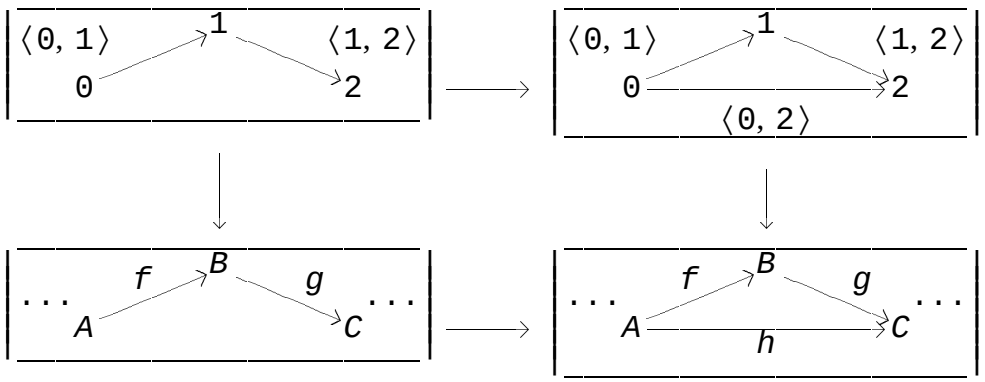
§3. Formal deductions of sketch–entailments

For motivation, we first consider the simple example of a possible way the category $\langle S \rangle$ *presented* by a c-sketch S may be constructed. $\langle S \rangle$ comes with a canonical "embedding" $\gamma : S \longrightarrow \langle S \rangle$; note that we are identifying the category $\langle S \rangle$ with the corresponding sketch as in the previous section. γ is defined by the familiar universal property: $\gamma : S \longrightarrow \langle S \rangle$ factors uniquely through any $S \longrightarrow D$, with D a (sketch derived from a) category. The construction of $\langle S \rangle$ can be performed in steps, providing successive extensions of sketches. In one step, e.g. the composite of two arrows, $f : A \to B$, $g : B \to C$, with matching codomain and domain, has to be added to T , the sketch of the previous step. This *can be* accomplished by taking the pushout

$$\begin{array}{ccc} R & \xrightarrow{r} & R' \\ {\scriptstyle\varphi}\downarrow & \square & \downarrow{\scriptstyle\varphi'} \\ T & \xrightarrow[t]{} & T' \end{array} \quad , \tag{1}$$

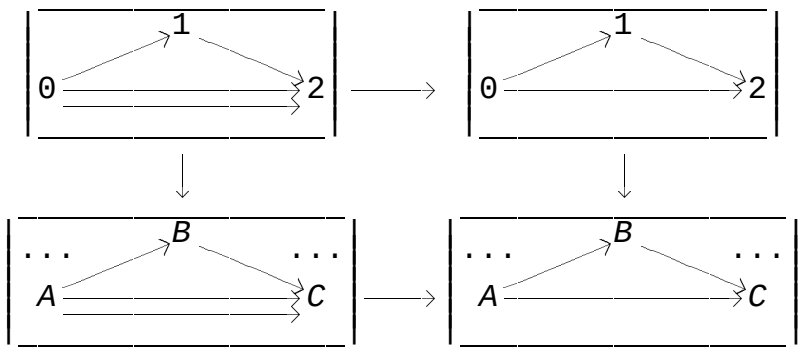
where r is "composition", the map ExComp of 1.1 and where $\varphi : R \longrightarrow T$ picks out the two arrows $\langle 0, 1 \rangle \mapsto f$, $\langle 1, 2 \rangle \mapsto g$:



This is a superfluous move if there already is in T a commutative triangle with f and g as two of the sides. But, applying another pushout

$$\begin{array}{ccc} R_1 & \xrightarrow{r} & R'_1 \\ {\scriptstyle\varphi_1}\downarrow & \square & \downarrow{\scriptstyle\varphi'_1} \\ T' & \longrightarrow & T'' \end{array} \quad ,$$

this time with r_1 being UnComp in 1.1, and φ_1 picking out the two commutative triangles concerned, $T"$ has the two composites identified; *nothing else happened to* $T"$:



With $r:R \longrightarrow R'$ any map of c-sketches, and $\varphi:R \longrightarrow T$ an arbitrary map, we write φ^*r for the pushout (1).

We apply the operations φ^*r iteratively; r ranges over the seven s-axioms for category in 1.1. We are tempted to call those seven sketch-maps the *templates* of the construction. The process goes through a possibly transfinite number of steps. Having made a limit-number of steps, we take the colimit of the chain of sketches of the previous steps as the construction of the next step. We will arrive at the desired category after a suitable number of steps without any particular care for what templates we use in what order *provided* we take care of unwanted "infinite loops". Note that we may get into the rut of adding more and more composites of f and g in the above example, not getting the opportunity for anything else, if we do not take the care to avoid doing so. Here is a general recipe:

apply the operation φ^*r *only if* φ *does not factor through* r .

Certainly, this is reasonable since if φ does factor through r , the job that we want to do with the operation φ^*r is already done; e.g., if r is the "uniqueness" map UnComp , the operation φ^*r will identify two arrows that are already equal. I claim that

any transfinite sequence of completion steps as described above that avoids superfluous steps as specified will terminate in $\langle S \rangle$ *in an ordinal number of steps whose cardinality is less than or equal* $\max(\aleph_0, \#(S))$.

I will not take the time to prove the claim; the essence of the argument will be contained in

the proof of 6. Corollary below. Suffice it to say that the *finite segments* of the construction of $\langle S \rangle$ will act as the *formal deductions* for the doctrine specified by $\mathtt{cSk}$ and $\mathcal{R}_{\mathtt{cat}}$.

Let $\mathbf{S}$ be a category having pushouts, $\mathcal{R}$ a set of arrows in it. We will talk about objects of $\mathbf{S}$ as "sketches", because of the applications. In what follows, $\mathbf{S}$ and $\mathcal{R}$ are fixed.

An entailment (an arrow in $\mathbf{S}$) $u : U \longrightarrow U'$ is *immediately deducible* (*from* $\mathcal{R}$) if it is a pushout of some $r \in \mathcal{R}$: there is a pushout-diagram of the form

$$\begin{array}{ccc} R & \xrightarrow{r} & R' \\ {\scriptstyle\varphi}\downarrow & \square & \downarrow{\scriptstyle\varphi'} \\ U & \xrightarrow[u]{} & U' \end{array} \qquad (1')$$

in $\mathbf{S}$, with r some element of $\mathcal{R}$. Further, $t : S \longrightarrow T$ is *directly deducible* (*from* $\mathcal{R}$) if it is the composite of finitely many immediately deducible ones. Finally, $s : S \longrightarrow S'$ is *deducible* (*from* $\mathcal{R}$), in notation $\vdash s$, or more specifically $\vdash_{\mathcal{R}} s$, if an extension of s is directly deducible: there is a $t' : S' \longrightarrow T$ such that $t' \circ s : S \longrightarrow T$ is directly deducible.

For the notion of formal deduction, there are a couple of slightly different candidates. In the most explicit version, a (*formal*) *deduction of* $s : S \to S'$ (*from* $\mathcal{R}$) consists of a map $t' : S' \longrightarrow T$, together with a finite sequence of pushout squares (1') with each r from $\mathcal{R}$ such that the u-legs of the squares are composable, and they give $t' \circ s$ as their composite. In a less explicit, but more economical version, a *deduction of* s is given by a sequence $\langle U_i \rangle_{i<n}$ of objects and a sequence $\langle u_i : U_i \to U_{i+1} \rangle_{i+1<n}$ of arrows such that each u_i is immediately deducible, and such that *there is* a map $t' : S' \longrightarrow T$ satisfying $t' \circ s = u_{n-1} \circ u_{n-2} \circ \ldots \circ u_0$.

We discuss effectiveness notions (decidable, effectively enumerable (e.e.), ...) for the use with sketch-based syntax. We take as the basic space of finite entities the set $\mathbb{HF}$ of hereditarily finite (hf) sets; $\mathbb{HF} = \mathbf{V}_\omega$, the ωth level in the von-Neumann hierarchy of pure sets. We have the standard notions of decidable (= recursive), e.e. (= recursively enumerable), etc. for subsets of $\mathbb{HF}$; they coincide with the sets that are Δ_1-definable, resp. Σ_1-definable in $(\mathbb{HF}, \in \restriction \mathbb{HF})$. The good thing about $\mathbb{HF}$ is that it is closed under the operations of taking ordered pairs,

finite sets, and finite sequences of elements; any morphism $f:A\to B$ between hf sets A, B is an hf set ($f = (A, B.\mathtt{graph}(f))$), *etc*.

In our determination of specific entities (specification types and sketch axioms), we take care of making them hf sets. Notice that the specification types for $\mathtt{cSk}$, and the axioms in $\mathcal{R}_{\mathtt{cat}}$ are hf sets; similar control will be exercised in the later examples too.

The general theory of sketches, as any reasonable categorical theory, has all its concepts invariant under isomorphism. For example, the notions of "finite", "exactness property" (a notion introduced later), "deducible from given set of axioms", etc. are all invariant under isomorphism in suitable senses. This invariance helps us give a precise meaning to effectiveness notions. In general, we say that a collection $\mathcal{C}$ of finite entities (sketches, sketch-entailments, sequences of sketch-entailments (deductions), ...) is *decidable*, or *e.e.*, if the subcollection $\mathcal{C}_{\mathtt{hf}}$ of its hf elements is decidable, resp. e.e.. Since $\mathcal{C}$ is (usually) closed under isomorphism, the subcollection $\mathcal{C}_{\mathtt{hf}}$ is representative: every entity in $\mathcal{C}$ has an isomorphic copy in $\mathcal{C}_{\mathtt{hf}}$.

E.g., if the finite category $\boldsymbol{C}$ is in fact an hf set (and clearly, every finite category is isomorphic to one which is an hf set), then every finite object of $\mathtt{Set}^{\boldsymbol{C}}$ is isomorphic to an (in fact, several) hf object (a functor $\boldsymbol{C}\longrightarrow\mathtt{Set}$ that takes hf values; such a functor can clearly be identified with an appropriate hf set). Thus, every finite sketch, and every finite sketch-entailment, in $\mathtt{Set}^{\boldsymbol{C}}$ is isomorphic to one which is hf.

I have made explicit certain aspects of effectiveness in the algebraic context. When people talk e.g. about an r.e. set of finite groups, they have in mind (or should have in mind) a concept like the one described above. One difference might be that people would tend to talk about Gödel numbering by natural numbers "in the usual way"; using hf sets removes this kind of vagueness.

Suppose that $\boldsymbol{S}$ is a finite sketch-category, with all specification types being hf sets. Then the predicate $\mathtt{Ded}(d, s, \mathcal{R})$ with the variables d, s and $\mathcal{R}$ ranging over $\mathbb{HF}$:

" $\mathcal{R}$ is a (finite) set of sketch-entailments in $\boldsymbol{S}$, s is a sketch-entailment in $\boldsymbol{S}$, and d is a deduction of s from $\mathcal{R}$ " ,

in either sense of "deduction", is decidable. This is seen by inspection, given our experience with similar statements in many contexts (among others, in "effective algebra").

It immediately follows that the set of syntactical consequences $\mathcal{D}[\mathcal{R}] = \{s : \vdash_{\mathcal{R}} s\}$ of a finite set $\mathcal{R}$ of sketch-entailments is r.e.. In fact, since $\mathcal{D}[\mathcal{R}] = \bigcup_{\substack{\mathcal{R}' \subset \mathcal{R} \\ \mathcal{R}' \text{finite}}} \mathcal{D}[\mathcal{R}']$, the same conclusion follows for any r.e. set $\mathcal{R}$ of axioms.

Similar assertions can be made in the more general case of finitary doctrine specifications, provided they are "countably and effectively" given.

The notion of formal deduction is a codification of that of *diagram manipulation*. In a sketch-category, a deduction (with a finite premise) involves a succession of finite sketches, that is, finite diagrams with certain distinguished parts designated to have special qualities (product, exponential, etc.). Each sketch in the succession is obtained by an application of a *rule*, that is, an arrow in $\mathcal{R}$, where an application of the rule $r : R \longrightarrow R'$ to a sketch U consists in selecting an R-figure $\varphi : R \longrightarrow U$ in U, and amalgamating R' with U with respect to the maps r and φ via a pushout. The last amalgamation, a pushout, effects only a *local* change in U, because of the nature of pushouts in a finite topos, deriving as they do from pushouts in $\mathbf{Set}$. These local changes consist in adding a bounded number of new entities (objects, arrows, designations of parts as "product", etc.), and/or identifying certain bounded number of entities (here "bounded" means "bounded by an integer depending only on $\mathcal{R}$, not on S ").

For an entailment (arrow in $\mathbf{S}$) s, let us say that s is *valid* if $\vDash_{\mathcal{R}} s$, and *deducible* if $\vdash_{\mathcal{R}} s$; let $\mathcal{V} = \mathcal{V}[\mathcal{R}]$ be the class of valid entailments, $\mathcal{D} = \mathcal{D}[\mathcal{R}]$ that of the deducible ones.

1. Lemma. For either of the classes $\mathcal{X} = \mathcal{V}$, $\mathcal{X} = \mathcal{D}$, the following properties hold.

(i) Every isomorphism belongs to $\mathcal{X}$.

(ii) The composite of two composable elements of $\mathcal{X}$ belongs to $\mathcal{X}$.

(iii) If $g \circ f \in \mathcal{X}$, then $f \in \mathcal{X}$.

(iv) Any pushout of a member of $\mathcal{X}$ is again a member of $\mathcal{X}$: if in the pushout (1'), $r \in \mathcal{X}$, then $u \in \mathcal{X}$.

The easy proof is left to the reader.

The class $\mathcal{D}$ is clearly contained in the least class $\mathcal{X}$ closed under (i), (iii) and (iv); by the lemma, applied to $\mathcal{X} = \mathcal{D}$, it is the same as that least class, as well as the least class satisfying all four of the conditions in the lemma. The lemma, applied to $\mathcal{X} = \mathcal{V}$ now, thus gives

2. Corollary (**Soundness Theorem**). Each deducible arrow is valid; $\mathcal{D} \subset \mathcal{V}$.

In what follows, we assume that $\boldsymbol{S}$ is a locally finitely presentable (lfp) category.

Next, we recall some elementary facts concerning comma-categories.

Let $S \in \boldsymbol{S}$. The comma-category $S \downarrow \boldsymbol{S}$ (with objects arrows $S \longrightarrow T$, and arrows $(S \longrightarrow T) \longrightarrow (S \longrightarrow U)$ the arrows $T \longrightarrow U$, making the triangle commute; see [23]) is also lfp, and the forgetful functor $S \downarrow \boldsymbol{S} \longrightarrow \boldsymbol{S}$ creates (small) colimits.

An fp object of $S \downarrow \boldsymbol{S}$ is called a *relatively finite* (rf) arrow in $\boldsymbol{S}$. The rf arrows are pushouts of fp arrows of $\boldsymbol{S}$: if $S \to T$ is rf, there exists pushout diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & \square & \downarrow \\ S & \longrightarrow & T \end{array}$$

with R, R' fp objects.

Connecting two comma-categories, we note that the composite of two (composable) rf arrows in $\boldsymbol{S}$ is rf again.

3. Theorem (**General Completeness Theorem**) Let $\boldsymbol{S}$ be an lfp category, and $\mathcal{R}$ a set of arrows between fp objects in $\boldsymbol{S}$. For any relatively finite entailment $s : S \longrightarrow S'$, $\vdash_{\mathcal{R}} s \iff \vDash_{\mathcal{R}} s$; that is, for such s, validity and deducibility coincide.

For the proof, we need some preliminaries. Throughout this section, $\mathbf{S}$ and $\mathcal{R}$ are as in the statement of 3.

Recall the cardinal $\lambda=\lambda_{\mathbf{S},\mathcal{R}}$ from §2. Let $S\in\mathbf{S}$. We denote by κ_S the least uncountable regular cardinal κ such that S is κ-presentable and $\lambda<\kappa$. When $\mathbf{S}$ is a finite sketch-category, $\kappa_S=\aleph_1$ for S (finite or) countable, and $\kappa_S=(\#(S))^+$ otherwise.

A transfinite sequence

$$\mathcal{U}=(\langle U_\beta\rangle_{\beta\le\alpha},\langle u_{\beta\gamma}:U_\beta\to U_\gamma\rangle_{\beta\le\gamma\le\alpha}) \qquad (2)$$

of sketches, with a system of connecting maps, is called an *infinitary deduction* (∞-deduction) if it is *compatible* ($u_{\gamma\delta}\circ u_{\beta\gamma}=u_{\beta\delta}$ $(\beta\le\gamma\le\delta\le\alpha)$, $u_{\beta\beta}=1_{U_\beta}$ $(\beta\le\alpha)$), *continuous* (for each limit ordinal $\beta\le\alpha$, U_β is the colimit of the restricted system with indices $<\beta$, with colimit coprojections the $u_{\gamma\beta}$ for $\gamma\le\beta$), and each $u_{\beta,\,\beta+1}$ ($\beta+1\le\alpha$) is an immediately deducible arrow; it is *of length* α ; it is *based on* the sketch S if $U_0=S$. Appropriately, a deduction is an ∞-deduction of finite length.

∞-deductions can be composed. If $\mathcal{U}$ is as above, and

$$\mathcal{V}=(\langle V_\nu\rangle_{\nu\le\lambda}\,,\ \langle v_{\mu\nu}:V_\mu\to V_\nu\rangle_{\mu\le\nu\le\lambda})$$

is another ∞-deduction such that $V_0=U_\alpha$, then $\mathcal{V}\circ\mathcal{U}$ is the ∞-deduction

$$\mathcal{V}\circ\mathcal{U}=(\langle W_\xi\rangle_{\xi\le\alpha+\lambda},\ \langle w_{\zeta\xi}:W_\zeta\to W_\xi\rangle_{\zeta\le\xi\le\alpha+\lambda})$$

of length $\alpha+\lambda$ for which $W_\beta=U_\beta$ $(\beta\le\alpha)$, $W_{\alpha+\nu}=V_\nu$ $(\nu\le\lambda)$, $w_{\beta\gamma}=u_{\beta\gamma}$ $(\beta\le\gamma\le\alpha)$, $w_{\beta,\,\alpha+\nu}=v_{0\nu}\circ u_{\beta\alpha}$ $(\beta\le\alpha,\ \nu\le\lambda)$ and $w_{\alpha+\mu,\,\alpha+\nu}=v_{\mu\nu}$ $(\mu\le\nu\le\lambda)$.

Let us call an immediately deducible arrow $u:U\longrightarrow U'$ *proper* if it results from some pushout (1'), with $r\in\mathcal{R}$, in which φ does not factor through r . In particular, this implies that u is not an isomorphism; but in general, the latter condition is weaker than u being a proper immediately deducible arrow.

The ∞-deduction (2) is *proper* if each $u_{\beta,\,\beta+1}$ ($\beta+1\le\alpha$) is a proper immediately deducible arrow.

4. Proposition. For any sketch S , all proper ∞-deductions based on S are of length $<\kappa_S$.

Proof. Suppose that, contrary to the assertion, there is a proper ∞-deduction $(\langle U_\alpha\rangle_{\alpha\le\kappa}, \langle u_{\beta,\,\alpha}: U_\beta \longrightarrow U_\alpha\rangle_{\beta\le\alpha\le\kappa})$ with $U_0 = S$, $\kappa = \kappa_S$. For every ordinal $\alpha<\kappa$, there is $r_\alpha \in \mathcal{R}$ such that $u_{\alpha,\,\alpha+1}$ is a pushout of r_α:

$$\begin{array}{ccc} R_\alpha & \xrightarrow{\quad r_\alpha \quad} & R'_\alpha \\ {\scriptstyle \varphi_\alpha}\downarrow & \square & \downarrow{\scriptstyle \varphi'_\alpha} \\ U_\alpha & \xrightarrow[\quad u_{\alpha,\,\alpha+1} \quad]{} & U_{\alpha+1} \end{array} ,$$

and such that

(3) φ_α does not factor through r_α.

Furthermore, if α is a limit ordinal,

(4) $U_\alpha = \mathrm{colim}(\langle u_{\gamma,\,\beta}: U_\gamma \longrightarrow U_\beta\rangle_{\gamma\le\beta<\alpha})$ with coprojections $\langle u_{\beta,\,\alpha}\rangle_{\beta<\alpha}$.

By induction on $\alpha<\kappa$, we see that U_α is κ-presentable; the reason is that a colimit of a diagram of size $<\kappa$ of κ-presentable objects is κ-presentable. Since also $\lambda<\kappa$, it follows by remarks in §2 that for all fp R and $\alpha<\kappa$,

(5) $\hom(R, U_\alpha)$ has cardinality $<\kappa$.

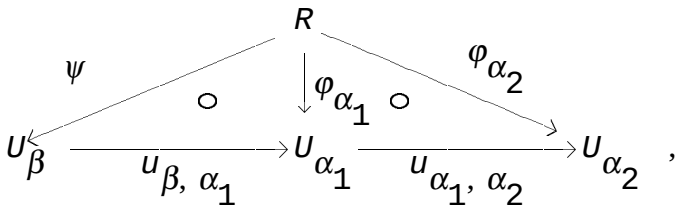
Let $\alpha<\kappa$ be any limit ordinal. Since R_α is fp, by (4) there is $\beta_\alpha<\alpha$ such that φ_α factors through $u_{\beta_\alpha,\,\alpha}$: we have

$$\begin{array}{ccc} & & R_\alpha \\ & {\scriptstyle \psi_\alpha}\swarrow \quad \circ & \downarrow{\scriptstyle \varphi_\alpha} \\ U_{\beta_\alpha} & \xrightarrow[\quad u_{\beta_\alpha,\,\alpha} \quad]{} & U_\alpha \end{array} \qquad (6)$$

.

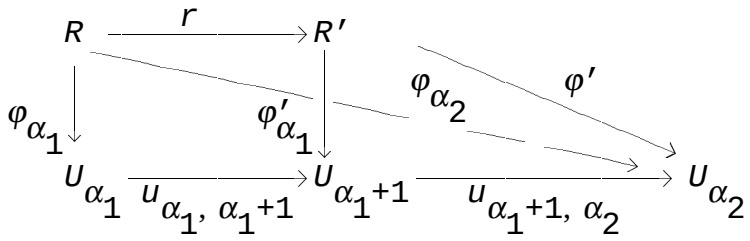
Since κ is regular and uncountable, by the regressive function theorem (Fodor's Lemma, [15]), there is $\beta<\kappa$ and an unbounded(-below-κ) set A of limit ordinals $<\kappa$ such that for all $\alpha\in A$, $\beta_\alpha=\beta$. Since $\#\mathcal{R}\le\lambda<\kappa$, and κ is regular, there are $(R \xrightarrow{r} R')\in\mathcal{R}$ and an

unbounded subset A' of A such that for all $\alpha\in A'$, $r_\alpha = r$, in particular, $R_\alpha = R$. By (5), $\hom(R, U_\beta)$ has cardinality $< \kappa$. Therefore, there are $\psi: R \longrightarrow U_\beta$ and an unbounded subset A'' of A' such that for all $\alpha\in A''$, $\psi_\alpha = \psi$. Let $\alpha_1 < \alpha_2$ be two distinct elements of A''. Applying (5) to $\alpha = \alpha_1$ and α_2, we get that in



(6')

the right-hand triangle commutes. Consider



in which $\varphi' \mathrel{\overline{\text{def}}} u_{\alpha_1+1,\, \alpha_2} \circ \varphi'_{\alpha_1}$. We obtain that

$$\varphi' \circ r = u_{\alpha_1+1,\, \alpha_2} \circ u_{\alpha_1,\, \alpha_1+1} \circ \varphi_{\alpha_1} = u_{\alpha_1,\, \alpha_2} \circ \varphi_{\alpha_1} = \varphi_{\alpha_2},$$

(the last equality by (6')), that is, φ_{α_2} factors through $r = r_{\alpha_2}$, contrary to (3) (properness). This contradiction completes the proof.

The ∞-deduction (2) is an *extension* of another one, say $\mathcal{U}'$, if, for some $\alpha' \le \alpha$, $\mathcal{U}' = \mathcal{U} \restriction \alpha' \mathrel{\overline{\text{def}}} (\langle U_\beta \rangle_{\beta\le\alpha'}, \langle u_{\beta\gamma}: U_\beta \to U_\gamma \rangle_{\beta\le\gamma\le\alpha'}) : \mathcal{U}$ is a *proper* extension of $\mathcal{U}'$ if, in addition, $\alpha' < \alpha$. A proper ∞-deduction is *maximal* if it does not have any proper extension which is a proper ∞-deduction. Note that all these concepts are relative to a given set $\mathcal{R}$ of sketch-axioms.

It is immediately seen that $\mathcal{U}$ as in (2) is maximal if and only if $\ulcorner \mathcal{U} \urcorner \mathrel{\overline{\text{def}}} U_\alpha$ satisfies every $r\in\mathcal{R}$, i.e. if and only if $\ulcorner \mathcal{U} \urcorner \in \boldsymbol{S}: \mathcal{R}$.

5. Corollary. Any proper ∞-deduction can be extended to a maximal one.

Proof. Consider the class C of proper ∞-deductions based on a fixed sketch S. Each $\mathcal{U} \in C$ has a length $< \kappa = \kappa_S$, and as noted in the previous proof, every object in $\mathcal{U}$ is κ-presentable. Let B be a small set of of κ-presentable objects such that every κ-presentable object is isomorphic to one in B. Let C' be the set of $\mathcal{U} \in C$ such that all the objects of $\mathcal{U}$ come from B. Clearly, C' is small. A *chain* X in C' is a subset of C' such that for any two elements of X, one is an extension of the other. The union of any chain in C' is clearly again an element of C'. By Zorn's lemma, for any $\mathcal{U} \in C'$, there is $\mathcal{U}' \in C'$ extending $\mathcal{U}$ which is maximal in C'. Clearly, if $\mathcal{U} \in C'$ is not maximal, then it is not maximal in C' either, that is, it has a proper extension in C'. Hence, any $\mathcal{U} \in C'$ has $\mathcal{U}' \in C'$ extending $\mathcal{U}$ such that $\mathcal{U}'$ is maximal in C', that is, maximal.

We call any sketch of the form $\ulcorner \mathcal{U} \urcorner$, with $\mathcal{U}$ a maximal proper ∞-deduction based on S, a *doctrinal hull* of S; the corresponding *canonical map* $\varphi_{\mathcal{U}} : S \longrightarrow \ulcorner \mathcal{U} \urcorner$ is $u_{0\alpha}$ when $\mathcal{U}$ is as in (2). Each doctrinal hull belongs to the doctrine $\mathbf{S} : \mathcal{R}$.

6. Corollary. Any sketch has at least one doctrinal hull.

An ∞-deduction $\mathcal{U}$ as in (2) based on S gives rise to the ∞-*directly deducible* arrow $\varphi_{\mathcal{U}} : S \longrightarrow \ulcorner \mathcal{U} \urcorner = U_\alpha$, $\varphi_{\mathcal{U}} = u_{0\alpha}$. An arrow $s : S \longrightarrow T$ is ∞-*deducible* if an extension of it is ∞-directly deducible: for some $t : T \longrightarrow T'$, $t \circ s$ is ∞-directly deducible. Note that to say that $s : S \longrightarrow T$ is ∞-deducible is the same as to say that there exists a doctrinal hull $\ulcorner \mathcal{U} \urcorner$ of S such that $\ulcorner \mathcal{U} \urcorner \vDash s$.

The *length* of an ∞-directly deducible arrow s is the least ordinal α for which there is an ∞-deduction $\mathcal{U}$ of length α with $\varphi_{\mathcal{U}} = s$.

7. Lemma. The length of an ∞-directly deducible arrow is either finite, or a limit ordinal.

Proof. First, we show that if $\mathcal{U} = (\langle U_\beta \rangle_{\beta \le \alpha+1}, \langle u_{\beta\gamma} : U_\beta \to U_\gamma \rangle_{\beta \le \gamma \le \alpha+1})$ is an ∞-deduction of length $\alpha+1$, with α a limit ordinal, then there is an ∞-deduction $\mathcal{U}^*$ of length α such that $\varphi_{\mathcal{U}} = \varphi_{\mathcal{U}^*}$. A consequence will be that the length of an ∞-directly deducible arrow cannot be of the form $\alpha+1$, with α a limit ordinal.

Let $\mathcal{U}$ as stated. By assumption, there is a pushout square

$$\begin{array}{ccc} R & \xrightarrow{\psi_\alpha} & U_\alpha \\ {\scriptstyle r}\downarrow & \square & \downarrow{\scriptstyle u_{\alpha,\,\alpha+1}} \\ R' & \xrightarrow[\psi'_\alpha]{} & U_{\alpha+1} \end{array} \qquad (7)$$

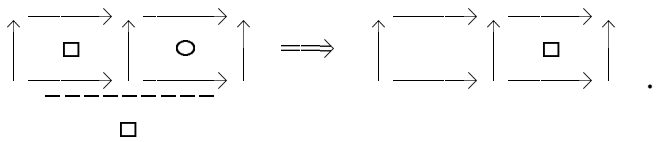
with $r \in \mathcal{R}$. Since R is fp, and α is a limit ordinal, there are $\beta<\alpha$ and $\psi_\beta : R \to U_\beta$ such that $\psi_\alpha = \psi_\beta \circ u_{\beta\alpha}$. Let, for γ with $\beta \le \gamma \le \alpha$, $\psi_\gamma = u_{\beta\gamma} \circ \psi_\beta$ (this is compatible with the previous notation for $\gamma=\beta$ and $\gamma=\alpha$). Consider, for $\beta \le \gamma \le \alpha$, the pushout

$$\begin{array}{ccc} R & \xrightarrow{\psi_\gamma} & U_\gamma \\ {\scriptstyle r}\downarrow & \square & \downarrow{\scriptstyle v_\gamma} \\ R' & \xrightarrow[\psi'_\gamma]{} & U'_\gamma \end{array} \quad ; \qquad (8)$$

for $\gamma=\alpha$, define (8) to be the already available (7). For $\beta \le \gamma \le \delta \le \alpha$, we have

$$\begin{array}{ccccc} R & \xrightarrow{\psi_\gamma} & U_\gamma & \xrightarrow{u_{\gamma\delta}} & U_\delta \\ {\scriptstyle r}\downarrow & \square & \downarrow{\scriptstyle v_\gamma} & \square & \downarrow{\scriptstyle v_\delta} \\ R' & \xrightarrow[\psi'_\gamma]{} & U'_\gamma & \xrightarrow[u'_{\gamma\delta}]{} & U'_\delta \end{array} \qquad \begin{array}{ccc} U_\delta & \xrightarrow{u_{\delta\alpha}} & U_\alpha \\ \downarrow{\scriptstyle v_\delta} & \square & \downarrow{\scriptstyle v_\alpha = u_{\alpha,\,\alpha+1}} \\ U'_\delta & \xrightarrow[u'_{\delta\alpha}]{} & U'_\alpha = U_{\alpha+1} \end{array} \quad ; \qquad (9)$$

namely, the universal property of (8) gives $u'_{\gamma\delta}$ to make the middle square (and the right-hand square) commute, and then it follows that middle square (and the right-hand square) are pushouts, by the principle



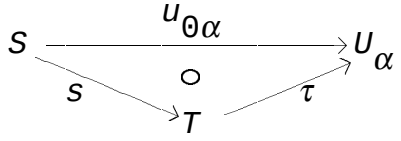
I claim that $\mathcal{U}' = (\langle U'_\gamma \rangle_{\beta\le\gamma\le\alpha}, \langle u'_{\beta\gamma} : U'_\beta \to U'_\gamma \rangle_{\beta\le\gamma\le\alpha})$ is an ∞-deduction. It is clear that $\mathcal{U}'$ is compatible. The fact that it is continuous follows from the fact that pushouts commute with filtered colimits. Finally, each $u'_{\gamma,\ \gamma+1}$ ($\beta\le\gamma<\gamma+1\le\alpha$) is immediately deducible, since it is (by (9)) a pushout of the immediately deducible arrow $u_{\gamma,\ \gamma+1}$.

Consider the composite ∞-deduction $\mathcal{U}^* = \mathcal{U}' \circ \langle v_\beta \rangle \circ \mathcal{U}\restriction\beta$; notice that $v_\beta : U_\beta \to U'_\beta$ gives a one-element deduction by (8) (for $\gamma=\beta$). $\varphi_{\mathcal{U}^*} = u'_{\beta\alpha} \circ v_\beta \circ u_{0\beta} = u_{\alpha,\ \alpha+1} \circ u_{\beta\alpha} \circ u_{0\beta} = \varphi_{\mathcal{U}}$, and the length of $\mathcal{U}^*$ is $\beta+1+(\alpha-\beta) = \beta+(1+(\alpha-\beta)) = \beta+(\alpha-\beta) = \alpha$. We have shown our assertion.

Let α be the length of some ∞-directly deducible arrow. We show by induction on α that α is either finite, or a limit ordinal. If $\alpha=0$ or α is limit, there is nothing to show. Assume $\alpha=\beta+1$; let $\mathcal{U}$ be an ∞-deduction of length $\beta+1$. If there were an ∞-deduction $\mathcal{V}$ of length $<\beta$ such that $\varphi_{\mathcal{V}} = \varphi_{\mathcal{U}\restriction\beta}$, then $\mathcal{W} = \langle u_{\beta,\ \beta+1} \rangle \circ \mathcal{V}$ would be an ∞-deduction of length $< \beta+1$ for which $\varphi_{\mathcal{W}}=\varphi_{\mathcal{U}}$, contradicting the minimality of α. We obtained that β is the length of $\varphi_{\mathcal{U}\restriction\beta}$. By induction hypothesis, therefore β is either finite or limit. In the first case, $\alpha=\beta+1$ is finite. In the second case, $\beta+1$ is a length, and β is limit, in contradiction to our first-proved assertion.

8. Lemma. Any relatively finite ∞-deducible entailment is deducible.

Proof. Suppose $s : S \to T$ is an rf entailment. By induction on the ordinal α, we show that if an ∞-directly deducible arrow of length α and with domain S factors through s, then s is (finitely) deducible. By 7., α is either finite, or a limit ordinal. If α is finite, there is nothing to prove. If α is limit, $\mathcal{U}$ is an ∞-deduction as in (2), $U_0=S$, and



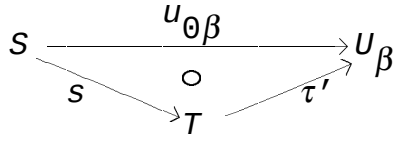
then, since, in $S\downarrow\mathbf{S}$, the object $(u_{0\alpha}:S\to U_\alpha)$ is the (filtered) colimit of the objects $(u_{0\beta}:S\to U_\beta)$ $(\beta<\alpha)$, with connecting arrows the $u_{\gamma\beta}$, and s as an object in $S\downarrow\mathbf{S}$ is fp, there is $\beta<\gamma$ such that the arrow

$$\tau : (s:S\to T)\longrightarrow(u_{0\alpha}:S\to U_\alpha)$$

factors through

$$u_{\beta\alpha} : (u_{0\beta}:S\to U_\beta)\longrightarrow(u_{0\alpha}:S\to U_\alpha) ;$$

there is $\tau' : (S\to T)\longrightarrow(u_{0\beta}:S\to U_\beta)$ with $\tau=u_{\beta\alpha}\circ\tau'$. We can ignore the last equality; even without that, we have that



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$\mathcal{U}\restriction\beta$ is an ∞-deduction of length $\leq\beta<\alpha$; by the induction hypothesis, s is deducible.

Proof of 3. Theorem. Suppose the relatively finite entailment $s:S\longrightarrow T$ is valid. Then, for some (any) (see 6. Cor.) doctrinal hull $\ulcorner\mathcal{U}\urcorner$ of S, $\ulcorner\mathcal{U}\urcorner\models s$. Since $\mathcal{U}_S$ is in particular an ∞-deduction, s is ∞-deducible. By 7. Lemma, it is deducible.

For later reference, we summarize some properties of the doctrinal hull (see before 6. Cor.).

9. Proposition. Let $\ulcorner\mathcal{U}\urcorner$ be a doctrinal hull of the sketch S, with $\varphi_{\mathcal{U}}:S\to\ulcorner\mathcal{U}\urcorner$ the canonical map. Then

(i) for any regular $\kappa>\lambda_{\mathbf{S},\,\mathcal{R}}$, if S is κ-presentable, so is $\ulcorner\mathcal{U}\urcorner$.

(ii) $\varphi_{\mathcal{U}}$ has the following mapping property: for any $A\in\mathbf{S}:\mathcal{R}$, any $\varphi:S\to A$ factors through $\varphi_{\mathcal{U}}$; that is, there is $\psi:\ulcorner\mathcal{U}\urcorner\longrightarrow A$ such that $\varphi=\psi\circ\varphi_{\mathcal{U}}$.

Proof. $\ulcorner\mathcal{U}\urcorner$ is the colimit of a maximal proper ∞-deduction (2), $S=U_0$, and $\varphi_{\mathcal{U}}$ is

$u_{0\alpha}: U_0 \to U_\alpha = \ulcorner \mathcal{U} \urcorner$. Assume that S is κ-presentable; $\kappa \geq \kappa_S$ (see before 4.). In the proof of 3., we noted that each U_β ($\beta < \alpha$) is κ-presentable (κ *loc.cit.* is κ_S ; but $\kappa \geq \kappa_S$). By 4., $\alpha < \kappa$; as a colimit of a diagram of size $< \kappa$ of κ-presentable objects, $\ulcorner \mathcal{U} \urcorner$ is κ-presentable. This shows (i).

To show (ii), for a given $\varphi: S \to A$, by recursion on $\gamma \leq \alpha$, we construct $\psi_\gamma: U_\gamma \to A$ such that $\psi = \psi_0$, and $\psi_\gamma \circ u_{\beta\gamma} = \psi_\beta$ whenever $\beta < \gamma \leq \alpha$. When γ is a limit ordinal, ψ_γ is uniquely given from the ψ_β , $\beta < \gamma$, by the universal property of U_γ as the colimit of the U_β , $\beta < \gamma$. When $\gamma = \beta + 1$, there is a pushout

$$\begin{array}{ccc} R & \xrightarrow{\quad r \quad} & R' \\ {\scriptstyle\theta}\downarrow & \square & \downarrow \\ U_\beta & \xrightarrow[u_{\beta\gamma}]{} & U_\gamma \end{array}$$

with $r \in \mathcal{R}$. Since $A \models r$, there is $\tau: R' \longrightarrow A$ with $\psi_\beta \circ \theta = \tau \circ r$. By the universal property of the pushout U_γ , there is $\psi_\gamma: U_\gamma \to A$ with (among others) $\psi_\gamma \circ u_{\beta\gamma} = \psi_\beta$. It follows that $\psi_\gamma \circ u_{\beta'\gamma} = \psi_{\beta'}$ for all $\beta' < \gamma$.

For $\gamma = \alpha$, ψ_α is the desired ψ .