§9. Sketch-semantics versus Tarskian semantics

In this section, I explain how Tarski-type semantics can be related to sketch-semantics. This may contribute to a better understanding of sketch-semantics. We will see that compactness in the usual sense for a generalized quantifier with a specific semantics is equivalent to compactness in the sense of the previous section for appropriately chosen parameters. In the case when we have axiomatizability (as for instance we do for the case of the "there are uncountably many" quantifier), we will establish the existence of a recursive translation in both directions between deductions in the usual sense and in the sketch-based sense. We will stop short of spelling out an explicit translation of one kind of deduction into the other, although it would not be difficult to do it.

For specificity, let us consider (full, finitary, possibly multi-sorted) first order logic with (sorted) equality, with a new quantifier $\mathcal{F}x$ added. Thus, formulas are formed as in first order logic, together with a new formation-rule, allowing $\mathcal{F}x\varphi$ as a formula for any formula φ. For specificity, we consider $\mathbf{t}$ (true; 0-ary operator), $\wedge$, $\neg$ and $\exists$ as primitive, and the other first-order operators as abbreviations in the usual way. The notions of free and bounded variables, alphabetic variants, proper substitution and the like remain essentially the same as in ordinary first order logic; the $\mathcal{F}$-quantifier is handled syntactically just as $\exists x$ and $\forall x$ are.

A *semantics* of the $\mathcal{F}$ quantifier is specified by giving a class $\mathcal{S}$ of pairs (X, A) of sets, with A being a subset of X, subject to the condition of *invariance under isomorphism*: whenever $f: X \xrightarrow{\cong} Y$ and $(X, A) \in \mathcal{S}$, then $(Y, f[A]) \in \mathcal{S}$. We write $\mathcal{S}[X] \underset{\text{def}}{=} \{A \subset X : (X, A) \in \mathcal{S}\}$.

We have the example, with any fixed cardinal κ, of the semantics $\mathcal{S}_\kappa$ consisting of those (X, A) in which the cardinality of A is less than κ. With $\kappa = \aleph_1$, this is the semantics we had in mind in §5 in connection with "$\mathcal{F}$-categories". A "dual" semantics $\mathcal{S}$ would have $(X, A) \in \mathcal{S}$ iff $\#(X - A) < \kappa$; still another dual would have $(X, A) \in \mathcal{S}$ iff $\#A \geq \kappa$. Both of these latter quantifiers are expressible using $\mathcal{F}$ in the first sense.

More "interesting" quantifiers can be introduced, by allowing the quantifier to bound more than one variable at a time, or even allowing quantification of variables ranging over subsets of the universe of the model, etc; see [4b]. The translations and conclusions given below have

counterparts in the more involved situations as well.

Given a semantics $\mathcal{S}$ as described, we can evaluate formulas in any structure. From now on, we consider $\mathcal{S}$ fixed.

Let L be a language in the usual sense, that is, a collection of sorts, and sorted relation-symbols; for the sake of simplicity of exposition, we disallow operation-symbols. The notion of L-structure is the usual one. $\vec{x}$, $\vec{y}$, ... always denote a proper variable sequence, that is, a finite sequence of *distinct*, sorted, variables. $\vec{x}$ is *of type* $\vec{X}$ if $\vec{x} = \langle x_i \rangle_{1 \le i \le n}$, $\vec{X} = \langle X_i \rangle_{1 \le i \le n}$ and $x_i : X_i$ (X_i is of sort x_i).

An entity $[\vec{x} : \varphi]$ (actually, simply an ordered pair $(\vec{x}, \varphi)$), where φ is a formula with all free variables in $\vec{x}$, is called an (L-)*formal set*, or L-*set*. Two formal sets $[\vec{x} : \varphi]$ and $[\vec{y} : \psi]$ are identified if $\vec{x}$, $\vec{y}$ are of the same type, and $\varphi[\vec{y}/\vec{x}]$, the result of properly (possibly renaming bound variables to avoid clashes) substituting $\vec{y}$ for $\vec{x}$, is an alphabetic variant of ψ (is obtained by properly renaming bound variables). The *type of* $[\vec{x} : \varphi]$ is the type of $\vec{x}$.

Given any L-structure M, $M[\vec{x}]$ denotes the Cartesian product $\prod_{i=1}^{n} M(X_i)$, where $\vec{x}$ is of type $\vec{X} = \langle X_i \rangle_{1 \le i \le n}$. The semantics $\mathcal{S}$ provides the definition $M[\vec{x} : \varphi]$, or $M_{(\mathcal{S})}[\vec{x} : \varphi]$, the ($\mathcal{S}$-)*meaning* of $[\vec{x} : \varphi]$ in M, a subset of $M[\vec{x}]$, by recursion on the structure of φ. The clauses with respect to the atomic formulas and the first-order logical operators are the usual ones. The specific clause related to $\mathcal{S}$ is this:

$$\vec{a} \in M[\vec{x} : \mathcal{F} y \varphi] \iff \{b : \vec{a}b \in M[\vec{x}y : \varphi]\} \in \mathcal{S}[M[\vec{x}]]$$

(here, it is assumed that y is not among $\vec{x}$; because of the possibility of renaming bound variables, this is not a real restriction; of course, $\vec{a}b$ denotes the concatenation $\vec{a}^\frown\langle b \rangle$, and similarly for $\vec{x}y$). We say that M is an $\mathcal{S}$-*model* of the sentence σ, $M \models_{\mathcal{S}} \sigma$, or $M \models \sigma$ when $\mathcal{S}$ is understood, if $M[\emptyset : \sigma] = M[\emptyset : \mathbf{t}] = 1 = \mathbf{true}$. Models of T form a groupoid

(category) $\mathtt{Mod}(T)$ (or $\mathtt{Mod}_{(\mathcal{S})}(T)$), with morphisms the *iso*morphisms of $\mathtt{L}_T$-structures; we ignore all morphisms but the isomorphisms.

We consider $\mathcal{F}$-sketches, objects in the sketch-category $\mathcal{F}\mathtt{Sk}$ defined in §6; of course, $\mathcal{F}\mathtt{Sk}$ is independent of the semantics $\mathcal{S}$. However, $\mathcal{S}$ gives rise to a specific object $\mathtt{Set}_{(\mathcal{S})}$ in $\mathcal{F}\mathtt{Sk}$. As far as the specification-types for $\mathtt{BoolSk}$ (= $\mathtt{cohSk}$) are concerned, $\mathtt{Set}_{(\mathcal{S})}$ is given as the Boolean category $\mathtt{Set}$. For the specification-type $\hat{\mathcal{F}}$ (see §6), we set

$(\Phi:\hat{\mathcal{F}}\longrightarrow\mathtt{Set})\in\hat{\mathcal{F}}[\mathtt{Set}_{(\mathcal{S})}]$ iff for all $y\in\Phi Y$,

$$\exists y'\in\Phi B.\,\Phi b y'=y. \iff \{x\in\Phi X:\,\exists x'\in\Phi A.\,\Phi a x'=x \,\&\, \Phi f x=y.\}\in\mathcal{S}[\Phi X]$$

(when Φa and Φb are inclusions of sets, which may be assumed without essential loss of generality, this becomes

$$y\in\Phi B \iff \{x\in\Phi A:\Phi f x=y\}\in\mathcal{S}[\Phi X]\ ).$$

For any (small) $\mathcal{F}$-sketch S, the collection of sketch-maps $S\to Set_{(\mathcal{S})}$, the $\mathcal{S}$-*models* of S, form a groupoid $\mathtt{Mod}(S)$ (or more explicitly, $\mathtt{Mod}_{(\mathcal{S})}(S)$), where a morphism $h:M\longrightarrow N$ is, by definition, a (natural) *iso*morphism $h:M\xrightarrow{\cong}N$ between M and N as maps from the underlying graph $\|S\|$ to the category $\mathtt{Set}$ (thus, we ignore all but the isomorphism mappings of models).

For L a language, an L-*fragment* F is a set of L-sets such that

$[\vec{x}:\exists y\varphi]\in F \Longrightarrow [\vec{x}y:\varphi]\in F$,

$[\vec{x}:\varphi\wedge\psi]\in F \Longrightarrow [\vec{x}:\varphi]\in F \,\&\, [\vec{x}:\psi]\in F$,

$[\vec{x}:\neg\varphi]\in F \Longrightarrow [\vec{x}:\varphi]\in F$,

$[xx': x=_X x']\in F$ for all sorts X ($x, x':X$),

and

$[\emptyset:\mathbf{t}]\in F$.

An *augmented theory* is a triple $T=(L, F, \Sigma)=(\mathtt{L}_T, \mathtt{F}_T, \Sigma_T)$, with L a language, F an L-fragment, Σ a set of L-sentences such that for $\sigma\in\Sigma$, $[\emptyset:\sigma]\in F$. (In contrast, a *simple theory* is a pair (L, Σ) of a language L and a set Σ of sentences; in what follows, until

further notice, theories are augmented). A sequence $\vec{X}=\langle X_i\rangle_{1\le i\le n}$ of sorts *occurs* in T if either L_T has a relation-symbol R sorted as "$R \rightarrowtail X_1\times\ldots\times X_n$", or if F_T contains a formal set $[\vec{x}:\varphi]$ of type $\vec{X}$. A *model* of T is a model of the corresponding simple theory; the notation $\mathsf{Mod}(T)$ is used accordingly.

With any augmented theory T, we associate an $\mathcal{F}$-sketch $S = \mathsf{S}[T]$; S consists of the following data (1.1) to (1.12).

(1.1) For every sort $X\in L$, an object X in S.

(1.2) For every $\vec{X}=\langle X_i\rangle_{1\le i\le n}$ occurring in T, an object denoted $\vec{X}$, or $\prod_{i=1}^{n} X_i$, together with arrows $\pi_i : \prod_{i=1}^{n} X_i \longrightarrow X_i$ ($1\le i\le n$), and the specification to the effect that $\langle \pi_i\rangle_i$ "is a product diagram". When $\vec{x}$ is type $\vec{X}$, $[\vec{x}]$ means $\vec{X}$.

(1.3) For every relation-symbol "$R \rightarrowtail X_1\times\ldots\times X_n$" in L, an object R and an arrow $R \overset{\mathsf{m}_R}{\rightarrowtail} \prod_{i=1}^{n} X_i$ in S, together with the specification that m_R "is a monomorphism".

(1.4) For every formal set $[\vec{x}:\varphi]$ in F, an object $[\vec{x}:\varphi]$ and a "monomorphism" $\mathsf{m}_{[\vec{x}:\varphi]} : [\vec{x}:\varphi] \longrightarrow [\vec{x}]$.

(1.5) When $\Phi = [\vec{x}:\varphi] = [xx' : x=_X x']$, the specifications to the effect that $\mathsf{m}_\Phi : \Phi \rightarrowtail X\times X$ "is the diagonal".

(1.6) When $\Phi = [\vec{x}:\varphi]$ with φ the atomic formula $Ru_1\ldots u_k$, with "$R \longrightarrow U_1\times\ldots\times U_k$", $\vec{x}=\langle x_i\rangle_{1\le i\le n}$, $x_i : X_i$, $U_j = X_{i_j}$, $u_j = x_{i_j}$, a "pullback diagram"

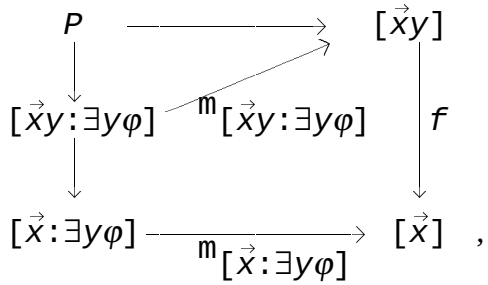
$$\begin{array}{ccc} \Phi & \overset{\mathsf{m}_\Phi}{\rightarrowtail} & [\vec{x}] \\ \downarrow & & \downarrow \pi \\ R & \underset{\mathsf{m}_R}{\rightarrowtail} & \vec{U} \end{array}$$

together with the specification that π "is the morphism

$$\langle \pi_{i_j} \rangle_j : \prod_{i=1}^{n} X_i \longrightarrow \prod_{j=1}^{k} U_j \ ,$$

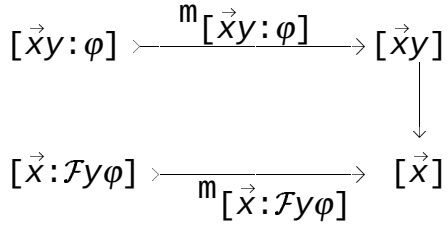
for the projections $\pi_{i_j} : \prod_{i=1}^{n} X_i \longrightarrow X_{i_j}$ ".

(1.7) For every formal set of the form $[\vec{x}:\exists y\varphi]$ in F_T, a diagram



together with the specification that the right-most vertical "is a product projection", and the data (in the form of the obvious map $\hat{\exists} \to |S|$ being an element of $\hat{\exists}[S]$) that the diagram is a $\exists$-specification.

(1.8) For every formal set of the form $[\vec{x}:\mathcal{F}y\varphi]$ in F_T, the diagram



understood as a map $\hat{\mathcal{F}} \to |S|$ in the obvious way is in $\hat{\mathcal{F}}[S]$.

(1.9), (1.10) and (1.11): Data explaining $[\vec{x}:\mathbf{t}]$, and ones relating $[\vec{x}:\varphi\wedge\psi]$ and $[\vec{x}:\neg\varphi]$ to $[\vec{x}:\varphi]$, $[\vec{x}:\psi]$ in the style of (1.7) and (1.8).

(1.12) For every $\tau\in\Sigma$, the specification that $\mathsf{m}_{[\emptyset:\tau]}:[\emptyset:\tau] \rightarrowtail [\emptyset:\mathbf{t}]$ "is an isomorphism".

Note that if T is finite (meaning that L_T , F_T , Σ_T are all finite), then $\mathsf{S}[T]$ is a finite sketch. Moreover, the map $T \mapsto \mathsf{S}[T]$, restricted to hf augmented theories T , is a recursive one.

We have a close connection between models of T and models of $\mathtt{S}[T]$. Every model M of T gives rise to a model $\hat{M}:\mathtt{S}[T]\to\mathtt{Set}_{(\mathcal{S})}$, for which

$\hat{M}X$ is MX for sorts X,

$\hat{M}(\prod_{i=1}^{n} X_i)$ is chosen to be the standard Cartesian product $\prod_{i=1}^{n} MX_i$ for the products $\prod_{i=1}^{n} X_i$ under (1.2),

for "$R \rightarrowtail \prod_{i=1}^{n} X_i$", $\hat{M}R$ is MR ($\subset \prod_{i=1}^{n} MX_i$), and $\hat{M}(\mathtt{m}_R)$ is the inclusion $MR \rightarrowtail \prod_{i=1}^{n} MX_i$,

each $\hat{M}[\vec{x}:\varphi]$ is chosen to be equal to $M[\vec{x}:\varphi]$,

and the rest of the items for $\hat{M}$ also are chosen in the "standard" ways.

This map $M \longmapsto \hat{M}$ extends to a functor

$$\mathtt{Mod}(T) \longrightarrow \mathtt{Mod}(\mathtt{S}[T]). \qquad (2)$$

To see this, we first characterize (iso)morphisms in $\mathtt{Mod}(\mathtt{S}[T])$.

(3) *Let* I, $J \in \mathtt{Mod}(\mathtt{S}[T])$. *Assume* $\langle k_X : IX \xrightarrow{\cong} JX\rangle_X$ *is a family of bijections indexed by the sorts in* $\mathtt{L}_T$ *such that, for every* "$R \rightarrowtail X_1\times\ldots\times X_n$" *in* L, *there is a (necessarily unique)* k_R *making*

$$\begin{array}{ccccc} IR & \xrightarrow{\mathtt{Im}_R} & I(\prod_{i=1}^{n} X_i) & \cong & \prod_{i=1}^{n} IX_i \\ {\scriptstyle k_R}\downarrow & \circ & \downarrow & \circ & \downarrow{\scriptstyle \prod_{i=1}^{n} k_{X_i}} \\ JR & \xrightarrow[\mathtt{Jm}_R]{} & J(\prod_{i=1}^{n} X_i) & \cong & \prod_{i=1}^{n} JX_i \end{array} \qquad (4)$$

commute. Then there is a unique $h:I \xrightarrow{\cong} J$ *for which* $h_X = k_X$ *for all* $X \in \mathtt{Sort}[\mathtt{L}_T]$.

(Clearly, the condition on $\langle k_X \rangle_X$ is necessary for the existence of such h.)

Indeed, we define $h_X = k_X$, $h_{\vec{X}} = \prod_{i=1}^{n} k_{X_i}$, $h_R = k_R$ from (4). Next, we define the components $h_{[\vec{x}:\varphi]} : I[\vec{x}:\varphi] \xrightarrow{\cong} J[\vec{x}:\varphi]$ by recursion on the structure of $[\vec{x}:\varphi]$ so that

$$\begin{array}{ccc} I[\vec{x}:\varphi] & \xrightarrow{Im_{[\vec{x}:\varphi]}} & I(\prod_{i=1}^{n} X_i) \cong \prod_{i=1}^{n} IX_i \\ \downarrow{\scriptstyle k_R} & \circ & \downarrow \quad \circ \quad \downarrow \prod_{i=1}^{n} k_{X_i} \\ J[\vec{x}:\varphi] & \xrightarrow[Jm_{[\vec{x}:\varphi]}]{} & J(\prod_{i=1}^{n} X_i) \cong \prod_{i=1}^{n} JX_i \end{array} \quad ;$$

for this, we use the items (1.7) to (1.11), and the fact that the diagrams in those items are mapped by I and J into such diagrams in $\mathtt{Set}$ with known properties. Note especially that the recursion-step for the case of $[\vec{x}:\mathcal{F}y\varphi]$ uses the assumption of isomorphism invariance of $\mathcal{S}$. Finally, we need to define the components of h at certain auxiliary objects that come up in items (1.7) to (1.11), e.g. the P in (1.7); these are also uniquely determined under the naturality requirement.

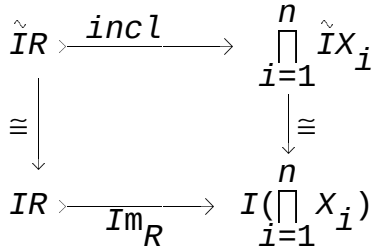
To define the functor (2), we note that if $h : M \xrightarrow{\cong} N$, we can put $k_X = h_X$ and have that each diagram (4) commutes for suitable k_R since it is the same as

$$\begin{array}{ccc} MR & \xrightarrow{incl} & \prod_{i=1}^{n} MX_i \\ \downarrow{\scriptstyle h_R} & \circ & \downarrow \prod_{i=1}^{n} h_{X_i} \\ NR & \xrightarrow[incl]{} & \prod_{i=1}^{n} NX_i \end{array} \quad .$$

$\hat{h} : \hat{M} \longrightarrow \hat{N}$ is given by (3) such that $\hat{h}_X = h_X$ for sorts X.

We have that the functor (2) is an equivalence of categories. In fact, we have that every model $I : T \to \mathtt{Set}_{(\mathcal{S})}$ gives rise to the $\mathtt{L}_T$-structure $\hat{\hat{I}}$, for which $\hat{\hat{I}}X = IX$, and for

" $R \rightarrowtail \prod_{i=1}^{n} X_i$ ", $\tilde{I}R$ is defined by the diagram



in $\mathbf{Set}$ where the right-hand vertical arrow in the canonical bijection between a standard Cartesian product and another product in $\mathbf{Set}$. Moreover, $\tilde{I}$ is a model of T, and $\iota : (\tilde{I})\hat{} \cong I$ by a unique isomorphism ι whose components ι_X, X a sort, are identities; the unique existence of ι is ensured by (3).

Let now T be a simple theory $T=(\mathbf{L}_T, \Sigma_T)$, and σ an $\mathbf{L}_T$-sentence. As usual, we write $T \models_{\mathcal{S}} \sigma$ for saying that σ is a consequence of T under the $\mathcal{S}$-semantics, that is, every $\mathcal{S}$-model M of T is an $\mathcal{S}$-model of σ.

Consider the augmented theory $T|\sigma \stackrel{\text{def}}{=} (\mathbf{L}_T, F, \Sigma_T)$, where F is the least fragment for which $[\emptyset : \tau] \in F$ for all $\tau \in \Sigma$, and $[\emptyset : \sigma] \in F$; let $T\|\sigma \stackrel{\text{def}}{=} (\mathbf{L}_T, F, \Sigma_T \cup \{\sigma\})$ with the same F. Consider the sketches $\mathbf{S}[T|\sigma]$, $\mathbf{S}[T\|\sigma]$. Note that the second is obtained from the first by adding a single isomorphism specification, specifying that $\mathbf{m}_{[\emptyset:\sigma]} : [\emptyset:\sigma] \longrightarrow [\emptyset:\mathbf{t}]$ is an isomorphism. Let $\alpha_{T,\sigma}$ be the inclusion $\mathbf{S}[T|\sigma] \longrightarrow \mathbf{S}[T\|\sigma]$. Note that $\alpha_{T,\sigma}$ is an $\{\ulcorner\mathbf{Iso}\urcorner\}$-statement. I claim that

$$T \models_{\mathcal{S}} \sigma \iff \mathbf{Set}_{(\mathcal{S})} \models \alpha_{T,\sigma}. \qquad (5)$$

First of all, we note that for $I \in \mathbf{Mod}(\mathbf{S}[T|\sigma])$ to have $\mathbf{Set}_{(\mathcal{S})} \models_I \alpha_{T,\sigma}$ is the same as to have that $I(\mathbf{m}_{[\emptyset:\sigma]})$ is an isomorphism. It is clear that this property is invariant under isomorphism in $\mathbf{Mod}(\mathbf{S}[T|\sigma])$; if $I \cong J$, and $\mathbf{Set}_{(\mathcal{S})} \models_I \alpha_{T,\sigma}$, then $\mathbf{Set}_{(\mathcal{S})} \models_J \alpha_{T,\sigma}$. Now, assume that $T \models_{\mathcal{S}} \sigma$, and $I \in \mathbf{Mod}(\mathbf{S}[T|\sigma])$. Let $M = \tilde{I}$. Since $M \models T$, we have that $M \models \sigma$. Consider $J = \hat{M}$. By the definition of $\hat{M}$, and $M \models \sigma$, we have that $J(\mathbf{m}_{[\emptyset:\sigma]})$ is an isomorphism, that is, $\mathbf{Set}_{(\mathcal{S})} \models_J \alpha_{T,\sigma}$. But, $J \cong I$, hence also

$\mathtt{Set}_{(\mathcal{S})} \models_I \alpha_{T,\sigma}$ as was to be shown. The converse is similar and simpler.

Let us pause to consider the equivalence (5).

First of all, we have a version for pure first order logic, without the $\mathcal{F}$-quantifier; to obtain this, in the above, ignore all references to $\mathcal{F}$ and $\mathcal{S}$. We have

$$T \models \sigma \iff \mathtt{Set} \models \alpha_{T,\sigma}\ , \tag{6}$$

for any first-order theory T, sentence σ in the language of T, and the corresponding $\mathtt{BoolSk}$-sketch entailment $\alpha_{T.\sigma}$. Now, let us apply *both* the Gödel-Mal'cev completeness theorem and the sketch-based completeness theorem 8.(18) for first-order logic. First, we apply Gödel's to conclude that

$$T \vdash \sigma \iff T \models \sigma\ , \tag{7}$$

where " $T \vdash \sigma$ " means that " σ is deducible in T " in the standard formal system. Second, 8.(18) says that

$$\vdash_{\mathcal{R}[\mathtt{Bool}]} \alpha_{T,\sigma} \iff \mathtt{Set} \models \alpha_{T,\sigma}\ . \tag{8}$$

We conclude that

$$T \vdash \sigma \iff \vdash_{\mathcal{R}[\mathtt{Bool}]} \alpha_{T,\sigma}\ . \tag{9}$$

In other words, there is a deduction d of σ in T in the ordinary sense iff there is a sketch-deduction d' of $\alpha_{T,\sigma}$ from the rules $\mathcal{R}[\mathtt{Bool}]$. Given that the construction of $\alpha_{T,\sigma}$ from T and σ is effective, and both notions of deduction are effective, we find that there are recursive functions assigning a d' to any d, and another one, assigning a d to any d'; in other words,

(10) *there are recursive functions of the variables T and σ*, a finite theory and a sentence, respectively, in classical first order logic, *assigning a sketch-deduction of $\alpha_{T,\sigma}$ from $\mathcal{R}[\mathtt{Bool}]$ to any ordinary deduction of σ from T, and vice versa.*

It is possible to describe such recursive functions quite explicitly, by extending the translation process started above to deductions.
The above translation of symbolic logic into sketch-logic allows us to infer compactness or abstract completeness for symbolic logic from the same-named property (understood in the sense of the last section) for sketch-logic.

A *subtheory* of the (simple) theory T is any (simple) theory T' such that $\mathbb{L}_{T'} \subset \mathbb{L}_T$ and $\Sigma_{T'} \subset \Sigma_T$. Let us write $T' \leq T$ if this holds. For any theory T, the finite subtheories of T, ordered under $\leq$, form a directed poset. Let T and the $\mathbb{L}_T$-sentence σ be fixed, and let us write $T' \leq_\sigma T$ if $T' \leq T$, T' is finite, and σ is an $\mathbb{L}_{T'}$-sentence. For any $T' \leq_\sigma T$, we have the inclusion maps $\iota_{T'} : \mathbb{S}[T' \mid \sigma] \longrightarrow \mathbb{S}[T \mid \sigma]$, $\tilde{\iota}_{T'} : \mathbb{S}[T' \| \sigma] \longrightarrow \mathbb{S}[T \| \sigma]$. It is easy to see that

$$(11)\quad \mathbb{S}[T \mid \sigma] = \operatorname*{colim}_{T' \leq_\sigma T} \mathbb{S}[T' \mid \sigma],$$

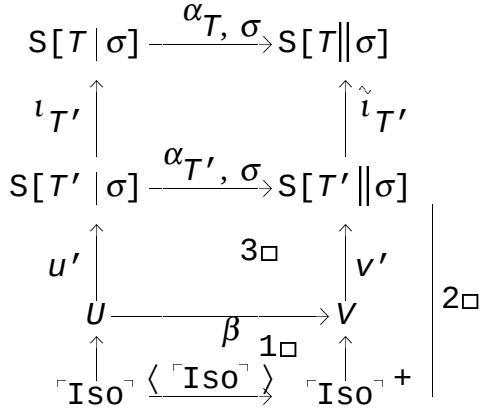
with colimit coprojections the $\iota_{T'}$, and similarly for $\mathbb{S}[T \| \sigma]$. Now, let κ be a cardinal, possibly ∞, and let us consider the property of *κ-compactness* for the $\mathcal{S}$-semantics: this holds if for any theory T of cardinality $< \kappa$ in first-order logic extended with $\mathcal{F}$, and any $\mathbb{L}_T$-sentence σ, $T \models_{\mathcal{S}} \sigma$ iff for some $T' \leq_\sigma T$ (in particular, T' is *finite*), $T' \models_{\mathcal{S}} \sigma$. We have that, for κ any uncountable regular cardinal,

(12) *κ-compactness for $\mathcal{S}$-semantics is implied by* $\boldsymbol{S}_\kappa$-$\{\ulcorner \mathbb{I}\mathrm{so} \urcorner\}$*-compactness for* $\{\mathbb{S}\mathrm{et}_{(\mathcal{S})}\}$ *in* $\mathcal{F}\mathbb{S}\mathrm{k}$.

Indeed, assume $\boldsymbol{S}_\kappa$-$\{\ulcorner \mathbb{I}\mathrm{so} \urcorner\}$-compactness for $\{\mathbb{S}\mathrm{et}_{(\mathcal{S})}\}$ in $\mathcal{F}\mathbb{S}\mathrm{k}$, and let T be a theory of cardinality $< \kappa$ such that $T \models_{\mathcal{S}} \sigma$. Then, by (5), $\mathbb{S}\mathrm{et}_{(\mathcal{S})} \models \alpha_{T,\sigma}$. $\alpha_{T,\sigma}$ is a $\boldsymbol{S}_\kappa$-$\{\ulcorner \mathbb{I}\mathrm{so} \urcorner\}$-statement. Hence, there is a finite $\{\ulcorner \mathbb{I}\mathrm{so} \urcorner\}$-approximation $\beta : U \to V$ of $\alpha_{T,\sigma}$,

$$\begin{array}{ccc} \mathbb{S}[T \mid \sigma] & \xrightarrow{\alpha_{T,\sigma}} & \mathbb{S}[T \| \sigma] \\ {\scriptstyle u}\uparrow & \square & \uparrow{\scriptstyle v} \\ U & \xrightarrow[\beta]{} & V \end{array},$$

such that $\mathrm{Set}_{(\mathcal{S})} \models \beta$. By (11), and since U is finite, there is $T' \leq_\sigma T$ such that we have the diagram



in which $\iota_{T'} \circ u' = u$ and $\tilde{\iota}_{T'} \circ v' = v$; the fact of the pushout marked 3 follows from those marked 1 and 2. We conclude that $\mathrm{Set}_{(\mathcal{S})} \models \alpha_{T',\,\sigma}$, which, by (5) again, means $T' \models_{\mathcal{S}} \sigma$ as desired.

We also have, almost immediately, from (5) that

(13) *Abstract completeness for first-order logic with the $\mathcal{S}$-quantifier is implied by abstract completeness for* $\boldsymbol{S} = \mathcal{F}\mathrm{Sk}$, $\mathcal{K} = \mathcal{K}[\{\ulcorner\mathrm{Iso}\urcorner\}]$, $\boldsymbol{P} = \boldsymbol{S}_{\aleph_0}$, *and* $\boldsymbol{C} = \{\mathrm{Set}_{(\mathcal{S})}\}$.

We also have the converses of (12) and (13) (obtained by replacing "is implied by" with "implies"). To prove these statements, we need to develop a translation of sketch-semantics into Tarskian semantics.

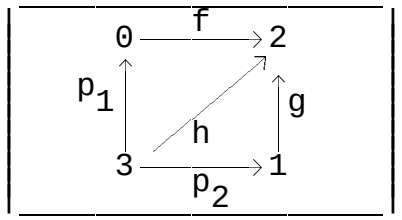
To any $\mathcal{F}\mathrm{Sk}$-sketch S, we associate a simple theory $\mathrm{T}[S] = (\mathrm{L}_S, \Sigma_S)$ such that we have an *isomorphism*

$$\mathrm{Mod}_{(\mathcal{S})}(S) \cong \mathrm{Mod}_{(\mathcal{S})}(\mathrm{T}[S]) \qquad (14)$$

of groupoids. The language L_S of $\mathrm{T}[S]$ is, by definition, the underlying graph of S; that is, the sorts of L_S are the objects of S, L_S has a unary operation symbol "$f : A \to B$"

for each arrow $f:A\to B$ of S, and L_S has no other symbols. The axioms of $T[S]$ correspond to the specifications in S; each specification of S is "expressed" by an axiom. In $\mathcal{F}\mathrm{Sk}$, we have the specification-types $\hat{t}$, $\hat{\mathrm{Pb}}$, $\ulcorner\mathrm{Iso}\urcorner$, $\ulcorner\mathrm{mon}\urcorner$, $\hat{\exists}$, $\hat{\mathrm{i}}$, $\hat{\mathrm{join}}$ and $\hat{\mathcal{F}}$. We will give the axioms corresponding to the $\hat{\mathrm{Pb}}$- and the $\hat{\mathcal{F}}$-specifications. For the rest, the reader may consult [27], pp. 90-92, where axioms are used for a similar task of "expressing".

Recall that $\hat{\mathrm{Pb}}$ is



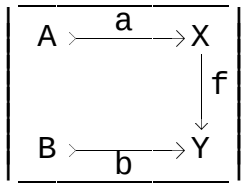
.

$\mathrm{CT}[]=\{fp_1=h, gp_2=h\}$

For each $\varphi\in\hat{\mathrm{Pb}}[S]$, we adjoin the following axiom to Σ_S (here and below, for easier reading, we write $\bar{a}$ for $\varphi(a)$ for all arguments a of φ):

$$\forall z:\bar{3}.\bar{f}\bar{p}_1 z=\bar{h}z\&\bar{g}\bar{p}_2 z=\bar{h}z.\ \&\ \forall x:\bar{0}.\forall y:\bar{1}.(\bar{f}x=\bar{g}y\to\exists!z:\bar{3}.\bar{p}_1 z=x\&\bar{p}_2 z=y.)\ .$$

Recall that $\hat{\mathcal{F}}$ is



.

$\mathrm{mon}[]=\{a,b\}$

Given $\varphi\in\hat{\mathcal{F}}[S]$, $\tilde{\mathrm{A}}x$ abbreviates the formula $\exists x'\in\bar{\mathrm{A}}.\bar{a}x'=x$, and similarly for $\tilde{\mathrm{B}}y$. For each $\varphi\in\hat{\mathcal{F}}[S]$, we adjoin the following axiom to Σ_S:

$$\forall y:\bar{\mathrm{Y}}(\tilde{\mathrm{B}}y\longleftrightarrow\mathcal{F}x:\bar{\mathrm{X}}(\tilde{\mathrm{A}}x\&\bar{f}x=y))\ .$$

Notice that both $\mathtt{Mod}_{(\mathcal{S})}(S)$ and $\mathtt{Mod}_{(\mathcal{S})}(\mathtt{T}[S])$ are subcategories of $\mathtt{Hom}(\|S\|, \mathtt{Set}) = \mathtt{Str}(\mathtt{L}_S)$; the graph-maps $\|S\| \to \mathtt{Set}$ are the same things as the $\mathtt{L}_S$-structures. Inspection shows that $\mathtt{Mod}_{(\mathcal{S})}(S)$ and $\mathtt{Mod}_{(\mathcal{S})}(\mathtt{T}[S])$ are *identical* as subcategories of $\mathtt{Hom}(\|S\|, \mathtt{Set}) = \mathtt{Str}(\mathtt{L}_S)$; that is, in (14), the isomorphism is the identity.

The $\{\ulcorner\mathtt{Iso}\urcorner\}$-statements in $\mathcal{F}\mathtt{Sk}$ are the sketch-entailments of the form $\mathtt{Inv}[S, f]: S \longrightarrow S'$, where S is any sketch, f is an arrow in S , and S' is obtained from S by adding a specification saying that " f is an isomorphism" (see §5). Let $\mathtt{Iso}[f]$ denote the $\mathtt{L}_S$-sentence $\forall b{:}B \exists !a{:}A.f(a)=b$; here, $f{:}A \to B$. Then, for any $M \in \mathtt{Mod}_{(\mathcal{S})}(S) = \mathtt{Mod}_{(\mathcal{S})}(\mathtt{T}[S])$, $\mathtt{Set}_{(\mathcal{S})} \models_M \mathtt{Inv}[S, f]$ iff $M \models \mathtt{Iso}[f]$. It immediately follows that

$$\mathtt{T}[S] \models_{\mathcal{S}} \mathtt{Iso}[f] \iff \mathtt{Set}_{(\mathcal{S})} \models \mathtt{Inv}[S, f] . \qquad (15)$$

Let us write $S' \leq_f S$ for S' is a finite sub-sketch of S and f belongs to S' . When $S' \leq_f S$, $\mathtt{Inv}[S', f]$ is a finite approximation of $\mathtt{Inv}[S, f]$ and $\mathtt{T}[S'] \leq_{\mathtt{Iso}[f]} \mathtt{T}[S]$. $\mathtt{T}[S]$ is the directed union (colimit) of the theories $\mathtt{T}[S']$ for S' ranging over all S' such that $S' \leq_f S$.

Now assume that κ-compactness holds for the $\mathcal{S}$-semantics, and $\mathtt{Set}_{(\mathcal{S})} \models \mathtt{Inv}[S, f]$. Then, by (15), $\mathtt{T}[S] \models_{\mathcal{S}} \mathtt{Iso}[f]$, and there is some $S' \leq_f S$ for which $\mathtt{T[S']} \models_{\mathcal{S}} \mathtt{Iso}[f]$, and by (15) , $\mathtt{Set}_{(\mathcal{S})} \models \mathtt{Inv}[S', f]$, which shows that

(16) *κ-compactness for $\mathcal{S}$-semantics implies $\boldsymbol{S}_{\kappa}$-$\{\ulcorner\mathtt{Iso}\urcorner\}$-compactness for $\{\mathtt{Set}_{(\mathcal{S})}\}$ in $\mathcal{F}\mathtt{Sk}$.*

(15) immediately gives that

(17) *Abstract completeness for first-order logic with the $\mathcal{S}$-quantifier implies abstract completeness for $\boldsymbol{S} = \mathcal{F}\mathtt{Sk}$, $\mathcal{K} = \mathcal{K}[\{\ulcorner\mathtt{Iso}\urcorner\}]$, $\boldsymbol{P} = \boldsymbol{S}_{\aleph_0}$, and $\boldsymbol{C} = \{\mathtt{Set}_{(\mathcal{S})}\}$.*