§5. Exactness properties

Exactness properties of structured categories abound in category theory; "coproducts are stable under pullback", "equivalence relations are effective" are examples. There are important ones involving infinitary operations (e.g., "filtered colimits of mono's is a mono"), but in this paper we look at finitary operations only. I am not aware of a general definition of "exactness property" in the literature. The framework of (generalized) sketches introduced in this paper provides a simple definition of "exactness property", which seems to cover all examples.

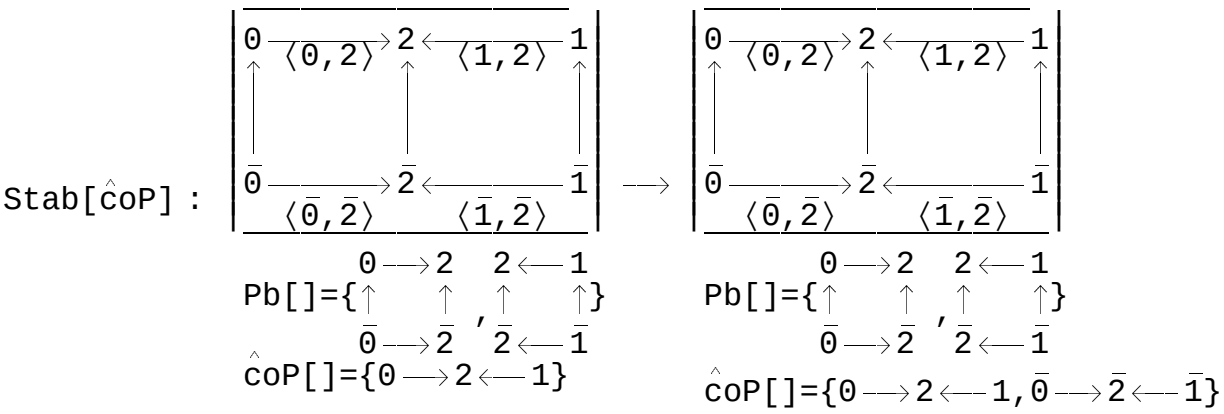
Let $\mathbb{S}$ be a sketch-category based on the (arbitrary) category $\mathbb{G}$ (see the previous section). An arrow $\varphi: S \longrightarrow T$ between finite objects in $\mathbb{S}$ is an *exactness property* (of any doctrine specified by $(\mathbb{S}, \mathcal{R})$ for some set $\mathcal{R}$ of sketch-axioms in $\mathbb{S}$) if the underlying map $\|\varphi\| : \|S\| \longrightarrow \|T\|$ (in $\mathbb{G}$) is an identity map; thus, the notion depends on $\mathbb{G}$ as well as $\mathbb{S}$. This means that T differs from S only by having more specifications; the underlying "graphs" ($\mathbb{G}$-objects) of S and T are the same.

Let us see some examples. First, we assume that $\mathbb{G}$ is `flSk`; let $\mathbb{S}$ be a sketch-category based on `flSk`.

Recall from the last section the notion of a "pullback natural transformation" (pbnt) in $\mathbb{S}$. Also, recall the constructions $T \underset{S}{\vec{\sqcup}} T^+$, $T^+ \underset{S}{\vec{\sqcup}} T^+$ in $\mathbb{S}|\{T\}$, based on an arrow $\varphi: S \longrightarrow T$ in $\mathbb{S}$; this uses natural transformations (nt's) in $\mathbb{S}$. Let us take S to be the initial object $\emptyset$, and let us replace nt's by pbnt's in the construction of $T \underset{S}{\vec{\sqcup}} T^+$ and $T^+ \underset{S}{\vec{\sqcup}} T^+$; we get $T \dot{\vec{\sqcup}} T^+$ and $T^+ \dot{\vec{\sqcup}} T^+$.

The inclusion-arrow $\mathtt{Stab}[T]: T \dot{\vec{\sqcup}} T^+ \longrightarrow T^+ \dot{\vec{\sqcup}} T^+$ expresses stability under pullback of T-diagrams.

As an example, stability of binary coproduct under pullback is expressed by



In particular, a category with finite limits and binary coproducts, construed as a sketch in $\texttt{flSk}\,|\,\{\hat{\texttt{coP}}\}$, satisfies $\texttt{Stab}[\hat{\texttt{coP}}]$ iff binary coproducts in it are stable under pullback.

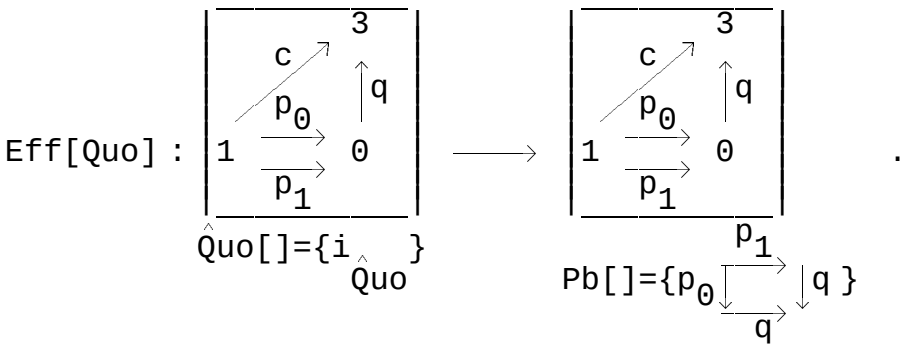
Regular categories are specified in the sketch-category $\texttt{regSk} \underset{\texttt{def}}{=} \texttt{flSk}\,|\,\{\ulcorner\texttt{mon}\urcorner\}\,|\,\{\hat{\exists}\}$ by the axioms $\mathcal{R}[\texttt{Reg}] \underset{\texttt{def}}{=} \texttt{Def}[\texttt{mon}] \cup \mathcal{R}[\texttt{fl}] \cup \mathcal{R}_{\ell}[\exists] \cup \texttt{Stab}[\hat{\exists}]$. Here, we have used notation from the previous section. The specification formalizes the definition of "regular category" saying that a regular category is one that has finite limits, and a left adjoint $\exists_f$ to each $f^*\!:\!S(Y) \longrightarrow S(X)$ satisfying the Beck-Chevalley condition. The reader may want to verify that $\texttt{Stab}[\hat{\exists}]$ indeed expresses the latter condition.

Coherent categories (see [27]) are specified in $\texttt{cohSk} \underset{\texttt{def}}{=} \texttt{regSk}\,|\,\{\texttt{i},\texttt{join}\}$ by

$$\mathcal{R}[\texttt{Coh}] \underset{\texttt{def}}{=} \mathcal{R}[\texttt{Reg}] \cup \mathcal{R}_{\ell}[\texttt{join}] \cup \texttt{Stab}[\hat{\texttt{i}}] \cup \texttt{Stab}[\hat{\texttt{join}}]\,.$$

Next, note that for $T = \hat{\texttt{Quo}}$, $\texttt{Stab}[\hat{\texttt{Quo}}]$ holds in any lex category with quotients of equivalence relations iff those quotients are stable under pullback; equivalently, if regular epi's are stable under pullback.

For the notion of Barr-exact category, we need one more sketh-axiom, $\texttt{Eff}[\texttt{Quo}]$, the effectiveness of quotients (effectiveness of equivalence relations). This is



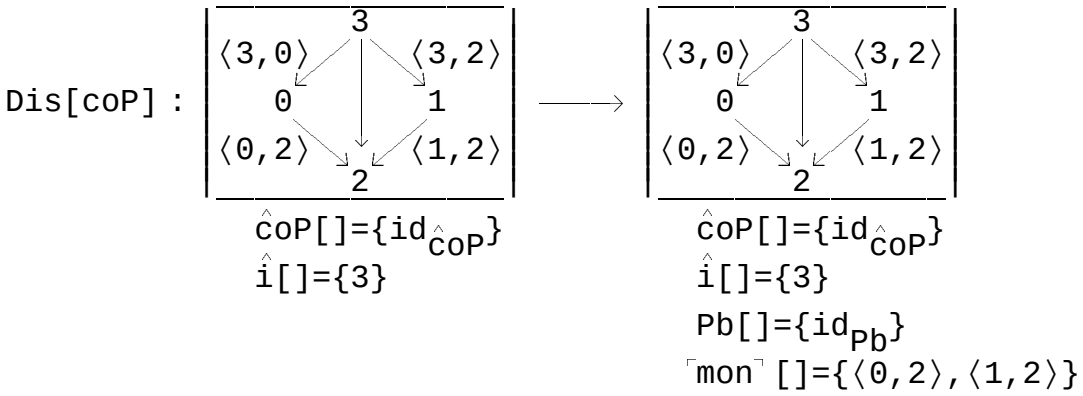
The sketch-category for the specification of **Barr–exact categories** (lex categories with effective equivalence relations and regular epi's stable under pullback) is

$$\mathtt{exSk} \underset{\mathtt{def}}{=} \mathtt{flSk}\,|\,\{\ulcorner\mathtt{EqRel}\urcorner\}\,|\,\{\hat{\mathtt{Quo}}\}\,,$$

and the axioms, beyond those for lex category, are $\mathtt{Def[EqRel]}$, $\mathcal{R}_{\ell}\mathtt{[Quo]}$, $\mathtt{Stab[Quo]}$ and $\mathtt{Eff[Quo]}$. Identifying a Barr-exact category with a sketch in $\mathtt{exSk}$, we find that

a sketch in $\mathtt{exSk}$ *is a Barr-exact category iff it satisfies the sketch-axioms for Barr-exact category.*

The category of sketches for distributive categories, $\mathtt{dSk}$, is $\mathtt{flSk}\,|\,\{\hat{\mathtt{i}},\hat{\mathtt{coP}},\ulcorner\mathtt{mon}\urcorner\}$. The sketch-axioms $\mathcal{R}\mathtt{[dis]}$ for **distributive categories** are $\mathcal{R}\mathtt{[Lex]}$, $\mathcal{R}_{\ell}\mathtt{[i]}$, $\mathcal{R}_{\ell}\mathtt{[coP]}$, $\mathtt{Stab}[\hat{\mathtt{i}}]$, $\mathtt{Stab}[\hat{\mathtt{coP}}]$, $\mathtt{Def[mon]}$ and $\mathtt{Dis[coP]}$, with the last defined as follows:



We have an obvious forgetful functor from the category of distributive categories (ones with

finite limits, initial object, binary coproduct, in which the latter two are stable under pullback, and binary coproducts are disjoint sums) to $\mathtt{dSk}$ which is full and faithful, and

whose image consists of those objects of $\mathtt{dSk}$ *that satisfy* $\mathcal{R}[\mathtt{dis}]$.

For **pretoposes** (see [27]), we consider the category $\mathtt{pretopSk}$ of *pretopos-sketches*, which is, by definition,

$$\mathtt{exSk} \underset{\mathtt{flSk}}{\times} \mathtt{dSk} = \mathtt{flSk}\,|\,\{\ulcorner \mathtt{EqRel} \urcorner, \hat{\mathtt{i}}, \hat{\mathtt{coP}}, \ulcorner \mathtt{mon} \urcorner\}\,|\,\{\hat{\mathtt{Quo}}\};$$

the sketch-axioms of pretopos are $\mathcal{R}[\mathtt{pretop}] \underset{\mathtt{def}}{=} \mathcal{R}[\mathtt{ex}] \cup \mathcal{R}[\mathtt{dis}]$. Clearly,

$(\mathtt{pretopSk}, \mathcal{R}[\mathtt{pretop}])$ *is a specification for the category of pretoposes; the latter is isomorphic to* $\mathtt{pretopSk}{:}\mathcal{R}[\mathtt{pretop}]$.

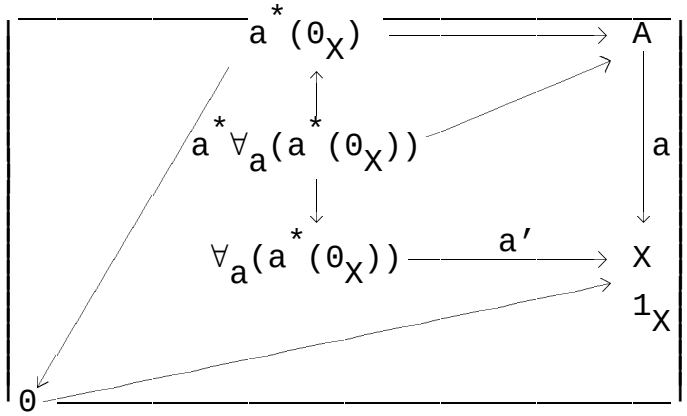
A Heyting category is a coherent category in which each $f^*{:}\mathtt{S}(Y) \longrightarrow \mathtt{S}(X)$ has a right adjoint $\forall_f$. **Heyting categories** are specified by $(\mathtt{HeySk}, \mathcal{R}[\mathtt{Hey}]) \underset{\mathtt{def}}{=} (\mathtt{cohSk}\,|\,\{\hat{\forall}\}, \mathcal{R}[\mathtt{Coh}] \cup \mathcal{R}_{\mathsf{r}}[\forall])$. A biHeyting category is a Heyting category in which the operation of coimplication $b \backslash a$ can be performed in every subobject lattice. **BiHeyting categories** are specified by

$$(\mathtt{biHeySk}, \mathcal{R}[\mathtt{biHey}]) \underset{\mathtt{def}}{=} (\mathtt{HeySk}\,|\,\{\hat{\mathtt{coimp}}\}, \mathcal{R}[\mathtt{Hey}] \cup \mathcal{R}_{\ell}[\mathtt{coimp}]).$$

In a Heyting category, for mono's $A \overset{a}{\rightarrowtail} X$, $B \overset{b}{\rightarrowtail} X$, the Heyting implication $[a] \to [b]$ of the subobjects $[a]$, $[b]$ of X, itself a subobject of X, can be given as $\forall_a(a^*([b]))$. In particular, the negation $\neg[a] = [a] \to 0_A$, with $0_X = [0 \to X]$, with 0 the initial object, is $\forall_a(a^*(0_X))$. Therefore, the principle of the Excluded Middle can be put in this form:

$$[a] \vee_X \forall_a(a^*(0_X)) = 1_X;$$

here, $\vee_X$ is the join-operation in $S(X)$, 1_X is the maximal element of $S(X)$. The exactness property expressing this is $E: \ulcorner E \urcorner \longrightarrow \hat{E}$, where $\ulcorner E \urcorner$ is:



$\hat{\forall}[] = \{$obvious$\}$

$$\{\ a^*(0_X) \longrightarrow A \ \downarrow \ 0 \longrightarrow X\ \} \in Pb[]$$

$\hat{i}[] = \{0\}$

$I[] = \{1_X\}$

(note that the $\hat{\forall}[]$ specification contains a $Pb[]$-specification, and a $mon[]$-specfication; thus, Pb[$\ulcorner E \urcorner$] has two elements), and $\hat{E}$ is $\ulcorner E \urcorner$ with the added specification $Join[] = \{"[a] \vee [a'] = [1_X]"\}$. Here I used a hybrid notation for easier understanding. The complex notation used to denote the objects is optional, and can be replaced by mere symbols.

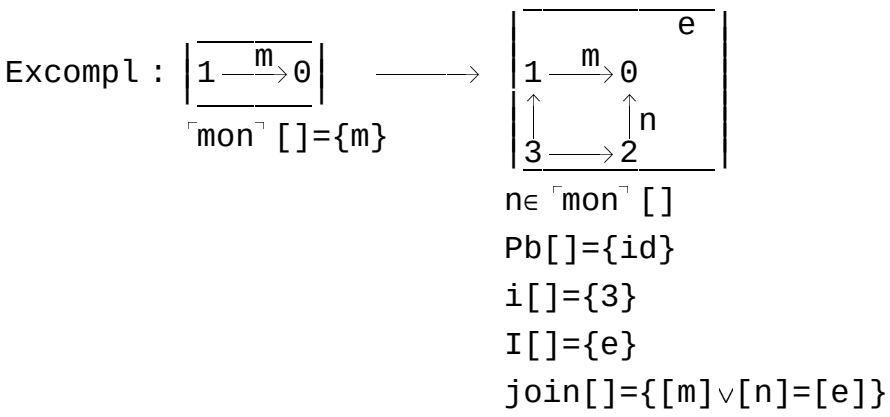
Boolean categories can be specified by

$$(BoolSk, \mathcal{R}[Bool]) \ \bar{def} \ (HeySk, \mathcal{R}[Hey] \cup \{E\}) .$$

This "says" that a Boolean category is a Heyting category in which negation satisfies the Law of the Excluded Middle. We could have first introduced negation as a defined operation in arbitrary Heyting categories, and required the law (exactness property) of the excluded middle for it in a second step.

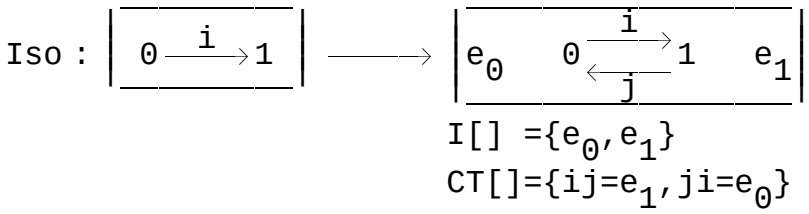
Alternatively, Boolean complements can be introduced directly on top of the coherent doctrine.

In cohSk , let Excompl be the following map:



The alternative specification for Boolean categories is (cohSk,ℜ[coh]∪{Excompl}) . Notice that, in this conception, BoolSk is equal to cohSk ; that is, no new type of specification is used with respect to the ones inherent in the coherent doctrine. This is in agreement with the fact that coherent functors between Boolean categories are Boolean: they preserve Boolean complements.

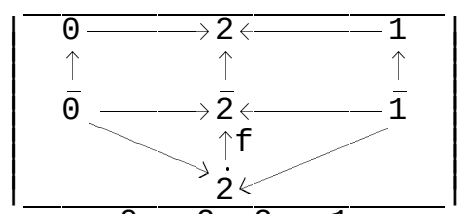
There is one type of exactness property that has to be discussed separately. This is when a morphism is asserted to be an isomorphism, on the basis of a situation given in a sketch. This will fit into our general framework only if we have a *specification* of an arrow being an isomorphism. Consider the following sketch-entailment Iso: ⌜Iso⌝ ⟶ $\hat{\text{Iso}}$ in cSk :



saying that i is an isomorphism. We decide to adjoin the specification-type ⌜Iso⌝ to all doctrinal specifications based on Graph , together with the axioms Def[Iso] (see the previous section). This amounts to considering each object (structured category) of the doctrine as having all its isomorphisms marked as such. Since any functor preserves isomorphisms, this will have no effect on the extent of the doctrines specified.

Suppose $\boldsymbol{S}$ is a sketch-category based on $\mathtt{Graph}$, with $\ulcorner\mathtt{Iso}\urcorner$ as one of its specification-types. Then for any sketch $S\in\boldsymbol{S}$ with a particular arrow $f\in\mathtt{Arr}\|S\|$ ($\|S\|$ being the underlying graph of S), we have the particular exactness property $\mathtt{Inv}[S, f]:S\longrightarrow S'$, where S' is obtained by adding to S the single specification $\ulcorner f\urcorner:\ulcorner\mathtt{Iso}\urcorner\longrightarrow\|S\|$ for which $\ulcorner f\urcorner(\mathtt{i})=f$ to $\ulcorner\mathtt{Iso}\urcorner[S]$.

It is well-known that *almost* all exactness properties can be equivalently reformulated so as to assert that a certain arrow is an isomorphism. For instance, with S the sketch



$$\mathtt{Pb}[]=\left\{\begin{matrix}0\longrightarrow 2 & 2\longleftarrow 1\\ \uparrow & \uparrow\\ \bar{0}\longrightarrow\bar{2} & \bar{2}\longleftarrow\bar{1}\end{matrix}\right\}$$

$$\hat{\mathtt{coP}}[]=\{0\longrightarrow 2\longleftarrow 1,\bar{0}\longrightarrow\dot{2}\longleftarrow\bar{1}\}$$

$$\bar{0}2\dot{2},\bar{1}2\dot{2}\in\mathtt{CT}[]\qquad,$$

$\mathtt{Inv}[S, f]$ is equivalent to $\mathtt{Stab}[\mathtt{coP}]$ provided $\mathcal{R}_{\ell}[\mathtt{coP}]$ is assumed. An important exception is when the exactness property concludes that a diagram commutes. Even such exactness properties can be equivalently replaced by ones of the form $\mathtt{Inv}[S, f]$ provided equalizers or coequalizers are present. However, for the doctrine of Cartesian closed categories, asserting a conditional commutativity apparently cannot always be replaced by a condition of the form $\mathtt{Inv}[S, f]$.

A frequently occurring type of exactness property asserts that, under certain definite circumstances, one subobject A of an object X is $\leq$ another subobject B of the same object X. This, apparently, requires the introduction of a new arrow mediating the order-relation. However, assuming we have pullbacks, we always have $A\wedge B\leq A$, and $A\leq B$ iff $A\wedge B=A$. Thus, we can express the exactness property by asserting that the morphism mediating $A\wedge B\leq A$ is an isomorphism.

Recall the notion of **S4 category** from [28]; it is also briefly described in the previous section. This is a doctrine based on $\mathtt{Graph}^{\rightarrow}$. To describe a sketch-specification of S4-categories, I introduce some terminology.

For any graph G, by $G_{(1)}$ and $G_{(2)}$, I mean the objects $G \longrightarrow \dot{G}$, $\emptyset \longrightarrow G$, respectively, of $\mathtt{Graph}^{\rightarrow}$; here $\dot{G}$ is the same as the graph G, except that each object and arrow of G carries a dot on it; the map $G \longrightarrow \dot{G}$ is "dotting". Note that what we denoted as $\mathtt{I}_1$, $\mathtt{I}_2$, $\mathtt{CT}_1$ and $\mathtt{CT}_2$ in the previous section are $\mathtt{I}_{(1)}$, $\mathtt{I}_{(2)}$, $\mathtt{CT}_{(1)}$ and $\mathtt{CT}_{(2)}$, *resp*. The operations $(\,)_{(1)}$, $(\,)_{(2)}$ extend in an obvious way to morphisms of graphs.

For $S \in \mathtt{cSk}$, $S_{(1)}$ and $S_{(2)}$ are the objects of $\mathtt{fSk}$ for which $|S_{(1)}|$, $|S_{(2)}|$ are $|S|_{(1)}$, $|S|_{(2)}$, *resp*., and for which $\mathtt{CT}_1[S_{(1)}] = \{\varphi_{(1)} : \varphi \in \mathtt{CT}[S]\}$, etc. For the specification-types for the coherent doctrine,

$$\hat{\mathtt{t}}\,,\ \hat{\mathtt{Pb}}\,,\ \hat{\mathtt{i}}\,,\ \ulcorner\mathtt{mon}\urcorner\,,\ \ulcorner\mathtt{Iso}\urcorner\,,\ \hat{\exists} \text{ and } \hat{\mathtt{join}} \qquad (1)$$

(with the latter two using $\ulcorner\mathtt{mon}\urcorner$ as an ingredient), we then have their subscripted versions $S_{(1)}$, $S_{(2)}$; all of the latter and $\ulcorner\Box\urcorner$ (see the end of the previous section) adjoined to $\mathtt{fSk}$ result in $\mathtt{S4Sk}$, the category of *S4-sketches*.

The (1) and (2)-subscripted versions of the axioms in $\mathcal{R}[\mathtt{coh}]$, $\mathcal{R}[\mathtt{coh}]_{(1)} \cup \mathcal{R}[\mathtt{coh}]_{(2)}$, express that in an S4-category $\begin{smallmatrix} \boldsymbol{B} \\ \uparrow \cdot \\ \boldsymbol{C} \end{smallmatrix}$, $\boldsymbol{C}$ and $\boldsymbol{B}$ are coherent categories.

With S any of the items in (1), $S^{+}_{(1)}$ denotes the $\mathtt{S4Sk}$-sketch obtained by adding the specification the identity $S_{(1)} \rightarrow S_{(1)}$ to $S_{(1)}$; that is, $S_{(1)}[S^{+}_{(1)}] = \{\mathtt{id}_{S_{(1)}}\}$. $S^{++}_{(1)}$ denotes the sketch obtained by adding the specification

$$\alpha \equiv S_{(2)} \longrightarrow S_{(1)} \equiv \begin{array}{ccc} S & \xrightarrow{(\,)^{\cdot}} & \dot{S} \\ \uparrow & & \uparrow \\ \emptyset & \xrightarrow[(\,)^{\cdot}]{} & S \end{array}$$

to $S^{+}_{(1)}$; $S_{(2)}[S^{++}_{(1)}] = \{\alpha\}$. The inclusion $S^{+}_{(1)} \longrightarrow S^{++}_{(1)}$ expresses (as a sketch-entailment) that in $\begin{smallmatrix} \boldsymbol{B} \\ \uparrow \cdot \\ \boldsymbol{C} \end{smallmatrix}$, the dot is a functor preserving the operation specified by S;

thus the set $\mathcal{R}[\cdot\mathrm{coh}]$ consisting of all $S^{+}_{(1)} \longrightarrow S^{++}_{(1)}$, S from (1), expresses that $()^{\cdot}$ is a coherent functor.

The axiom

$$\mathrm{Cons}: \left|\begin{array}{c} \dot{0} \xrightarrow{\dot{\alpha}} \dot{1} \\ \hline 0 \xrightarrow[\alpha]{} 1 \end{array}\right| \longrightarrow \left|\begin{array}{c} \dot{0} \xrightarrow{\dot{\alpha}} \dot{1} \\ \hline 0 \xrightarrow[\alpha]{} 1 \end{array}\right|$$

$$\mathrm{Iso}_{(2)}[]=\{\dot{\alpha}\} \qquad \mathrm{Iso}_{(2)}[]=\{\dot{\alpha}\}$$

$$\mathrm{Iso}_{(1)}[] \text{ "=" } \{\alpha\}$$

expresses that $()^{\cdot}$ is conservative.

The sketch-axioms for S4-categories are those in $\mathcal{R}[\mathrm{fun}]$, $\mathcal{R}[\mathrm{coh}]_{(1)} \cup \mathcal{R}[\mathrm{coh}]_{(2)}$, $\mathcal{R}[\cdot\mathrm{coh}]$, $\{\mathrm{Cons}\}$ and $\mathcal{R}_{\mathrm{r}}[\Box]$ (see the previous section).