§4. Specifying operations defined as adjoints

F. W. Lawvere's discovery that underlies categorical logic is that "all logical operations arise as adjoints", possibly in an iterative fashion. For the last qualification, think of the notion of exponentiation (in a Cartesian closed category) that arises as an adjoint of a functor derived from binary product, itself obtained as an adjoint to a "primitive" functor. Let us call, informally, an *adjointness doctrine* any notion of structured category, or even of a finite diagram of categories connected by functors, with structure defined by adjoints, that is, by universal properties. In this and the next sections, I will attempt to show in a systematic fashion how adjointness doctrines can be specified in a natural way via sketch-semantics. The proof-theory of the last section then gives a *complete* notion of formal deducibility for each adjointness doctrine so specified. The overall point will be the uniform character, uniform over all adjointness doctrines, of the notion of formal deduction, directly arising from specification, and ultimately, from the common wording of the definition of the doctrine.

As is well-known, some of the adjointness doctrines are closely related to logics in the symbolic tradition; the doctrine of elementary toposes to higher order logic, the doctrine of Boolean categories to first order logic, etc. From the results of this paper, together with certain facts in categorical logic, it follows that there exists a translation from the proof-systems of those logics into sketch-based proof-systems of the corresponding doctrines, and vice versa. It is then possible to spell out these translations in a completely explicit manner. For more details on this point, see §9.

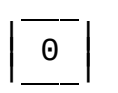
One more general remark. We will be dealing with structured categories with operations defined, via universal properties, up to isomorphism only. The corresponding notion of functor will be that preserving only the universal property of distinguished diagrams (products, exponentials, etc) (as opposed to the notion involving preserving operations on the nose which in fact in this case makes no sense since there are no strictly defined operations).

In this section, the sketch-specification of operations defined by universal properties is the subject. The next section deals with the second ingredient of adjointness doctrines, *exactness properties*.

Let us start with the example of **Cartesian categories**: categories with finite products. In this example, we start with the category $\mathbf{G} = \mathtt{cSk}$ introduced in the previous section. We

introduce the following specific c-sketches $\hat{t}$, $\hat{P}$ (objects in $\mathbf{G}$) (we reserve the symbols t , P for some other, but related things):

$\hat{t}$:

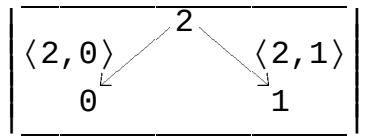


$I[]=\emptyset$

$CT[]=\emptyset$

("terminal object")

$\hat{P}$:



.

$I[]=\emptyset$

$CT[]=\emptyset$

("product")

$|\hat{t}|$, the underlying graph of $\hat{t}$, has one object 0 , and no arrows. Both specification-sets $I[\hat{t}]$ and $CT[\hat{t}]$ are empty. $\hat{P}$ has the underlying graph shown, and (again) empty specification sets as shown.

With $\mathbf{G} = \mathrm{cSk}$, and $\mathcal{G} = \{\hat{t},\hat{P}\}$, we put $\mathrm{fpSk} \underset{\mathrm{def}}{=} \mathbf{G}\,|\,\mathcal{G} = \mathrm{cSk}\,|\,\{\hat{t},\hat{P}\}$. In detail, a *finite-product (fp) sketch* S is a c-sketch $|S|$, together with a set $\hat{t}[S]$, written more simply as $t[S]$, of maps of c-sketches $\hat{t} \longrightarrow |S|$ (*terminal objects in* S), and a set $P[S]$ of maps of c-sketches $\hat{P} \longrightarrow |S|$ (*product diagrams in* S). The sets $t[S]$, $P[S]$ are the *fp-specification sets* of S . The category of (small) fp-sketches is denoted fpSk .

The (*specific*) *s*(*ketch*)*-axioms for Cartesian category* are the following eight fp-sketch maps (arrows in fpSk):

$Ex[t]$:

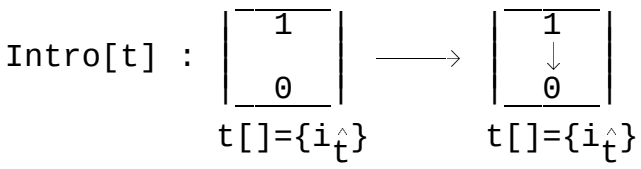
empty graph

$I[]=\emptyset$

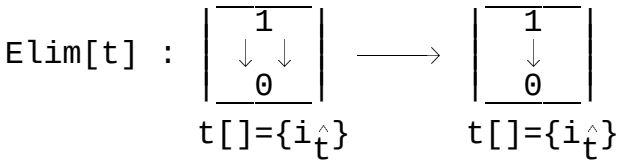
$CT[]=\emptyset$

$t[]=\{id_t\}$

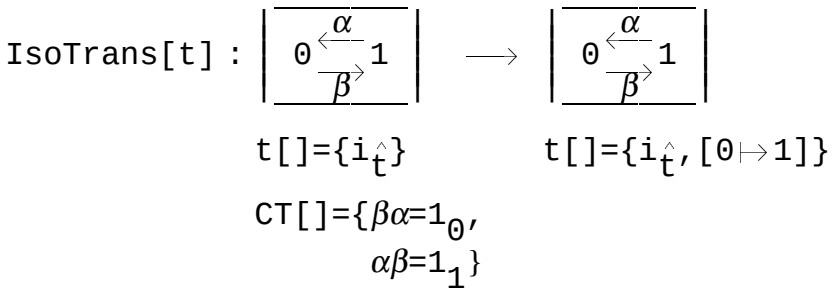
("Existence of terminal object".)



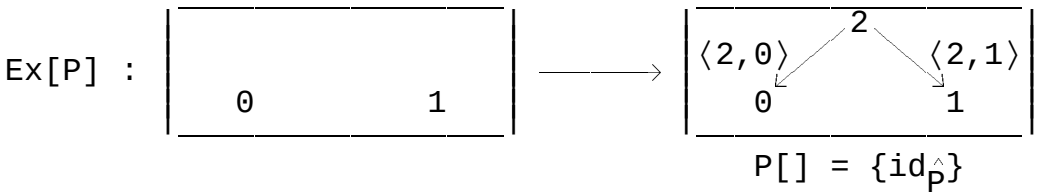
("Existence-part of universal property of terminal object")



("Uniqueness-part of universal property of terminal object".)

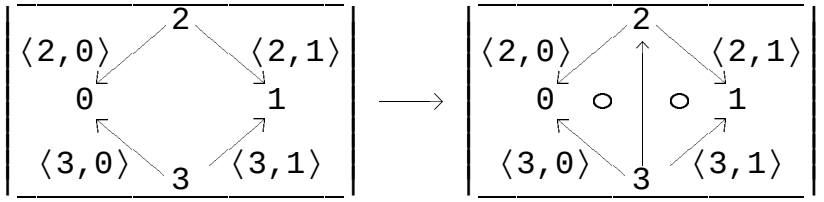


("An object isomorphic to a terminal object is terminal".)



("Existence of product".)

Intro[P] :

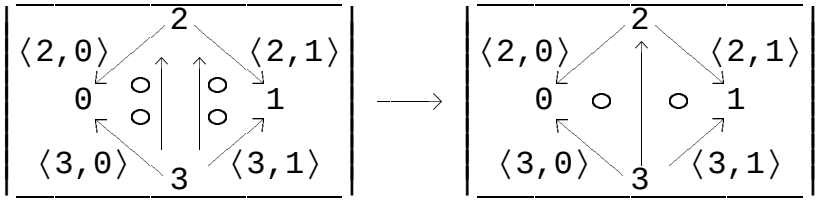


P[] = {$i_{\hat{P}}$}

P[] = {$i_{\hat{P}}$}
CT[] = {320,321}

("Existence-part of universal property of product".)

Elim[P] :

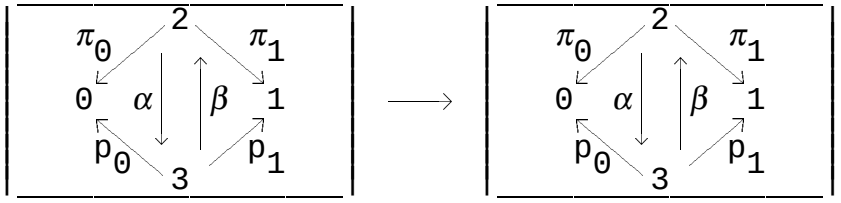


P[] = {$i_{\hat{P}}$}
CT[]={all four triangles}

P[] = {$i_{\hat{P}}$}
CT[] = {320,321}

("Uniqueness-part of universal property of product".)

IsoTrans[P] :



CT[]={$\beta\alpha=id_2$, $\alpha\beta=id_3$, $\pi_0\beta=p_0$, $\pi_1\beta=p_1$}
P[] = {$i_{\hat{P}}$}

P[] = {$i_{\hat{P}}$,j}
$j(2) = 3$

(Here, $\pi_0 = \langle 2, 0 \rangle$, etc. The "rule" "expresses" that a diagram isomorphic to a product diagram is itself a product diagram.)

The notation used is somewhat abbreviated; in the case of `IsoTrans[t]` and `IsoTrans[P]`, we omitted the specifications `CT[]` in the codomain sketch. The obvious convention is that any specification appearing in the domain sketch gives rise to one, its

image, in the codomain sketch, even if this is not shown in the notation. Thus, in the two mentioned cases, the CT[] specifications in the codomains should be repetitions of those in the respective domains.

Every c-sketch may, and is, considered also to be an fp-sketch, with empty fp-specification sets; maps of c-sketches become maps of fp-sketches. Let us also consider the sketch-axioms for category as maps of fp-sketches. By the "sketch-axioms for Cartesian category" (without the qualification "specific") we mean the specific ones together with the ones for category (as maps of fp-sketches). We write $\mathcal{A}$[Car] for the set of all sketch-axioms for Cartesian category.

A (small) category $\mathbf{C}$ gives rise to an fp-sketch $\mathbf{C}_{(s)}$, in which the fp-specification sets are the obvious ones; e.g., P[$\mathbf{C}_{(s)}$] is the set of all (binary) product diagrams in $\mathbf{C}$, construed as maps from $\hat{P}$ to $\mathbf{C}$. A finite-product preserving functor $F:\mathbf{C}\longrightarrow\mathbf{D}$ gives rise to an fp-sketch map $\mathbf{C}_{(s)}\longrightarrow\mathbf{D}_{(s)}$, and in fact, the resulting map hom($\mathbf{C}$,$\mathbf{D}$)$\longrightarrow$hom($\mathbf{C}_{(s)}$,$\mathbf{D}_{(s)}$) is a bijection. In particular, we may apply this construction to Cartesian categories $\mathbf{C}$; we have a fully faithful functor from the category Car of small Cartesian categories to the category fpSk of fp sketches which is also one-to-one on objects. We *identify* every cartesian category with the corresponding fp-sketch.

We have

An fp sketch is a Cartesian category iff it satisfies all s-axioms of Cartesian category; the Cartesian functors are the same as sketch-maps between the Cartesian categories as fp sketches.

This proposition spells out in what sense we can *specify* the concept of Cartesian category, together with the corresponding ("non-strict") notion of morphism of Cartesian categories, with the help of that of fp-sketch, using sketch-semantics. The proof of the proposition is essentially by inspection; the verbal comments given above at the individual sketch-axioms describe what part in the definition of "Cartesian category" those axioms ensure. The one surprising element is the need for the two IsoTrans axioms. They are used to ensure that *all* terminal objects and product diagrams according to usual definition via the universal properties are indeed marked in the sketch as such. The more general proposition (6) stated and proved below will cover the just stated case of Cartesian categories.

Let us make a comparison of the Ehresmannian semantics of sketches and our sketch-semantics. The notion of finite-product sketch in Ehresmann's sense, and that of our fp sketch are the same. According to Ehresmann, a *model* of an fp sketch S in a category $\mathbf{C}$ (a $\mathbf{C}$-model of S) is a graph-map from the underlying graph of S to that of $\mathbf{C}$ that transforms all identity-arrow, commutative-triangle, terminal-object and binary-product specifications in S into a real identity arrow, commutative square, terminal object and binary product diagram, respectively, in $\mathbf{C}$. Let us say that the sketch-entailment (sketch-arrow) $r:R\to R'$ of sketches is *true* in the category $\mathbf{C}$ if for any $\mathbf{C}$-model φ of R, there is a $\mathbf{C}$-model φ' of R' such that $\varphi=\varphi'\circ r$ (equality and composition of graph-maps). We have the following variant of the last proposition:

A category $\mathbf{C}$ *is Cartesian iff the sketch-entailments* `Ex[t]`, `Intro[t]`, `Elim[t]`, `Ex[P]`, `Intro[P]`, `Elim[P]` *are true in* $\mathbf{C}$.

The truth of this proposition is seen by inspection. However, note the difference with respect to the previous statement; now, we do not refer to the category axioms, neither to the `IsoTrans` axioms. The "explanation" is, of course, that these axioms are automatically true in any category (note this especially for `Isotrans[t]` and `Isotrans[P]`).

Next, observe the obvious fact that a $\mathbf{C}$-model of S is the same thing as a sketch-map $S\longrightarrow\mathbf{C}_{(s)}$, with $\mathbf{C}_{(s)}$ the sketch associated to $\mathbf{C}$ above. This allows us to formulate the case for the Cartesian categories of the General Completeness Theorem (GCT) of the last section in terms of models of sketches, as follows.

A (*relatively*) *finite sketch-entailment is true in all Cartesian categories (in the sense of the definition in terms of models) if and only if it is deducible from the axioms* $\mathcal{R}$`[Car]` *for Cartesian category.*

Note, however, that it would be wrong to replace $\mathcal{R}$`[Car]` here by the reduced set of axioms of the previous statement.

The last displayed statement may be called the *trivial completeness theorem for Cartesian categories*. It expresses the intuitive idea that any (finite) diagrammatic truth about Cartesian categories can be shown by a finite amount of obviously allowable diagram manipulation. We have similar "trivial completeness theorems" for the other examples treated later. In §8, we will make contact with "non-trivial" completeness theorems.

Next, I will give a framework for the presentation of other examples. This framework is not strictly necessary; any one example in what follows could be given directly, without the framework. On the other hand, the framework is not general enough to grasp all the important adjointness doctrines; modifications of it will be needed to deal with the final examples.

We are going to iterate the construction of $\boldsymbol{G}\,|\,\boldsymbol{K}$. With a fixed category $\boldsymbol{G}$, we talk about a *category of sketches based on* $\boldsymbol{G}$ when we mean one obtained by iterating (finitely often) the construction $\boldsymbol{G'} \mapsto \boldsymbol{G'}\,|\,\boldsymbol{K'}$ starting with $\boldsymbol{G}$. In other words, $\boldsymbol{G}$ itself is a sketch-category based on $\boldsymbol{G}$, and if $\boldsymbol{G'}$ is based on $\boldsymbol{G}$, so is $\boldsymbol{G'}\,|\,\boldsymbol{K'}$, for any $\boldsymbol{K'} \subset \mathtt{Ob}(\boldsymbol{G'})$. (Note, of course, that the same $\boldsymbol{S}$ may be based on more than one $\boldsymbol{G}$, namely all the intermediate stages of the construction of $\boldsymbol{S}$. Note also that, for now, we have excluded the use of the construction $\boldsymbol{G'}\,\|\boldsymbol{K'}$.) For any $\boldsymbol{S}$ based on $\boldsymbol{G}$, we have a forgetful functor $\|\ \| : \boldsymbol{S} \longrightarrow \boldsymbol{G}$. Moreover, if here always $\boldsymbol{K'}$ is a finite set of fp objects of $\boldsymbol{G'}$, we say that $\boldsymbol{S}$ is *finite* over $\boldsymbol{G}$. In the next family of examples, we will consider sketch-categories all finite over $\mathtt{cSk}$, the category of category sketches; in particular, all these sketch categories will be finite over $\mathtt{Graph}$, the category of graphs. $\mathtt{fpSk}$, in the modified formulation, is a sketch-category based on $\mathtt{cSk}$, and further, on $\mathtt{Graph}$ as well.

Suppose $\boldsymbol{S}$ is a sketch-category based on $\boldsymbol{G}$; $S \in \mathtt{Ob}(\boldsymbol{S})$. A *specification-type* of S is any K, where $K \in \boldsymbol{K'}$, $\boldsymbol{G'}\,|\,\boldsymbol{K'}$ is any intermediate stage of the construction of $\boldsymbol{S}$ from $\boldsymbol{G}$; a *specification-set* of S is any set of the form $K[S']$, with K as above, and S' the underlying object S in $\boldsymbol{G'}$. Without causing confusion, we may write $K[S]$ for $K[S']$ here. For $S \in \mathtt{fpSk}$, with the latter considered as based on $\mathtt{Graph}$, S has four specification sets, $\mathtt{I}[S]$, $\mathtt{CT}[S]$, $\mathtt{t}[S]$ and $\mathtt{P}[S]$; when $\mathtt{fpSk}$ is considered as based on $\mathtt{cSk}$, S has two specification-sets, $\mathtt{t}[S]$ and $\mathtt{P}[S]$.

Let us put ourselves in the category $\mathtt{cSk}$. Let $S \in \mathtt{cSk}$. A *commutative square-specification* (css) in S is a pair (s_0, s_1) of commutative triangle specifications $s_0, s_1 \in \mathtt{CT}[S]$ such that ($s_0(0) = s_1(0)$, $s_0(2) = s_1(2)$ and) $s_0(\langle 0, 2\rangle) = s_1(\langle 0, 2\rangle)$ (see the specification of the graph $\mathtt{CT}$). The css $s = (s_0, s_1)$ *specifies that the square*

$$\begin{array}{ccc} s_0(0) & \xrightarrow{\quad s_1(\langle 0,1\rangle) \quad} & s_1(1) \\ {\scriptstyle s_0(\langle 0,1\rangle)}\downarrow & & \downarrow{\scriptstyle s_1(\langle 1,2\rangle)} \\ s_0(1) & \xrightarrow[\quad s_0(\langle 1,2\rangle) \quad]{} & s_0(2) \end{array}$$

commutes. We write $\mathtt{CT}^{(4)}[S]$ for the set of css's in S. There are similar other definitions, of commutativity specifications of other shapes. Note that any $\varphi:S\longrightarrow T$ maps $\mathtt{CT}^{(4)}[S]$ to $\mathtt{CT}^{(4)}[T]$ in the obvious way.

Let $U\overset{\varphi}{\underset{\psi}{\rightrightarrows}}V$ be objects and arrows in $\mathtt{cSk}$. A *natural transformation* (nt) $\alpha:\varphi\longrightarrow\psi$ is a pair

$$\alpha=(\langle\alpha_X:\varphi X\to\psi X\rangle_{X\in\mathtt{Ob}(U)},\ \langle s_f\in\mathtt{CT}^{(4)}[V]\rangle_{(f:X\to Y)\in\mathtt{Arr}(U)})$$

of a families of arrows and commutative-square specifications in V as shown such that, for every $(f:X\to Y)\in\mathtt{Arr}(U)$, s_f specifies that the diagram

$$\begin{array}{ccc}\varphi X & \xrightarrow{\ \alpha_X\ } & \psi X\\ {\scriptstyle\varphi f}\downarrow & & \downarrow{\scriptstyle\psi f}\\ \varphi Y & \xrightarrow[\ \alpha_Y\ ]{} & \psi Y\end{array}$$

in the category-sketch V commutes.

Note that, if we have $T\xrightarrow{\theta}U\overset{\varphi}{\underset{\psi}{\rightrightarrows}}V\xrightarrow{\tau}W$, then both nt's $T\overset{\varphi\theta}{\underset{\psi\theta}{\rightrightarrows}}V$ and $U\overset{\tau\varphi}{\underset{\tau\psi}{\rightrightarrows}}W$ are defined, in the obvious way.

With any $U\overset{\varphi}{\underset{\varphi}{\rightrightarrows}}V$, that is, a natural *endo*map $\alpha:\varphi\longrightarrow\varphi$, a *specification of* "$\alpha=1_\varphi$" is a family $\langle s_X\in\mathtt{I}[V]\rangle_{X\in\mathtt{Ob}(U)}$ such that $s_X(\langle 0,0\rangle)=\alpha_X$ for all $X\in\mathtt{Ob}(U)$.

Let $\varphi:S\longrightarrow T$ be a fixed map in $\mathtt{cSk}$. Consider all "diagrams" in $\mathtt{cSk}$ of the form

$$S\xrightarrow{\varphi}T\overset{\psi}{\underset{\sigma}{\rightrightarrows}}U\quad\text{such that }\psi\varphi=\sigma\varphi,$$

together with a specification s of "$\alpha\varphi=1_{\psi\varphi}$". Temporarily, with fixed $S\xrightarrow{\varphi}T$ and U but

with all other data variable, let us denote the set of all such pairs $(S \xrightarrow{\varphi} T \underset{\sigma}{\overset{\psi}{\Downarrow \alpha}} U,\ s)$ by $\mathcal{C}_U$.

With any $C \in \mathcal{C}_U$, C as displayed, and any map $U \xrightarrow{\theta} V$, we have $\theta C \in \mathcal{C}_V$:

$$\theta C = (S \xrightarrow{\varphi} T \underset{\theta\sigma}{\overset{\theta\psi}{\Downarrow \theta\alpha}} V,\ \theta s),$$

with θs having the obvious meaning. We may consider the universal property of $(S \xrightarrow{\varphi} T \underset{\sigma}{\overset{\psi}{\Downarrow \alpha}} U,\ s)$ according to which the map $\theta \mapsto \theta C$ is a bijection $\mathrm{hom}(U, V) \longrightarrow \mathcal{C}_V$. In fact, there are U and $C \in \mathcal{C}_U$ having the described universal property; we denote U by $T \underset{S}{\overset{\rightarrow}{\sqcup}} T$ and C by

$$S \xrightarrow{\varphi} T \underset{\iota_1}{\overset{\iota_0}{\Downarrow \alpha}} T \underset{S}{\overset{\rightarrow}{\sqcup}} T,$$

and call it the *arrow-cokernel* of φ. The construction of these items is the expected one. $T \underset{S}{\overset{\rightarrow}{\sqcup}} T$ is obtained by taking the disjoint sum $T \sqcup T$, and adding some arrows and specifications; notably, arrows $\alpha_X : \iota_0 X \to \iota_1 X$ ($X \in \mathrm{Ob}(T)$), further arrows $\alpha_f : \iota_0 X \to \iota_1 Y$ one for each $f : X \to Y$ in T (the latter are needed for the css's making sure that α is an nt), and the obvious specifications needed for making α an nt. Of course, $T \underset{S}{\overset{\rightarrow}{\sqcup}} T$, given as it is by a universal property, is determined up to a unique isomorphism in the usual way.

The above concepts and constructions are generalized to any sketch-category $\boldsymbol{G}$ based on cSk in a straight-forward manner. An nt $\alpha : \varphi \longrightarrow \psi$, for $U \underset{\psi}{\overset{\varphi}{\rightrightarrows}} V$ in $\boldsymbol{G}$, is the same as $\alpha : \|\varphi\| \longrightarrow \|\psi\|$. We obtain, for any $\varphi : S \longrightarrow T$ in $\boldsymbol{G}$, the arrow-cokernel

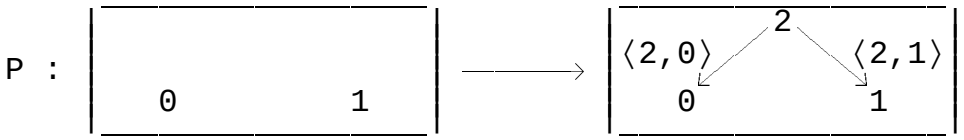
$$S \xrightarrow{\varphi} T \underset{\iota_1}{\overset{\iota_0}{\Downarrow \alpha}} T \underset{S}{\overset{\rightarrow}{\sqcup}} T \qquad (1)$$

in $\boldsymbol{G}$, with an appropriate specification of "$\alpha\varphi$=1" and with the appropriate universal property; the forgetful functor applied to this gives the arrow-cokernel on $\|\varphi\|$. In fact, it is easily seen that adding specifications to $\|T\| \underset{\|S\|}{\vec{\sqcup}} \|T\|$ so that ι_0 and ι_1 are 1-1, and jointly surjective, maps on specifications, results in an appropriate $T \underset{S}{\vec{\sqcup}} T$.

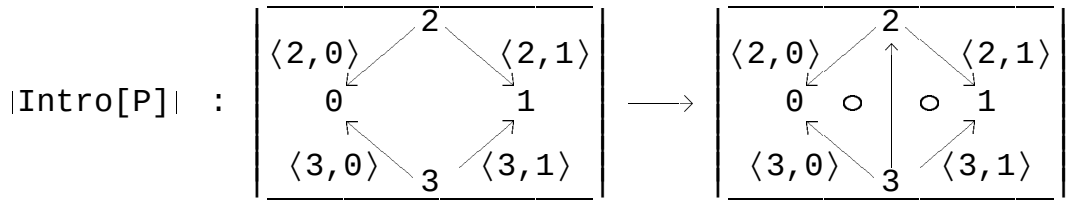
With the same $\varphi : S \longrightarrow T$, and with the cokernel $S \longrightarrow T \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} T \underset{S}{\sqcup} T$, we have (by the universal property of the pushout), a canonical comparison map

$$\iota : T \underset{S}{\sqcup} T \longrightarrow T \underset{S}{\vec{\sqcup}} T \,. \qquad (2)$$

Let us apply the just-described constructions to $\boldsymbol{G}$ = cSk and the map (inclusion)



(P is identical to |Ex[P]| ; see 1.2 ; this item deserves the simple name P rather than $\hat{P}$, which is the codomain of P) as $\varphi : S \longrightarrow T$ in cSk . Then $\iota : T \underset{S}{\sqcup} T \longrightarrow T \underset{S}{\vec{\sqcup}} T$ is (up to isomorphism) the map



in cSk with Intro[P] as in 1.2, except for the inessential difference that the two commutative triangles are to be replaced by commutative squares, by using additional identity arrows on 0 and 1 .

We consider another construction based on an arrow $\varphi : S \longrightarrow T$ in a sketch-category $\boldsymbol{G}$

based on cSk , that of the *double arrow-cokernel*,

$$S \xrightarrow{\varphi} T \underset{\iota_1}{\overset{\iota_0}{\downarrow\beta \quad \downarrow\gamma}} T \underset{S}{\overset{\rightarrow}{\sqcup}} T \qquad (3)$$

with two, rather than one, nt's. Both the universal property defining the double cokernel, and its construction should be clear. Note that we have a canonical map (given by the universal property of the double cokernel)

$$\iota : T \underset{S}{\overset{\rightarrow}{\sqcup}} T \longrightarrow T \underset{S}{\overset{\rightarrow}{\sqcup}} T \qquad (4)$$

identifying the two natural transformations, that is, $\iota\beta = \iota\gamma = \alpha$.

In the example just considered, the map $\iota : T \underset{S}{\overset{\rightarrow}{\sqcup}} T \longrightarrow T \underset{S}{\overset{\rightarrow}{\sqcup}} T$ is (essentially) the same as |Elim[P]| .

Finally, for the purposes of the "isomorphism transfer axioms", let us introduce, with any $\varphi : S \longrightarrow T$ as before, the *isomorphism cokernel*

$$S \xrightarrow{\varphi} T \underset{\iota_1}{\overset{\iota_0}{\downarrow\alpha \quad \uparrow\alpha^{-1}}} T \underset{S}{\overset{\cong}{\sqcup}} T \qquad (5)$$

in the expected way; this involves two nt's α and α^{-1} , and two specifications, one to the effect that "$\alpha \circ \alpha^{-1} = 1_{\iota_1}$" , the other that "$\alpha^{-1} \circ \alpha = 1_{\iota_0}$" .

Here is then a general scheme of introducing a new operation, with a universal property. Let $\boldsymbol{G}$ be any sketch-category based on cSk . Consider any specific map $\varphi : S \longrightarrow T$ of sketches in $\boldsymbol{G}$. Consider $\boldsymbol{S} = \boldsymbol{G} | \{T\}$. Define the sketches T^+ , $T \underset{S}{\overset{\rightarrow}{\sqcup}} T^+$, $T \underset{S}{\overset{\rightarrow}{\sqcup}} T^+$, $T \underset{S}{\overset{\cong}{\sqcup}} T^+$, $T^+ \underset{S}{\overset{\cong}{\sqcup}} T^+$ in $\boldsymbol{S}$ as follows. $|T^+| = T$, $T[T^+] = 1_T$; $|T \underset{S}{\overset{\rightarrow}{\sqcup}} T^+| = T \underset{S}{\overset{\rightarrow}{\sqcup}} T$, $T[T \underset{S}{\overset{\rightarrow}{\sqcup}} T^+] = \iota_1$ from (1);

$|T\underset{S}{\coprod^{\rightarrow}}T^{+}| = T\underset{S}{\coprod^{\rightarrow}}T$, $T[T\underset{S}{\coprod^{\rightarrow}}T^{+}] = \iota_1$ from (3); $|T\underset{S}{\coprod^{\cong}}T^{+}| = T\underset{S}{\coprod^{\cong}}T$, $T[T\underset{S}{\coprod^{\cong}}T^{+}] = \iota_1$ from (5);
$|T^{+}\underset{S}{\coprod^{\cong}}T^{+}| = T\underset{S}{\coprod^{\cong}}T$, $T[T^{+}\underset{S}{\coprod^{\cong}}T^{+}] = \{\iota_0, \iota_1\}$ from (5). Further, we have the following maps

$$\mathtt{Ex}[\varphi] : S \longrightarrow T^{+},$$

$$\mathtt{Intro}_{\mathtt{r}}[\varphi] : T\underset{S}{\coprod}T^{+} \longrightarrow T\underset{S}{\coprod^{\rightarrow}}T^{+},$$

$$\mathtt{Elim}_{\mathtt{r}}[\varphi] : T\underset{S}{\coprod^{\rightarrow}}T^{+} \longrightarrow T\underset{S}{\coprod^{\rightarrow}}T^{+},$$

$$\mathtt{IsoTrans}[\varphi] : T\underset{S}{\coprod^{\cong}}T^{+} \longrightarrow T^{+}\underset{S}{\coprod^{\cong}}T^{+};$$

on the level of $\boldsymbol{G}$, the first is φ, the second is the canonical map (2), the third is (4), the last is the identity. The variants when $T\underset{S}{\coprod^{\rightarrow}}T^{+}$ and $T\underset{S}{\coprod^{\rightarrow}}T^{+}$ are replaced by $T^{+}\underset{S}{\coprod^{\rightarrow}}T$ and $T^{+}\underset{S}{\coprod^{\rightarrow}}T$, defined in the expected way, are also important; they result in

$$\mathtt{Intro}_{\ell}[\varphi] : T^{+}\underset{S}{\coprod}T \longrightarrow T^{+}\underset{S}{\coprod^{\rightarrow}}T,$$

$$\mathtt{Elim}_{\ell}[\varphi] : T^{+}\underset{S}{\coprod^{\rightarrow}}T \longrightarrow T^{+}\underset{S}{\coprod^{\rightarrow}}T.$$

When we apply this construction to $\boldsymbol{G} = \mathtt{cSk}$, $T = \hat{\mathtt{P}}$, $\varphi = \mathtt{P}:S \longrightarrow T$, we get the four axioms for product among the sketch-axioms for Cartesian category. Let, in general,

$$\mathcal{R}_{\mathtt{r}}[\varphi] \underset{\mathtt{def}}{=} \{\mathtt{Ex}[\varphi], \mathtt{Intro}_{\mathtt{r}}[\varphi], \mathtt{Elim}_{\mathtt{r}}[\varphi], \mathtt{IsoTrans}[\varphi]\},$$

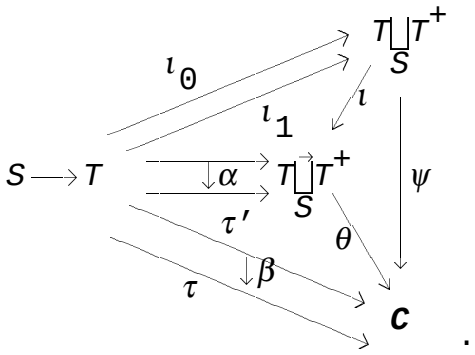
and $\mathcal{R}_{\ell}[\varphi]$ similarly defined.

What does $\mathcal{R}_{\mathtt{r}}[\varphi]$ do ($\varphi:S \to T$ is as before)? Suppose $\boldsymbol{G}$ is a sketch-category based on $\mathtt{cSk}$; assume $\boldsymbol{C} \in \boldsymbol{G}\,|\,\{T\}$, and also that $\boldsymbol{C} \vDash \mathcal{R}_{\mathtt{cat}}$, that is, $\boldsymbol{C}$ is, among others, a category.

With any S-type diagram $\sigma : S \longrightarrow \mathbf{C}$, we have the category $\langle \varphi, \sigma \rangle$ of all T-type extensions of σ: the objects are $\tau : T \longrightarrow \mathbf{C}$ with $\tau \circ \varphi = \sigma$, with arrows $\beta : \tau' \longrightarrow \tau$ defined as nt's $\beta : \tau' \longrightarrow \tau$ (in the sense of "nt" in $\mathbf{G}$ defined above) satisfying $\beta\varphi = 1_\sigma$ (since $\mathbf{C}$ is a category, the last makes sense, and it is equivalent to saying that $\mathbf{C}$ as a sketch contains a specification of "$\beta\varphi = 1_\sigma$"). We have

(6) $\mathbf{C} \models \mathcal{R}_r[\varphi]$ *iff* (i) *for every* S*-diagram* $\sigma : S \longrightarrow \mathbf{C}$, *the category* $\langle \varphi, \sigma \rangle$ *has a terminal object* and (ii) $T[\mathbf{C}]$ *is equal to the set of diagrams* $\tau : T \longrightarrow \mathbf{C}$ *which are terminal in* $\langle \varphi, \tau\varphi \rangle$.

For the "only if" part, assume $\mathbf{C} \models \mathcal{R}_r[\varphi]$ and let $\sigma : S \longrightarrow \mathbf{C}$. I claim that any $\tau \in T[\mathbf{C}]$ is terminal in $\langle \varphi, \sigma \rangle$ for $\sigma = \tau\varphi$. Let $\tau' \in \langle \varphi, \sigma \rangle$. Consider the diagram



Here, ι is $\mathrm{Intro}_r[\varphi]$. ψ is defined by the universal property of $T \underset{S}{\sqcup} T^+$, using the fact that $\tau \in T[\mathbf{C}]$; $\psi\iota_0 = \tau'$, $\psi\iota_1 = \tau$. Using $\mathbf{C} \models \mathrm{Intro}_r[\varphi]$, we have θ with $\theta\iota = \psi$. Finally, we let $\beta = \theta\alpha$. This shows that there is an arrow $\beta : \tau' \longrightarrow \tau$. The uniqueness of β is shown by using $\mathbf{C} \models \mathrm{Elim}_r[\varphi]$. We have shown that τ is terminal in $\langle \varphi, \sigma \rangle$. Thus, one of the two inclusions in (ii) is proved.

Turning to (i), by $\mathbf{C} \models \mathrm{Ex}[\varphi]$, there is $\tau^+ : T^+ \longrightarrow \mathbf{C}$ such that $\tau^+ \circ \mathrm{Ex}[\varphi] = \sigma$. Let $\tau : T \longrightarrow \mathbf{C}$ be the underlying arrow of τ^+; $\tau \in T[\mathbf{C}]$ and $\tau\varphi = \sigma$. By what we already know, (i) follows.

Finally, for the other inclusion in (ii), assume τ is terminal in $\langle \varphi, \sigma \rangle$ for $\sigma = \tau\varphi$. By the

previous paragraph, there is $\tau' \in T[\mathbf{C}]$ with $\tau'\varphi = \sigma$. Both τ and τ' are terminal in $\langle \varphi,\sigma \rangle$ (τ' by what we saw above). Therefore there is a (unique) isomorphism $\beta : \tau' \xrightarrow{\cong} \tau$ in $\langle \varphi,\sigma \rangle$. Using $\mathbf{C} \models \mathtt{IsoTrans}[\varphi]$, it is easy to conclude that $\tau \in T[\mathbf{C}]$.

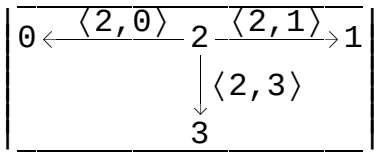
We have shown the "only if" part of (6); the "if" part is left to the reader.

There is an obvious analog of (6) for initial objects in place of terminal ones, using the ℓ-subscripted variants of the sketch-axioms used in (6).

The just-described framework is suitable for the introduction of many operations defined by universal properties, for the use in "adjointness doctrines", but not of all such operations.

Let us discuss **Cartesian closed categories**. We start with $\mathbf{G} = \mathtt{fpSk}$, already a sketch-category; we define $\mathtt{CcSk} = \mathtt{fpSk}\,|\{\hat{\mathtt{E}}\mathtt{xp}\}$, with $\hat{\mathtt{E}}\mathtt{xp}$ the following specific fp-sketch:

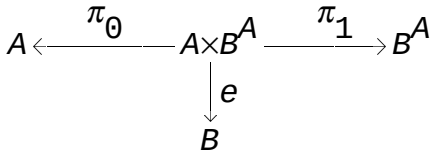
$\hat{\mathtt{E}}\mathtt{xp}$:



.

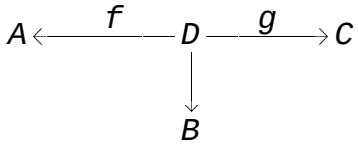
($\mathtt{i}_{\mathtt{P}}$ is the inclusion of the graph $\mathtt{P}$ in $\hat{\mathtt{E}}\mathtt{xp}$. As a general convention, each unspecified specification-set is understood as defined to be the empty set; e.g., $\mathtt{CT}[\hat{\mathtt{E}}\mathtt{xp}]$ is equal to $\emptyset$.)

In other words, a Cc-sketch (an object of $\mathtt{CcSk}$) S is an fp-sketch $|S|$, together with a set $\mathtt{Exp}[S]$ of fp-sketch maps $\hat{\mathtt{E}}\mathtt{xp} \longrightarrow |S|$. In connection with $\hat{\mathtt{E}}\mathtt{xp}$, one should think of

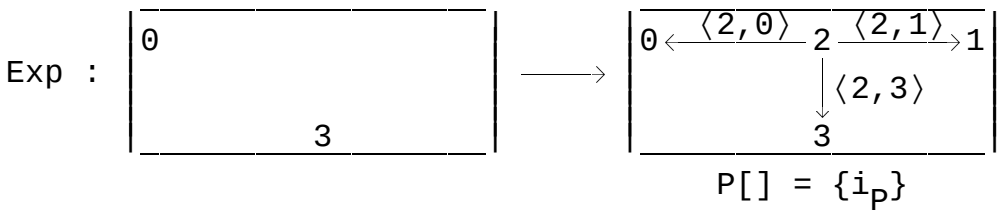


with π_0, π_1 product-projections, e the evaluation; indeed, $\hat{\mathtt{Exp}}$ will be used to pick out such diagrams in Cartesian closed categories. The point here is that the specification of an exponential diagram *involves* the specification of a product diagram; therefore, it is natural to define Cc sketches *starting with* fp-sketches.

Our way of looking at Cc categories is based on the obviously correct formulation that says that a category is Cc if it is Cartesian, and for every pair (A, B) of objects, among the diagrams



such that (f, g) is a product (with A , B fixed, and all other items varying), there is a terminal one. In a more explicit form, we can put this as follows. We consider the following inclusion map, called $\mathtt{Exp}: \ulcorner\mathtt{Exp}\urcorner \longrightarrow \hat{\mathtt{Exp}}$, in $\mathtt{fpSk}$:



Then, a Cartesian category $\boldsymbol{C}$ is Cc iff for any $\sigma: \ulcorner\mathtt{Exp}\urcorner \longrightarrow \boldsymbol{C}$, given by $A=\sigma(1)$, $B=\sigma(3)$, the category $\langle \mathtt{Exp}, \sigma \rangle$ (see before (6) above) has a terminal object.

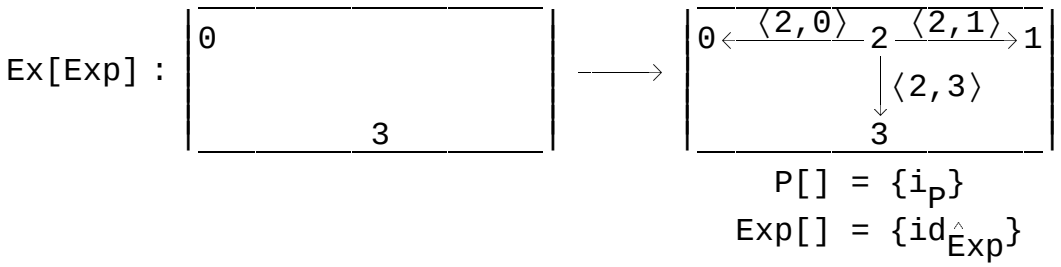
Let us define $\mathcal{R}[\mathtt{Cc}] \underset{\mathtt{def}}{=} \mathcal{R}[\mathtt{Car}] \cup \mathcal{R}_{\mathtt{r}}[\mathtt{Exp}]$; the last two terms were defined before.

For any (small) category $\boldsymbol{C}$, we have the Cc-sketch $\boldsymbol{C}_{(\mathtt{s})} \in \mathtt{CcSk}$ defined in the obvious way. We have a fully faithful functor from $\mathtt{Cc}$, the category of small Cc categories, to $\mathtt{CcSk}$; this maps $\boldsymbol{C}$ to $\boldsymbol{C}_{(\mathtt{s})}$. We *identify* the Cc category $\boldsymbol{C}$ with $\boldsymbol{C}_{(\mathtt{s})}$. On the basis of our preceding remarks, using (6) for $\varphi = \mathtt{Exp}$, and using the corresponding fact for Cartesian categories, we obtain that

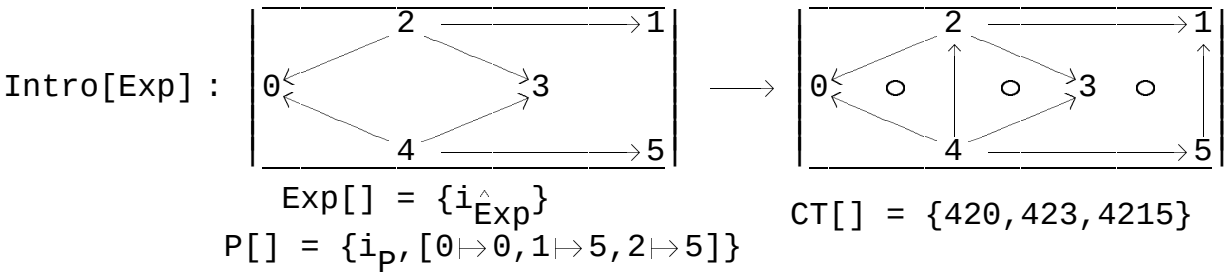
a Cc-sketch in $\mathtt{CcSk}$ *is a Cc category iff it satisfies all axioms of Cc category.*

Thus, the doctrine of Cc categories can be seen to have been defined by the concept of Cc-sketch, and the sketch-axioms of (category, fp-category and) Cc category.

For the sake of concreteness, let us spell out the four *specific s-axioms of Cc category*, `Ex[Exp], Intro[Exp], Elim[Exp], IsoTrans[Exp]` (we omitted the subscript `r`), the elements of $\mathcal{R}$`[Exp]` . As a consequence of the General Completeness Theorem of the last section, these axioms together with the rules for category and Cartesian category are sufficient to deduce, in the specific sense of the preceding section, any entailment coached in the form of a sketch-map in `CcSk` which is correct for all Cc categories.

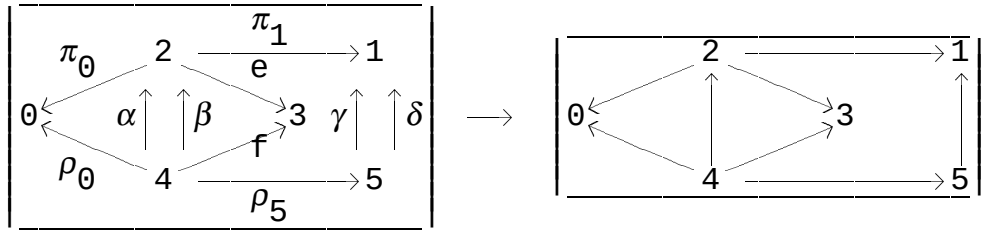


("Existence of the exponential".)



("Existence-part of the universal property of the exponential". The abbreviation `4215` refers to a "commutative square", involving an additional arrow $4 \to 1$ (not shown), and two commutative triangles.)

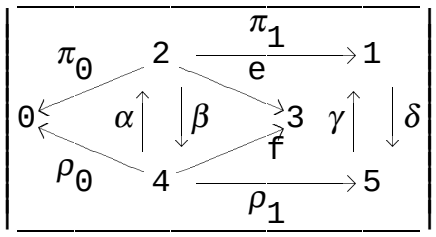
Elim[Exp] :



$CT[]=\{\pi_0\alpha=\rho_0, \pi_0\beta=\rho_0, \pi_1\alpha=\gamma\rho_5,$
$\pi_1\beta=\gamma\rho_5, e\alpha=f, e\beta=f\}$
$Exp[] = \{i_{\hat{Exp}}\}$

("Uniqueness part of the universal property of the exponential".)

IsoTrans[Exp]:



$|\overline{same}|$

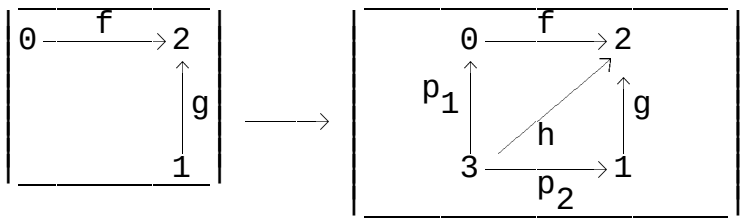
plus:

$(\rho_0, \rho_1, f) \in Exp[]$

$CT[]=\{\pi_0\alpha=\rho_0, \beta\alpha=1_4, \alpha\beta=1_2,$
$\pi_1\alpha=\gamma\rho_5, e\alpha=f, \delta\gamma=1_5, \gamma\delta=1_1\}$
$Exp[]=\{i_{\hat{Exp}}\},\ P[]=\{(\pi_0,\pi_1),(\rho_0,\rho_1)\}$

We turn to finite limits, in particular, pullbacks. Let $Pb: \ulcorner Pb \urcorner \longrightarrow \hat{Pb}$ be the following sketch-map in cSk :

Pb :



$CT[]=\{fp_1=h, gp_2=h\}$

with the letters denoting fixed entities such as $f = \langle 0,2 \rangle$, etc. Let us use $t: \ulcorner t \urcorner \longrightarrow \hat{t}$ as in Cartesian categories (thus, $Ex[t]$ is the item named the same way there). Let us write

$\mathtt{flSk}$ for $\mathtt{cSk}|\{\hat{\mathtt{t}},\hat{\mathtt{Pb}}\}$, and $\mathcal{R}[\mathtt{fl}]$ for $\mathcal{R}[\mathtt{cat}]\cup\mathcal{R}_{\mathtt{r}}[\mathtt{t}]\cup\mathcal{R}_{\mathtt{r}}[\mathtt{Pb}]$. The specification $(\mathtt{flSk},\mathcal{R}[\mathtt{fl}])$ specifies, up to isomorphism of categories, the category of categories with terminal object and pullback, that is, finitely complete categories.

The objects of $\mathtt{flSk}$ are, essentially, the finite-limit sketches of Ehresmann. Let us use $\mathtt{i}$ for the map $\emptyset\longrightarrow\hat{\mathtt{i}}$ in $\mathtt{Graph}$, where $\hat{\mathtt{i}}$ is the graph with the only object $\mathtt{1}$ (we need to distinguish $\mathtt{i}$ from $\mathtt{t}$; that is why the object $\mathtt{0}$ is replaced by $\mathtt{1}$), and no arrows, and let $\mathtt{Po}$ be the same as $\mathtt{Pb}$ with all arrows reversed. Then $\mathtt{flcSk}\ \underline{\underline{\mathtt{def}}}\ \mathtt{cSk}|\{\hat{\mathtt{t}},\hat{\mathtt{Pb}},\hat{\mathtt{i}},\hat{\mathtt{Po}}\}$ is the category of all finite-limit/colimit sketches of Ehresmann, and the set

$$\mathcal{R}[\mathtt{flc}]\ \underline{\underline{\mathtt{def}}}\ \mathcal{R}[\mathtt{cat}]\cup\mathcal{R}_{\mathtt{r}}[\mathtt{t}]\cup\mathcal{R}_{\mathtt{r}}[\mathtt{Pb}]\cup\mathcal{R}_{\ell}[\mathtt{i}]\cup\mathcal{R}_{\ell}[\mathtt{Po}]$$

of rules in the last sketch-category specifies the doctrine (category) of all categories with finite limits and colimits.

For the purposes of further examples, it is convenient to introduce another general construction, related to *explicit definition* in ordinary logic: the introduction of a new symbol standing as an abbreviation for a complex expression.

Let $\boldsymbol{G}$ be any (sketch-)category, $(\varepsilon\colon\ulcorner\varepsilon\urcorner\longrightarrow\hat{\varepsilon})\in\mathtt{Arr}(\boldsymbol{G})$. Let us consider the sketch-category $\boldsymbol{G}|\{\ulcorner\varepsilon\urcorner\}$; we define two particular sketch-entailments in $\boldsymbol{G}|\{\ulcorner\varepsilon\urcorner\}$. Let ε^{+} be the sketch in $\boldsymbol{G}|\{\ulcorner\varepsilon\urcorner\}$ for which $|\varepsilon^{+}|=\ulcorner\varepsilon\urcorner$ and $\ulcorner\varepsilon\urcorner[\varepsilon^{+}]=\{\mathtt{id}_{\ulcorner\varepsilon\urcorner}\}$. Let ε^{*} be the sketch for which $|\varepsilon^{*}|=\hat{\varepsilon}$ and $\ulcorner\varepsilon\urcorner[\varepsilon^{*}]=\{\varepsilon\}$. We have the maps

$$\mathtt{E}[\varepsilon]\ \underline{\underline{\mathtt{def}}}\ \mathtt{Elimdef}[\varepsilon]\ :\ \varepsilon^{+}\longrightarrow\varepsilon^{*},$$

$$\mathtt{I}[\varepsilon]\ \underline{\underline{\mathtt{def}}}\ \mathtt{Introdef}[\varepsilon]\ :\ \hat{\varepsilon}\longrightarrow\varepsilon^{*};$$

on the level of $\boldsymbol{G}$, the first map is equal to ε, the second the identity; these stipulations clearly determine them uniquely (in $\mathtt{I}[\varepsilon]$, $\hat{\varepsilon}$ is meant as the sketch in $\boldsymbol{G}|\{\ulcorner\varepsilon\urcorner\}$ whose underlying $\boldsymbol{G}$-object is $\hat{\varepsilon}$, and for which $\ulcorner\varepsilon\urcorner[\hat{\varepsilon}]=\emptyset$).

The role of these sketch-axioms is as follows. A sketch $S \in \boldsymbol{G} \mid \{ \ulcorner \varepsilon \urcorner \}$ satisfies the axioms in the set $\mathtt{Def}[\varepsilon] \underset{\mathrm{def}}{=} \{\mathtt{E}[\varepsilon], \mathtt{I}[\varepsilon]\}$ iff the set $\ulcorner \varepsilon \urcorner [S]$ is equal to the set $\langle \varepsilon \rangle S$ of those maps $\varphi \colon \ulcorner \varepsilon \urcorner \longrightarrow |S|$ which factor through ε, *i.e.*, for which there is a commutative diagram

$$\begin{array}{ccc} \ulcorner \varepsilon \urcorner & \xrightarrow{\ \varepsilon\ } & \hat{\varepsilon} \\ & {\scriptstyle \varphi} \searrow \quad \circ \quad \swarrow {\scriptstyle \psi} & \\ & |S| & \end{array} \qquad . \tag{7}$$

Indeed, $S \models \mathtt{E}[\varepsilon]$ iff $\ulcorner \varepsilon \urcorner [S] \subset \langle \varepsilon \rangle S$, $S \models \mathtt{I}[\varepsilon]$ iff $\langle \varepsilon \rangle S \subset \ulcorner \varepsilon \urcorner [S]$, as is easily seen. In other words, the axioms $\mathtt{Def}[\varepsilon]$ pick out those instances $\varphi \colon \ulcorner \varepsilon \urcorner \to |S|$ *at which* the axiom ε holds.

As a consequence, we have that the maps in $\boldsymbol{G} \mid \{ \ulcorner \varepsilon \urcorner \}$ between sketches satisfying $\mathtt{Def}[\varepsilon]$ are the same as the maps in $\boldsymbol{G}$; more precisely, if $S, T \in \boldsymbol{G}[\ulcorner \varepsilon \urcorner]$, and both S and T satisfy $\mathtt{Def}[\varepsilon]$, then any $\sigma \colon |S| \longrightarrow |T|$ is (gives rise to a map) $S \longrightarrow T$. For this, we only need to see that if $(\varphi \colon \ulcorner \varepsilon \urcorner \to |S|) \in \ulcorner \varepsilon \urcorner [S]$, then $(\sigma\varphi \colon \ulcorner \varepsilon \urcorner \to |T|) \in \ulcorner \varepsilon \urcorner [T]$. But the assumption means that we have (7) for some ψ; then

$$\begin{array}{ccc} \ulcorner \varepsilon \urcorner & \xrightarrow{\ \varepsilon\ } & \hat{\varepsilon} \\ & {\scriptstyle \sigma\varphi} \searrow \quad \circ \quad \swarrow {\scriptstyle \sigma\psi} & \\ & |T| & \end{array} \qquad ,$$

which shows that $\sigma\varphi \in \ulcorner \varepsilon \urcorner [T]$.

The use of the previous construction is to introduce defined concepts. E.g., in $\boldsymbol{G} = \mathtt{flSk}$, let $\mathtt{mon} \colon \ulcorner \mathtt{mon} \urcorner \longrightarrow \hat{\mathtt{m}}\mathtt{on}$ be the map

$$\mathtt{mon} : \left| \begin{array}{c} \overline{\phantom{0 \xrightarrow{m} 1}} \\ 0 \xrightarrow{m} 1 \\ \overline{\phantom{0 \xrightarrow{m} 1}} \end{array} \right| \longrightarrow \left| \begin{array}{c} \overline{\phantom{0 \xrightarrow{m} 1}} \\ \begin{array}{c} e \\ 0 \xrightarrow{m} 1 \end{array} \\ \overline{\phantom{0 \xrightarrow{m} 1}} \end{array} \right|$$

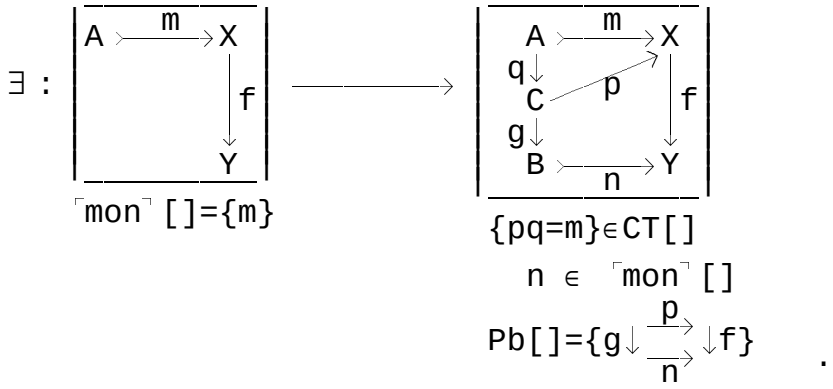
$$\mathtt{I}[\,]=\{e\}$$

$$\mathtt{Pb}[\,]=\{e \underset{m}{\overset{e}{\rightrightarrows}} \downarrow m\}$$

and let $\boldsymbol{S} = \boldsymbol{G} \mid \{ \ulcorner \mathtt{mon} \urcorner \}$. For $\boldsymbol{C} \in \boldsymbol{S}$ such that $|\boldsymbol{C}|$ is a category with finite limits (regarded

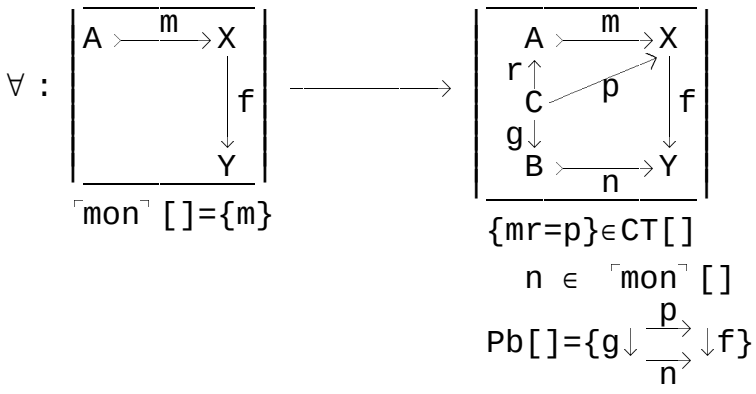
as a sketch in $\mathbf{G}$), $\mathbf{C} \models \mathtt{Def[mon]}$ iff $\ulcorner\mathtt{mon}\urcorner[\mathbf{C}]$ is the collection of monomorphisms in $|\mathbf{C}|$. Thus, the specification $(\mathtt{flSk}\,|\,\{\ulcorner\mathtt{mon}\urcorner\}, \mathcal{R}[\mathtt{fl}] \cup \mathtt{Def[mon]})$ specifies the doctrine of categories with finite limits, with the monomorphisms *named as such* in them.

Let us consider how to sketch-specify the quantifiers. Let $\mathbf{G} = \mathtt{flSk}\,|\,\{\ulcorner\mathtt{mon}\urcorner\}$. Define $\exists : \ulcorner\exists\urcorner \longrightarrow \hat{\exists}$ to be the following inclusion arrow in $\mathbf{G}$:

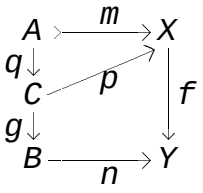


(As before, all letters are completely fixed entities; implicitly, $\mathtt{CT}[\hat{\exists}]$ is determined to have three elements, two of which, together with another arrow $\mathtt{C} \to \mathtt{Y}$, are not mentioned; these are contained in the pullback-specification.)

The map $\forall : \ulcorner\forall\urcorner \longrightarrow \hat{\forall}$ is the inclusion



If $\mathbf{C} \in \mathbf{G}\,|\,\{\ulcorner\exists\urcorner\}$ has $|\mathbf{C}|$ a category with finite limits, then $\mathbf{C} \models \mathcal{R}_{\ell}[\exists]$ iff for all $f : X \longrightarrow Y$, $f^{*} : \mathtt{S}(Y) \longrightarrow \mathtt{S}(X)$ has a left adjoint $\exists_{f}$. Moreover, the diagram



belongs to $\hat{\exists}[\boldsymbol{C}]$ iff, besides the diagram having the qualities involved in being a diagram of type $\hat{\exists}$, we have that $[n]$, the subobject $\in \mathtt{S}(Y)$ determined by n, is $\exists_f([m])$. (In fact, the axiom $\mathtt{Elim}_\ell[\exists]$ is redundant.)

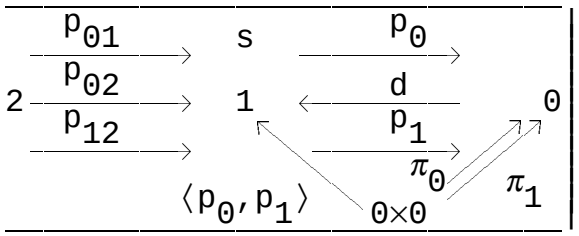
The role of $\forall$ is similar with respect to the right adjoint $\forall_f$ of f^*; of course, now we use $\mathcal{R}_\mathtt{r}[]$ rather than $\mathcal{R}_\ell[]$.

The quantifiers are used in doctrines such as regular categories, coherent categories and Heyting categories; however, the complete specification of these types of categories requires the treatment of exactness properties, and it is deferred to the next section.

For the purposes of the doctrines of Barr-exact categories and of pretoposes, we need to handle equivalence relations and their quotients. Let $\hat{\mathtt{EqRel}}$ denote the following fl-sketch:

$\hat{\mathtt{EqRel}}$:

$|\hat{\mathtt{EqRel}}|$ =



with $1_1:1\to 1$, $1_0:0\to 0$

with specification-sets

$$\mathtt{I}[\hat{\mathtt{EqRel}}] = \{1_1:1\to 1, 1_0:0\to 0\}$$

$$\mathtt{CT}[\hat{\mathtt{EqRel}}] = \{\ \pi_0\langle p_0,p_1\rangle = p_0,\ \pi_1\langle p_0,p_1\rangle = p_1,\ p_0 s = p_1,\ p_1 s = p_0,\ p_0 d = 1_0,\ p_1 d = 1_0,\ p_0 p_{02} = p_0 p_{01},\ p_1 p_{02} = p_1 p_{12}\ \}$$

$$\mathtt{P}[\hat{\mathtt{E}}\mathtt{qRel}] = \{\ 0\times 0 \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} 0\ \}$$

$$\mathtt{Pb}[\hat{\mathtt{E}}\mathtt{qRel}] = \left\{ \begin{array}{ccc} 2 & \xrightarrow{p_{01}} & 1 \\ {\scriptstyle p_{12}}\downarrow & & \downarrow{\scriptstyle p_1} \\ 1 & \xrightarrow[p_0]{} & 0 \end{array} ,\ \begin{array}{ccc} 1 & \xrightarrow{1_1} & 1 \\ {\scriptstyle 1_1}\downarrow & & \downarrow{\scriptstyle \langle p_0, p_1\rangle} \\ 1 & \xrightarrow[\langle p_0, p_1\rangle]{} & 0\times 0 \end{array} \right\}$$

We define

$$\ulcorner \mathtt{EqRel} \urcorner \ \underset{\mathrm{def}}{=} \left| \begin{array}{c} \overline{\phantom{1 \underset{p_1}{\overset{p_0}{\rightrightarrows}} 0}} \\ 1 \underset{p_1}{\overset{p_0}{\rightrightarrows}} 0 \\ \underline{\phantom{1 \underset{p_1}{\overset{p_0}{\rightrightarrows}} 0}} \end{array} \right| ,$$

and by $\mathtt{EqRel}$, we denote the inclusion-map

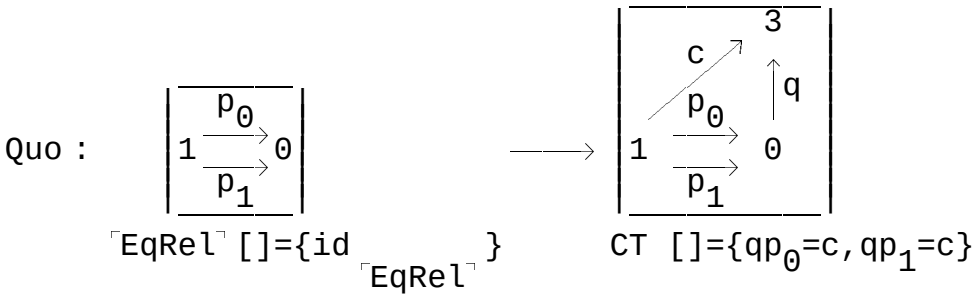
$$\mathtt{EqRel} : \ulcorner \mathtt{EqRel} \urcorner \longrightarrow \hat{\mathtt{E}}\mathtt{qRel} .$$

Clearly, in a category $\boldsymbol{C}$ with finite limits, $R \underset{g}{\overset{f}{\rightrightarrows}} A = (\varphi : \ulcorner \mathtt{EqRel} \urcorner \longrightarrow \boldsymbol{C})$ is an equivalence relation (see, e.g., [18]) iff there is $\psi : \hat{\mathtt{E}}\mathtt{qRel} \longrightarrow \boldsymbol{C}$ for which

$$\begin{array}{ccc} \ulcorner \mathtt{EqRel} \urcorner & \xrightarrow{\mathtt{EqRel}} & \hat{\mathtt{E}}\mathtt{qRel} \\ & \circ & \\ & {\scriptstyle \varphi}\searrow \quad \swarrow{\scriptstyle \psi} & \\ & \boldsymbol{C} & \end{array}$$

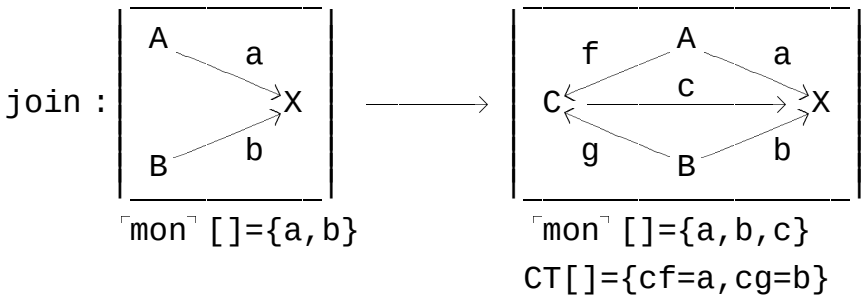
commutes. Thus, $(\mathtt{flSk} \mid \{\ulcorner \mathtt{EqRel} \urcorner\}, \mathcal{R}[\mathtt{fl}] \cup \mathtt{Def}[\mathtt{EqRel}])$ is a specification of categories with finite limits, with equivalence relations distinguished.

Let $\boldsymbol{G} = \mathtt{flSk} \mid \{\ulcorner \mathtt{EqRel} \urcorner\}$, the category of sketches with finite-limit and equivalence-relation specifications. By $\mathtt{Quo} : \ulcorner \mathtt{Quo} \urcorner \longrightarrow \hat{\mathtt{Q}}\mathtt{uo}$ we denote the map



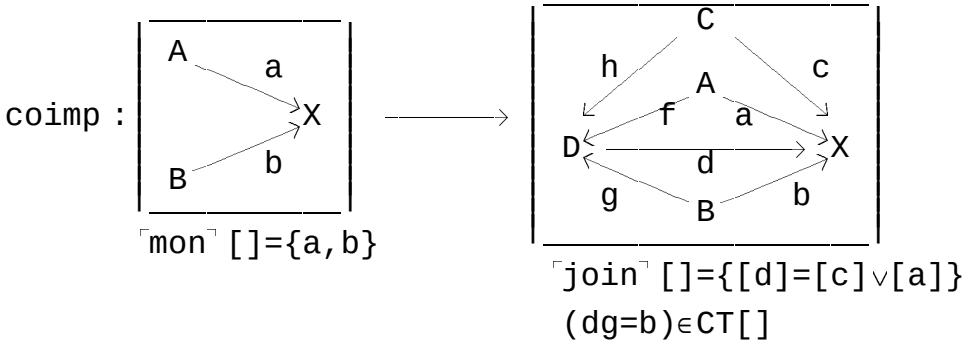
in $\boldsymbol{G}$ (as usual, Quo is an inclusion; there is (necessarily) an ⌜EqRel⌝ specification in $\hat{\text{Quo}}$, which is not marked). Next, $\boldsymbol{S} \stackrel{\text{def}}{=} \boldsymbol{G}\,|\,\{\hat{\text{Quo}}\}$. We have the axioms Def[EqRel] and $\mathcal{R}_\ell$[Quo], the latter expressing the ("left-adjoint") universal property of q as coequalizer of (p_0, p_1). Adding these to the specification of lex categories, we obtain the specification for lex categories with coequalizers of equivalence relations. To get the specification for Barr-exact categories, we need to add certain *exactness properties* in the form of additional sketch-axioms, to be discussed in the next section.

For coherent and Heyting categories, we need the (binary) join of subobjects. We specify this operation over the doctrine of finite limits (with monomorphisms distinguished). We consider join: ⌜join⌝ $\longrightarrow$ $\hat{\text{join}}$ defined as follows:



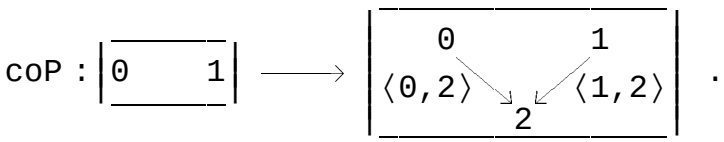
Then the operation of join is specified by the axioms $\mathcal{R}_\ell$[join] (again, Elim_ℓ[join] is redundant).

In a lattice, the operation ( )\a of coimplication is the left adjoint of $(\,)\vee a$: $b \backslash a \le c \iff b \le c \vee a$. We can specify coimplication in subobject lattices over any doctrine containing finite limits and joins as a left-adjoint operation by using the map

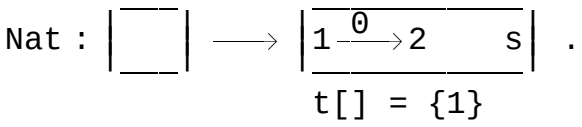


and the axioms $\mathcal{R}_{\ell}$[coimp] .

(Binary) coproducts are used in distributive categories and pretoposes. Coproduct is a left adjoint operation, specified (over Graph) by the use of



The Natural Numbers Object, a left adjoint concept, is specified (over fpSk) by using



The adjoint operations to be considered next need a modified set-up.

The subobject classifier $t:1 \rightarrowtail \Omega$ is the terminal object among all monomorphism $X \rightarrowtail A$, where an arrow between two monomorphisms is a pullback-square. Accordingly, we need a modification of the notion of nt.

A *pullback natural transformation* (pbnt) is defined in any sketch-category based on flSk . With reference to the notation where the concept of nt was introduced, a pbnt $\alpha:\varphi \longrightarrow \psi$ is an nt α , together with a specification $s_f \in$ Pb[V] , one for each $f \in$ Arr(U) , that specifies that the naturality square (specified by s_f to commute) is in fact a pullback.

Above, we defined $\mathcal{R}_r[\varphi]$ for any $\varphi: S \longrightarrow T$ in $\boldsymbol{G}$, a set of four entailments in $\boldsymbol{G}[T]$. Changing in $\mathcal{R}_r[\varphi]$ all references to nt's to ones to pbnt's results in $\dot{\mathcal{R}}_r[\varphi]$. Now, consider

$$\mathtt{Omega}: \left|\overline{\quad}\right| \longrightarrow \left|\overline{0 \xrightarrow{t} 1}\right| \qquad .$$
$$\ulcorner \mathtt{mon} \urcorner\, [] = \{t\}$$

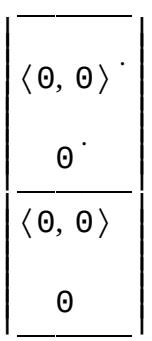
Then, $\dot{\mathcal{R}}_r[\mathtt{Omega}]$ will serve to specify the subobject classifier over finite limits.

Elementary toposes are specified by $(\mathtt{topSk}, \mathcal{R}[\mathtt{Top}]) \;\bar{\bar{\mathtt{def}}}\; (\mathtt{CcSk} \mid \{\hat{\mathtt{Omega}}\}, \mathcal{R}[\mathtt{Cc}] \cup \dot{\mathcal{R}}_r[\mathtt{Omega}])$, and **elementary toposes with a natural number object** by $(\mathtt{topSk} \mid \{\hat{\mathtt{Nat}}\}, \mathcal{R}[\mathtt{Top}] \cup \mathcal{R}_\ell[\mathtt{Nat}])$.

I will now incorporate certain modal predicate logics in the framework of adjointness doctrines. The difference with respect to the foregoing is that the doctrines to be considered are not based on $\mathtt{Graph}$. I recall from [19] that an *S4-category* is a pair $\begin{smallmatrix}\boldsymbol{B}\\ \uparrow \cdot \\ \boldsymbol{C}\end{smallmatrix}$ of categories $\boldsymbol{C}$, $\boldsymbol{B}$ connected with a functor $(\,)^{\cdot}$ such that $\boldsymbol{C}$ is coherent, $\boldsymbol{B}$ is Boolean, $(\,)^{\cdot}$ is a conservative coherent functor, and for every $X \in \boldsymbol{C}$, the induced map $(\,)^{\cdot}: \mathtt{S}(X) \longrightarrow \mathtt{S}(\dot{X})$ has a right adjoint (*necessity*) $\Box_X: \mathtt{S}(\dot{X}) \longrightarrow \mathtt{S}(X)$. Sketches for S4-categories are based on the category $\mathtt{Graph}^{\rightarrow}$ whose objects are maps $\begin{smallmatrix}U_2\\ \uparrow \\ U_1\end{smallmatrix}$ of graphs, with arrows commutative squares of graph-maps. Of course, $\mathtt{Graph}^{\rightarrow}$ is a finite topos too. When depicting an object $\begin{smallmatrix}U_2\\ \uparrow \\ U_1\end{smallmatrix}$ of $\mathtt{Graph}^{\rightarrow}$, we draw two boxes, the one for the graph U_1 below the one for U_2. To indicate the image in U_2 of an object or an arrow in U_1, we use a dot (that is, the map is $(\,)^{\cdot}: U_1 \to U_2$).

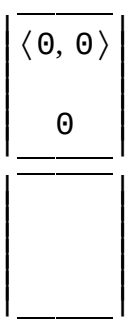
The appropriate notion of "sketch for specifying functors", or *functor-sketch*, f-sketch for short, is given by the sketch-category $\mathtt{fSk} \;\bar{\bar{\mathtt{def}}}\; \boldsymbol{G} \mid \{\mathtt{I}_1, \mathtt{CT}_1, \mathtt{I}_2, \mathtt{CT}_2\}$ where the four listed objects of $\mathtt{Graph}^{\rightarrow}$ are as follows.

I_1 :



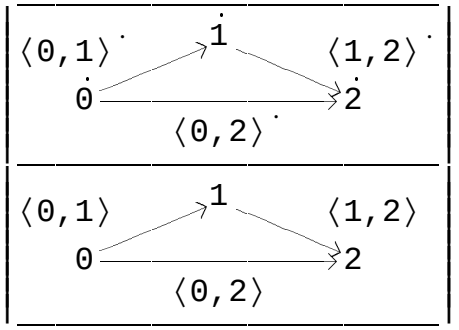
("identity in $(\ )_1$")

I_2 :



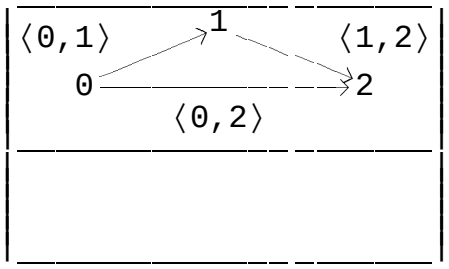
("identity in $(\ )_2$")

CT_1 :



(commutative triangle in $(\ )_1$)

CT_2 :



(commutative triangle in $(\ )_2$)

There are sketch-entailments in $\mathtt{fSk}$, collectively called $\mathcal{R}[\mathtt{fun}]$, that have the property that a sketch in $\mathtt{fSk}$ satisfies them iff it *is* a functor between categories; I leave the spelling out of these to the reader.

We have the forgetful functors $(\ )_1, (\ )_2 : \mathtt{fSk} \longrightarrow \mathtt{cSk}$; $\begin{array}{c} U_2 \\ \uparrow \\ U_1 \end{array} \longmapsto (\ )_1 U_1$, $\begin{array}{c} U_2 \\ \uparrow \\ U_1 \end{array} \longmapsto (\ )_2 U_2$.

Given $U \underset{\psi}{\overset{\varphi}{\rightrightarrows}} V$ in $\mathtt{fSk}$, $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$, a *natural transformation* (nt) $\alpha : \varphi \longrightarrow \psi$ is a pair $\alpha = (\alpha_1, \alpha_2)$ of nt's $\alpha_i : \varphi_i \longrightarrow \psi_i$ (i=1, 2) (where "nt" in the latter position is understood in the sense of $\mathtt{cSk}$) *such that* $\alpha_2(\ )_U^{\cdot} = (\ )_V^{\cdot}\alpha_1$. Again, given $T \xrightarrow{\tau} U \underset{\psi}{\overset{\varphi}{\rightrightarrows}} V \xrightarrow{\theta} W$ in $\mathtt{fSk}$, the definitions of $T \underset{\psi\tau}{\overset{\varphi\tau}{\rightrightarrows}} V$ and $U \underset{\theta\psi}{\overset{\theta\varphi}{\rightrightarrows}} W$ are straightforward.

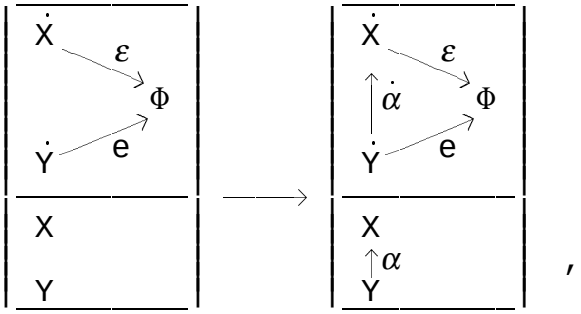
With this notion of nt, and the straightforward derived notion of "isomorphism nt", we now can define, for any $\varphi : S \longrightarrow T$ in $\mathtt{fSk}$, the sketch-axioms $\mathtt{Ex}[\varphi]$, $\mathtt{Intro}_{\mathtt{r}(\ell)}[\varphi]$, $\mathtt{Elim}_{\mathtt{r}(\ell)}[\varphi]$ and $\mathtt{Isotrans}[\varphi]$ analogously to the foregoing. As before, the set of these four is denoted by $\mathcal{R}_{\mathtt{r}}[\varphi]$ ($\mathcal{R}_{\ell}[\varphi]$).

The simplest example showing the use of these is the specification of the right and left adjoints of a(n arbitrary) functor. Consider $\mathtt{RA} : \ulcorner \mathtt{RA} \urcorner \longrightarrow \hat{\mathtt{RA}}$ given as

$$\mathtt{RA} : \left| \begin{array}{|c|} \hline \Phi \\ \hline \\ \hline \end{array} \right| \longrightarrow \left| \begin{array}{|c|} \hline \dot{X} \xrightarrow{\varepsilon} \Phi \\ \hline X \\ \hline \end{array} \right| .$$

The sketches in $\mathtt{fSk}\,|\,\{\ulcorner \mathtt{RA} \urcorner\}$ satisfying $\mathcal{R}[\mathtt{fun}] \cup \mathcal{R}_{\mathtt{r}}[\mathtt{RA}]$ are exactly those functors $F : \boldsymbol{C}_1 \longrightarrow \boldsymbol{C}_2$ that have a right adjoint, and for which $\hat{\mathtt{RA}}[F]$ consists of those $(X \in \boldsymbol{C}_1, (\varepsilon : FX \longrightarrow \Phi) \in \boldsymbol{C}_2)$ in which ε satisfies the universal property for the counit at X for the right adjoint of F. To illustrate, $\mathtt{Intro}_{\mathtt{r}}[\mathtt{RA}]$ looks like this:

$\mathtt{Intro}_r[\mathtt{RA}]$:

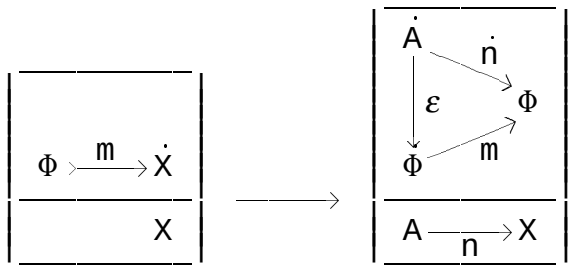


$\ulcorner \mathtt{RA} \urcorner [] = \{\varepsilon\}$ $\quad$ $\ulcorner \mathtt{RA} \urcorner [] = \{\varepsilon\}$ $\quad$ $\mathtt{CT}[] = \{\varepsilon\dot{\alpha} = \mathtt{e}\}$,

expressing as it does the existence-part of the universal property of ε .

The necessity operator $\Box$ mentioned above, a "local" right adjoint (operating on subobjects only), is specified by $\mathcal{R}_r[\Box]$, where $\Box$ is

$\Box$:



$\ulcorner \mathtt{mon} \urcorner [] = \{\mathtt{m}\}$ $\quad$ $\ulcorner \mathtt{mon} \urcorner [] = \{\mathtt{m},\mathtt{n}\}$ $\quad$ $\mathtt{CT}[] = \{\mathtt{m}\varepsilon = \dot{\mathtt{n}}\}$.

Closing this section, let me mention that there is a framework for a completely general concept of adjoint operation. Since all the interesting examples can be handled in the manner detailed above, I will refrain from describing it.