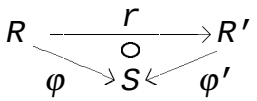
§2. Sketch semantics

Let $\mathbf{S}$ be an arbitrary category; we talk about "sketches" when referring to objects of $\mathbf{S}$ since in the applications S will be a sketch-category obtained by the constructions described in §1. Let $r:R \longrightarrow R'$ be an arbitrary arrow in $\mathbf{S}$. We say that a sketch $S \in \mathbf{S}$ *satisfies* r, in symbols $S \models r$, if

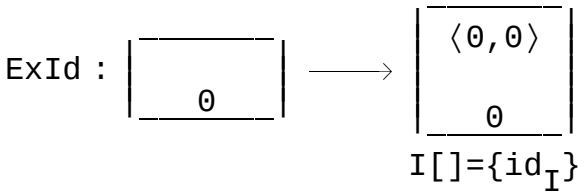
(*) any $\varphi:R \longrightarrow S$ factors through r; that is, for any $\varphi:R \longrightarrow S$, there is $\varphi':R' \longrightarrow S$ such that



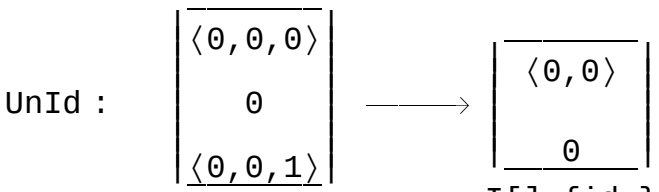
commutes.

Because of their use in this definition, the maps of $\mathbf{S}$ are sometimes called *sketch-axioms*, or *sketch-entailments*.

To show the use of the definition, let us return to the example of $\mathbf{S} = \mathtt{cSk}$ in §1; objects of $\mathtt{cSk}$ are called *c-sketches*. We define seven specific maps in $\mathtt{cSk}$, the *s(ketch)-axioms for category*. Here and later, the specifications will occasionally be sketchy (pun unintended), but I hope that there is no serious possibility of misunderstanding. The maps are always "almost inclusions"; entities in the domain are mapped to the same-named entities in the codomain, unless there is a collapse of two entities into one.



("Existence of identity". Here, we have two c-sketches, and a map between them. The domain sketch has no arrows, and has both specification-sets $\mathtt{I[]}$, $\mathtt{CT[]}$ empty. The codomain sketch has $\mathtt{I[]} = \{\mathtt{id}_{\mathtt{I}}\}$.)



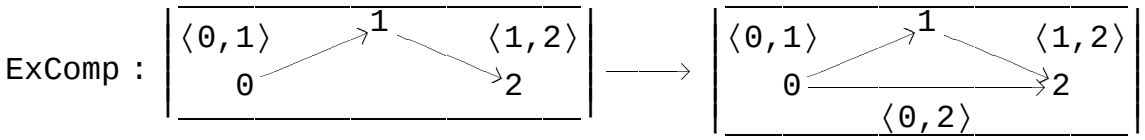
$I[]=\{i_0,i_1\}$

$i_0(\langle 0,0\rangle)=\langle 0,0,0\rangle$

$i_1(\langle 0,0\rangle)=\langle 0,0,1\rangle$

$I[]=\{\mathrm{id}_I\}$

("Uniqueness of identity"; actually, superfluous.)



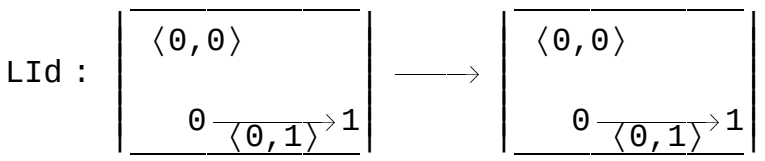
$CT[] = \{\mathrm{id}_{CT}\}$

("Existence of composite"; in the domain, `I[]` and `CT[]` are empty; in the codomain, `I[]` = ∅ , and `CT[]` = $\{\mathrm{id}_{CT}\}$.)

$CT[]=\{[\langle 0,1\rangle\mapsto f,\langle 1,2\rangle\mapsto g,\langle 0,2\rangle\mapsto h],$

$[\langle 0,1\rangle\mapsto f,\langle 1,2\rangle\mapsto g,\langle 0,2\rangle\mapsto i]\}$

("Uniqueness of composite". The domain has two commutative triangles, the codomain one; the map is the obvious one, in particular, it identifies the parallel arrows. The letters `f , g , h, i` stand for completely fixed entities such as the ones used before (e.g. ⟨0, 1⟩), and they are employed only as a shorthand).



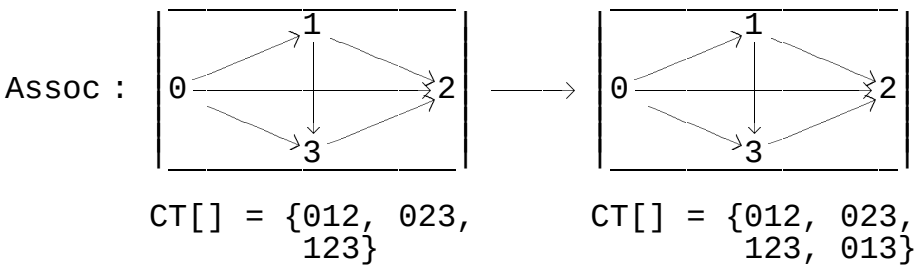
$I[] = \{i_I\}$ $\quad$ $I[] = \{i_I\}$

$CT[] = \emptyset$ $\quad$ $CT[] = \{\tau\}$

("Left identity law". In both the domain and the codomain, there is an identity specification $\in$`I[]` ; in the codomain C , we have `CT[`C`] = {`τ`}` with the map τ`:CT` $\longrightarrow$ `|`C`|` which maps `0, 1`$\mapsto$ `0` , `2`$\mapsto$`1` , $\langle$`0, 1`$\rangle\mapsto\langle$`0, 0`$\rangle$, $\langle$`1, 2`$\rangle\mapsto\langle$`0, 1`$\rangle$, $\langle$`0, 2`$\rangle\mapsto\langle$`0, 1`$\rangle$. Here, and below, a notation like $\mathtt{i}_{\mathtt{I}}$ means an inclusion-map of the entity in the subscript.)

`RId:`

("Right identity law"; similar to `LId` .)



("Associative law". The abbreviations used are self-explanatory.)

Any category $\boldsymbol{C}$ gives rise to a c-sketch, denoted by $\boldsymbol{C}_{(\mathtt{s})}$, whose underlying graph is the underlying graph of $\boldsymbol{C}$, and for which `I[`$\boldsymbol{C}_{(\mathtt{s})}$`]` , `CT[`$\boldsymbol{C}_{(\mathtt{s})}$`]` are the set of all identities and the set of all commutative triangles in the category, in the usual sense, construed as graph-arrows `I`$\rightarrow$ `|`$\boldsymbol{C}$`|` , `CT`$\rightarrow$ `|`$\boldsymbol{C}$`|` in the obvious ways. Also, with a functor F`:`$\boldsymbol{C}\longrightarrow\boldsymbol{D}$, we have the corresponding sketch-map $F_{(\mathtt{s})}$`:`$\boldsymbol{C}_{(\mathtt{s})}\longrightarrow\boldsymbol{D}_{(\mathtt{s})}$; as a mapping, $F_{(\mathtt{s})}$ is identical to F . If a c-sketch S is obtained from a(n obviously unique) category, we say S *is* a category. Let $\mathcal{R}_{\mathtt{cat}}$ denote the set of the seven sketch axioms `ExId, UnId, ExComp, UnComp, LId, RId, Assoc` ; these are called the *sketch-axioms for category*.

Now, here is a proposition:

A c-sketch is a category if and only if it satisfies each of the sketch-axioms for category. More precisely: the mapping $\boldsymbol{C}\mapsto\boldsymbol{C}_{(\mathtt{s})}, F\mapsto F_{(\mathtt{s})}$ *is a full and faithful functor of the category* `Cat` *of small categories into the category of c-sketches; it is also 1-1 on objects, and the range of it consists exactly of those sketches that satisfy the s-axioms for category.*

This is practically clear. Indeed, a c-sketch S is (arises from) a category iff (1) to any object $A \in |S|$, there is an arrow $1_A : A \longrightarrow A$ assigned to A, (2) which is unique; (3) to any pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ as shown, there is assigned a composite $gf : A \longrightarrow C$, (4) which is unique; (5&6) composites with identities behave in the right way; and (7) composition is associative. Now, here, each of the conditions (i) for $i=1$ to 7 is equivalent to saying that $S \vDash r_i$, with r_i for $i=1$ to 7 being the maps listed above in this order.

The contents of the last sentence are, of course, completely elementary; still, together with the examples to be shown later, it contains the main point of the paper: the fact that there is a natural way of expressing the individual parts of specifications of various notions (that of "category" in this case) in the form of sketch-axioms.

A sketch axiom $R \longrightarrow R'$ may be read as an *entailment* that says: "the situation described in R entails that described in R'". For instance, ExComp "says" that the situation consisting of two composable arrows entails that which extends the first one by having (at least one) composite; UnComp "says" that the situation of having two arrows which are composites of the same pair of arrows entails the one in which those two arrows are equal. The entailment $R \longrightarrow R'$ being satisfied by S, $S \vDash (R \longrightarrow R')$, means that no matter how we have a situation of the type R in S, we have another one in S of type R' *extending* the first one.

For a category $\mathcal{S}$ of sketches, and a sketch-entailment r, "$\mathcal{S} \vDash r$" stands for "for all $S \in \mathcal{S}$, $S \vDash r$". With a set $\mathcal{R}$ of sketch-entailments (arrows), $S \vDash \mathcal{R}$ means $S \vDash r$ for all $r \in \mathcal{R}$; $\mathcal{S} \vDash \mathcal{R}$ means that $S \vDash \mathcal{R}$ for all $S \in \mathcal{S}$; and finally, $\vDash_{\mathcal{R}} r$ means that for all S such that $S \vDash \mathcal{R}$, we have $S \vDash r$. These notations follow usual practice in logic.

Given a category $\mathbf{S}$ and a set $\mathcal{R}$ of arrows in it, by $\mathbf{S}{:}\mathcal{R}$ we denote the full subcategory of $\mathbf{S}$ consisting of $S \in \mathbf{S}$ for which $S \vDash \mathcal{R}$. $\mathbf{S}{:}\mathcal{R}$ is called the category (or *doctrine*) *specified* by $\mathbf{S}$ and $\mathcal{R}$. Above, up to isomorphism we exhibited Cat, the category of small categories, as $\mathbf{S}{:}\mathcal{R}$ for $\mathbf{S} = \mathrm{cSk}$ and $\mathcal{R} = \mathcal{R}_{\mathrm{cat}}$. Our contention is that many categories of structured categories can be naturally and profitable exhibited in the form $\mathbf{S}{:}\mathcal{R}$. For brevity, a *doctrine specification* is a pair $(\mathbf{S}, \mathcal{R})$, with $\mathbf{S}$ a sketch-category, with $\mathcal{R}$ a set of arrows in $\mathbf{S}$; it *specifies* $\mathbf{S}{:}\mathcal{R}$.

A doctrine specification $(\mathbf{S}, \mathcal{R})$ is *finitary* if $\mathbf{S}$ is a finitary sketch-category (see §1), and $\mathcal{R}$ is a set of arrows between fp objects of $\mathbf{S}$. The same is a *finite* doctrine specification if $\mathbf{S}$ is

a finite sketch-category, and $\mathcal{R}$, in addition to the previous requirement, is also a finite set. In this paper, we will restrict ourselves to finitary doctrine specification, although "infinitary" ones are equally relevant, in relation to doctrines of infinitary logic. In fact, with a single exception, all our examples will be finite doctrine specifications.

Note that the general construction $\mathbf{S}:\mathcal{R}$ is far from being new; $\mathbf{S}:\mathcal{R}$ is the category of objects that are *injective* relative to the class $\mathcal{R}$ of arrows; see e.g. [1]. Also, the construction has a very general scope that is well-known. E.g., the categories that can be obtained in the form $\mathbf{S}:\mathcal{R}$, with $\mathbf{S}$ a presheaf category $\mathtt{Set}^{\mathbf{C}}$ and $\mathcal{R}$ a small set of arrows, are, up to equivalence of categories, those accessible categories that have (small) products (see [1]), a class more comprehensive than that of the locally presentable categories. Thus, it is not the mere possibility of presenting a category in the form $\mathbf{S}:\mathcal{R}$ that matters; rather, what does is the actual presentation itself.

Let us note that, for the purposes of specification, the construction $\mathbf{G}\,|\,\mathbf{K}$ is not necessary, the construction $\mathbf{G}\,\|\,\mathbf{K}$ suffices. The most immediate thing to say is that the former is equivalent to a full subcategory of the latter singled out by sketch-axioms. With any $K\in|\mathbf{K}|$, consider the sketch-map $\mathtt{Uni}[K]:\langle\langle K\rangle\rangle\longrightarrow\langle K\rangle$ in $\mathbf{G}\,|\,|\,\mathbf{K}$ as follows. $\langle K\rangle$ has $|\langle K\rangle|=\bar{K}$, and $K[\langle K\rangle]=\{0\}$, with $\bar{0}=\mathrm{id}_{\bar{K}}$; all other $K'[\langle K\rangle]=\emptyset$; $\langle\langle K\rangle\rangle$ also has $|\langle\langle K\rangle\rangle|$ equal $\bar{K}$, but $K[\langle\langle K\rangle\rangle]$ equal to $\{0,1\}$, with $\bar{0}=\bar{1}=\mathrm{id}_{\bar{K}}$; the map $\mathtt{Uni}[K]$ is the identity on the level of $\mathbf{K}$. Clearly, the sketches satisfying each $\mathtt{Uni}[K]$ $(K\in\mathbf{K})$ are exactly the ones in which the maps $s\mapsto\bar{s}$ are each one-to-one; the latter form the essential image of the inclusion of $\mathbf{G}\,|\,\mathbf{K}$ in $\mathbf{G}\,\|\,\mathbf{K}$. (The existence in the satisfaction of the axiom $\mathtt{Uni}[K]$ is automatically a unique existence; this means that the objects of $\mathbf{G}\,|\,\mathbf{K}$ are singled out in $\mathbf{G}\,\|\,\mathbf{K}$ by the condition of being orthogonal (see [9]) to given arrows between fp objects; this also shows that $\mathbf{G}\,|\,\mathbf{K}$ is lfp if $\mathbf{G}\,\|\,\mathbf{K}$ is.)

Accordingly, we can restate our specifications, in the one example above, and the many examples later, to take place always in the context of a sketch-category of the form $\mathbf{G}\,\|\,\mathbf{K}$. Then $\mathtt{cSk}_{|}$ (as opposed to $\mathtt{cSk}$) is $\mathtt{Graph}\,\|\,\{\mathtt{I},\mathtt{CT}\}$. Then, we have to redefine the sketch-axioms for category, in the obvious way, by replacing each by its image under the inclusion functor $\mathbf{G}\,|\,\mathbf{K}\longrightarrow\mathbf{G}\,\|\,\mathbf{K}$. Finally, we add the sketch-axioms $\mathtt{Uni[I],\ Uni[CT]}$. The resulting specification gives $\mathtt{Cat}$ up to equivalence.

The category (doctrine) $\mathbb{S}:\mathcal{R}$ specified by $(\mathbb{S},\mathcal{R})$ is an accessible category having small products as remarked above. In general, we cannot say much more; e.g., $\mathbb{S}:\mathcal{R}$ is not locally presentable in general; actually, most of our examples are not locally presentable (equivalently, they do not have all (small) limits). (Most of the examples become *locally presentable as a bicategory*, with an additional structure of 2-cells; this is the main subject of the sequel [30]; here, we will not have anything to do with the higher dimensional aspects of doctrines.)

Assume that $\mathbb{S}$ is an lfp category, and $\mathcal{R}$ is a set of arrows between fp objects of $\mathbb{S}$ (these are the assumption on $(\mathbb{S},\mathcal{R})$ under which the general theory of this section, and sections 3 and 8 go through; a finitary doctrine specification satisfies these assumptions). Consider the following cardinal invariant of $(\mathbb{S},\mathcal{R})$. Let $\lambda = \lambda_{\mathbb{S},\mathcal{R}}$ be the maximum of the following three cardinals: $\aleph_0$, the cardinality of $\mathcal{R}$, and the cardinality of the isomorphism classes of fp objects in $\mathbb{S}$. When $(\mathbb{S},\mathcal{R})$ is a finite doctrine specification, λ is equal to $\aleph_0$; now, the fp objects are the finite objects; any hom-set of two finite objects is finite.

For any regular cardinal $\kappa > \lambda$ (hence, in the finite case, any uncountable regular cardinal), an object $S \in \mathbb{S}$ is κ-presentable iff $\mathrm{hom}(R, S)$ is of cardinality $< \kappa$ for all $R \in \mathbb{S}_{\aleph_0}$. This follows from 2.3.13 (p. 30) in [25] ; notice (to make the connection with *loc.cit.*) that, by 2.3.4. (p. 22) *loc.cit.*, $\aleph_0$ κ. *E.g.*, in the finite case, a sketch is κ-presentable, for any infinite regular cardinal κ, iff its cardinality (in the usual sense of the cardinality of a functor $\mathbb{C} \longrightarrow \mathrm{Set}$), is $< \kappa$; this is true for $\kappa = \aleph_0$ since in a finite topos, the finitely presentable objects are exactly the finite objects; and for $\kappa > \aleph_0$ by what we just said. We have

Let κ be a regular cardinal, $\kappa > \lambda_{\mathbb{S},\mathcal{R}}$. Then $\mathbb{S}:\mathcal{R}$ is κ-accessible, and an object of $\mathbb{S}:\mathcal{R}$ is κ-presentable in $\mathbb{S}:\mathcal{R}$ iff it is κ-presentable in $\mathbb{S}$. Moreover, $\mathbb{S}:\mathcal{R}$ has filtered colimits and arbitrary products, both preserved by the inclusion into $\mathbb{S}$.

Thus, for a finite doctrine specification $(\mathbb{S}, \mathcal{R})$, $\mathbb{S}:\mathcal{R}$ is $\aleph_1$-accessible, and κ-accessible for all uncountable regular cardinals κ.

I'll show the last-stated proposition only for a finitary $(\mathbb{S}, \mathcal{R})$. In this case, I'll show that one can write down a set of axioms in "regular logic", in a language for first order logic, whose category of models is isomorphic to $\mathbb{S}:\mathcal{R}$; both the language and the set of axioms are of

cardinality $\leq \lambda_{\mathbf{S},\mathcal{R}}$, and they are finite in the case of a finite doctrine specification. (An axiom in regular logic is a sentence of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$, with φ and ψ positive primitive formulas, of the form $\exists \vec{y} \bigwedge_{i<n} \theta_i(\vec{x}, \vec{y})$, with the θ_i atomic.)

The above conclusions on accessibility and filtered colimits will then follow by 3.3.5 (ii) in [25] (p. 56). (In fact, *loc.cit.* gives the desired conclusion on κ-accessibility only when κ is a successor cardinal. However, in the remaining case when κ is weakly inaccessible, $\kappa > \lambda_{\mathbf{S},\mathcal{R}}$, one directly checks, using that $\mathbf{S}:\mathcal{R}$ is κ'-accessible for all successor κ' such that $\lambda_{\mathbf{S},\mathcal{R}} < \kappa' < \kappa$, that the objects that are κ'-presentable for some $\kappa' < \kappa$ form a dense subcategory of $\mathbf{S}:\mathcal{R}$ via κ-filtered colimits.) The proof of 3.3.5 (ii) in [25] gives that the objects of $\mathbf{S}:\mathcal{R}$ that are κ-presentable in $\mathbf{S}$ are dense in $\mathbf{S}:\mathcal{R}$, from which the conclusion on presentability follows. The assertion about products is best verified directly from the definitions.

Let $\mathbf{S} = \mathrm{Set}^{\mathbf{C}}$, $\mathbf{C}$ a small category. Let L be the multi-sorted language which is the underlying graph of $\mathbf{C}$: its sorts are the objects of $\mathbf{C}$, its unary sorted operation symbols the arrows of $\mathbf{C}$. The axioms include the statements of the form $\forall x(g(f(x)) = (g \circ f)(x))$, for every composable pair (f, g) of arrows of $\mathbf{C}$, and the ones of the form $\forall x(1_C(x) = x)$ for each $C \in \mathrm{Ob}(\mathbf{C})$. Each of the rest of the axioms α_r is associated with a sketch-axiom $r: R \to R'$ in $\mathcal{R}$, as follows:

$$\forall\langle x_{(C,\,a)}\rangle_{C\in\boldsymbol{C},\,a\in R(C)}{}^{1}\Big(\bigwedge_{\substack{f:C_1\to C_2\\(Rf)(a_1)=a_2}} f(x_{(C_1,\,a_1)})=x_{(C_2,\,a_2)} \dashrightarrow$$

$$\exists\langle y_{(C,\,b)}\rangle_{C\in\boldsymbol{C},\,b\in R'(C)}{}^{2}\Big(\bigwedge_{\substack{f:C_1\to C_2\\b_1\in RC_1,\,b_2\in RC_2\\(Rf)(b_1)=b_2}} f(x_{(C_1,\,b_1)})=x_{(C_2,\,b_2)}$$

$$\wedge \bigwedge_{\substack{C\in\boldsymbol{C}\\a\in RC,\,b\in R'C\\r_C(a)=b}} x_{(C,\,a)}=y_{(C,\,b)}\ {}^{21}\Big)\Big) \ .$$

The class of objects of $\boldsymbol{S}:\mathcal{R}$ is precisely the class of L-models satisfying the axioms; the arrows of $\boldsymbol{S}:\mathcal{R}$ are the L-homomorphisms.