certain diagrams specified as pullbacks, and certain diagrams (sieves; finite sets of arrows with a common codomain) specified as coverings.

Here are the sketch-axioms for Grothendieck topology:

$$\mathtt{Isocov}: \left|\overline{\mathtt{B}_0 \xrightarrow{\mathtt{u}_0} \mathtt{A}}\right|_{\mathtt{Iso}[]=\{\mathtt{u}_0\}} \longrightarrow \left|\overline{\mathtt{B}_0 \xrightarrow{\mathtt{u}_0} \mathtt{A}}\right|_{\substack{\mathtt{Iso}[]=\{\mathtt{u}_0\} \\ \mathtt{Cov}_1[]=\{\mathtt{id}\}}},$$

expressing that a sieve consisting of an isomorphism is a covering;

$$\mathtt{Monot}_f : \left| \overline{\mathtt{Cov}_n} \right|_{\mathtt{Cov}_m[]=\{\bar{f}\}} \longrightarrow \left| \overline{\mathtt{Cov}_n} \right|_{\substack{\bar{f}\in \mathtt{Cov}_m[] \\ \mathrm{id}\in \mathtt{Cov}_n[]}}$$

one for each triple $(m\in\mathbb{N},\ n\in\mathbb{N},\ f:m\to n)$, with $\bar{f}:\mathtt{Cov}_m\to\mathtt{Cov}_n$ for which $\bar{f}(\mathtt{A})=\mathtt{A}$, $\bar{f}(\mathtt{B}_i)=\mathtt{B}_{f(i)}$; $\mathtt{Monot}_f$ expresses "monotonicity";

$$\mathtt{Comp}_{\vec{n}} : \left| \overline{(\mathtt{C}_{i,k} \xrightarrow{\mathtt{v}_{i,k}} \mathtt{B}_i \xrightarrow{\mathtt{u}_i} \mathtt{A})_{i<n,\ k<n_i}} \right|_{\substack{\mathtt{Cov}[]=\{(\mathrm{id}_{\mathtt{Cov}_n}\}\}\cup \\ \{\{\mathtt{v}_{i,k}:k<n_i\}:i<n\}}} \longrightarrow$$

$$\left| \overline{(\mathtt{C}_{i,k} \xrightarrow{\mathtt{v}_{i,k}} \mathtt{B}_i \xrightarrow{\mathtt{u}_i} \mathtt{A})_{i<n,k<n_i}} \right|_{\substack{\mathtt{Cov}[]=\{(\mathrm{id}_{\mathtt{Cov}_n}\}\}\cup \\ \{\{\mathtt{v}_{i,k}:k<n_i\}:i<n\}\cup \\ \{\{\mathtt{v}_{i,k}\circ\mathtt{u}_i:k<n_i,\ i<n\}}},$$

one for each $\vec{n}=\langle n_i\rangle_{i<n}$, expressing that a composition of coverings is a covering (here we used $\mathtt{Cov}[]$ without a subscript as the union of the specification sets $\mathtt{Cov}_m[]$, and used an informal notation for members of $\mathtt{Cov}[]$); and finally,

$$\mathtt{PbCov}_n : \left| \overline{\mathtt{Cov}_n \dot{\times}_{\vec{\to}} \mathtt{Cov}_n} \right|_{\mathtt{Cov}_n[]=\{\mathtt{i}_0\}} \longrightarrow \left| \overline{\mathtt{Cov}_n \dot{\times}_{\vec{\to}} \mathtt{Cov}_n} \right|_{\mathtt{Cov}_n[]=\{\mathtt{i}_0,\ \mathtt{i}_1\}},$$

one for each $n\in\mathbb{N}$, expressing that a pullback of a covering is a covering. We are using notation introduced at the beginning of Section 4; the dot refers to the presence of pullback squares; $\mathtt{i}_0$, $\mathtt{i}_1$ are the canonical maps $\mathtt{Cov}_n \rightrightarrows \mathtt{Cov}_n \dot{\times}_{\vec{\to}} \mathtt{Cov}_n$.

Let $\mathcal{R}[\mathtt{Grtopfin}]$ contain all sketch-axioms in $\mathcal{R}_r[\mathtt{Pb}]$, and all the above axioms for sites. With any finitary site, that is, category with pullbacks and a finitary Grothendieck

topology (one generated by finite sieves), one associates, in an obvious way, a sketch in GrtopfinSk ; this maps finitary sites onto the sketches in GrtopfinSk that satisfy $\mathcal{R}$[Grtopfin] . Also, the maps of sites (pullback-preserving functors taking covers to covers) correspond precisely to the maps of sketches.

What does the General Completeness Theorem say in the case of the doctrine of Grothendieck topologies? Suppose we have a Grtopfin-sketch S , describing a finite amount of information on certain objects and arrows forming commutative diagrams, pullbacks, and certain sieves being coverings. Suppose that a particular sieve $\mathcal{S}$ in the same diagram is *necessarily* a covering as a consequence, meaning that in any site "realizing" S , $\mathcal{S}$ is a cover. Then there is a finite sequence of steps each consisting either of producing new pullbacks, or of deriving consequences of certain diagrams being pullbacks, or, finally, deriving new coverings out of old ones by one of the four closure conditions defining Grothendieck topologies that produces the desired conclusion that $\mathcal{S}$ is a cover. This conclusion is obvious in the modified form with an underlying category with pullbacks used as "background". In the form we get it here the conclusion is not entirely obvious. However, the really interesting conclusion is the true completeness theorem, pointed out in §8, drawing the same conclusion from a hypothesis about surjective families in Set .

Next, we consider an example of a logic with a generalized quantifier, the quantifier Qx read "there are uncountably many x such that ... ".

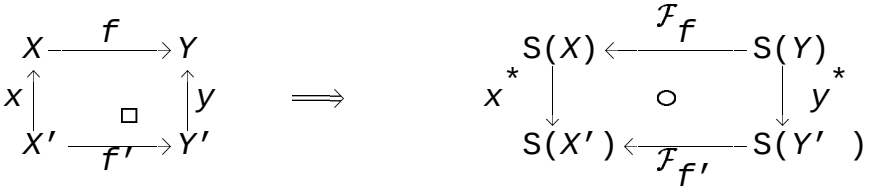
We will take the specification (BoolSk,$\mathcal{R}$[Bool]) of Boolean categories, and adjoin the quantifier Qx with suitable axioms. Actually, we will use the dual quantifier $\mathcal{F}x \equiv \neg$Q$x\neg$, read "there are only *few* x ...", where "few" means, in the central interpretation, "countably many".

An $\mathcal{F}$-category $\boldsymbol{C}$ is, by definition, a Boolean category, also written $\boldsymbol{C}$, together with an order-preserving operation

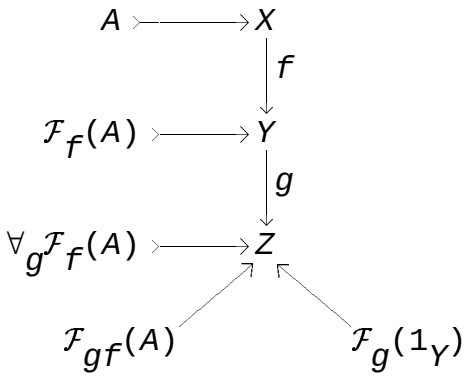
$$\mathcal{F}_f : \mathrm{S}(Y) \longrightarrow \mathrm{S}(X)$$

for every $f:X \longrightarrow Y$ in $\boldsymbol{C}$ such that the following two conditions (Stab) and (Reg) hold:

(Stab)



(Reg) With reference to data as in



we have $\mathcal{F}_g(1_Y) \wedge \forall_g \mathcal{F}_f(A) \leq \mathcal{F}_{gf}(A)$.

(We have used the abbreviated notation of subobjects by their domains). The standard $\mathcal{F}$-category is $\mathtt{Set}$, with $\mathcal{F}$ having the meaning

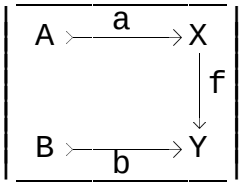
$$y \in \mathcal{F}_f(A) \iff \{x \in X : f(x)=y \ \& \ x \in A\} \text{ has cardinality } < \kappa ,$$

with a fixed infinite regular cardinal κ ; the axiom (Reg) is then true as a consequence of the definition of regularity: the union of $<\kappa$ sets each of cardinality $<\kappa$ is of cardinality $<\kappa$. (Specifically,

$z \in \mathcal{F}_g(1_Y)$ $\iff$ there are only few $y \in Y$ over z ,
$z \in \forall_g \mathcal{F}_f(A)$ $\iff$ all $y \in Y$ over z have few $x \in A \subset X$ over them,
$z \in \mathcal{F}_{gf}(A)$ $\iff$ there are only few $x \in A \subset X$ over z ;

here, " y is over z " is " $g(y)=z$ " , etc.). The case $\kappa = \aleph_1$ is especially interesting because of Keisler's completeness theorem [17]; see also §7 and §8 .

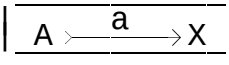
The sketch-specification of $\mathcal{F}$-categories starts with $\mathtt{BoolSk}$, and adjoins

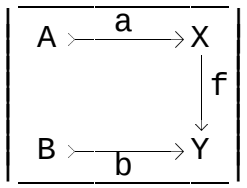
$\hat{\mathcal{F}}$ = 

;

mon[]={a,b}

that is, $\mathcal{F}$Sk $\stackrel{=}{\text{def}}$ BooleSk | {$\hat{\mathcal{F}}$} . The sketch-axioms are as follows:

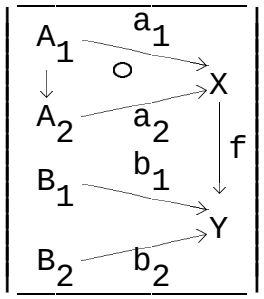
Ex[$\mathcal{F}$] : 

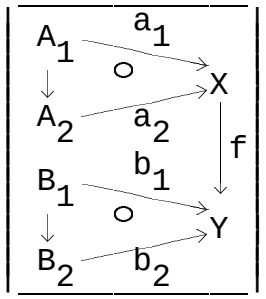
⟶ 

⌜mon⌝ []={a}

⌜mon⌝ []={a,b}

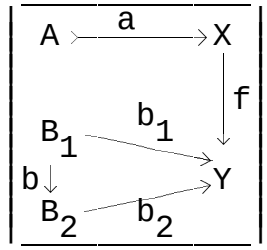
$\hat{\mathcal{F}}$[] = {id}

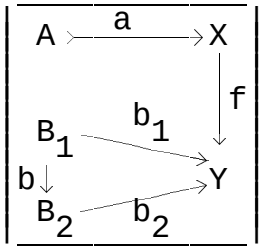
Mon[$\mathcal{F}$] : 

⟶ 

$\hat{\mathcal{F}}$[]= {(f,a_1,b_1),(f,a_2,b_2)}

$\hat{\mathcal{F}}$[]= {(f,a_1,b_1),(f,a_2,b_2)}

IsoTrans[$\mathcal{F}$] : 

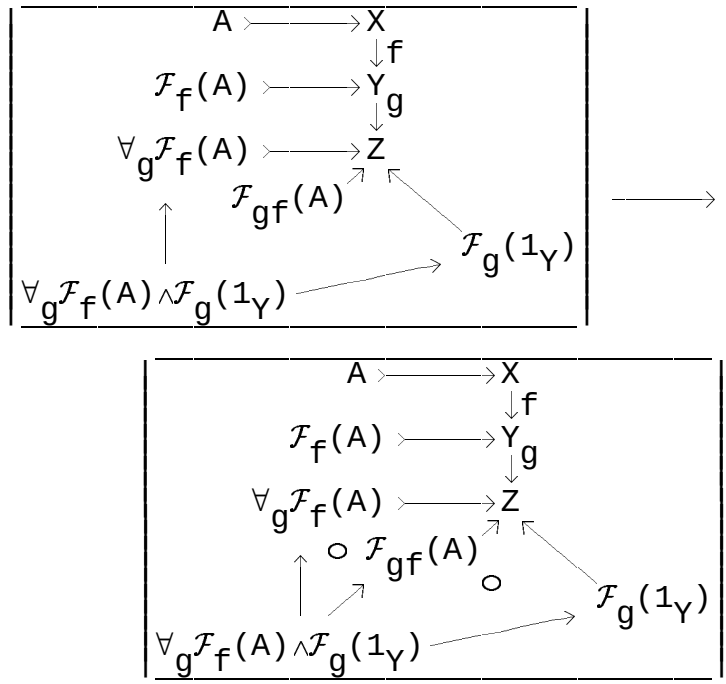
⟶ 

$\hat{\mathcal{F}}$[]={(f,a,b_1)}

{(f,a,b_2)}∈$\hat{\mathcal{F}}$[]

Iso[]={b}

Reg[$\mathcal{F}$] :



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here, we used, I hope, a self-explanatory notation. (To make the last axiom conform to the norms of exactness properties in the previous section, we should use a remark in that section on the order of subobjects appearing in an exactness property).

Clearly, with $\mathcal{R}$[$\mathcal{F}$] consisting of the axioms in $\mathcal{R}$[Boole], the ones listed above, and Stab[$\hat{\mathcal{F}}$], with Stab[] as given in Section 4, we have that (fewSk,$\mathcal{R}$[$\mathcal{F}$]) specifies $\mathcal{F}$-categories, with functors preserving the Boolean structure and the $\mathcal{F}_f$'s.

There is a close correspondence between theories in L(Q) subject to Keisler's axioms [17] and $\mathcal{F}$-categories. The connection is entirely analogous to that of theories in full first order logic and Boolean categories, explained in [27]. For more about this, see the last section of the paper.