Introduction

A completeness theorem asserts the equality of the formal deducibility relation and the semantic consequence relation in a particular logic. Algebraic Logic, and in particular, Categorical Logic replaces a completeness theorem by a representation theorem. Witness the examples of the Stone representation theorem for Boolean algebras, which is the algebraic equivalent of the (strong) completeness theorem for classical propositional logic; the representation theorems for cylindric [14] and polyadic [12] algebras that correspond similarly to the Gödel-Mal'cev completeness theorem for Classical First Order Logic (CFOL); or the categorical representation theorems, also to be mentioned below, similarly corresponding to the same completeness theorem and to other completeness theorems in logic. A common element in these correspondences is the fact that the algebraic and the logical formulations are *translation-equivalent*; given either the algebraic or the logical formulation, the other can be deduced by a formal argument that is a routine translation, all whose elements are in view already when the algebraic version of the logic is introduced. In particular, it (the formal argument) can be carried out in a weak metatheory, not going, in strength, beyond a fragment of first order arithmetic.

Thus, categorical logic already has a framework for expressing completeness theorems. However, it can be legitimately argued that completeness-as-representation does not express the full intended meaning of completeness. The fact that truth is carried adequately by finite combinatorial objects, namely the formal deductions, has, apparently, disappeared in the statement of the representation theorem.

The Abstract Completeness Theorem (ACT) for CFOL, saying that the validities in CFOL form a recursively enumerable set, is an expression of the last-stated fact in an abstract form. Now, the ACT is *not apparent* on the form of either of the representation theorems related to CFOL as in fact it is on that of any of the usual variants of the completeness theorem. The fact remains that the ACT *can be derived* from representation; however, this derivation, going through the translation to completeness expressed in the usual syntax of CFOL, requires stepping out of the framework of categorical logic; apparently, categorical logic is *not sufficient* in itself for expressing the ACT.

To rectify this situation, what is needed is a *syntax directly related* to categorical logic, which provides an adequate notion of *formal deduction*. Such a syntax is described in this paper, not

just for CFOL, but in a uniform way, for a whole family of *doctrines*, logics in a categorical formulation. As a result, we will obtain an analysis of the idea of completeness, applicable, again in a uniform manner, to several logics such as intuitionistic and modal logics, and generalized quantifiers. The question whether the ACT is valid in a given situation becomes a precise question, provided the situation has been given a syntactic formulation as described here.

The syntax to be given is a natural codification of the idea of *diagram manipulation*, the informal idea of "syntax" for category theory. The syntax is based on a generalization of C. Ehresmann's concept of *sketch* [7], [11], [20], [10], [34].

Three things are done with the concept of sketch in this paper. One is a simple but sweeping generalization of the concept, the effect of which is that sketches become instrumental in essentially any (categorical) doctrine, not just ones based on limits and colimits as with the original, standard, notion of sketch. The second is the use of sketches in the *specification* of doctrines via *sketch-axioms*, which are *morphisms of sketches*. Finally, the third is the formulation of a proof-theory, specifically a notion of formal deduction. The new deductions deduce *sketch-entailments*, morphisms of sketches; the sketch-axioms are particular sketch-entailments. The new notion of deducibility serves to formulate a *General Completeness Theorem* (GCT), valid in any sketch-specified doctrine.

The "real", or *Specific Completeness Theorems* (SCT's) result from the combination of Representation Theorems (RT's) and the appropriate GCT. Each RT depends not just on the doctrine at hand, but also the choice of the *representing objects* in the doctrine. For instance, in the doctrine of Intuitionistic First Order Logic, we have the RT in which the representing objects are the presheaf categories (this is related to Kripke's completeness theorem [19]), and another one in which the representing objects are the categories of sheaves over complete Heyting algebras (compare [32], [33]). One form of the RT related to the standard completeness theorem for CFOL refers to the doctrine of coherent categories, and `Set` , the category of sets, as the (single) representing object in it. Accordingly, a SCT refers to a doctrine as well as a choice of *semantical objects* (as the "representing objects" can alternatively be called) in it; the GCT refers to a doctrine alone. Note that the role of representation theorems, the hallmark of Algebraic Logic, remains integral in the theory.

An *Abstract Completeness Theorem* (ACT) will express the property of a class of objects (sometimes a single object) within a doctrine, namely the fact that the *exactness properties* of

the class for the operations codified in the doctrine form a recursively enumerable set. The ACT of this paper formulated for the categorical doctrine of first order logic, and the distinguished object the category of sets, is translation equivalent to the usual ACT for CFOL.

There is an adequate expression of *compactness* in this context as well; see §8.

Next, I give a more detailed account of the basic ideas of the paper, and their connections with the literature. The remarks that follow will perhaps be more easily understood after having seen the basic concepts in the first three sections.

According to Ehresmann, a sketch is a graph with certain cones and cocones distinguished in it. A simple-minded generalization is to allow arbitrary types of distinguished diagrams in place of cones and cocones. The intention is, as with Ehresmann, to associate certain definite qualities with the distinguished diagrams; in Ehresmann's case these are "limit cone" and "colimit cocone". In the case of Ehresmann's sketches, this intention is realized by the fixing of the intended semantics, which stipulates that a *model* of a sketch, a graph map from the underlying category of the sketch into a category, should turn the distinguished cones and cocones into *limit* cones and *colimit* cocones, respectively.

The Ehresmannian semantics has an obvious and potentially vast extension to the generalized sketches, by associating essentially arbitrary qualities of diagrams in categories with the distinguished diagrams, the only restriction being that the quality associated with a diagram has to match the shape of the diagram. For example, in Cartesian closed categories, the operation of exponentiation involves a kind of diagram that contains all the data for any particular exponential, in much the same way as a limit cone contains the necessary data for a particular instance of limit; since a product is involved in the exponential, the data for that product are part of the diagram. Thus, we have a corresponding kind of (generalized) sketch *for* Cartesian closed categories. In this kind of semantic determination, it is not necessary to limit ourselves to qualities given by universal properties; the target categories may need to have additional structure to support the semantics.

This paper adopts the above-described simple-minded extension of the notion of sketch. However, it restricts the possible semantics to a certain "internal" form, in contrast to the "purely external", and in principle arbitrary, determination in Ehresmann's style of the qualities talked about above.

C. Lair's *trames* (webs) [21] are generalized sketches, for the purposes of dealing with categorical operations defined by universal properties. These "adjoint operations", F. W. Lawvere's most important contribution to the foundational applications of category theory, are defined in a hierarchical fashion, one operation serving as a basis for another's adjoint definition. An example is the above-mentioned exponentiation, which uses binary product in its definition. Lair's *trames* are explicitly built in a hierarchical fashion conforming to the character of the operations they intend to specify. They represent a substantial generalization of the notion of sketch, sufficient for the specification of all adjoint operations. In relation to the (generalized) sketches of this paper, they are "special purpose" sketches. They have a specific form, among others with a negative part and a positive part distinguished, corresponding to "left-adjoint" and "right-adjoint" type operations. The present paper's generalized sketches are more general and more simple-minded; once again, in this paper the weight of the responsibility for specificity is placed on sketch-semantics in a new sense, rather than on the notion of sketch.

In the examples, this paper also adopts the hierarchical aspect of sketch-building, inasmuch sketches for Cartesian closed categories in this paper are based on sketches for Cartesian categories. However, in the general concept there is no need to carry along the hierarchical character.

The semantics of *trames* is of the same "external" kind as that of Ehresmann's sketches.

Another notion of generalized sketch is C. Wells' *forms* [35]. A form is derived from a pre-supposed specification of any particular doctrine (kind of structured category); this specification is assumed to be in the form of a finite limit sketch. Thus, if one has a finite limit sketch whose models are Cartesian closed categories (which does exist for the notion of Cartesian closed category with specified operations and morphisms preserving the operations on the nose), then Wells' theory gives a notion of "Cartesian closed form"; any one Cartesian closed form specifies (is the presentation of) a particular Cartesian closed category.

The chief difference between the sketches of this paper and the forms of Wells is that the former arise directly from the common definition of the doctrine considered, whereas the second needs the mediation of the finite-limit specification of the doctrine. Wells' forms use the language of finite-limit sketches as a specification language; this paper's sketches use a direct, and simple, specification language. Another difference is that the Wells forms apply only to doctrines that are finite-limit specifiable. The doctrine of Cartesian closed categories,

with morphisms preserving the operations only up to isomorphism, is not finite-limit specifiable. The present paper's approach is not affected by this fact, and it applies equally well to the two versions of the doctrine of Cartesian closed categories.

As already indicated, the main idea of the paper is a *sketch-semantics* in a new sense. This is given by the well-known notion of relative injectivity (an object being injective relative to an arrow, or a set of arrows; see [1]) in an arbitrary category. The category in this case is the category of all sketches of a certain particular kind. Thus, we are talking about a sketch S satisfying an arrow r of sketches, the latter pictured as an entailment, in the sense of "the domain situation entailing the codomain situation". The definition is that $S \models r$ iff all arrows from the domain of r to S factor through r; that is, iff S is injective relative to r.

The novelty is certainly not in the general form of the semantics, which is injectivity. A strong form of injectivity, orthogonality ("factors through a unique arrow" above) appears already in [9] as a tool of semantics; e.g., the sheaves with respect to a given topology are selected by orthogonality conditions from among the pre-sheaves. For a history of orthogonality and injectivity, see the historical remarks to Chapters 1 and 4 in [1]. As another example, H. Andréka and I. Németi [2] use cone-injectivity (a generalized form of injectivity) as a formal tool for semantics, in their case explicitly for the purposes of first order logic. However, Andréka's and Németi's application of injectivity and the ones in this paper take place in entirely different categories. In the Andréka-Németi case, the application takes place in a category containing the category of models of a first order theory as a subcategory. In this paper, it takes place in the category of sketches. Note that sketches correspond to theories in the conceptual framework of logic. Thus, we have a semantics, where *theories* themselves are being models of statements that are *morphisms of sketches* that is, morphisms of theories.

Sketch-axioms (sketch-entailments used as axioms) are used to specify a doctrine in the form as the full subcategory of sketches satisfying the sketch-axioms. Thus the structured categories that are members of the doctrine appear as sketches and they are selected by the axioms. Semantics in Ehresmann's sense, the concept of a model of a sketch, is recaptured simply in the form of a morphism of the sketch into a member of the doctrine.

I am not aware of the existence in the literature of an anticipation of the idea of using the category of sketches in the manner just described.

As was said above, sketch-semantics represents an internalization, and a consequent restriction,

of a potentially arbitrary type of external sketch-semantics. It is rather clear that arbitrary external sketch-semantics as was indicated above will not be capable of being formulated in the sketch-semantics of this paper. The bulk of the paper consists in the demonstration, in individual examples and classes of examples, that, on the other hand, sketch-semantics is adequate for apparently all of the usual categorical doctrines, including ones that involve operations not defined by universal properties, such as monoidal categories. The fact that the sketch-semantic specifications are available for the doctrines in question is not entirely immediate. E.g., the necessary presence of the rule (sketch-entailment) of the form `IsoTrans` for the specification of adjoint operations (see §4) was a surprise to me, and the treatment of monoidal categories (with not necessarily strict monoidal functors as morphisms!) and similar doctrines requires a new, mathematically interesting way of looking at these doctrines.

As indicated above, the main contribution of the paper is a syntactic calculus, providing an adequate (sound and complete) formalization of sketch-semantics. Although I had the basic ideas of the syntax of sketches for some years, the impetus to work them out was provided by Charles Wells's talk at the October '92 Montreal Category Meeting about a sketch-theoretic approach to logic aiming at proof theory. An extended abstract [5] of the projected work, "Graph-based logic and logic" by him and A. Bagchi, was made available on the Category Theory Network. It seems to me that the approach of the present paper is substantially different from that of Bagchi and Wells, which is based on Wells' forms mentioned above.

The proof of the General Completeness Theorem, expressing the adequacy of the deductive calculus, combines the classical construction injective hulls in a general setting, with a finiteness argument that exploits the specifics of the construction. The construction itself is, needless to say, far from being new. E.g., it appears in M. Barr's [4a], where the author refers to Grothendieck's construction of enough injectives in AB5 categories as the source for his own proof.

The technical aspects of this paper are rather minor. On the other hand, I am struck by the conceptual importance of the scheme, which shows, to me at least, that categorical logic is, to a great degree, autonomous, even in matters *syntactical*. I believe that in the literature there is no *general* theory of completeness in logic (not just *categorical* logic) comparable to the present one in scope and conceptual unity.

The paper [29] is a summary of related material. The present paper is a working out of certain

of the details of sections 1, 2, 3 and 5 of [29]. [29] contains ideas for work in proof theory proper in the context of the sketch-based syntax. Another paper [30] is in preparation, which, within the framework of the present one, gives an algebraic theory of doctrines, concentrating on their bicategorical aspects.

When item m. in Section n. is referred to in another section, I use the notation n.m.

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