CMS Summer Meeting 2001 Réunion d'été de la SMC

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McGill

Logic and D.G. Quillen's homotopical algebra (*)

Algèbre en théorie des modèles

Model Theoretic Algebra

University of Saskatchewan - June 2 - 4, 2001

(*) Misleading title! More truthful:
"Introduction to FOLDS"
Here, I simply never got to the connections to Quillen (which ∃).

FOLDS: First Order Logic with Dependent Sorts

- A multi-sorted f.o. logic;
  sorts (types) depend on variables;
  variables typed; ∴ variables (may) depend on other variables

- Introduced in 1995 by myself:
  [M1] = [M. M.; Towards a Categorical Foundation of Mathematics, Logic Colloquium '95 (Haifa), Lecture Notes in Logic, v. 11; Springer 1998; pp. 153–190]

- Dependent types occurred previously in:
  → [P. Martin-Löf, An Intuitionist Theory of Types. Proc. Bristol Logic Colloquium, North-Holland, 1973]:
  (a specific formal theory; FOLDS is a multi-purpose logic, like ordinary f.o. logic.

  and → [J. Cartmell, Generalised algebraic theories and contextual categories, Annals of Pure & Applied Logic 32 (1986), 209–243]:
  (no quantification in this; it is an equational logic.

→ In the important special case of the first-order language of categories: [P. Freyd, Properties invariant within equivalence types of categories, In Algebra, Topology and Category Theories Academic Press, 1976; pp 55–61] — in a disguised form.

;) Monograph:

[M2] = [M.M., First Order Logic with Dependent Sorts, with Applications to Category Theory. Preprint, 1995.]

Accepted for Springer Lecture Notes in Logic in 1996.

Unpublished; under revision.

- In FOLDS, we have:

  Signatures $L, \ldots$: categories of a special kind;

  L-structures $M, N, \ldots$: $M : L \to Set$ (functor);

  L-(FOLDS-) formulas $\Phi, \Psi, \ldots$;

  satisfaction-relation : $M \models \Phi[\alpha]$

  for α an M-context : finite system of elements of M, of a special kind.

- New element: for any L, the relation of L-(FOLDS-) equivalence ; $M \simeq_L N$ (M, N: L-structures).

  In the model-theory of FOLDS, FOLDS-equivalence takes the place of ordinary isomorphism in ordinary model theory.

  L-formulas are "invariant" under L-equivalence – and there is a strong converse to this statement (see [M1] & [M2]).

- Main application: a new universe of 'set-like' entities, of various (dependent) types, each with its own concept of identity; with a formal theory formulated in FOLDS. See:

[M3] [M.M., The Multitopic ω-Category of All Multitopic ω-Categories. Preprint. September 1999]

and

[C. Hermida, M.M. & J. Power] On higher dimensional categories. J. Pure and Applied Algebra; Part 1: v. 153 (2000), 221–246; Part 2: v. 157 (2001), 247–277; Part 3: in print]

⊙ Features of the syntax of FOLDS:

1) No operation-symbols.

2) In fact, no relation-symbols; all relations are made into dependent types.

3) Formulas are closed under all Boolean operations. Only atomic formula: $\mathbf{t}$ (true; 0-ary Boolean); $\mathbf{f} =_{def} \neg \mathbf{t}$.

4) Quantification is over a variable of a fixed type only (of course).

(!) 5) Cannot quantify a variable x if there is a free variable left which depends on x.

⊙ For any FOLDS signature L, one has

$$L_{\omega\omega} :=: L_{\omega\omega}(\text{FOLDS}) :$$

set of finitary first-order formulas

but also:

$$L_{\omega\omega_1},\quad L_{\omega\infty} \text{ and even } L_{\infty\infty}$$

[see Appendix 1, for the full statement of the syntax of FOLDS]

$\boxed{\text{Example 1.}}$ (to show how ordinary FOL is subsumed in FOLDS)

Signature L_0:

$$\begin{tikzcd} R \arrow[d, bend right, "\pi_0"'] \arrow[d, bend left, "\pi_1"] \\ U \end{tikzcd}$$

L_0-structure M: $\quad M : \underbrace{L_0}_{\text{as a (trivial) category}} \longrightarrow \mathrm{Set}$ (functor)

For $u, v \in \underbrace{MU}_{M(U) = \underbrace{U^M}_{\text{[model-theory notation]}}}$, $\quad MR(u,v) \underset{\text{def}}{=} \{ r \in MR : M\pi_0 r = u, \; M\pi_1 r = v \}$

The L_0-structures M s.t. each R-fiber $MR(u,v)$ is either empty or the singleton $\{0\}$ (M is called normal then) are in one-to-one correspondence with ordinary structures of the form $(\underset{\substack{\uparrow \\ \text{set}}}{U}, \underset{\substack{\uparrow \\ \text{binary rel'n on } U}}{R})$.

Syntax: variables : $u, v, \dots : U$

$\qquad\qquad\qquad\qquad r, \dots : R(u,v)$

Examples for formulas:

⊙ $\forall u{:}U.\ \exists v{:}U.\ \exists r{:}R(u,v).\ \mathit{tt}$

[expressing $\forall u\, \exists v\, R(u,v)$]

presentation/abbreviation: of last formula:

$$\begin{cases} \langle u{:}U,\ v{:}V,\ r{:}R(u,v)\rangle \quad : \quad \text{context} \\ \qquad (\text{system of variable declarations}) \\ \forall u\, \exists v\, \exists r.\ \mathit{tt} \end{cases}$$

⊙ $\forall u{:}U.\ \forall v{\in}U.\ \forall r{:}R(u,v).\ \forall s{:}R(v,u).\ \overbrace{\neg\mathit{tt}}^{\text{false}}$

[expresses $\forall u \forall v\, (Ruv \to \neg Rvu)$]

⊙ Any ordinary FOL formula with a single binary relation R (and no equality) becomes an L_0-FOLDS-formula upon replacing each quantifier $\forall u$, ($\exists u$) by $\forall u{:}U$ ($\exists u{:}U$), and $R(u,v)$ by $\exists r{:}R(u,v).\ \mathit{tt}$;

— and that's all!

$\boxed{\text{Example 2}}$ (part of the main example to come later)

Signature L_1: the category, with <u>presentation</u>:

$$\begin{tikzcd} S_2 \arrow[d, shift right=6, "\partial_0"'] \arrow[d, "\partial_1"'] \arrow[d, shift left=6, "\partial_2"] \\ S_1 \arrow[d, shift right=3, "\partial_0"'] \arrow[d, shift left=3, "\partial_1"] \\ S_0 \end{tikzcd}$$

$$\partial_1\partial_2 = \partial_1\partial_1 \overset{\text{def}}{=} \rho_0$$
$$\partial_0\partial_2 = \partial_1\partial_0 \overset{\text{def}}{=} \rho_1$$
$$\partial_0\partial_0 = \partial_0\partial_1 \overset{\text{def}}{=} \rho_2$$

L_1-<u>structure</u> :: $M : L_1 \longrightarrow \mathrm{Set}$

(<u>imagine</u>: each of S_0, S_1, S_2 is a set,
each ∂_i a function;
they satisfy the stated composition identities).

Let: M be an L_1-structure; write S_0 for MS_0, etc.

Use: $x, y, \dots \in S_0$; $f, g, \dots \in S_1$; $a, b, \dots \in S_2$.

Then: picture of two (M-)<u>contexts</u>
(of elements of M):

[an element of S_i : i-<u>simplex</u>; $i = 0, 1, 2$ for now]

$\boxed{2}$

$$\begin{tikzcd}[column sep=large] & {\textcircled{1}\ y = \rho_1 a} \arrow[dr, "g = \partial_0 a"] & \\ {\rho_0 a = x\ \textcircled{0}} \arrow[ur, "f = \partial_2 a"] \arrow[rr, "h = \partial_1 a"'] \arrow[rr, phantom, bend left=20, "\textcircled{a}"] & & {z = \rho_2 a\ \textcircled{2}} \end{tikzcd}$$

$$\begin{tikzcd} & \partial_0\partial_2 = \partial_1\partial_0 \arrow[dr, "\partial_0"] & \\ \partial_1\partial_2 = \partial_1\partial_1 \arrow[ur, "\partial_2"] \arrow[rr, "\partial_1"'] & & \partial_0\partial_0 = \partial_0\partial_1 \end{tikzcd}$$

$\updownarrow$

$\boxed{1}$

$$\partial_1 f = x \xrightarrow{f} y = \partial_0 f$$

Example for a (variable-) context: 'same' as $\boxed{2}$ above:

$$\langle x : S_0,\ y : S_0,\ z : S_0,\ f : S_1(\overset{\partial_1}{x}, \overset{\partial_0}{y}),\ g : S_1(y, z),\ h : S_1(x, z),$$

$$a : S_2(\underset{\rho_0}{x}, \underset{\rho_1}{y}, \underset{\rho_2}{z}, \underset{\partial_2}{f}, \underset{\partial_0}{g}, \underset{\partial_1}{h})\ \rangle$$

Examples for sentences:

$\forall x\, \forall y\, \forall z\, \forall f\, \forall g\, \exists h\, \exists a.\, \mathbb{t}$ $\qquad (\ \Lambda^1(2)\ )$

$\forall x\, \forall y\, \forall z\, \forall f\, \forall h\, \exists g\, \exists a.\, \mathbb{t}$ $\qquad (\ \Lambda^0(2)\ )$

$\forall x\, \forall y\, \forall z\, \forall g\, \forall h\, \exists f\, \exists a.\, \mathbb{t}$ $\qquad (\ \Lambda^2(2)\ )$

non-FOLDS-formula:

$$\boxed{\forall x\ \exists a .\ tt}$$

in more detail:

$$\forall x : S_0 \,.\, \exists a : S_2(x, y, z, f, g, h) .\ tt$$

f: free (!) and depends on x:

$$f : S_1(x, y)$$

illegal (see 5) above) p13

— · —

The type $S_2(x, y, z, f, g, h)$ is well-formed

only if the variables $x, \ldots, h$ 'line up'

as in the picture; i.e., given as in the first line of the context displayed.

Identification allowed:

$$\langle x : S_0 ,\ f : S_1(x, x) ,\ a : S_2(x, x, x, f, f, f) \rangle$$

is a legal context.

— · —

General remark: $Var(\Phi)$, the set of free variables of $\Phi \in$ FOLDS is always a (finite) context of variables:

$$x \in Var(\Phi) \ \&\ \underbrace{x \text{ depends on } y} \Rightarrow y \in Var(\Phi)$$

$$\boxed{\text{for } x : K(\langle y_p \rangle_{p \in |K|}),\ y = y_p \text{ for some } p}$$

Example 3 (extension of Ex. 2):

FOLDS-language for "category":

("commutative triangle")

L_3:

$$\begin{tikzcd}
I \arrow[dr, "i"'] & S_2 \arrow[d, shift left=3, "\partial_2"] \arrow[d, "\partial_1"] \arrow[d, shift right=3, "\partial_0"'] & E \arrow[dl, shift left=1, "e_0"'] \arrow[dl, shift right=1, "e_1"] \\
& S_1 \arrow[d, shift right=1, "c=\partial_0"'] \arrow[d, shift left=1, "\partial_1 = d"] & \\
& S_0 &
\end{tikzcd}$$

(for identity arrows) → I

E ← (equality of parallel arrows)

S_1 (= A (for "arrow"))

(codomain) $c = \partial_0$, $\partial_1 = d$ (domain)

S_0 (= O (for "object"))

additional relations:

$$\partial_0 i = \partial_1 i$$
$$\partial_0 e_0 = \partial_0 e_1$$
$$\partial_1 e_0 = \partial_1 e_1$$

Example: (to illustrate the expressive power of FOLDS):
Sentence expressing existence of binary products:

"$z = x \times y$"

$$\begin{tikzcd}
& z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
x \arrow[r, phantom, "a"] & \arrow[r, phantom, "b"] & y \\
& w \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "h"'] &
\end{tikzcd}$$

$\langle x, y, z, w : O;$
$f : A(w, x),\ g : A(w, y), \ldots$
$h, h_1, h_2 : A(w, z);$
$a : S_2(w, z, x;\ h, p, f),$
$b : \ldots;\ a_1 : \ldots,\ \ldots$
$e : E(w, z;\ h_1, h_2) \rangle$

$$\forall x\, \forall y\, \exists z\, \exists p\, \exists q\, \forall f\, \forall g\, [(\exists h\, \exists a\, \exists b.\, \mathrm{tt})$$
$$\wedge (\forall h_1\, \forall h_2\, \forall a_1\, \forall a_2\, \forall b_1\, \forall b_2\, \exists e.\, \mathrm{tt})]$$

(Some) General definitions:

L: category; $K, K', \ldots$: objects of L;

arrow $f: K \to K'$ is **proper** if $f \neq id_K$;

D L is a **FOLDS-signature** if

1) L is **skeletal**: $K \cong K' \Rightarrow K = K'$;

2) L is **one-way**: no proper $K \to K$;
same

3) L has **finite fan-out**: for all K,
$\{f \in Arr(L) : dom(f) = K\}$ is finite

Consequences:

4) For any given K, the set of $n \in \mathbb{N}$ such that there exists

$$K = K_n \to K_{n-1} \to \cdots \to K_0$$

all proper

is finite; the maximum such n is called: $\dim(K)$.

5) $K \xrightarrow{p} K'$, p proper $\Rightarrow \dim(K) > \dim(K')$

⊙ L-structure : functor $M : L \to \mathit{Set}$

morphism of L-structures $P \xrightarrow{m} M$

natural transformation (= homomorphism):

$$L \underset{M}{\overset{P}{\rightrightarrows}} \mathit{Set}\,, \quad \Downarrow m$$

L-Str : category of L-structures

$$L\text{-}\underline{\mathrm{Str}} = \mathit{Set}^L$$

⊙ Auxiliary notation & terminology:

Fix FOLDS-signature L.

- $p, p', q, \dots$: proper arrows in L.
- $\underline{d}p$ = domain(p) ; K_p = codomain(p)
- $|K| \underset{\mathrm{def}}{=} \{p : \underline{d}p = K\}$ = set of arrows out of K
- $M, N, P, \dots$: L-structures
- $K, K', \dots$: objects of L
- A context for K in M:

$$\alpha = \langle \alpha(p) \rangle_{p \in |K|} \quad \text{such that}$$

$$\alpha(p) \in K_p \quad (p \in |K|)$$

and

$$\begin{tikzcd} & K_p \arrow[dd, "r"] \\ K \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\circ"] & \\ & K_q \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \alpha(p) \in MK_p \arrow[d, mapsto, "M(r)"] \\ \alpha(q) \end{tikzcd} \quad \begin{array}{l} (p, q \in |K|; \\ r \in Arr(L)) \end{array}$$

(α: 'compatible' $|K|$-indexed family of elements of M).

- $M[K] \underset{def}{=}$ set of all contexts for K in M.

Note: for any $a \in MK$,

- $[a] \underset{def}{=} \langle (\dot{M}p)(a) \rangle_{p \in |K|} \in M[K]$

context of a

- For $\alpha \in M[K]$:

$$MK(\alpha) \underset{\text{def}}{=} \{ a \in MK : [a] = \alpha \}$$

$$= \underset{\sim}{\text{fiber}} \text{ of } MK \text{ over } \alpha.$$

- Have: projection
$$\begin{tikzcd} MK \arrow[d, "\pi_K"'] \\ M[K] \end{tikzcd} : \begin{tikzcd} a \arrow[d, mapsto] \\ [a] \end{tikzcd}$$

- $$MK \overset{\checkmark}{=} \dot{\bigcup_{\alpha \in M[K]}} MK(\alpha)$$

- For $m : P \longrightarrow M$, $K \in L$,

have: induced map $m_{[K]} : P[K] \longrightarrow M[K]$

and:
$$\begin{tikzcd} PK \arrow[r, "m_K"'] \arrow[d, "\pi_K"'] \arrow[dr, phantom, "\circ"] & MK \arrow[d, "\pi_K"] \\ P[K] \arrow[r, "m_{[K]}"'] & M[K] \end{tikzcd}$$

— and, therefore, for any $\xi \in P[K]$

and for $\alpha \underset{\text{def}}{=} m_{[K]}(\xi) \in M[K]$

the induced map:

$$\boxed{m_{\xi} : PK(\xi) \longrightarrow MK(\alpha)}$$

• • $P \xrightarrow{m} M$ is called <u>fiberwise</u> <u>surjective</u> (FS)

if m_{ξ} is surjective for all $\xi \in P[K]$

for all $K \in L$.

• An L-<u>equivalence</u> of L-structures M, N

is a triple $E = (P, m, n)$:

$$\begin{tikzcd} & P \arrow[dl, two heads, "m"'] \arrow[dr, two heads, "n"] & \\ M & & N \end{tikzcd}$$

(<u>span</u> in L-Str)

such that both m and n are FS.

<u>Notation</u>: $\boxed{E : M \simeq_L N}$;

⊙ Main concept: L-equivalence ($\simeq_L$):

$$M \simeq_L N \quad \underset{def}{\Longleftrightarrow} \quad \exists E \text{ s.t. } E : M \simeq_L N.$$

Remark: sometimes $\simeq$ is written for $\simeq_L$.

⊙ $\simeq_L$ is an equivalence relation.

Transitivity: $(Q, m, n) : M \simeq N$ & $(R, n', p) : N \simeq P$

$$\Rightarrow \; ? : M \simeq P$$

$$\begin{tikzcd}
& & S \arrow[dl, "q"'] \arrow[dr, two heads, "r"] & & \\
& Q \arrow[dl, "m"'] \arrow[dr, two heads, "n"] & & R \arrow[dl, "n'"'] \arrow[dr, two heads, "p"] & \\
M & & N & & P
\end{tikzcd}$$

$\Leftarrow$ defined as pullback of n & n'

Pullback of FS morphism is FS: n' FS $\Rightarrow q$ FS

n FS $\Rightarrow r$ FS

Composite of FS morphisms is FS:

mq & pr are FS

$\therefore (S, mq, pr) : M \simeq P$.

$\odot$ $M \cong N \implies M \simeq N$

$\uparrow$ ordinary isomorphism

(existence of

$$M \underset{g}{\overset{f}{\rightleftarrows}} N$$

$gf = id_M$; $fg = id_N$)

- <u>Important general observation:</u>

a context $\mathfrak{X}$ of variables is essentially the same as an L-structure $\hat{\mathfrak{X}} = \hat{C} : L \to Set$;

(For instance, in Example 2, when

$\mathfrak{X} = \langle x, y, z : S_0 ;\ f : S_1(x,y),\ g : S_2(y,z),\ h : S_3(x,z) \rangle$

(pictorially:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

)

then $\hat{\mathfrak{X}}$ is the functor

$$\begin{array}{c} \emptyset \\ \left(\partial_0 \downarrow \partial_1 \downarrow \ \downarrow \partial_2 \right) \\ \{g, h, f\} \\ \partial_0 \downarrow\downarrow \partial_1 \\ \{z, y, x\} \end{array} \quad : \quad \begin{array}{ccc} \partial_0 g = z & \partial_0 h = z & \partial_0 f = y \\ \partial_1 g = y & \partial_1 h = x & \partial_1 f = x \end{array}$$

)

An evaluation of $\mathcal{X}$ in M is the same as a morphism $\hat{\mathcal{X}} \xrightarrow{\mu} M$.

(This "functorial language of contexts" is a great help ...)

Any $\hat{\mathcal{X}} \xrightarrow{\mu} M$ is a system of M-elements "of the shape $\mathcal{X}$".

• Notation: for a context of variables $\mathcal{X}$, and $\mathcal{X}$-systems $\hat{\mathcal{X}} \xrightarrow{\mu} M$, $\hat{\mathcal{X}} \xrightarrow{\nu} N$, we write

$$\boxed{(P, m, n) : (M, \mu) \simeq (N, \nu)}$$

(read: "the equivalence (P, m, n) takes μ to ν")

if there exists $\pi : \hat{\mathcal{X}} \longrightarrow P$ such that

$$\begin{tikzcd}
& P \arrow[dl, two heads, "m"'] \arrow[dr, two heads, "n"] & \\
M \arrow[r, phantom, "\circ"] & {} \arrow[r, phantom, "\circ"] & N \\
& \hat{\mathcal{X}} \arrow[ul, "\mu"] \arrow[uu, "\pi"] \arrow[ur, "\nu"'] &
\end{tikzcd}$$

For single elements $a \in MK$, $b \in NK$

$$(P,m,n) : (M,a) \simeq (N,b)$$

means (by definition):

$(P,m,n) : M \simeq N$

and there is $x \in PK$ such that

$$\begin{tikzcd} & x \arrow[dl, mapsto, "m_K"'] \arrow[dr, mapsto, "n_K"] & \\ a & & b \end{tikzcd}$$

This is equivalent to

$$(P,m,n) : (M, \langle a \rangle) \simeq (N, \langle b \rangle)$$

for $\hat{\mathfrak{X}} \xrightarrow{\langle a \rangle} M$, $\hat{\mathfrak{X}} \xrightarrow{\langle b \rangle} N$

where $\hat{\mathfrak{X}} = YK$, and $\langle a \rangle$ is $[a]$

"Yoneda"

above, together with: $\langle a \rangle(id_K) = a$.

p.12, last line

⊙ <u>Fact</u>: for any $L_{\underbrace{\infty\infty}_{!}}$ (FOLDS)

formula Φ with free variables

in the context $\mathcal{X}$,

$$(M, \mu) \simeq (N, \nu) \Rightarrow \left[M \models \Phi[\mu] \Leftrightarrow N \models \Phi[\nu] \right]$$

(<u>invariance</u> under equivalence).

⊙ Invariance theorems:

"essentially, $\underset{\sim}{\text{only}}$ L(FOLDS)-formulas

are preserved under L-equivalence".

See [M1] and [M2].

— · —

Examples :

⊙ Example 1 continued (from p. 4):

(i) • Every $M \in L_0$-Str is L_0-equivalent to a normal one (for each (u,v), discard all but one $r \in R(u,v)$).

(ii) For $u, v \in MU$, written just U, define

$$u \approx v \underset{\text{def}}{\Longleftrightarrow} \forall w((\underbrace{Ruw}_{\text{abbreviation of } \exists r : R(u,v).} \leftrightarrow Rvw) \wedge (Rwu \leftrightarrow Rwv))$$

M is skeletal iff$_{\text{def}}$ $u \approx v \iff u = v$ $(u, v \in U)$

Then:

- every $M \in L_0$-Str is L_0-equivalent to a normal and skeletal structure.

(iii) A normal $M \in L_0$-Str is the same as an ordinary structure $(U; R \subseteq U \times U)$.

- For M, N normal and skeletal:

$$M \simeq_{L_0} N \iff M \cong N$$

ordinary isomorphism

<u>Remark</u>: for a (normal) $M = (U; R)$, which is a pre-order, $u \approx v \iff Ruv \wedge Rvu$, the usual equiv. rel'n induced by R. The skeletal version of M is the corresponding partial order.

⊙ Example 2, continued (from p. 6):

Let: $P \xrightarrow{m} M$; $x, \dots \in M$; $\bar{x}, \dots \in P$.

"m is FS" means $\underset{S_0}{\boxed{1}}$ & $\underset{S_1}{\boxed{2}}$ & $\underset{S_2}{\boxed{3}}$:

$$P \overset{m}{\twoheadrightarrow} M$$

$\boxed{1}$

For $K = S_0$:

$\forall$: x

$\exists$: $\bar{x} \longmapsto x$

$\boxed{2}$

For $K = S_1$:

$$\forall: \left\{ \begin{array}{lcl} \bar{x} & \longmapsto & x \\ & & \downarrow f \\ \bar{y} & \longmapsto & y \end{array} \right.$$

$$\exists: \left\{ \begin{array}{l} \bar{x} \\ \downarrow \bar{f} \\ \bar{y} \end{array} \right. \longmapsto \begin{array}{l} x \\ \downarrow f \\ y \end{array}$$

$\boxed{3}$ For $K = S_2$:

$$\forall: \left\{ \begin{tikzcd} & \bar{y} \arrow[dr, "\bar{g}"] & \\ \bar{x} \arrow[ur, "\bar{f}"] \arrow[rr, bend right, "\bar{h}"'] & & \bar{z} \end{tikzcd} \right. \longmapsto \begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "a"] & \\ x \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & {} & z \end{tikzcd} \quad \left[a \in \underbrace{MS_2(x,y,z,f,g,h)}_{=[a]}\right]$$

$$\exists: \left\{ \begin{tikzcd} & \bar{y} \arrow[dr, "\bar{g}"] \arrow[d, phantom, "\bar{a}"] & \\ \bar{x} \arrow[ur, "\bar{f}"] \arrow[rr, bend right, "\bar{h}"'] & {} & \bar{z} \end{tikzcd} \right. \longmapsto a$$

Example 3 continued (from p. 9):

- Every category $\mathbb{C}$ may be regarded, in a canonical way, as an L_3-structure $M = \widetilde{\mathbb{C}}$.

$$\left(\begin{array}{l} \text{E.g., } M(I) = \{ id_X : X \in Ob(\mathbb{C}) \}, \\ \qquad id_X \xmapsto[M(i)]{} X \end{array}\right)$$

- There is a finite set Σ_{cat} of L_3-FOLDS sentences such that, for any L_3-structure M

$$M \models \Sigma_{cat} \iff \text{there is a category } \mathbb{C} \text{ such that } M \simeq_{L_3} \widetilde{\mathbb{C}}.$$

$\therefore$ The concept of "category" is FOLDS-definable.

- For categories $\mathbb{C}, \mathbb{D}$:

$$\widetilde{\mathbb{C}} \simeq_{L_3} \widetilde{\mathbb{D}} \iff \mathbb{C} \underset{\uparrow}{\simeq} \mathbb{D}$$

(ordinary) category equivalence

$$\left(\mathbb{C} \simeq \mathbb{D} \overset{}{\underset{def}{\iff}} \exists : \; GF \cong id_M \; \mathbb{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathbb{D} \; id_N \cong FG \right)$$

Model theory for FOLDS

$$L_{\infty\infty}(\text{FOLDS}) \subset L_{\infty\infty}(\text{full FOL w/} =)$$

Model theory for FOLDS uses only $L_{\infty\infty}(\text{FOLDS})$

– and replaces isomorphism by L-equivalence

No equality – for a replacement, see later!

In full FOL, it is trivial that every structure can be characterized up to isomorphism by a sentence in $L_{\infty\infty}$ – but this uses equality heavily – I don't know what happens in $L_{\infty\infty}(\text{FOLDS})$ & L-equivalence,

But the "Scott-sentence" constructions in $L_{\infty\omega}$ work.

$\boxed{\text{DEF}}$ There is a concept:

$$M \overset{\text{weak}}{\simeq}_L N$$

"back and forth equivalence"

whose definition is similar in style to that of

$\simeq_L$ – but using functors

$$\mathbb{B} = \mathbb{B}[L] \longrightarrow \mathcal{S}et$$

for $\mathbb{B}[L] = ((\mathcal{S}et^L)_{\text{finite}})^{op}$

($\mathbb{B}[L]$ is the finite-limit completion of L,

so $\dfrac{M : L \xrightarrow{!} \mathcal{S}et}{\hat{M} : \mathbb{P} \xrightarrow{\text{lex}} \mathcal{S}et}$)

PROP

$$M \equiv^{\text{FOLDS}}_{\infty\omega} N \iff M \overset{\text{weak}}{\simeq}_{L} N$$

$$\boxed{\begin{array}{l} M \models \Phi \iff N \models \Phi \\ \text{for all } L_{\infty\omega}\text{(FOLDS) sentences } \Phi \end{array}}$$

PROP When L, M, N are countable,

$$M \overset{\text{weak}}{\simeq}_L N \Rightarrow M \simeq_L N;$$

hence,

$$M \equiv^{\text{FOLDS}}_{\infty\omega} N \iff M \simeq_L N.$$

PROP For L, M countable, $\alpha < \omega_1$, there is $\sigma_\alpha[M] \in L_{\omega\omega_1}(\text{FOLDS})$ of quantifier complexity α such that for any L-structure N,

$$N \equiv^{\alpha\,(\text{FOLDS})}_{\infty\omega} M \iff N \models \sigma_\alpha[M].$$

elem. equiv. for sentences of quant. compl. $\leq \alpha$

PROP For L, M countable, there is $\alpha < \omega_1$ such that

$$N \equiv^{\alpha\,(\text{FOLDS})}_{\infty\omega} M \Rightarrow N \equiv^{(\text{FOLDS})}_{\infty\omega} M$$

PROP Hence, for L, M countable, there is $\sigma[M] \in L_{\omega_1\omega}(\text{FOLDS})$ ("Scott sentence") such that, for ctble N, $N \models \sigma[M] \iff N \simeq_L M$.

PROP. Gödel-completeness for $L_{\omega\omega}$ (FOLDS) and $L_{\omega\omega_1}$ (FOLDS) work, by usual Henkin-type arguments.

Remarks: Just like a variable, each individual constant c has its context $[c]$; the set of constants involved in $[c]$ is denoted by $|c|$.

In FOLDS, only (contextually) closed sets of individual constants are usable: C is such if for all $c \in C$, $|c| \subseteq C$.

Fact: Suppose $C_1 \subset C_2$, both closed. Then there is a (possibly transfinite) sequence $\langle c_\gamma \rangle_{\gamma<\alpha}$ such that $\{c_\gamma : \gamma<\alpha\} = C_2 - C_1$ and for all $\beta \leq \alpha$, $C_1 \cup \{c_\gamma : \gamma < \beta\}$ is closed.

"There are enough closed sets of ind. constants."

PROP Compactness, Łoś ultraproduct theorem for FOLDS are trivial consequences of the same things for ordinary FOL (unlike Gödel completeness).

[See top of p. 19]

$\boxed{\text{DEF}}$: FOLDS-formulas are closed under the Boolean operations. Therefore, for any given FOLDS theory T in $L_{\omega\omega}$ ($T = (L, \Sigma)$, Σ: set of axioms), and for any given (mostly, but not necessarily, finite) context $\mathcal{X}$ of variables, we have the Boolean algebra $F_T(\mathcal{X})$ of (equiv. classes of) formulas with free variables in $\mathcal{X}$,

$\therefore$ We have the notions of consistent and of complete types $p(\mathcal{X})$ in the variables $\mathcal{X}$ over T.

$\therefore$ We have types over a model as types over the complete (FOLDS-) diagram of the model

$\therefore$ We have the notions of saturated model, homogeneous model, ...

we have concepts of stability: stable, totally transcendental, ...

$\boxed{\text{PROP}}$ M, N saturated of the same cardinality:

$$M \equiv^{(\text{FOLDS})}_{\omega\omega} N \iff M \simeq N.$$

(This can be used to prove the "invariance theorem")

$\boxed{\text{DEF}}$, Instead of "elementary embedding", the useful concept is: span in L-Str

$$\begin{tikzcd} & P \arrow[dl, "m"'] \arrow[dr] & \\ M & & N \end{tikzcd}, \quad m : FS$$

s.t. for all finite $\gamma : \hat{\mathcal{X}} \to P$ & all Φ, $M \models \Phi[m \circ \gamma] \iff N \models \Phi[n \circ \gamma]$

(used in a proof of the intuitionistic version of the "invariance theorem")

$\boxed{\text{PROP}}$ Omitting Types Theorem : OK.

$\boxed{\text{DEF}}$ Because of lack of equality, cardinality is not an $\simeq$-invariant. But, for a FOLDS-theory $T = (L, \Sigma)$, we may call it $\aleph_0'$-<u>categorical</u> if $M, N \models T \implies M \equiv_{\infty\omega} N$ (right?)

$\boxed{\text{PROP}}$ <u>Ryll-Nardzewski's</u> thm holds:

For a countable FOLDS theory $T=(L,\Sigma)$,

T is $\aleph_0$-categorical iff $F_T(\mathcal{X})$ is

finite for every finite context $\mathcal{X}$.

$\boxed{\cdot}$ Intrinsic identity

Suppose: $E = (P, m, n) : M \simeq M$

We say that E extends the identity if there exists $M \xrightarrow{p} P$

Such that:

$$\begin{tikzcd}
 & M \arrow[dl, "id_M"'] \arrow[d, "p"] \arrow[dr, "id_M"] & \\
M & P \arrow[l, two heads] \arrow[r, two heads] & M
\end{tikzcd}$$

(This implies that E takes a to itself (too!):

$$E : (M, a) \simeq (M, a)$$

for any $a \in |M|$.)

Definition. Suppose a and b are elements of M of the same kind: $a, b \in MK$.

We say: a & b are intrinsically identical:

$a \approx_M b \iff$ there exists $E : M \simeq M$

(i) extending the identity

and

(ii) taking a to b.

For systems $\hat{\mathfrak{X}} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$

of the same shape $\mathfrak{X}$, the definition

of $\alpha \approx_M \beta$ is analogous.

The full diagram: $\alpha \approx_M \beta$ iff:

$\exists$:

$$\begin{tikzcd}
 & M \arrow[dl, "id_M"'] \arrow[d, "p"] \arrow[dr, "id_M"] & \\
M \arrow[r, phantom, "\circ"] & P \arrow[l, two heads, "m"'] \arrow[r, two heads, "n"] \arrow[d, phantom, "\circ"] & M \arrow[l, phantom, "\circ"] \\
 & \hat{\mathfrak{X}} \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$\boxed{\text{PROP}}$: Intrinsic identity is an equivalence relation.

$\boxed{\text{PROP}}$: Intrinsic identity is invariant under equivalence: if $E : M \simeq N$, and E takes $\alpha \in M$ to $\alpha' \in N$, $\beta \in M$ to $\beta' \in N$, then $\alpha \approx_M \beta \Rightarrow \alpha' \approx_N \beta'$.

$\boxed{\text{DEF}}$ $L' \underset{\text{convex}}{\subset} L$ means : L' is a

full subcategory of L, and

$K \in L'$ & $K \xrightarrow{\Gamma} K' \Rightarrow K' \in L'$

For any $n \in \mathbb{N}$, $\underbrace{L\restriction \leq n}_{\text{objects: of dim } \leq n} \underset{\text{convex}}{\subset} L$.

$(P, m, n) : M \simeq N$ is L'-*fixed*

(for $L' \underset{\text{convex}}{\subset} L$) if $P\restriction L' = M\restriction L'$

and $m\restriction L' : P\restriction L' \longrightarrow M\restriction L'$ is $id_{M\restriction L'}$

– and similarly for n.

$\alpha \approx_{L', M} \beta$: defined similarly as $\alpha \approx_M \beta$

$\boxed{L'\text{-}\underline{\text{fixed}} \text{ intrinsic identity}}$

but with $(P, m, n) : M \simeq M$

being L'-*fixed*

$\boxed{\text{PROP}}$ L'-fixed $\approx$ is an equivalence relation; it is invariant under L-equivalence.

Examples:

Example 1 continued:

the relation $\approx$ defined on p. 16 is now intr. id. (for elements $u, v \in \mathcal{U}$).

Example 3 continued:

for L_3-structures in general: not much to say – but for models of (L_3, Σ_{cat}) ('categories, essentially):

$\boxed{X \overset{\text{isomorphic}}{\cong} y}$ $\Downarrow$

For $x, y \in O$ (objects)

$$x \approx y \iff \exists : \left\{ \begin{array}{l} x \underset{g}{\overset{f}{\rightleftarrows}} y \\ ((gf)E(\mathrm{id}_x),\ (fg)E(\mathrm{id}_y)) \end{array} \right.$$

$\iff \exists f : A(x,y).\ \exists g : A(y,x).\ \exists i : I(x).$
$\exists j : I(y).\ \exists h : A(x,x).\ \exists a : S_2(x,y,x,f,g,h).$
$\exists k : A(y,y).\ \exists b : S_2(y,x,y,g,f,k)\ \exists e : E(x,x,h,i)$
$\exists e' : E(y,y,k,i).\ \mathbf{t}$

For $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$,

$f \approx f'$ iff $\exists$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circ"] & y \arrow[d, "\cong"] \\ x' \arrow[r, "f'"'] & y' \end{tikzcd}$$

and

$f \approx_{L\Gamma \leq 0} f'$ iff $x = x'$, $y = y'$

and $f \mathrel{E} f'$

— . —

$\boxed{\text{DEF}}$ Let: $s : M \xrightarrow{\cong} N$,

$E = (P, m, n) : M \simeq_L N$.

E <u>coincides</u> with s if

$E : (M, a) \simeq (N, b) \iff b = s(a)$.

$\boxed{\text{DEF}}$ M is <u>skeletal</u> if

for all $n \in \mathbb{N}$

$K \in L$, $\dim K = n$

$a, b \in MK$:

$$a \underset{L\restriction <n}{\approx} b \quad \Rightarrow \quad a = b$$

$\boxed{\text{PROP}}$ <u>Have:</u> for M, N <u>skeletal</u> :

$$E : M \simeq N \implies E \text{ coincides with an isomorphism } M \xrightarrow{\cong} N$$

$\boxed{\text{DEF}}$ Let: $\mathbb{K}$ be a class of L-structures closed under L-equivalence. $\mathbb{K}$ is <u>skeletalizable</u> if for every $M \in \mathbb{K}$, there is skeletal $M^{\#}$ ($\in \mathbb{K}$) such that $M \simeq M^{\#}$. [$M^{\#}$ is determined up to <u>isomorphism</u>]

Many (all?) interesting classes are skeletalizable.

So are: $\mathbb{K} = L_0\text{-Str}$ (Ex 1),

$\mathbb{K} = \mathrm{Mod}(L_3, \Sigma_{cat}) = \underline{\mathrm{Cat}}$ (Ex 3)

$\boxed{\text{DEF}}$ Class $\mathbb{K} \subset L\text{-Str}$ has <u>definable</u> <u>intrinsic</u> <u>identities</u> if:

for each finite context $\mathcal{X}$, there is formula $\Phi(\mathcal{X}, \mathcal{X}')$

↑ disjoint copy of $\mathcal{X}$

such that for $M \in \mathbb{K}$, $\underbrace{\alpha \in M^{\mathcal{X}}}_{\alpha : \hat{\mathcal{X}} \to M}$, $\beta \in M^{\mathcal{X}}$:

$$\alpha \approx_M \beta \iff M \models \Phi[\alpha, \beta].$$

<u>Ex's:</u> (Example 1) L_0-Str

and

(Example 3) $\mathrm{Mod}(L_3, \Sigma_{cat})$

have definable intrinsic identities (see above!)

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When $\mathbb{K}$ is skeletizable and has definable intrinsic identities, then the FOLDS-theory of $\mathbb{K}$ is reduced to the ordinary model theory of its skeletal part.

$\boxed{\text{PROP}}$ Suppose $\mathbb{K} \subset L\text{-Str}$ is axiomatizable, not necessarily in FOLDS, and possibly with additional predicates (PC_Δ class). Suppose $\mathbb{K}$ has definable intrinsic identities. Then for any finite context $\mathcal{X}$, formula $\Lambda(\mathcal{X})$ in full first-order logic (with equality) over L, there is L-FOLDS formula $\Phi(\mathcal{X})$ such that:

in every skeletal $M \in \mathbb{K}$,

$$M \models \forall \mathcal{X} \left(\Lambda(\mathcal{X}) \leftrightarrow \Phi(\mathcal{X}) \right).$$

$\boxed{\cdot}$ Simplicial sets

- Category Δ :

  objects : $[n] = \{0, 1, \ldots, n\}$ $(n \in \mathbb{N})$

  arrows:

  $[m] \xrightarrow{f} [n]$: $\leq$-preserving function

- $\Delta^{\uparrow}$: same objects as Δ

  arrows : injective ($<$-preserving) arrows of Δ.

- $L \stackrel{def}{=} (\Delta^{\uparrow})^{op}$: FOLDS-signature

Example 2 : L_2 is the ≤ 2-dimensional part of L.

- $\underline{\text{Simp}}\,\underline{\text{Set}} \underset{def}{=} \text{Set}^{\Delta^{op}}$

- For $S \in$ simp Set, $S^{\uparrow}$ is its reduct to L.

- $\underline{\mathrm{Top}}_K \underset{\mathrm{def}}{=}$ category of compactly generated topological spaces

Have: functors

- $$\begin{tikzcd} \underline{\text{simp Set}} \arrow[r, shift left, "|-|"'] & \underline{\mathrm{Top}}_K \arrow[l, shift left, "\mathrm{Sing}"'] \end{tikzcd}$$

$$\boxed{|-| \dashv \mathrm{Sing}}$$

$|-|$ left adjoint to Sing

$|-|$: realization functor

Sing : singular-complex functor

- Homotopy : $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in Top_K

$f \sim g \underset{\mathrm{def}}{\Longleftrightarrow} \exists\, h :$

homotopic

$$\begin{tikzcd} X \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & I \times X \arrow[r, "h"] & Y \end{tikzcd}$$

I : unit interval $= [0,1]$.

• <u>Homotopy</u> for simplicial maps: $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in <u>simp Set</u>:

$f \sim g \underset{\text{def}}{\Longleftrightarrow} \cdots$ formally <u>same</u> definition, with I

replaced by $\Delta[1]$

$$= Y[1] = \hom_{\Delta}[-,[1]].$$

— • —

• $S \in$ simp Set is a <u>Kan</u>-complex if

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, tail] & S \\
\Delta[n] \arrow[ur, dashed, "\exists"'] \arrow[u, phantom, "\circ" near end, shift left=6ex] &
\end{tikzcd}$$

$\Lambda^k[n]$: subfunctor of

$$\Delta[n] = \hom_{\Delta}(-,[n])$$

missing the <u>two</u> elements: $[n] \xrightarrow{id} [n]$;

and $[n-1] \underset{\text{injective}}{\xrightarrow{\delta_k}} [n]$, $k \notin \operatorname{Im}(\delta_k)$

("omit k")

Significance: For $T, S \in \underline{\text{simpSet}}$,

the realization functor $\underline{\text{simpSet}} \longrightarrow \underline{\text{Top}}_K$

induces

$$\hom_{\text{simpSet}}(T, S) \xrightarrow[\varphi]{} \hom_{\text{Top}_K}(|T|, |S|)$$

If S is Kan, then φ induces a bijection of the homotopy classes of maps $T \to S$, and the homotopy classes of maps $|T| \longrightarrow |S|$.

"The homotopy category of Kan-complexes is the same as the homotopy category of spaces"

DEF Kan-complexes S and T are <u>homotopy-equivalent</u>, $S \sim T$, if $\exists$:

$$gf \sim id_S \circlearrowright S \underset{f}{\overset{g}{\leftrightarrows}} T \circlearrowleft id_T \sim fg$$

Proposition S, T: Kan-complexes

1. $S \sim T \iff S^{\uparrow} \simeq_L T^{\uparrow}$.

2. Intrinsic identity in $S^{\uparrow}$ coincides with homotopy of maps into S.

3. For $M \in L\text{-}Str$,

$$M \simeq_L S^{\uparrow} \Rightarrow \exists T \text{ Kan complex}$$
$$M = T^{\uparrow}$$

4. Any two degeneracy structures on $S^{\uparrow}$ are intrinsically identical (homotopic).