§8. Kinds of completeness

For our purposes here, we will have to be specific about *size-restrictions*. In our previous definitions of categories, we restricted objects to be *small*; we had in mind (as usual) a certain fixed but arbitrary Grothendieck universe $\mathcal{U}_0$, and "small" meaning "belonging to $\mathcal{U}_0$". However, we want certain standard "large" categories such as the category of sets to be members of our doctrines. Therefore, we choose ("Grothendieck's axiom" [3]) another Grothendieck universe $\mathcal{U}_1$ with $\mathcal{U}_0 \in \mathcal{U}_1$, and take each of our sketch-categories and doctrines to have their objects restricted to $\mathcal{U}_1$ rather than $\mathcal{U}_0$; after all, $\mathcal{U}_0$ was arbitrary, the notion of "small" is relative. In other words, we decide that in the previous sections, "small" should be understood as $\mathcal{U}_1$-small; in particular, $\mathtt{Set}$, the category of small sets as it appeared in the previous sections, should be understood as $\mathtt{Set}_{\mathcal{U}_1}$, the category of $\mathcal{U}_1$-small sets. For the category of $\mathcal{U}_1$-small sets, and for that of $\mathcal{U}_0$-small sets, I will write $\mathtt{SET}$ and $\mathtt{Set}$, respectively. Note that concepts such as "locally presentable", "accessible", etc., depend on the choice of the meaning of "small"; the said notions will be understood here with "small" meaning $\mathcal{U}_1$-small. Under these conventions, $\mathtt{Set}$ will be an object of the standard doctrines.

However, when a category is said to be lfp, we will insist that the class of isomorphism types of fp objects is (bijectively related to) a $\mathcal{U}_0$-small set. Likewise, in the definition of $(\mathbb{S}, \mathcal{R})$ being a finitary doctrine specification, we require that the data going into its description of $\mathbb{S}$ (the exponent-category $\mathbb{X}$ in the initial presheaf category $\mathtt{SET}^{\mathbb{X}}$, the sets of specification names) and the rule-set $\mathcal{R}$ are $\mathcal{U}_0$-small.

In this section, the word "small" will be meant as "$\mathcal{U}_0$-small". With all other specific notations for categories ($\mathtt{Graph}$, $\mathtt{cSk}$, ...) appearing in the previous sections, we stick to the decision that they consist of the $\mathcal{U}_1$-small objects of the corresponding kind. Thus, the underlying graph of $\mathtt{Set}$ is an object of $\mathtt{Graph}$; the category sketch associated with $\mathtt{Set}$ is in $\mathtt{cSk}$, etc.

We could be very explicit about these things (in the style of [3]), and carry the universe-information with all our notations; however, this seemed burdensome to me.

These complications arise because in categorical representation theorems, categories of various

sizes appear. E.g., the embedding theorems for Abelian categories refer to arbitrary *small* Abelian categories, and the specific *large* categories of modules.

In §4, we gave the specification $(\mathtt{cohSk}, \mathcal{R}[\mathtt{Coh}])$ for coherent categories. We know (for references, see below) that Gödel's completeness theorem is translation-equivalent to the representation-theorem for (small) coherent categories; it is now my task to point out a *completeness theorem in terms of the sketch-based syntax* which is translation-equivalent to Gödel's completeness theorem.

There is an obvious guess we can make. $\mathtt{Set}$, the category of small sets, is a member of the coherent doctrine $\mathtt{cohSk}:\mathcal{R}[\mathtt{Coh}]$; therefore, for any sketch-entailment σ in $\mathtt{cohSk}$,

$$\vdash_{\mathcal{R}[\mathtt{Coh}]} \sigma \Longrightarrow \mathtt{Set} \vDash \sigma .$$

If the reverse implication held, we would have a perfect completeness theorem! Unfortunately, it does not; not even for finite σ. Take the sketch-entailment

$$\left|\begin{array}{c}\overline{}\\ A\\ f\downarrow\\ B\\ \underline{}\end{array}\right| \longrightarrow \left|\begin{array}{c}\overline{}\\ A\\ f\downarrow\uparrow g\\ B\\ \underline{}\end{array}\right|$$

$$\hat{\exists}[]=\{\exists_f(1_A)=1_B\} \qquad (gf=1_B)\in \mathtt{CT}[]$$

expressing that "every surjective map has a section". This is valid in $\mathtt{Set}$ (axiom of choice), but it is not valid in all coherent categories; in particular, it is not deducible from $\mathcal{R}[\mathtt{Coh}]$. The way out is to suitably restrict the class of entailments forming the range of σ. Recall exactness properties from §5. We have that

(1) *For any exactness property* σ *in* $\mathtt{cohSk}$, *we have*

$$\vdash_{\mathcal{R}[\mathtt{Coh}]} \sigma \Longleftrightarrow \mathtt{Set} \vDash \sigma .$$

In what follows, among others we will obtain a proof (1). In the last section, we will indicate why (1) is translation-equivalent to the Gödel completeness theorem (in the form saying that a sentence is deducible in the standard Hilbert-type formal system for first order logic iff it is

valid).

In fact, we develop a framework for completeness theorems like (1), involving representation theorems. We will see, in a general context, that (1) follows from the General Completeness Theorem of §2, and the representation theorem for coherent categories:

(2) *For any small coherent category* $\mathbf{C}$ *, there is a conservative (isomorphism reflecting) coherent functor* $\mathbf{C} \longrightarrow \mathtt{Set}^{\mathtt{I}}$ *into a suitable Cartesian power* $\mathtt{Set}^{\mathtt{I}}$ *of* Set .

(For references concerning (2), see below.)

Let $\mathbf{S}$ be an lfp category; of course, in practice, $\mathbf{S}$ will be a finitary sketch-category. Let $\mathcal{K}$ be a set of arrows between fp objects of $\mathbf{S}$. Let us call an arrow in $\mathbf{S}$ a *$\mathcal{K}$-statement*, if it is a pushout of an arrow in $\mathcal{K}$. We introduce a notation to deal with the latter notion; with given

$$\begin{array}{ccc} S & & \\ \psi\uparrow & & \\ U & \xrightarrow[w]{} & V \end{array} , \qquad (3)$$

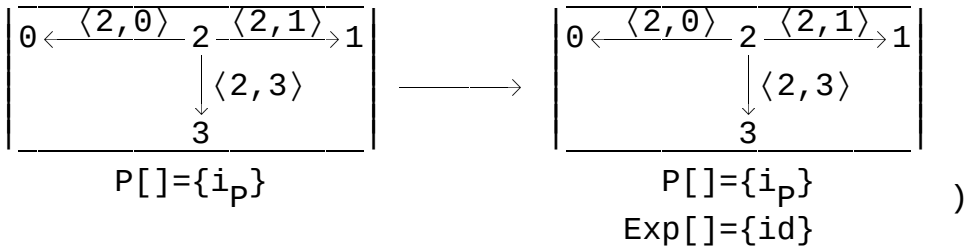
the (a chosen) pushout of the data is denoted as

$$\begin{array}{ccc} S & \xrightarrow{\psi[w]} & S[w] \\ \psi\uparrow & \square & \uparrow\psi' \\ U & \xrightarrow[w]{} & V \end{array} ; \qquad (3')$$

a $\mathcal{K}$-statement is (up to isomorphism) one of the form $\psi[w]$ for some $w \in \mathcal{K}$ and arbitrary ψ . Note that every $\mathcal{K}$-statement is relatively finite (see §3).

Here is a class of examples for $\mathcal{K}$. Suppose that the sketch-category $\mathbf{S}$ is based on $\mathbf{G}$ (recall from §4 what this means). Consider the arrows of $\mathbf{S}$ of the form $\langle K \rangle \underline{\underline{\mathtt{def}}}\ K \to K^{+}$, where K is a specification-type of $\mathbf{S}$ (over $\mathbf{G}$), and K^{+} is obtained by adjoining a single element, the identity, to K in the specification set $K[]$; that is, $K[K^{+}] = \{\mathtt{id}_{K}\}$, and the arrow is

the inclusion. We may call the arrows described as the *specification-arrows* of $\mathbf{S}$ (over $\mathbf{G}$). (E.g., when $\mathbf{S} = \mathtt{CcSk}$, the sketch-category for Cc categories, and $\mathbf{G} = \mathtt{Graph}$, then we have five specification-arrows, one for each of the specification types $\hat{\mathtt{I}}$, $\hat{\mathtt{CT}}$, $\hat{\mathtt{t}}$, $\hat{\mathtt{P}}$ and $\hat{\mathtt{E}}\mathtt{xp}$; the specification-arrow corresponding to $\hat{\mathtt{E}}\mathtt{xp}$ is the inclusion



)

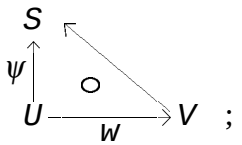
Let $\mathbf{K}$ be a set of specification-types of $\mathbf{S}$, and let $\mathcal{K} = \mathcal{K}[\mathbf{K}]$ be the set of all finite coproducts

$$\coprod_{i<n} \varphi_i : \coprod_{i<n} \ulcorner \varphi_i \urcorner \longrightarrow \coprod_{i<n} \hat{\varphi}_i$$

of specification-arrows $\varphi_i : \ulcorner \varphi_i \urcorner \to \hat{\varphi}_i$. It is easy to see that when $\mathbf{K}$ is the set of all specification-types, the finite $\mathcal{K}[\mathbf{K}]$-statements are exactly the exactness properties of $\mathbf{S}$ (over $\mathbf{G}$), in the sense of §5. When $\mathcal{K}=\mathcal{K}[\mathbf{K}]$, we say " $\mathbf{K}$-statement" for " $\mathcal{K}$-statement".

For a class $\mathbf{P}$ of sketches, the *$\mathbf{P}$-$\mathcal{K}$-statements* are those $\mathcal{K}$-statements whose domain is in $\mathbf{P}$. When $\mathcal{K}=\mathcal{K}[\mathbf{K}]$, we say " $\mathbf{P}$-$\mathbf{K}$-statement" for " $\mathbf{P}$-$\mathcal{K}$-statement".

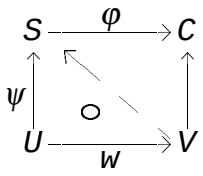
For data as in (3), let us write $S \vDash_\psi w$ to mean that there exists a commutative diagram



;

in other words, $S \vDash w$ iff, for all $\psi : U \to S$, $S \vDash_\psi w$.

Let us say that the arrow $\varphi : S \to C$ is *$\mathcal{K}$-conservative*, if for all $(w : U \to V) \in \mathcal{K}$ and for all $\psi : U \to S$, $C \vDash_{\varphi \circ \psi} w$ implies that $S \vDash_\psi w$. The picture for this is as follows:



;

for any $V \to C$ making the square commutative, there is a diagonal making the left triangle commutative. We recognize the notion of *purity*; see [1]; in other words, φ is $\mathcal{K}$-conservative is to say that φ is pure with respect to the arrows in $\mathcal{K}$, similarly to the terminology of [1]. We say "$\boldsymbol{K}$-conservative" for "$\mathcal{K}[\boldsymbol{K}]$-conservative".

In case $\mathcal{K} = \mathcal{K}[\boldsymbol{K}]$, this means that φ reflects the K-specifications for $K \in \boldsymbol{K}$; that is, for any specification-type $K \in \boldsymbol{K}$ and $\gamma : K \to S$, if $\varphi \circ \gamma \in K[C]$, then $\gamma \in K[S]$. If $\boldsymbol{K}$ consists of the single specification-type $\ulcorner \mathtt{Iso} \urcorner$ (see §5), then φ being $\boldsymbol{K}$-conservative means that φ reflects isomorphisms.

A family $\langle \varphi_i : S \to C_i \rangle_{i \in I}$ of arrows with a fixed domain S is *$\mathcal{K}$-conservative* if the induced arrow $S \longrightarrow \prod_{i \in I} C_i$ is $\mathcal{K}$-conservative. This means that for all $(w : U \to V) \in \mathcal{K}$ and for all $\psi : U \to S$, if for all $i \in I$, $C_i \vDash_{\varphi_i \circ \psi} w$, then $S \vDash_{\psi} w$.

Let $\boldsymbol{C}$ and $\boldsymbol{A}$ be classes of objects of $\boldsymbol{S}$. We say that $\boldsymbol{C}$ is *$\mathcal{K}$-representative with respect to $\boldsymbol{A}$*, if for all $S \in \boldsymbol{A}$, the class of all arrows $S \longrightarrow C$ with $C \in \boldsymbol{C}$ is $\mathcal{K}$-conservative. Spelled out, this means the following:

for any $(w : U \to V) \in \mathcal{K}$, $S \in \boldsymbol{A}$ and $\psi : U \to A$, if
for all $\varphi : S \to C$ with $C \in \boldsymbol{C}$, we have $C \vDash_{\varphi \circ \psi} w$, (4)
then $S \vDash_{\psi} w$.

We say "$\boldsymbol{K}$-representative" for "$\mathcal{K}[\boldsymbol{K}]$-representative".

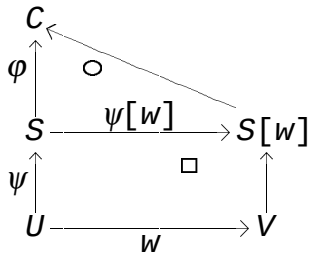
Thus, when $\boldsymbol{S} = \mathtt{cohSk}$, $\boldsymbol{K} = \{\ulcorner \mathtt{Iso} \urcorner\}$, and $\boldsymbol{A}$ consists of the small object of $\mathtt{cohSk} : \mathcal{R}[\mathtt{Coh}]$, then $\boldsymbol{C} = \{\mathtt{Set}\}$ is $\mathcal{K}$-representative with respect to $\boldsymbol{A}$; this is the representation theorem (2) above. Note that in this case we could have chosen $\boldsymbol{K}$ as the set of all specification types, with equivalent results.

Let us make two observations. The first is that the condition in line (4) is equivalent to

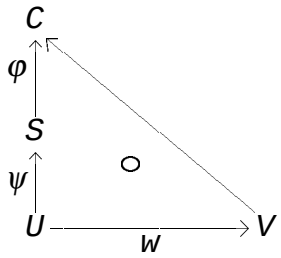
$\mathbf{C} \models \psi[w]$;

$$\forall C \in \mathbf{C}\ ,\ \forall \varphi : S \to C\ \ (C \models_{\varphi \circ \psi} w) \iff \mathbf{C} \models \psi[w] \qquad (5)$$

(the data for this statement are as in (3)). To see this, assume the left-hand side of (5), and consider the diagram



in which we want the existence of the slanted arrow to make the triangle commute. But the assumption gives $V \to C$ making



commute; the required $S[w] \longrightarrow C$ results from the universal property of the pushout. The converse direction of (5) is (even more) obvious. As a result of this observation, to say that $\mathbf{C}$ is $\mathcal{K}$-representative with respect to $\mathbf{A}$ is to say that, for data as in (3), and $S \in \mathbf{A}$,

$$\mathbf{C} \models \psi[w] \iff S \models_{\psi} w\ ;$$

moreover, in this equivalence, the right-to-left implication is automatic; we have

(6) $\mathbf{C}$ *is* $\mathcal{K}$*-representative with respect to* $\mathbf{A}$ *iff for any* $S \in \mathbf{A}$, $(w : U \to V) \in \mathcal{K}$ *and* $\psi : U \to S$ (see (3)), *the implication*

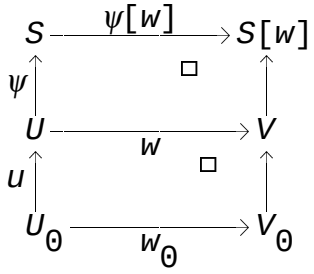
$$\mathbf{C} \models \psi[w] \implies S \models_{\psi} w\ . \qquad (6')$$

holds.

The second observation is that

(7) *if* $\mathbf{C}$ *is* $\mathcal{K}$*-representative for* $\mathbf{A}$*, then* $\mathbf{C}$ *is* $\mathcal{K}^*$*-representative for* $\mathbf{A}$*, where* $\mathcal{K}^*$ *is the class of all* $\mathcal{K}$*-statements.*

To see this, we show (6') for $w \in \mathcal{K}^*$. We have a pushout as in the lower half of the following diagram, with $w_0 \in \mathcal{K}$:



But $\psi[w] = (\psi u)[w_0]$, and

$$\mathbf{C} \models \psi[w] \Longrightarrow \mathbf{C} \models (\psi u)[w_0] \Longrightarrow S \models_{\psi u} w_0 \Longrightarrow S \models_{\psi} w ,$$

where the second implication is by the assumed $\mathcal{K}$-representability, and the third by the universal property of the upper pushout.

Now, we introduce another parameter: a class $\mathcal{R}$ of finite entailments (arrows of $\mathbf{S}$); we will be interested in the doctrine $\mathbf{S}:\mathcal{R}$. Let $\mathbf{P}$ be a class of objects of $\mathbf{S}$. We say that the subclass $\mathbf{C}$ of $\mathbf{S}:\mathcal{R}$ *gives a complete semantics in* $\mathbf{S}:\mathcal{R}$ *for* $\mathbf{P}$*-*$\mathcal{K}$*-statements*, or that $\mathbf{P}$*-*$\mathcal{K}$*-completeness holds in* $\mathbf{S}:\mathcal{R}$ *with respect to* $\mathbf{C}$, if for any $\mathbf{P}$-$\mathcal{K}$-statement, deducibility from $\mathcal{R}$ is equivalent to validity in $\mathbf{C}$:

$$\vdash_{\mathcal{R}} \gamma \iff \mathbf{C} \models \gamma \qquad (8)$$

for any $\mathbf{P}$-$\mathcal{K}$-statement γ .

As an example, (1) says that in $\mathtt{cohSk}:\mathcal{R}[\mathtt{Coh}]$, $\boldsymbol{P}$-$\mathcal{K}$-completeness holds with respect to $\{\mathtt{Set}\}$, with $\boldsymbol{P}$ the set of finite sketches, and $\mathcal{K} = \{\langle \mathtt{Iso} \rangle\}$.

A class of examples is obtained by choosing a infinite regular cardinal κ, and putting $\boldsymbol{P} = \boldsymbol{S}_\kappa$, the class of κ-presentable objects of $\boldsymbol{S}$. In the previous example, $\kappa = \aleph_0$.

We are going to state a result asserting the equivalence of completeness and representability in suitable senses. To have the result in sufficient generality for the applications, we bring in the concept of doctrinal hull; see 3.8. Let $\varphi_{\mathcal{U}} : S \to \ulcorner \mathcal{U} \urcorner$ be a doctrinal hull of S, and let $\gamma : S \to T$. We note the equivalence

$$\vDash_{\mathcal{R}} \gamma \iff \ulcorner \mathcal{U} \urcorner \vDash_{\varphi_{\mathcal{U}}} \gamma \ . \tag{9}$$

Indeed, the left-to-right implication holds since $\ulcorner \mathcal{U} \urcorner \in \boldsymbol{S}:\mathcal{R}$. Conversely, assume $\ulcorner \mathcal{U} \urcorner \vDash_{\varphi_{\mathcal{U}}} \gamma$, $A \in \boldsymbol{S}:\mathcal{R}$, and $\varphi : S \to A$. By $\ulcorner \mathcal{U} \urcorner \vDash_{\varphi_{\mathcal{U}}} \gamma$, we have $\theta : T \to \ulcorner \mathcal{U} \urcorner$ such that $\theta \circ \gamma = \varphi_{\mathcal{U}}$. By 3.8.(ii), there is $\psi : \ulcorner \mathcal{U} \urcorner \to A$ such that $\psi \circ \varphi_{\mathcal{U}} = \varphi$. Thus, $(\psi \circ \theta) \circ \gamma = \varphi$, showing that $A \vDash_{\varphi} \gamma$.

1. Proposition. Suppose $(\boldsymbol{S}, \mathcal{R})$ is a finitary doctrine specification, and $\mathcal{K}$ is a set of arrows between fp objects in $\boldsymbol{S}$. Let $\boldsymbol{P}$ be a class of objects in $\boldsymbol{S}$ such that for any $S \in \boldsymbol{P}$, any doctrinal hull of S (with respect to $\mathcal{R}$) is also in $\boldsymbol{P}$. Let $\boldsymbol{C}$ be a subclass of $\boldsymbol{S}:\mathcal{R}$. Then, for $\mathtt{h}_{\mathcal{R}}(\boldsymbol{P})$ the class of doctrinal hulls of members of $\boldsymbol{P}$,

$$\boldsymbol{C} \text{ is } \mathcal{K}\text{-representative with respect to } \mathtt{h}_{\mathcal{R}}(\boldsymbol{P}) \tag{10}$$

if and only if

$$\boldsymbol{P}\text{-}\mathcal{K}\text{-completeness holds in } \boldsymbol{S}:\mathcal{R} \text{ with respect to } \boldsymbol{C}\ . \tag{11}$$

Proof. Assume (11), to show (10). Because of (6), assume $S \in \mathtt{h}_{\mathcal{R}}(\boldsymbol{P})$, $w : U \to V$, $\psi : U \to S$ and $\boldsymbol{C} \vDash \psi[w]$, to show $S \vDash_{\psi} w$. By the right-to-left implication in (8), we have $\vdash_{\mathcal{R}} \psi[w]$ since by assumption, $S \in \mathtt{h}_{\mathcal{R}}(\boldsymbol{P}) \subset \boldsymbol{P}$, and thus $\psi[w]$ is a $\boldsymbol{P}$-$\mathcal{K}$-statement. By soundness (see 3.2.; the $\vdash_{\mathcal{R}} s \Rightarrow \vDash_{\mathcal{R}} s$ direction of 3.3.), and $S \in \mathtt{h}_{\mathcal{R}}(\boldsymbol{P}) \subset \boldsymbol{S}:\mathcal{R}$, it follows that $S \vDash \psi[w]$,

hence $S \models_{\psi} w$ (see (3'), with an arrow $\sigma : S[w] \to S$ witnessing $S \models_{\mathrm{id}_S} \psi[w]$, which gives $\sigma \circ \psi' : V \to S$ witnessing $S \models_{\psi} w$) as desired.

Assume (10), to show (11); let $\gamma : S \to T$ be a $\boldsymbol{P}$-$\mathcal{K}$-statement; we want to see that the equivalence (8) holds. The left-to-right implication follows from soundness since $\boldsymbol{C} \subset \boldsymbol{S}:\mathcal{R}$. Assume $\boldsymbol{C} \models \gamma$. Let $\varphi_{\mathcal{U}} : S \to \ulcorner \mathcal{U} \urcorner$ be a doctrinal hull of S with respect to $\mathcal{R}$. By 3.1.(iv), $\boldsymbol{C} \models \varphi_{\mathcal{U}}[\gamma]$. Applying (7), in particular, (6') in the form

$$\boldsymbol{C} \models \varphi_{\mathcal{U}}[\gamma] \implies \ulcorner \mathcal{U} \urcorner \models_{\varphi_{\mathcal{U}}} \gamma ,$$

we obtain $\ulcorner \mathcal{U} \urcorner \models_{\varphi_{\mathcal{U}}} \gamma$. Applying (9), we get $\models_{\mathcal{R}} \gamma$. The General Completeness Theorem (3.3.) gives $\vdash_{\mathcal{R}} \gamma$ as desired.

Recall from §2 the (small) cardinal $\lambda_{\boldsymbol{S},\,\mathcal{R}}$ associated with any finitary doctrine specification $(\boldsymbol{S}, \mathcal{R})$. When $\boldsymbol{P} = \boldsymbol{S}_{\kappa}$, and $\kappa > \lambda_{\boldsymbol{S},\,\mathcal{R}}$, then the condition $\mathsf{h}_{\mathcal{R}}(\boldsymbol{P}) \subset \boldsymbol{P}$ of 1. is satisfied, by 3.8.(i). In fact, in this case $\mathsf{h}_{\mathcal{R}}(\boldsymbol{P}) = \boldsymbol{S}_{\kappa} \cap (\boldsymbol{S}:\mathcal{R}) \;\mathsf{def}= (\boldsymbol{S}:\mathcal{R})_{\kappa}$, since for every $S \in \boldsymbol{S}:\mathcal{R}$, S is its own (unique) doctrinal hull. Note that, by §2, $\boldsymbol{S}:\mathcal{R}$ is κ-accessible, and in fact,

(12) *every object in* $\boldsymbol{S}:\mathcal{R}$ *is a* κ*-filtered colimit of objects in* $(\boldsymbol{S}:\mathcal{R})_{\kappa}$.

As a corollary, we arrive at the following simpler version of 1.

2. Corollary. For $(\boldsymbol{S}, \mathcal{R})$, $\mathcal{K}$ and $\boldsymbol{C}$ as above, and κ a regular cardinal, $\kappa > \lambda_{\boldsymbol{S},\,\mathcal{R}}$, we have that $\boldsymbol{C}$ is $\mathcal{K}$-representative with respect to $\boldsymbol{S}_{\kappa}$ if and only if $\boldsymbol{S}_{\kappa}$-$\mathcal{K}$-completeness holds with respect to $\boldsymbol{C}$.

The "only if" part of the Corollary gives that (2) implies (1), in fact, more generally, (1) for σ an arbitrary small "exactness property", any small $\{\ulcorner \mathrm{Iso} \urcorner\}$-statement.

Above we said that (1) results from (2) and the General Completeness Theorem (GCT); here,

via the proof of the Proposition, we have brought in doctrinal hulls again, in addition to the GCT. Here is a proof of the "only if" part of the Corollary without reference to doctrinal hulls (excepting the implicit ones in the proof of the GCT).

First, we observe that, under the conditions of the Corollary,

$$\vdash_{\mathcal{R}} \gamma \iff (\mathbf{S}:\mathcal{R})_{\kappa} \vDash \gamma \qquad (13)$$

for all $\mathbf{S}_{\kappa}$-$\mathcal{K}$-statements γ. The left-to-right implication is obvious. By the GCT, $\mathbf{S}:\mathcal{R} \vDash \gamma \Rightarrow \vdash_{\mathcal{R}} \gamma$. But by (12) any arrow from the domain of γ, a κ-presentable object in $\mathbf{S}$, into an object in $\mathbf{S}:\mathcal{R}$ factors through an object in $(\mathbf{S}:\mathcal{R})_{\kappa}$; hence, $(\mathbf{S}:\mathcal{R})_{\kappa} \vDash \gamma \Rightarrow \mathbf{S}:\mathcal{R} \vDash \gamma$. The right-to-left implication in (13) follows.

Assume that $\mathbf{C}$ is $\mathcal{K}$-representative with respect to $\mathbf{S}_{\kappa}$, γ is a $\mathbf{S}_{\kappa}$-$\mathcal{K}$-statement, $W \in (\mathbf{S}:\mathcal{R})_{\kappa}$, and $\theta:\mathrm{dom}(\gamma) \to W$. We have

$$\mathbf{C} \vDash \gamma \underset{3.1}{\Longrightarrow} \mathbf{C} \vDash \theta[\gamma] \underset{(7)}{\Longrightarrow} W \vDash_{\theta} \gamma .$$

This shows that

$$\mathbf{C} \vDash \gamma \Longrightarrow (\mathbf{S}:\mathcal{R})_{\kappa} \vDash \gamma .$$

By (13), it follows that

$$\mathbf{C} \vDash \gamma \Longrightarrow \vdash_{\mathcal{R}} \gamma ,$$

which was to be shown.

The most frequent case of 2. in practice is obtained by taking $\kappa=\infty$, the least non-$\mathcal{U}_0$-small (strongly inaccessible) cardinal; the resulting assertion also follows from the conjunction of the cases of 2. for all $\kappa<\infty$.

3. Corollary. Suppose $(\mathbf{S}, \mathcal{R})$ and $\mathcal{K}$ are as in 1. Then any class $\mathbf{C}$ objects in $\mathbf{S}:\mathcal{R}$ is $\mathcal{K}$-representative with respect to the small members of the doctrine $\mathbf{S}:\mathcal{R}$ if and only if $\mathbf{C}$

provides a complete semantics in $\mathbf{S}:\mathcal{R}$ with respect to all small $\mathcal{K}$-statements.

I want to discuss the content of the last corollary in the case of first order logic. For this purpose, I bring in, besides the coherent doctrine (figuring in (1) and (2)), the doctrine of Boolean categories $\mathtt{BoolSk}:\mathcal{R}[\mathtt{Bool}]$ (§5), and that of finitary Grothendieck topologies $\mathtt{GrtopfinSk}:\mathcal{R}[\mathtt{Grtopfin}]$ (§6). Statement (2) above expresses that $\{\mathtt{Set}\}$ is $\{\langle \mathtt{Iso}\rangle\}$-representative with respect to the small members of the coherent doctrine. We have the further *representation theorems*

(14) $\{\mathtt{Set}\}$ *is* $\{\ulcorner \mathtt{Iso} \urcorner\}$*-representative with respect to the small members of the Boolean doctrine*;

and

(15) $\{\mathtt{Set}\}$ *is* $\{\mathtt{Cov}_n:n\in\mathbb{N}\}$*-representative with respect to the small members of the doctrine of finitary Grothendieck topologies.*

For (15), note that a morphism of finitary Grothendieck topologies is $\{\mathtt{Cov}_n:n\in\mathbb{N}\}$-conservative iff every finite sieve (a family of maps with a common codomain) in the domain-category which is mapped into a cover by the morphism is a cover itself; briefly, "coverings are reflected". In (15), $\mathtt{Set}$ means the finitary Grothendieck site on the category $\mathtt{Set}$ consisting of the finite surjective sieves. Thus, (15) says that in a small finitary Grothendieck site $\mathcal{C}$, a finite sieve is a cover iff it becomes a cover under all site-maps from $\mathcal{C}$ to $\mathtt{Set}$.

(15) is essentially identical to Deligne's theorem asserting that coherent toposes have enough points. Deligne's theorem appears as an Appendix to Exposé 6 of [3], and dates from 1963. It and its proof have nothing overtly to do with logic. In [24], a direct proof of (15) is given, without the detour through toposes. The proof in [24] is "purely representation theoretic", without overt reference to logic; in fact, it can be regarded as a "higher dimensional" variant of the standard proof of Stone's representation theorem for distributive lattices (in particular, for Boolean algebras).

Categorical logic, initiated by Lawvere, gradually arrived at the insight, mainly due to A. Joyal, that Deligne's theorem is *equivalent* to Gödel's completeness, more precisely, to the uncountable generalization of it given by Mal'cev in the 1930's. In [27], the proof of Deligne's theorem is given by applying Gödel completeness through a two-way *translation* process between logical operations and categorical operations; the precise spelling-out of the translation is in fact the main point of [27]. See also [16] for a reformulation of Deligne's proof, as well as a sketch of the argument of [27] (announced in [26]).

The representation theorem (2) for the coherent doctrine can be easily derived from (15); this is done in [24]. It also follows rather directly from Deligne's original theorem (via the "classifying topos" of the coherent category). (14) is a *special case* of (2), similarly to the fact that the Stone representation theorem for Boolean algebras is a special case of that for distributive lattices.

By 3., and as a consequence of (2) , (14) and (15), respectively, we have

(16) *For any small* $\{\ulcorner \mathtt{Iso} \urcorner\}$*-statement* σ *in* $\mathtt{cohSk}$,

$$\vdash_{\mathcal{R}[\mathtt{Coh}]} \sigma \iff \mathtt{Set} \vDash \sigma \,.$$

(17) *For any small* $\{\mathtt{Cov}_n : n \in \mathbb{N}\}$*-statement* σ *in* $\mathtt{GrtopfinSk}$,

$$\vdash_{\mathcal{R}[\mathtt{Grtopfin}]} \sigma \iff \mathtt{Set} \vDash \sigma \,.$$

(18) *For any small* $\{\ulcorner \mathtt{Iso} \urcorner\}$*-statement* σ *in* $\mathtt{BoolSk}$,

$$\vdash_{\mathcal{R}[\mathtt{Bool}]} \sigma \iff \mathtt{Set} \vDash \sigma \,.$$

From the representation theorem (15) for Grothendieck topologies, we can also deduce that $\mathtt{Set}$ is representative with respect to the small members of the doctrines $\mathtt{regSk}:\mathcal{R}[\mathtt{Reg}]$ (the doctrine of regular categories), $\mathtt{exSk}:\mathcal{R}[\mathtt{ex}]$ (the doctrine of Barr-exact categories), $\mathtt{dSk}:\mathcal{R}[\mathtt{dis}]$ (the doctrine of distributive categories), $\mathtt{pretop}:\mathcal{R}[\mathtt{pretop}]$ (the doctrine of pretoposes). As consequences, we have the specific completeness theorems saying that in any of these doctrines, an (∞-)exactness property is deducible from the axioms of the doctrine

iff it is valid in $\mathtt{Set}$. The representation theorem with $\mathtt{Set}$ as the semantic object for $\mathtt{flSk:}\mathcal{R}\mathtt{[fl]}$, the doctrine of categories with finite limits, is essentially trivial; the resulting specific completeness theorem is less so.

To state further completeness theorems, some remarks on exactness properties. In a doctrine of structured categories, a morphism is faithful in the usual sense iff it is $\mathtt{\{CT\}}$-conservative, equivalently, $\mathtt{\{I,CT\}}$-conservative. For all doctrines of structured categories which contain in their specification the type of pullback as well as the axioms for pullback, and in which all operations are defined by universal properties (as in Section 4), $\{\ulcorner\mathtt{Iso}\urcorner\}$-conservativeness implies (hence, is equivalent to) full conservativeness, that is, $\boldsymbol{K}$-conservativeness for $\boldsymbol{K}$ the class of all specification-types. It requires more structure to have $\mathtt{\{CT\}}$-conservativeness (faithfulness) to be the same as full conservativeness; for pretoposes, this holds, but for coherent categories, it does not.

The conclusion in the sentence before the last is also valid for the doctrine of $\mathcal{F}$-categories (see Section 5); $\mathcal{F}$-categories form a doctrine of structured categories, but $\mathcal{F}$-quantification is not defined by a universal property.

The same, *mutatis mutandis*, can be said for the doctrine of S4-categories (Section 4); this is based on $\boldsymbol{G} = \mathtt{Graph}^{\rightarrow}$ rather than $\mathtt{Graph}$.

In the next two paragraphs, "representative" means "$\{\ulcorner\mathtt{Iso}\urcorner\}$-representative with respect to small objects", the most common concept of representation in categorical logic.

The class of Grothendieck toposes are representative in the doctrine $\mathtt{HeySk:}\mathcal{R}\mathtt{[Hey]}$ of Heyting categories, with respect to small objects and fully conservative maps. In fact, the subclass of presheaf-toposes $\mathtt{Set}^{\boldsymbol{C}}$ ($\boldsymbol{C}$ small category), and also the class of localic toposes $\mathtt{Sh(}\boldsymbol{L}\mathtt{)}$, with $\boldsymbol{L}$ a frame, is representative. These facts are essentially Kripke's completeness theorem [19] and Rasiowa's and Sikorski's completeness theorem for intuitionistic logic [32], [33], respectively. The class of presheaf toposes are representative in the doctrine $\mathtt{biHeySk:}\mathcal{R}\mathtt{[biHey]}$ of biHeyting categories; see [28].

The S4 categories (see §5) of the form $\begin{array}{c}\mathtt{Set}^{|\boldsymbol{C}|}\\ \uparrow\cdot\\ \mathtt{Set}^{\boldsymbol{C}}\end{array}$, with $\boldsymbol{C}$ a small category, $|\boldsymbol{C}|$ its set of

objects, and $()^{\cdot}$ the canonical functor, are representative in the doctrine $\mathtt{S4Sk}:\mathcal{R}[\Box]$ of S4-categories. This is related to Kripke's completeness theorem for S4 modal predicate logic [18], but mathematically is somewhat stronger; for the exact form quoted here, see [28].

Several other completeness theorems in [28] are stated in the form of representation theorems for doctrines whose sketch-specifications should be easy to produce on the basis of the examples in this paper.

Keisler's completeness theorem [17] can be translated into saying that the $\mathcal{F}$-category $\mathtt{Set}$ with the $\kappa=\aleph_1$ interpretation is representative with respect to the class $\mathcal{B}$ of *countable* $\mathcal{F}$-categories (and with respect to the fully conservative maps); see also the next section. The corresponding sketch-based completeness theorem results by 2., with the choice $\kappa=\aleph_1$.

Next, we describe a representation theorem concerning Cartesian closed categories due to D. Cubric [6], [7], and the corresponding specific completeness theorem; this uses 1., rather than 2. or 3.

In the case the doctrine of Cartesian closed categories (Ccc's), we have the notion of the free Ccc over any graph, and in fact, the *free Ccc over* any $\mathtt{CcSk}$-sketch S as the Ccc $\mathcal{F}(S)$, with the sketch-map $\mathtt{i}_S:S\to\mathcal{F}(S)$, such that $\mathtt{Ccc}^*\mathcal{F}(S), \boldsymbol{C})\longrightarrow\mathtt{CcSk}(S, \boldsymbol{C})$ defined by composition with $\mathtt{i}_S$ is an equivalence of categories. Here, $\mathtt{Ccc}^*(\mathcal{F}(S), \boldsymbol{C})$ is the category of all Ccc-morphisms $\mathcal{F}(S)\to\boldsymbol{C}$, with all natural *isomorphisms* as arrows; and $\mathtt{CcSk}(S, \boldsymbol{C})$ is the category whose objects are the sketch-maps $S\to\boldsymbol{C}$, and whose arrows are the natural *isomorphisms* between them as between maps from a graph to a category. $\mathcal{F}(S)$ is determined up to equivalence of categories.

It turns out that, up to equivalence of categories, the $\mathcal{F}(S)$ for a fixed S are the same as the doctrinal hulls of S. In other words,

(19) *if* $\mathtt{i}_{\mathcal{U}}:S\longrightarrow\ulcorner\mathcal{U}\urcorner$ *is any doctrinal hull of* S, *then* $\mathtt{i}_{\mathcal{U}}:S\longrightarrow\ulcorner\mathcal{U}\urcorner$ *has the universal property of* $\mathtt{i}_S:S\to\mathcal{F}(S)$.

This latter fact will be presented in [30] in a more general context.

Cubric's representation theorem concerns Ccc categories of the form $\mathcal{F}(G)$ for a *graph* G, that is, a CcSk-sketch with all specification-sets empty; let us call a Ccc of the form $\mathcal{F}(G)$ for a graph G a *free* Ccc. Cubric's theorem says that

(20) (Cubric) *Any free Ccc has a faithful Ccc functor to* Set.

Now, let us call a sketch S *free* if there are a graph G and a ∞-deduction $\psi: G \to S$. consider the following class $\boldsymbol{P}$ of small CcSk-sketches. Since the composite of two composable ∞-deductions is again an ∞-deduction; therefore in case $\psi: G \to S$ is an ∞-deduction, any doctrinal hull of S is a doctrinal hull of G as well. Thus, the class $\boldsymbol{S}_{\mathtt{free}}$ of free sketches is closed under taking doctrinal hulls. We conclude that 1. is applicable to obtain that

(21) $\mathcal{R}$[Cc] is a complete axiomatization with respect to $\boldsymbol{S}_{\mathtt{free}}$-{CT}-statements in CcSk true in Set.

(a paraphrase of " $\boldsymbol{S}_{\mathtt{free}}$-{CT}-completeness holds in CcSk:$\mathcal{R}$[Cc] with respect to {Set} ").

The finite $\boldsymbol{S}_{\mathtt{free}}$-{CT}-statements are obtained by taking a finite graph G, extending G by applying the rules of Ccc's to obtain, say, $G \to S$, and finally, arbitrarily equating two parallel arrows f and g in S, resulting in $G \to S \to S[f=g]$. The truth of the statement $G \to S[f=g]$ in a Ccc $\boldsymbol{C}$ is that " f and g are necessarily equal " in $\boldsymbol{C}$; that is, whenever G is interpreted in $\boldsymbol{C}$ via a map $\varphi: G \to \boldsymbol{C}$, inducing an essentially uniquely determined extension $\varphi': S \to \boldsymbol{C}$, then $\varphi'(f)=\varphi'(g)$. (21) says that if $G \to S[f=g]$ is true in Set, then it is deducible from the axioms for Ccc's .

It is well-known that the restriction to the $\boldsymbol{S}_{\mathtt{free}}$ cannot be removed from the Cubric completeness theorem; see [6], [7].

Let us turn to *compactness*. Consider a finitary sketch-category $\boldsymbol{S}$, and a class $\mathcal{K}$ of fp maps

in $\mathbf{S}$ as before, but not any rule-set $\mathcal{R}$ as yet. A finite $\mathcal{K}$-statement α is a *finite* ($\mathcal{K}$-)*approximation* of the $\mathcal{K}$-statement φ if φ is a pushout of α; notice that any pushout of a $\mathcal{K}$-statement is a $\mathcal{K}$-statement. Let $\mathbf{P}$ and $\mathbf{C}$ classes of objects in $\mathbf{S}$. We say that *$\mathbf{P}$-$\mathcal{K}$-compactness holds with respect to $\mathbf{C}$* if for any $\mathbf{P}$-$\mathcal{K}$-statement α, if α holds in $\mathbf{C}$, then there is a finite approximation of α which also holds in $\mathbf{C}$ (notice that the converse is automatic, and in fact that α is deducible from any approximation of it).

4. Proposition. *$\mathbf{P}$-$\mathcal{K}$-compactness holds with respect to $\mathbf{C}$ if and only if there is a set $\mathcal{R}$ of fp arrows in $\mathbf{S}$ such that $\mathbf{P}$-$\mathcal{K}$-completeness holds in $\mathbf{S}:\mathcal{R}$ with respect to $\mathbf{C}$.*

Proof. The "only if" part is obvious: take $\mathcal{R}$ to be the set of finite approximations of the $\mathbf{P}$-$\mathcal{K}$-statements that are true in $\mathbf{C}$.

For the converse, let $\mathcal{R}$ be a set of fp arrows. We show that if a $\mathcal{K}$-statement α is deducible from $\mathcal{R}$, then there a finite approximation of α that is also deducible from $\mathcal{R}$; this will clearly suffice.

Assume n is an integer ≥ 0, and $\vec{r} = \langle r_i : R_i \to R'_i \rangle_{1 \leq i \leq n}$ is a sequence of fp arrows. An *$\vec{r}$-system* consists of data of the form

$$S_0 \xrightarrow{\varphi_1} S_1 \xrightarrow{\varphi_2} S_2 \xrightarrow{\varphi_3} S_3 \to \ \ldots \ \to S_{n-1} \xrightarrow{\varphi_n} S_n \tag{22.1}$$

and for every $1 \leq i \leq n$, a pushout diagram

$$\begin{array}{ccc} S_{i-1} & \xrightarrow{\varphi_i} & S_i \\ {\scriptstyle \psi_i}\uparrow & \square & \uparrow{\scriptstyle \psi'_i} \\ R_i & \xrightarrow[r_i]{} & R'_i \end{array} \quad . \tag{22.2}$$

The displayed $\vec{r}$-system is *finite* if each S_i is an fp object.

An arrow to the displayed $\vec{r}$-system from the $\vec{r}$-system

$$\hat{S}_0 \xrightarrow{\hat{\varphi}_1} \hat{S}_1 \xrightarrow{\hat{\varphi}_2} \hat{S}_2 \xrightarrow{\hat{\varphi}_3} \hat{S}_3 \to \ \ldots \ \to \hat{S}_{n-1} \xrightarrow{\hat{\varphi}_n} \hat{S}_n ,$$

$$\begin{array}{ccc} \hat{S}_{i-1} & \xrightarrow{\hat{\varphi}_i} & \hat{S}_i \\ {\scriptstyle \hat{\psi}_i}\uparrow & \square & \uparrow{\scriptstyle \hat{\psi}'_i} \\ R_i & \xrightarrow[r_i]{} & R'_i \end{array}$$

is given by arrows $\sigma_i : \hat{S}_i \longrightarrow S_i$ for $i<n$ such that

$$\begin{array}{ccc} S_{i-1} & \xrightarrow{\varphi_i} & S_i \\ {\scriptstyle \sigma_{i-1}}\uparrow & \circ & \uparrow{\scriptstyle \sigma_i} \\ \hat{S}_{i-1} & \xrightarrow[\hat{\varphi}_i]{} & \hat{S}_i \end{array}$$

and such that $\psi_i = \sigma_{i-1} \circ \hat{\psi}_i$, $\psi'_i = \sigma_i \circ \hat{\psi}'_i$ for all $1 \le i \le n$. Composition of arrows of $\vec{r}$-systems is obvious; thus, $\vec{r}$-systems form a category $\mathrm{S}_{\vec{r}}$. It is pretty clear that filtered colimits in $\mathrm{S}_{\vec{r}}$ exist, and they are computed componentwise. We prove by induction on the length n of $\vec{r}$ that

(23) *every $\vec{r}$-system is a directed colimit of finite $\vec{r}$-systems.*

For $n=0$, the assertion holds since the object S is a directed colimit of fp objects. For the induction step, suppose the $\vec{r}$-system (22.1), (22.2) for $\vec{r}$ of length n , and assume (induction hypothesis) that the $\vec{r}\,' \underset{\mathrm{def}}{=} \langle r_i : R_i \to R'_i \rangle_{1 \le i \le n-1}$-system obtained by deleting the data indexed by n is the colimit of a diagram of the finite $\vec{r}\,'$-systems

$$S^p_0 \xrightarrow{\varphi^p_1} S^p_1 \xrightarrow{\varphi^p_2} S^p_2 \xrightarrow{\varphi^p_3} S^p_3 \to \ \ldots \ \to S^p_{n-2} \xrightarrow{\varphi^p_{n-1}} S^p_{n-1} ,$$

$$\begin{array}{ccc} S^p_{i-1} & \xrightarrow{\varphi^p_i} & S^p_i \\ {\scriptstyle \psi^p_i}\uparrow & \square & \uparrow{\scriptstyle \psi'^p_i} \\ R_i & \xrightarrow[r_i]{} & R'_i \end{array},$$

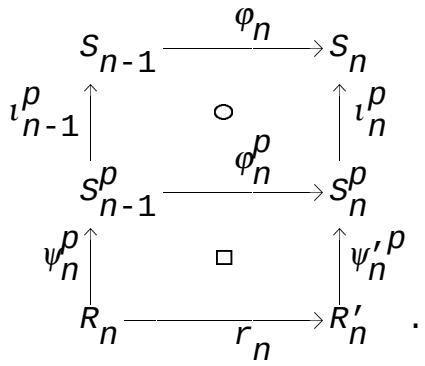
indexed by p ranging over a directed poset P. Then, in particular, $S_{n-1} = \operatorname{colim}_p S^p_{n-1}$, with colimit coprojections $\iota^p_{n-1}: S^p_{n-1} \to S_{n-1}$, say. Since R_n is fp, there is $p_0 \in P$ such that $\psi_n: R_n \to S_{n-1}$ factors through $\iota^{p_0}_{n-1}: S^{p_0}_{n-1} \to S_{n-1}$ as $\psi_n = \iota^{p_0}_{n-1} \circ \psi^{p_0}_n$ for a suitable $\psi^{p_0}_n: R_n \to S^{p_0}_{n-1}$. Let $P_0 = \{p \in P : p_0 \leq p\}$ be ordered as P is, a directed poset, and for $p \in P_0$,

$$\psi^p_n \;\bar{\text{def}}\; \sigma^{p_0 \leq p}_{n-1} \circ \psi^{p_0}_n : R_n \to S^p_{n-1},$$

where $\sigma^{p_0 \leq p}_{n-1} : S^{p_0}_{n-1} \longrightarrow S^p_{n-1}$ is the connecting map for the diagram of the S^p_{n-1}. Consider, for each $p \in P_0$, the pushout

$$\begin{array}{ccc} S^p_{n-1} & \xrightarrow{\varphi^p_n} & S^p_n \\ {\scriptstyle \psi^p_n}\uparrow & \square & \uparrow{\scriptstyle \psi'^p_n} \\ R_n & \xrightarrow[r_n]{} & R'_n \end{array}, \qquad (24)$$

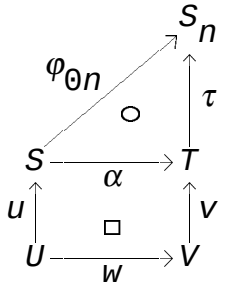
defining S^p_n, φ^p_n and ψ'^p_n; note that S^p_n is fp. By the universal property of the pushout (24), we have $\iota^p_n: S^p_n \longrightarrow S_n$ such that



We also have similarly defined connecting maps $\sigma_n^{p\le q}: S_n^p \longrightarrow S_n^q$. By (22.2) for $i=n$, it follows that $S_n = \operatorname*{colim}_{p\in P_0} S_n^p$ with coprojections ι_n^p.

We have all the data needed to show that the given $\vec{r}$-system is indeed a directed colimit of finite $\vec{r}$-systems, which completes the proof of (23).

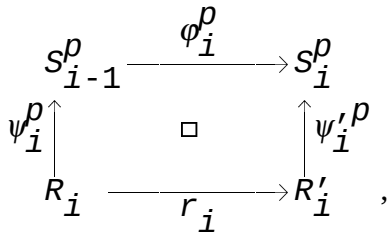
Now, suppose that the $\mathcal{K}$-statement $\alpha: S \to T$ is deducible from $\mathcal{R}$. Then, we have $n\ge 0$, a sequence $\vec{r} = \langle r_i: R_i \to R'_i \rangle_{1\le i\le n}$ of arrows $r_i \in \mathcal{R}$, an $\vec{r}$-system (22.1), (22.2) with $S_0 = S$, and a diagram



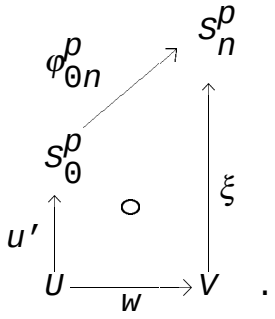
(25)

with $w\in\mathcal{K}$; here, $\varphi_{0n} = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$. Applying (23), let a P-indexed diagram of the finite $\vec{r}$-systems

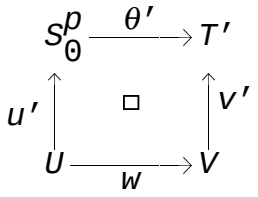
$$S_0^p \xrightarrow{\varphi_1^p} S_1^p \xrightarrow{\varphi_2^p} S_2^p \xrightarrow{\varphi_3^p} S_3^p \to \ \ldots \ \to S_{n-1} \xrightarrow{\varphi_n^p} S_n^p ,$$



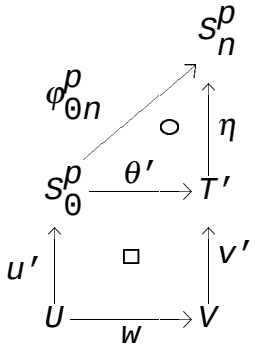
(one for each $p\in P$, P a directed poset) be such that its colimit is the given one, with coprojections $\iota_i^p : S_i^p \longrightarrow S_i$. Since U and V are fp, there is $p\in P$ such that $u:U\to S$ factors as $u=\iota_0^p\circ u'$ for some $u':U\to S_0^p$, and the composite $\tau v: V\to S_n$ factors as $\tau v=\iota_n^p\circ\xi$ for a suitable $\xi: R_n\to S_n^p$ such that

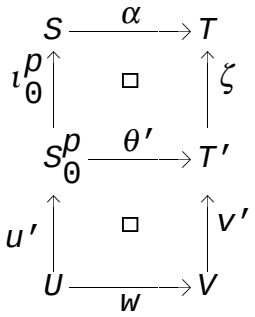


Let us consider the pushout



This fits into the following two diagrams:





(26)

($\varphi^p_{0n} = \varphi^p_n \circ \varphi^p_{n-1} \circ \ldots \circ \varphi^p_1$) such that, in addition, $\eta v' = \xi$ and $\zeta v' = v$ (the pushout character of the upper square of the second diagram follows from (25)). Then α' is an fp arrow, which is deducible from the first diagram in (26) (since φ^p_{0n} is directly deducible), and which is a finite $\mathcal{K}$-approximation of α by the second diagram in (26). This completes the proof of 4.

In the next section, we will show that in case the traditional concept of compactness is available, namely in the case of first order logic and its extensions with generalized quantifiers, our concept is equivalent to the traditional one.

4. says that compactness is equivalent to *axiomatizability* by a set of *finite* axioms. The corresponding assertion in the traditional context is a well-known, and trivial, fact. Of course, the completeness theorems mentioned above as examples each have a corollary in the way of a compactness assertion.

In the traditional context, by *abstract completeness* we mean the recursive enumerability of validities. Abstract completeness is held up as the essential content of completeness, independent of a specific axiomatization. Of course, compactness is also such an "essential content", but less "basic" in some sense (certainly less finitary). As we noted in §3, when we are in a finite sketch-category, the set $\mathcal{D}[\mathcal{R}] = \{s : \vdash_{\mathcal{R}} s\}$ of formal consequences of an r.e. set $\mathcal{R}$ of rules is r.e. As it happens, in most cases in practice we have a *finite* set $\mathcal{R}$ of rules that axiomatizes the concept at hand. Of course, in all these cases, abstract completeness holds, where, formally, we say that, with respect to any given parameters $\mathbf{S}$, $\mathcal{K}$, $\mathbf{P}$ and $\mathbf{C}$ as before, *abstract completeness holds* if the the set of finite $\mathbf{P}$-$\mathcal{K}$-statements valid in $\mathbf{C}$ form an r.e. set.

The value of the notions introduced in this section lies in their uniformity and conceptual simplicity. Now it is possible to ask, concerning categorically given logics, questions the kind of which the logicians used to ask about logics defined by generalized quantifiers. For instance, consider the doctrine of finite limits and finite colimits (thus, this extends the doctrine of pretoposes by having arbitrary coequalizers). Is the set of exactness properties with domain a free sketch (defined now in `flcSk` (see §4) as free sketches were in `CcSk`) that are true in `Set` recursively enumerable; finitely axiomatizable? There are a number of interesting questions of completeness concerning notions of monoidal categories. Our

framework does not materially help in solving these problems, it only gives a convenient formulation to them.