§1. Categories of sketches

Let $\mathbf{G}$ be a category, and $\mathbf{K} = \langle \bar{K} \rangle_{K \in |\mathbf{K}|}$ a (small) indexed set of objects $\bar{K}$ of $\mathbf{G}$. We define a new category $\mathbf{G}|\mathbf{K}$, the *category of sketches over* $\mathbf{G}$ with *specification names* the elements of $|\mathbf{K}|$, and *specification types* the objects $\bar{K}$ of $\mathbf{G}$ for $K \in |\mathbf{K}|$. The objects of $\mathbf{G}|\mathbf{K}$ are entities of the form $S = \langle\ |S|, \langle K[S] \rangle_{K \in |\mathbf{K}|}\ \rangle$ with $|S| \in \mathrm{Ob}(\mathbf{G})$, and $K[S] \subset \mathrm{hom}(\bar{K}, |S|)$ $(K \in |\mathbf{K}|)$. In other words, a *sketch* S is given by an *underlying* $\mathbf{G}$-object $|S|$, and, for each specification name $K \in |\mathbf{K}|$, by a set $\bar{K}[S]$ of *K-specifications*, each K-specification being a morphism $\bar{K} \longrightarrow |S|$ in $\mathbf{G}$. An arrow $\varphi : S \longrightarrow T$ in $\mathbf{G}|\mathbf{K}$ is an arrow $|\varphi| : |S| \longrightarrow |T|$ such that, for all $K \in |\mathbf{K}|$, $s \in K[S]$ we have $|\varphi| \circ s \in K[T]$; in other words, a map of sketches is a morphism on the underlying level that transforms specifications into specifications. Composition of arrows in $\mathbf{G}|\mathbf{K}$ is as in $\mathbf{G}$.

In our examples, we will always have that the map $K \mapsto \bar{K}$ is an identity; each specification type will be its own name. In this case, $\mathbf{K}$ is given by the set $|\mathbf{K}|$ of objects of $\mathbf{G}$, and we write $\mathbf{K}$ for this set $|\mathbf{K}|$; now, the construction gives a category $\mathbf{G}|\mathbf{K}$ on the basis of a category $\mathbf{G}$ and a set $\mathbf{K}$ of objects of $\mathbf{G}$. The reason for the general definition as it is given is to allow for the theoretical possibility of using the same specification type with more than one name. E.g., if $\mathbf{G}$ is the category of sets, and we want to use the empty set as a specification type with two different names, we have to use the first-stated definition with $K \mapsto \bar{K}$ not 1-1. However, with the same $\mathbf{G}$, if we avoid the empty set as a specification type, we can always find enough distinct isomorphic copies of prospective specification types to be able to use them as their own names. We will take advantage of the latter possibility in our examples, for underlying categories $\mathbf{G}$ different from, but related to, the category of sets.

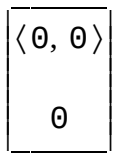
The reader familiar with Ehresmann's sketches [7] will immediately see how they arise as an example of the general construction; in particular, $\mathbf{G}$ in that case is the category of graphs. To aid the reader's intuition, I now give an even simpler example, "sketches to specify categories". Sections 4, 5, 6 and 7 are devoted to the systematic presentation of many other examples.

Graphs are directed graphs. A (small) graph is the same as an object of the category $\mathrm{Set}^{\mathbf{g}}$

(Set is the category of small sets and functions), with g the graph $A \underset{c}{\overset{d}{\rightrightarrows}} O$; for $G \in Set^{g}$, $Ob(G) = G(O)$ = the set of objects of G , $Arr(G) = G(A)$ = the set of arrows of G ; for $a \in Arr(G)$, $d(a)$ and $c(a)$ are the domain and codomain of a , respectively.

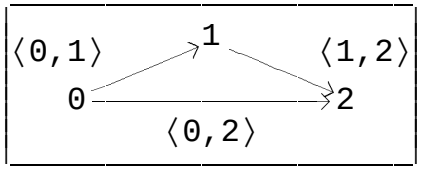
We single out the following two specific graphs I and CT as specification types:

I :



("identity")

CT ;



("commutative triangle")

I has one object 0 , and one arrow $\langle 0,0 \rangle : 0 \to 0$. CT has three objects and three arrows as shown. Let us stress that I and CT are given individually, and not just up to isomorphism. They will serve as *names* for copies of themselves in other graphs, and have to be fixed absolutely.

The general definition of $\boldsymbol{G}|\boldsymbol{K}$ is applied with $\boldsymbol{G} = Graph = Set^{g}$, the category of (small) graphs, and $\boldsymbol{K} = \{I, CT\}$. The resulting category, cSk , is called the category of *category sketches*, or c-sketches for short, for their role (to be described in the next section) in specifying categories. Spelled out, the definition is as follows. A *category-sketch* (c-sketch) S is a graph $|S|$, together with a set $I[S]$ of graph-maps $I \longrightarrow |S|$ (the *identity specifications* in S) and a set $CT[S]$ of graph-maps $CT \longrightarrow |S|$ (the *commutative triangle specifications* in S). A *map* $\varphi : S \longrightarrow T$ *of c-sketches* is a graph map $|\varphi| : |S| \longrightarrow |T|$ with the property that for any $\iota \in I[S]$, $\varphi \circ \iota \in I[T]$ and for any $\tau \in CT[S]$, $\varphi \circ \tau \in CT[T]$; a map of sketches takes an identity-specification or a commutative triangle specification in the domain into an entity of the same kind in the codomain.

We also consider a variant of the construction $\mathbb{G}|\mathbb{K}$. Given $\mathbb{G}$ and $\mathbb{K}=\langle \bar{K}\rangle_{K\in |\mathbb{K}|}$ as before, we define $\mathbb{G}\|\mathbb{K}$ to be the following category. An object S of $\mathbb{G}\|\mathbb{K}$ is given by an object $|S|$ of $\mathbb{G}$, and for each $K\in |\mathbb{K}|$, an abstract set $K[S]$ and an $K[S]$-indexed family $\langle \bar{s}\rangle_{s\in K[S]}$ of morphisms $\bar{s}:\bar{K}\longrightarrow |S|$. A morphism $\varphi:S\longrightarrow T$ of $\mathbb{G}\|\mathbb{K}$ is given by a morphism $|\varphi|:|S|\longrightarrow |T|$ and functions $K[\varphi]:K[S]\longrightarrow K[T]$ $(K\in |\mathbb{K}|)$ such that for all $K\in |\mathbb{K}|$ and for all $s\in K[S]$, we have $(K[\varphi](s))^{-}=|\varphi|\circ \bar{s}$; diagrammatically,

$$\begin{array}{ccc} K[S] & \xrightarrow{\ (\ )^{-}\ } & \mathbb{G}(\bar{K},\ |S|\ ) \\ {\scriptstyle K[\varphi]}\downarrow & \circ & \downarrow{\scriptstyle |\varphi|\circ(\ )} \\ K[T] & \xrightarrow[\ (\ )^{-}\ ]{} & \mathbb{G}(\bar{K},\ |T|\ ) \end{array}$$

(a circle in a diagram signifies commutativity). Composition of morphisms in $\mathbb{G}\|\mathbb{K}$ is defined componentwise. Let us add that the identity morphisms have all their components identities.

Once again, in our examples, the naming-function $K\mapsto\bar{K}$ will be the identity; we have the category $\mathbb{G}\|\mathbb{K}$ for a set $\mathbb{K}$ of objects of $\mathbb{G}$.

$\mathbb{G}|\mathbb{K}$ is, essentially, the part of $\mathbb{G}\|\mathbb{K}$ in which each indexing $s\longrightarrow |s|$ above is 1-1. Indeed, the obvious inclusion $\mathbb{G}|\mathbb{K}\longrightarrow\mathbb{G}\|\mathbb{K}$ is full and faithful, and its essential image consists of the objects of $\mathbb{G}\|\mathbb{K}$ with the said property of injectivity.

The second, more complicated, construction $\mathbb{G}\|\mathbb{K}$ is more basic than the first one, $\mathbb{G}|\mathbb{K}$. For one thing, the construction $\mathbb{G}\|\mathbb{K}$ is enough; the uses of the $\mathbb{G}|\mathbb{K}$ can be replaced by uses of the $\mathbb{G}\|\mathbb{K}$, but not (apparently) vice versa. For another, the categories obtained by the construction $\mathbb{G}\|\mathbb{K}$ are better behaved than the ones obtained by the other construction. (These remarks will be expanded on below.) Nevertheless, the simpler construction $\mathbb{G}|\mathbb{K}$ will be the one used more frequently.

The two constructions will be employed to produce a large class of sketch-categories, which in turn will be used for the specification of kinds of structured categories. The constructions will be used iteratively; we start with a simple category $\mathbb{G}$ (most commonly, $\mathtt{Graph}$), obtain $\mathbb{G}|\mathbb{K}$ (less frequently, $\mathbb{G}\|\mathbb{K}$) for a suitable $\mathbb{K}\subset \mathtt{Ob}(\mathbb{G})$, then use $\mathbb{G}|\mathbb{K}$ (or $\mathbb{G}\|\mathbb{K}$) as the category $\mathbb{G}'$ to construct a further sketch-category $\mathbb{G}'|\mathbb{K}'$ (or $\mathbb{G}'\|\mathbb{K}'$), and possibly

iterate the procedure.

As we will see, the structure, especially the colimit structure, of the sketch-categories will be important. In fact, all our sketch-categories will be very simple, familiar kinds of categories, with well-understood colimit structure, among other things. Specifically, if we only use the construction $\boldsymbol{G}\|\boldsymbol{K}$, and we start (as we do in the examples) with simple enough "base-"categories, as sketch-categories we get, up to isomorphism, functor categories of the form $\mathtt{Set}^{\boldsymbol{X}}$. If certain finiteness requirements in the data are met (as they are in *almost* all examples), here the category $\boldsymbol{X}$ is finite. When we also use the construction $\boldsymbol{G}\,|\,\boldsymbol{K}$, we obtain reflective subcategories with special properties of categories of the form $\mathtt{Set}^{\boldsymbol{X}}$. In what follows, we elaborate on these remarks.

A *presheaf topos* is a category of the form $\mathtt{Set}^{\boldsymbol{X}}$, with $\boldsymbol{X}$ any small category; a *finite topos* (the terminology is that of [4]) is any $\mathtt{Set}^{\boldsymbol{X}}$ where $\boldsymbol{X}$ is a finite category (having finitely many objects and arrows). Every presheaf topos is a locally finitely presentable (lfp) category (for concepts in the Gabriel-Ulmer theory of lfp categories, see [9] and [1]).

Note, to begin with, that the category $\mathtt{Graph} = \mathtt{Set}^{\mathtt{g}}$ is a finite topos; the graph $\mathtt{g}$ here can be replaced by the category it freely generates, which has only identity arrows in addition to those of $\mathtt{g}$. Similarly, the category of 2-graphs is a finite topos: $\mathtt{2\text{-}Graph} = \mathtt{Set}^{\mathtt{c}_2}$, where $\mathtt{c}_2$ is the category

$$\mathtt{2} \;\underset{\mathtt{c'}}{\overset{\mathtt{d'}}{\rightrightarrows}}\; \mathtt{1} \;\underset{\mathtt{c}}{\overset{\mathtt{d}}{\rightrightarrows}}\; \mathtt{0} \;\mathtt{;}\; \mathtt{dd'=dc'}\,\mathtt{,}\; \mathtt{cd'=cc'}$$

(only the non-identity arrows are shown). $\mathtt{2\text{-}Graph}$ gives rise to a sketch-category that can be used to specify the concept of 2-category (in the sense of sketch-semantics, see §2). We can make similar remarks on n-graphs and n-categories in general.

In a finite topos $\mathtt{Set}^{\boldsymbol{X}}$, the finitely presentable (fp) objects are exactly the *finite* ones, that is, the functors $G\mathtt{:}\boldsymbol{X}\longrightarrow\mathtt{Set}$ for which $G(X)$ is finite for all $X\in\boldsymbol{X}$. The reason is as follows. Regardless whether $\boldsymbol{X}$ is finite, if $G\mathtt{:}\boldsymbol{X}\longrightarrow\mathtt{Set}$, and $\langle (X_i, x_i\in GX_i)\rangle_{i\in I}$ is a family of

"elements" of G, then the subfunctor F of G generated by these elements has $F(Y)=\{(Gf)(x_i)\colon i\in I,\ (f\colon X_i\to Y)\in \mathrm{Arr}(\mathbf{X})\}$ for all $Y\in \mathrm{Ob}(\mathbf{X})$; if $\mathbf{X}$ and I are finite, F is finite (independently of G). It is clear that every fp functor is finitely generated (that is, it is as F above with I finite); in fact, this would hold in a much greater generality. Thus, in a finite topos, every fp object is finite; the converse is fairly clear.

More generally, for any lfp category $\mathbf{A}$, if $\mathbf{X}$ is a finite category, we have $(\mathbf{A}^{\mathbf{X}})_{\mathrm{fp}} = (\mathbf{A}_{\mathrm{fp}})^{\mathbf{X}}$.

I claim that

(1) *If $\mathbf{G}$ is a presheaf topos, then $\mathbf{G}/\!\!/\mathbf{K}$ is isomorphic to a presheaf topos; if $\mathbf{G}$ is a finite topos, $|\mathbf{K}|$ is a finite set, and each $\bar{K}$ ($K\in|\mathbf{K}|$) is a finite object in $\mathbf{G}$, then $\mathbf{G}/\!\!/\mathbf{K}$ is isomorphic to a finite topos.*

Proof. Let $\mathbf{G} = \mathrm{Set}^{\mathbf{X}}$, $\mathbf{K}=\langle \bar{K}\rangle_{K\in|\mathbf{K}|}$ an indexed set of objects of $\mathbf{G}$. We construct the following category $\mathbf{X}^*\mathbf{K}$. We let

$$\mathrm{Ob}(\mathbf{X}^*\mathbf{K}) \underset{\mathrm{def}}{=} \mathrm{Ob}(\mathbf{X}) \sqcup |\mathbf{K}|$$

(for simplicity, we assume that $\mathrm{Ob}(\mathbf{X})$ and $|\mathbf{K}|$ are disjoint sets, $\mathrm{Ob}(\mathbf{X}^*\mathbf{K}) = \mathrm{Ob}(\mathbf{X})\cup|\mathbf{K}|$.) The arrows of $\mathbf{X}^*\mathbf{K}$ are as follows. The arrows $X\longrightarrow Y$ in $\mathbf{X}^*\mathbf{K}$ for $X, Y\in\mathrm{Ob}(\mathbf{X})\subset\mathrm{Ob}(\mathbf{X}^*\mathbf{K})$ are those in $\mathbf{X}$. For $K\in|\mathbf{K}|$ and $X\in\mathrm{Ob}(\mathbf{X})$, the arrows $K\longrightarrow X$ in $\mathbf{X}^*\mathbf{K}$ are the triples $\langle K, X, a\rangle$, with $X\in\mathbf{X}$, $a\in KX$. The only arrows with domain and codomain both in $|\mathbf{K}|\subset\mathrm{Ob}(\mathbf{X}^*\mathbf{K})$ are identities. There are no arrows of the form $X\to K$ for $K\in|\mathbf{K}|$, $X\in\mathrm{Ob}(\mathbf{X})$.

Composition in $\mathbf{X}^*\mathbf{K}$ is given as follows. The arrows between objects in $\mathrm{Ob}(\mathbf{X})\subset\mathrm{Ob}(\mathbf{X}^*\mathbf{K})$ are composed as in $\mathbf{X}$. For composition of the form $K\to X\to Y$ ($X, Y\in\mathrm{Ob}(\mathbf{X})$, $K\in|\mathbf{K}|$), we have

$$(f\colon X\to Y)\circ\langle K, X, a\rangle \underset{\mathrm{def}}{=} \langle K, Y, \bar{K}(f)(a)\rangle\ . \qquad (2)$$

All other compositions involve identities, and thus are obvious. The associativity of the composition in $\boldsymbol{X}^*\boldsymbol{K}$ is a consequence of the same fact for $\boldsymbol{X}$, and the functoriality of the $\bar{K}:\boldsymbol{X}\to\mathtt{Set}$, $K\in|\boldsymbol{K}|$.

If $\boldsymbol{X}$, $|\boldsymbol{K}|$ and the functors $\bar{K}$ ($K\in|\boldsymbol{K}|$) are all finite, then $\boldsymbol{X}^*\boldsymbol{K}$ is a finite category.

With $\boldsymbol{G}=\mathtt{Set}^{\boldsymbol{X}}$, the category $\boldsymbol{G}\big\|\boldsymbol{K}$ is isomorphic (not just equivalent) to $\mathtt{Set}^{\boldsymbol{X}^*\boldsymbol{K}}$. The isomorphism

$$(\ )\hat{}:\boldsymbol{G}\big\|\boldsymbol{K}\xrightarrow{\cong}\mathtt{Set}^{\boldsymbol{X}^*\boldsymbol{K}} \tag{3}$$

sends $S\in\mathtt{Ob}(\boldsymbol{G}\big\|\boldsymbol{K})$ to $\hat{S}:\boldsymbol{X}^*\boldsymbol{K}\longrightarrow\mathtt{Set}$ where $\hat{S}(X)=|S|(X)$ ($X\in\mathtt{Ob}(\boldsymbol{X})$), $\hat{S}(K)=K[S]$ ($K\in|\boldsymbol{K}|$); for $f:X\to Y$, $\hat{S}(f)=|S|(f)$; and $\hat{S}(\langle K,X,a\rangle):\hat{S}(K)\longrightarrow\hat{S}(X)$, that is, $\hat{S}(\langle K,X,a\rangle):K[S]\longrightarrow|S|(X)$, is the mapping $s\mapsto\bar{s}_X(a)$. The fact that $\hat{S}$ is a functor, in particular that it preserves the instance (2) of composition, is a consequence of the naturality of $\bar{s}:\bar{K}\to|S|$.

Informally, but clearly, the information contained in $\hat{S}$ is precisely that contained in S; the functor $|S|$ is the restriction of $\hat{S}$ to the full subcategory on the objects in $\boldsymbol{X}$, and the map $\bar{s}_X:\bar{K}(X)\to|S|(X)$ is the same as $a\mapsto\hat{S}(\langle K,X,a\rangle)(s)$.

The functor (3) sends $\varphi:S\longrightarrow T$ to $\hat{\varphi}:\hat{S}\longrightarrow\hat{T}$ for which $\hat{\varphi}_X=\varphi_X$ and $\hat{\varphi}_K=K[\varphi]$; the naturality of $\hat{\varphi}$ with respect to the arrow $f:X\to Y$ is the naturality of φ with respect to the same f, and the naturality of $\hat{\varphi}$ with respect to the arrow $\langle K,X,a\rangle:\bar{K}\longrightarrow X$ is the condition $(|\varphi|,\langle\bar{K}[\varphi]\rangle_{K\in|\boldsymbol{K}|})$ has to satisfy to give rise to the arrow $\varphi:S\to T$.

The rest of the verification that (3) is an isomorphism of categories is left to the reader.

As I said above, the construction $\boldsymbol{G}\big\|\boldsymbol{K}$ would suffice, at the expense of a slight added complexity of the specifications. According to the last-proved proposition, all the

sketch-categories obtained by repeated use of the $\mathcal{G}\|\mathcal{K}$ construction, based on starting categories that are finite toposes, and using finite $\mathcal{K}$'s , are finite toposes. In any topos, coproducts are disjoint sums; colimits in general can be calculated in special ways; in presheaf toposes, colimits (and limits) are computed pointwise in Set .

The following easy propositions refer to more general situations. $\mathcal{G}$ and $\mathcal{K}$ are as in the first sentence of the section, otherwise arbitrary.

(4) *The collection of functors* $\{|\ |:\mathcal{G}\|\mathcal{K}\to\mathcal{G}\}\cup\{K[\,]:\mathcal{G}\|\mathcal{K}\to\mathrm{Set} : K\in|\mathcal{K}|\}$ *create limits and colimits in* $\mathcal{G}\|\mathcal{K}$.

(5) *The inclusion* $i:\mathcal{G}\,|\,\mathcal{K}\longrightarrow\mathcal{G}\|\mathcal{K}$ *has a left adjoint* δ for which $|\delta(S)| = |S|$ and $K[\delta(S)] = \{\bar{s}: s\in K[S]\}$; $\mathcal{G}\,|\,\mathcal{K}$ is a reflective subcategory of $\mathcal{G}\|\mathcal{K}$.

(6) i *is powerful: it induces bijections* $\mathrm{Sub}_{\mathcal{G}\,|\,|\mathcal{K}}(S)\to\mathrm{Sub}_{\mathcal{G}\,|\,\mathcal{K}}(iS)$ *on the subobject lattices.* δ *preserves monomorphisms. The unit components* $\eta_S:S\longrightarrow\delta(S)$ *are regular epimorphisms* (in $\mathcal{G}\|\mathcal{K}$; $\mathcal{G}\,|\,\mathcal{K}$ is regular-epi reflective in $\mathcal{G}\|\mathcal{K}$).

(7) *Assume that* $\mathcal{G}$ *is an lfp category, and for every* $K\in|\mathcal{K}|$, $\bar{K}$ *is an fp object of* $\mathcal{G}$ (in this case, we call the system $\mathcal{K} = \langle\bar{K}\rangle_{K\in|\mathcal{K}|}$ *finitary*). *Then* $\mathcal{G}\|\mathcal{K}$ *is an lfp category, and an object* S *of* $\mathcal{G}\|\mathcal{K}$ *is fp iff* $|S|$ *is fp in* $\mathcal{G}$ *, each specification-set* $K[S]$ ($K\in|\mathcal{K}|$) *is finite, and for all but finitely many* $K\in|\mathcal{K}|$, $K[S]$ *is empty*. (When in addition $|\mathcal{K}|$ is finite, in which case we call the system $\mathcal{K}$ *finite*, the last clause is superfluous.)

Proof of (7). By (4), if the category $\mathcal{G}$ has (small) colimits, then so does $\mathcal{G}\|\mathcal{K}$, and the forgetful functor $|\ |:\mathcal{G}\|\mathcal{K}\longrightarrow\mathcal{G}$ and the functors $K[\,]:\mathcal{G}\|\mathcal{K}\to\mathrm{Set}$ preserve them.

We will show that $\mathcal{G}\|\mathcal{K}$ is $\aleph_0$-accessible (see [1], [25]); since by the above, $\mathcal{G}\|\mathcal{K}$ is cocomplete, it will follow that $\mathcal{G}\|\mathcal{K}$ is lfp.

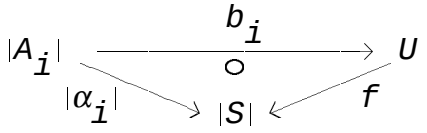
Let us call $S\in\mathcal{G}\|\mathcal{K}$ *finite* if $|S|$ is fp, each $K[S]$ ($K\in|\mathcal{G}|$) is a finite set, and all but finitely many $K[S]$ ($K\in|\mathcal{G}|$) are empty. Using what filtered colimits are in $\mathcal{G}\|\mathcal{K}$, it is easy to check that every finite object is fp. Next, I claim that every object of $\mathcal{G}\|\mathcal{K}$ is a filtered

colimit of finite objects. In fact, writing $\mathbf{A}$ for $\mathbf{G}\|\mathbf{K}$, and $\mathbf{A}_f$ for the full subcategory of $\mathbf{A}$ consisting of the finite objects, we can show that the canonical diagram

$$(\varphi : A \to S) \mapsto A : \mathbf{A}_f \downarrow S \longrightarrow \mathbf{A} \qquad (8)$$

is filtered, and has S as a colimit, with coprojections $\ell_\varphi = \varphi$.

To see that $\mathbf{A}_f \downarrow S$ is a filtered category, let $\mathcal{A} = \langle \alpha_i : A_i \to S,\ \alpha_\iota : A_i \to A_j \rangle_{\iota : i \to j}$ be a finite diagram in $\mathbf{A}_f \downarrow S$. Since S is a canonical filtered colimit of fp objects in $\mathbf{G}$, there is $f : U \to |S|$ in $\mathbf{G}$, with $U \in \mathbf{G}$ fp, such that for all i, we have a commutative diagram



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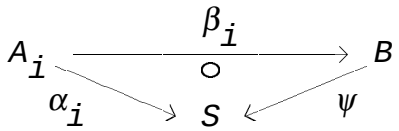
and such that $\langle b_i \rangle_i$ provides a cocone on the diagram $|\mathcal{A}|$, $\mathcal{A}$ composed with the forgetful functor $\mathbf{G}\|\mathbf{K} \longrightarrow \mathbf{G}$. For each i, $K \in |\mathbf{K}|$ and $s \in K[A_i]$, and for $s_i \overline{\overline{\text{def}}}\ K[\alpha_i](s)$, we have $\bar{s}_i : \bar{K} \to |S|$. Since there are altogether finitely many $\bar{s}_i$'s ($\bigsqcup_{i,\,K} K[A_i]$ is a finite set), and each $\bar{G}$ is fp, there is a factorization $f = hg$ with $g : U \to V$, V fp, such that each $\bar{s}_i$ factors as $\bar{s}_i = g\hat{s}_i$ for a suitable $\hat{s}_i : \bar{G} \to V$. Further, we may choose $g : U \to V$ (by composing the given g with a suitable $V \to V'$) so that, in addition, for every $\iota : i \to j$, and every $K \in |\mathbf{K}|$, $s \in G[A_i]$, we have $(K[\alpha_\iota](s))\hat{}_j = \hat{s}_i$.

Define $B \in \mathbf{G}\|\mathbf{K}$ by $|B| = V$, $K[B] = \{s_i : i \in I,\ s \in K[A_i]\}$, and $(s_i)\bar{} : \bar{K} \to |B|$ as $\hat{s}_i$. Define $\psi : B \to S$ by $|\psi| = h$, each $K[B] \to K[S]$ as an inclusion. Define $\beta_i : A_i \to B$ by $|\beta_i| = gb_i$, each $K[A_i] \to K[B]$ as $s \mapsto s_i$. Then we have



and $\langle \beta_i \rangle_i$ gives a cocone on $\mathcal{A}$ in $\mathbf{A}_f \downarrow S$ as desired.

Having given a flavor of the arguments involved, I leave the verification that S is canonically the colimit of (8) to the reader.

What we know now contains the information that $\mathbf{G}\|\mathbf{K}$ is lfp; also that every fp object is a retract of a finite one; since the retract of an fp object of $\mathbf{G}$ is fp, it is immediate that a retract of a finite object of $\mathbf{G}\|\mathbf{K}$ is finite.

(9) *Under the same hypotheses as in (7), the inclusion* $i:\mathbf{G}\,|\,\mathbf{K}\to\mathbf{G}\|\mathbf{K}$ *preserves filtered colimits; thus, by (5),* $\mathbf{G}\,|\,\mathbf{K}$ *is lfp as well. Moreover, the fp objects of* $\mathbf{G}\,|\,\mathbf{K}$ *coincide with the fp objects of* $\mathbf{G}\|\mathbf{K}$ *that are in* $\mathbf{G}\,|\,\mathbf{K}$ *.*

Proof of (9). By (5), or seen directly, $\mathbf{G}\,|\,\mathbf{K}$ has colimits; for $S=\operatorname{colim}_{i\in I} S_i$ we have $|S|=\operatorname{colim}_{i\in I}|S_i|$ as before, and $K[S]=\{\varphi_i\circ g\colon i\in I,\ (g:\bar{K}\to|S_i|)\in K[S_i]\}$, where the $\varphi_i\colon|S_i|\to|S|$ are colimit coprojections. It is not hard to check that the inclusion $i:\mathbf{G}\,|\,\mathbf{K}\longrightarrow\mathbf{G}\|\mathbf{K}$ preserves filtered colimits; note however that this needs the hypothesis that $\mathbf{K}$ is finitary. As a reflective subcategory of an lfp category, with the inclusion preserving filtered colimits, $\mathbf{G}\,|\,\mathbf{K}$ is lfp. Moreover, the fp objects of $\mathbf{G}\,|\,\mathbf{K}$ are retracts of the ones of the form $\delta(A)$ (δ from (5)), A fp in $\mathbf{G}\|\mathbf{K}$. However, any subobject of a finite object of $\mathbf{G}\|\mathbf{K}$ is finite; the desired conclusion concerning the fp objects of $\mathbf{G}\,|\,\mathbf{K}$ follows.

Let us call a category $\mathbf{G}$ *Noetherian* if it is lfp, the product of two fp objects is fp, and every subobject of an fp object is fp. It is clear that finite toposes are Noetherian. The characterization of fp objects in (7) shows that if $\mathbf{G}$ is Noetherian, and $\mathbf{K}$ is finitary, then $\mathbf{G}\|\mathbf{K}$ and $\mathbf{G}\,|\,\mathbf{K}$ are Noetherian as well.

Another property being inherited from $\mathbf{G}$ to $\mathbf{G}\|\mathbf{K}$ and $\mathbf{G}\,|\,\mathbf{K}$ is being a regular category; if $\mathbf{G}\|\mathbf{K}$ is regular, so are $\mathbf{G}\,|\,\mathbf{K}$ and the functor i ; this easily follows from (4), (5) and (6).

One notices that in a regular Noetherian category, any quotient of an fp object is fp, and that every object is the (canonical) filtered colimit of its fp subobjects.

Finally, we observe that if we have $\mathbf{G}\underset{i}{\overset{p}{\leftrightarrows}}\mathbf{G}'$, with i full, powerful and faithful and

$p \dashv i$, then, for any system $\boldsymbol{K} = \langle \bar{K} \rangle_{K \in |\boldsymbol{K}|}$ of objects of $\boldsymbol{G}$, and for $i\boldsymbol{K} \bar{\bar{\text{def}}} \langle i\bar{K} \rangle_{K \in |\boldsymbol{K}|}$, we have functors

$$\boldsymbol{G}\big\|\boldsymbol{K} \underset{i'}{\overset{p'}{\leftarrows}} \boldsymbol{G}'\big\|i\boldsymbol{K} \ ,$$

such that $p' \dashv i'$, and i' is full, powerful and faithful. In fact, when i is an inclusion, then so is i' ; and $|p'(S)| = p(|S|)$, $K[p'(S)] = K[S]$, and $\bar{s}(p'S) : \bar{G} \to |p'(S)|$ is $\eta_{|S|} \circ \bar{s}(S)$, for $\bar{s}(S) : \bar{K} \to |S|$ and $\eta_{|S|}$ the unit of the adjunction $p \dashv i$. Also, if p preserves monomorphisms, so does p' , and if $p \dashv i$ is a regular epi reflection, so is $p' \dashv i'$.

By a *finite* (resp., *finitary*) *sketch-category*, we mean a category obtained by starting with a finite topos (resp., presheaf topos), and applying the constructions $\boldsymbol{G}\big\|\boldsymbol{K}$, $\boldsymbol{G}\,|\,\boldsymbol{K}$ iteratively in a finite number of times, always with a finite (resp., finitary) system $\boldsymbol{K}$. We conclude

(10) *Every finitary sketch-category* $\boldsymbol{S}$ *is regular and Noetherian, and it is a full regular-epi-reflective subcategory of a presheaf topos* $\boldsymbol{E}$ *such that the inclusion is powerful and preserves filtered colimits, the reflector preserves monomorphisms, and the finite objects of* $\boldsymbol{S}$ *are the finite objects of* $\boldsymbol{E}$ *that lie in* $\boldsymbol{S}$ *. If* $\boldsymbol{S}$ *is a finite sketch-category,* $\boldsymbol{E}$ *can be chosen, in addition, to be a finite topos.*

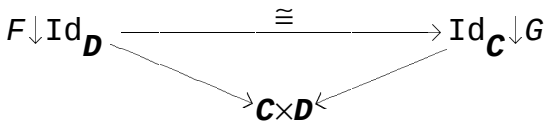
Finally in this section, I will describe a further analysis of the construction of sketch-categories that was kindly communicated to me by F. W. Lawvere.

First of all, Lawvere points out that the foregoing constructions form a special case of a more general situation. Given any adjoint pair

$$\boldsymbol{C} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{D} \ , \quad F \dashv G$$

of functors, we have that the comma-categories $F \downarrow \text{Id}_{\boldsymbol{D}}$, $\text{Id}_{\boldsymbol{C}} \downarrow G$ are isomorphic; in fact an adjunction between F and G is the same thing as an isomorphism of these categories making

the diagram



containing the forgetful functors commute. The category $F\downarrow \mathrm{Id}_{\boldsymbol{D}}$, equivalently $\mathrm{Id}_{\boldsymbol{C}}\downarrow G$, is called the *adjunction category* for the given adjunction. Lawvere observes that

(11) *The categories* $\boldsymbol{G}^{|\boldsymbol{K}|}$ *are precisely the adjunction categories for adjunctions*

$$\mathrm{Set}^{|\boldsymbol{K}|} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \boldsymbol{G} \ ;$$

with a discrete exponent $|\boldsymbol{K}|$ *on the left side.*

(Here, we are assuming that $\boldsymbol{G}$ has colimits; the functors $\mathrm{Set}^{|\boldsymbol{K}|} \longrightarrow \boldsymbol{G}$ having a right adjoint are precisely the left Kan extensions of functors $|\boldsymbol{K}|^{\mathrm{op}} \longrightarrow \boldsymbol{G}$.) In fact, with a given $\boldsymbol{K}=\langle \bar{K} \rangle_{K\in |\boldsymbol{K}|}$ as before, the functor F is the left Kan-extension (or, unique colimit preserving extension) of the functor $K \longmapsto \bar{K}$ from the discrete category $|\boldsymbol{K}|^{\mathrm{op}} = |\boldsymbol{K}|$ to $\boldsymbol{G}$ along the Yoneda functor $|\boldsymbol{K}| \longrightarrow \mathrm{Set}^{|\boldsymbol{K}|}$. Indeed, an object S of the adjunction category is a triple $(\Phi,\ |S|\ ,\ \Psi : F\Phi \to |S|\ )$; here, we may write $K \longmapsto K[S]$ $(K\in |\boldsymbol{K}|\ )$ for Φ; then Φ can be written in the form $\coprod_{K\in |\boldsymbol{K}|} K[S]\cdot |\boldsymbol{K}|\,(K,\ -\ )$; accordingly, $F\Phi$ is $\coprod_{K\in |\boldsymbol{K}|} K[S]\cdot \bar{K}$; and an arrow $F\Phi \to |S|$ is a family $\langle s \mapsto \bar{s} : K[S] \to \hom(\bar{K},\ |S|\ )\rangle_{K\in |\boldsymbol{K}|}$, which is precisely the data for a sketch $S\in \boldsymbol{G}^{|\boldsymbol{K}|}$. One sees that the notions of morphism also correspond to each other.

Next, Lawvere points out that

(12) *The adjunction category for an adjunction between presheaf toposes is isomorphic to a presheaf topos.*

Here is the construction for (12). Consider an adjunction

$$\mathtt{Set}^{\boldsymbol{A}} \underset{G}{\overset{F}{\rightleftarrows}} \mathtt{Set}^{\boldsymbol{X}} \quad , \quad F \dashv G$$

between presheaf toposes. The adjunction category is $\mathtt{Set}^{\boldsymbol{B}}$ for $\boldsymbol{B} = \boldsymbol{A}^*_F\boldsymbol{X}$ constructed as follows. $\mathtt{Ob}(\boldsymbol{B}) = \mathtt{Ob}(\boldsymbol{A}) \sqcup \mathtt{Ob}(\boldsymbol{X})$. Both $\boldsymbol{A}$ and $\boldsymbol{X}$ are full subcategories of $\boldsymbol{B}$. The arrows $A \to X$, with $A \in \boldsymbol{A}$, $X \in \boldsymbol{X}$, are the elements of $(F\hat{A})(X)$, with $\hat{A} = \boldsymbol{A}(A, \text{-})$. There are no arrows of the form $X \to A$, with $A \in \boldsymbol{A}$, $X \in \boldsymbol{X}$. The composites of $B \xrightarrow{f} A \xrightarrow{t} X$, $A \xrightarrow{t} X \xrightarrow{u} Y$ with $B, A \in \boldsymbol{A}$, $X, Y \in \boldsymbol{X}$ and $t \in (F\hat{A})(X)$ are $tf = (F\hat{f})_X(t)$, $ut = ((F\hat{A})(u))(t)$.

To explain the isomorphism between the adjunction category and $\mathtt{Set}^{\boldsymbol{B}}$, start with an object $(C, D, FC \xrightarrow{h} D)$ of the adjunction category; $C \in \mathtt{Set}^{\boldsymbol{A}}$, $D \in \mathtt{Set}^{\boldsymbol{X}}$. The Yoneda correspondence assigns $\hat{a} : \hat{A} \to C$ to any $a \in C(A)$; we have $F\hat{a} : F\hat{A} \to FC$, and composing with h , $h \circ F\hat{a} : F\hat{A} \to D$ and the function

$$(h \circ F\hat{a})_X : (F\hat{A})(X) \to D(X) \ .$$

The restrictions to $\boldsymbol{A}$ and $\boldsymbol{X}$ of the functor Φ corresponding to $(C, D, FC \xrightarrow{h} D)$ are C and D , respectively, and for $t : A \to X$, that is, $t \in (F\hat{A})(X)$, the value $\Phi t : \Phi A \to \Phi X$, i.e., $\Phi t : C(A) \to D(X)$, is the map

$$(a \in C(A)) \longmapsto (h \circ F\hat{a})_X(t) \ .$$

Of course, these determinations generalize the ones going into the proof of (1).

Further, Lawvere points out a special property of the categories $\boldsymbol{A}^*_F\boldsymbol{X}$ thus constructed. A category is called *one-way* (see [22]) if its endomorphism monoids are trivial (the only element of $\hom(X, X)$ is 1_X). The skeletal one-way categories are those without circuits: without positive-length composable paths of non-identity arrows starting and ending in the same object. If $\boldsymbol{X}$ is a finite skeletal one-way category, then for any $X \in \boldsymbol{X}$, we can set $\ell(X)$, the level of X , the maximal integer n such that there is a length-n composable path starting in

X of non-identity arrows. It is clear that all arrows go from an object to a lower level object. Conversely, if the set of objects of a category is partitioned into levels so that all arrows go from an object to a lower level object, then the category is clearly skeletal and one-way. Let the *height* of the category be the maximal level of any object in it.

In Lawvere's construction, if the categories $\mathbf{A}$ and $\mathbf{X}$ are one-way, then so is $\mathbf{A}^*_F\mathbf{X}$; the property "skeletal" is similarly inherited. In fact, if F is finite meaning that each $(F\hat{A})(X)$ ($A\in\mathbf{A}$, $X\in\mathbf{X}$) is a finite set, and if $\mathbf{A}$ and $\mathbf{X}$ are both finite, skeletal and one-way, then so is $\mathbf{A}^*_F\mathbf{X}$, and the height of $\mathbf{A}^*_F\mathbf{X}$ is less than or equal to the sum of heights of $\mathbf{A}$ and $\mathbf{X}$.

As was pointed out in (11), the construction of the sketch-category $\mathtt{Set}^{\mathbf{X}}\|\mathbf{K}$ corresponds to the construction $\mathbf{A}^*_F\mathbf{X}$ with $\mathbf{A} = |\mathbf{K}|$ a discrete category. Thus, every finite sketch-category built up by using only the construction $\mathbf{G}\|\mathbf{K}$ is of the form $\mathtt{Set}^{\mathbf{X}}$ for a finite, skeletal one-way category $\mathbf{X}$. Conversely, every such $\mathtt{Set}^{\mathbf{X}}$ is a finite sketch-category via the $\|$ construction starting with the terminal category $\mathbf{1}$; this is shown by an obvious induction on the height of $\mathbf{X}$. In conclusion: the finite sketch-categories via the $\|$ construction are precisely the finite toposes with a finite, skeletal, one-way exponent.

Note finally that $\mathtt{Graph}$, $\mathtt{2\text{-}Graph}$ and indeed, $n\text{-}\mathtt{Graph}$ in general are one-way toposes, finite toposes with one-way exponent categories.