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*Logic and D.G. Quillen's homotopical
algebra* *

Algèbre en théorie des modèles

Model Theoretic Algebra



University of Saskatchewan - June 2 - 4, 2001

* Misleading title! More truthful:
"Introduction to FOLDS"

Here, I simply never got to the connections to Quillen (which I).

FOLDS: First Order Logic with Dependent Sorts

① A multi-sorted f.o. logic;

Sorts (types) depend on variables;

variables typed; $\therefore$ variables (may) depend on other variables

② Introduced in 1995 by myself:

[MI] = [M.M., Towards a Categorical Foundation of Mathematics, Logic Colloquium '95 (Haifa), Lecture Notes in Logic, v. 11; Springer 1998; pp. 153-190]

③ Dependent types occurred previously in:

→ [P. Martin-Löf, An Intuitionist Theory of Types. Proc. Bristol Logic Colloquium, North-Holland, 1973]:

(a specific formal theory; FOLDS is a multi-purpose logic, like ordinary f.o. logic.

and
→ [J. Cartmell, Generalised algebraic theories and contextual categories, Annals of Pure & Applied Logic 32 (1986), 209-243]:

(no quantification in this; it is an equational logic..

→ In the important special case of the first-order language of categories: [P. Freyd, Properties invariant within equivalence types of categories, In Algebra, Topology and Category Theories Academic Press, 1976; pp 55-61] ~ in a disguised form.

2) Monograph:

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[M2] = [M.M., First Order Logic

with Dependent Sorts, with Applications
to Category Theory, Preprint, 1995.]

Accepted for Springer Lecture Notes in Logic in 1996.

Unpublished; under revision.



① In FOLDS, we have:

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Signatures $L, \dots$: categories of a special kind;

L-structures $M, N, \dots$: $M : L \rightarrow \text{Set}$ (functor);

L-(FOLDS-)formulas $\Phi, \Psi, \dots$;

satisfaction-relation : $M \models \Phi[\alpha]$

for α an M-context : $\bigvee^{\text{finite}}$ system of elements of M , of a special kind.

② New element : for any L , the relation of

L-(FOLDS-)equivalence ; $M \simeq_L N$

(M, N : L-structures).

In the model-theory of FOLDS, FOLDS-equivalence takes the place of ordinary isomorphism in ordinary model theory.

L-formulas are "invariant" under L-equivalence — and there is a strong converse to this statement (see [M1] & [M2]).

③ Main application : : a new universe of 'set-like' entities, of various (dependent) types, each with its own concept of identity; with a formal theory formulated in FOLDS. See:

[M3] [M.M., The Multitopic ω -Category of All Multitopic ω -Categories. Preprint. September 1999]

and

[C. Hermida, M.M. & J. Power] On higher dimensional categories. J. Pure and Applied Algebra; Part 1: v. 153 (2000), 221-246; Part 2: v. 157 (2001), 247-277; Part 3: in print]

① Features of the syntax of FOLDS:

- 1) No operation-symbols.
- 2) In fact, no relation-symbols; all relations are made into dependent types.
- 3) Formulas are closed under all Boolean operations.
Only atomic formula: $\underline{t}$ (true; 0-ary Boolean); $\underline{f} \stackrel{\text{def}}{=} \neg \underline{t}$.
- 4) Quantification is over a variable of a fixed type only (of course).
- (!) 5) Cannot quantify a variable x if there is a free variable left which depends on x .

① For any FOLDS signature L , one has

$$L_{\omega\omega} ::= L_{\omega\omega}(\text{FOLDS}) :$$

set of finitary first-order formulas

but also:

$$L_{\omega\omega_1}, \quad L_{\omega\omega\omega} \quad \text{and even} \quad L_{\omega\omega\omega\omega}$$

[see Appendix 1, for the full statement of the syntax of FOLDS]

Example 1. (to show how ordinary FOL is subsumed in FOLDS)

Signature L_0 :

$$\begin{array}{c} R \\ \pi_0 \left(\begin{array}{c} \\ \end{array} \right) \pi_1 \\ U \end{array}$$

L_0 -structure M : $M : \underline{L_0} \longrightarrow \text{Set}$ (functor)
as a (trivial) category

For $u, v \in \underline{MU}$, $MR(u, v) \stackrel{\text{def}}{=} \{ r \in MR : M\pi_0 r = u, M\pi_1 r = v \}$
 $M(U) = \underline{U^M}$
 [model-theory notation]

The L_0 -structures M s.t. each R -fiber $MR(u, v)$ is either empty - or the singleton $\{o\}$ (M is called normal then) are in one-to-one correspondence with ordinary structures of the form (U, R) .

$$\begin{array}{cc} & \uparrow \quad \uparrow \\ & \text{set} \quad \text{binary rel'n on } U \end{array}$$

Syntax: variables : $u, v, \dots : U$
 $r, \dots : R(u, v)$

Examples for formulas:

$$\textcircled{1} \quad \forall u:U. \exists v:U. \exists r:R(u,v). \#$$

[expressing $\forall u \exists v R(u,v)$]

presentation/abbreviation: of last formula:

$$\left\{ \begin{array}{l} \langle u:U, v:V, r:R(u,v) \rangle : \text{context} \\ \text{(system of variable declarations)} \\ \forall u \exists v \exists r. \# \end{array} \right.$$

$$\textcircled{2} \quad \forall u:U. \forall v \in U. \forall r:R(u,v). \forall s:R(v,u). \underbrace{\neg \#}_{\text{false}}$$

[expresses $\forall u \forall v (Ruv \rightarrow \neg Rvu)$]

$\textcircled{3}$ Any ordinary FOL formula with a single binary relation R (and no equality) becomes an L_0 -FOLDS-formula upon replacing each quantifier $\forall u, (\exists u)$ by $\forall u:U$ ($\exists u:U$), and $R(u,v)$ by $\exists r:R(u,v). \#$;

— and that's all!

Example 2 (part of the main example to come later)

Signature L_1 : the category, with presentation:

$$\begin{array}{ccc}
 & S_2 & \\
 \partial_0 \downarrow & \partial_1 \downarrow & \downarrow \partial_2 \\
 & S_1 & \\
 \partial_0 \downarrow & \downarrow \partial_1 & \\
 & S_0 &
 \end{array}
 \qquad
 \begin{array}{l}
 \partial_1 \partial_2 = \partial_1 \partial_1 \stackrel{\text{def}}{=} \beta_0 \\
 \partial_0 \partial_2 = \partial_1 \partial_0 \stackrel{\text{def}}{=} \beta_1 \\
 \partial_0 \partial_0 = \partial_0 \partial_1 \stackrel{\text{def}}{=} \beta_2
 \end{array}$$

L_1 -structure $:: M: L_1 \longrightarrow \text{Set}$

(imagine: each of S_0, S_1, S_2 is a set,

- each ∂_i a function;

they satisfy the stated composition identities).

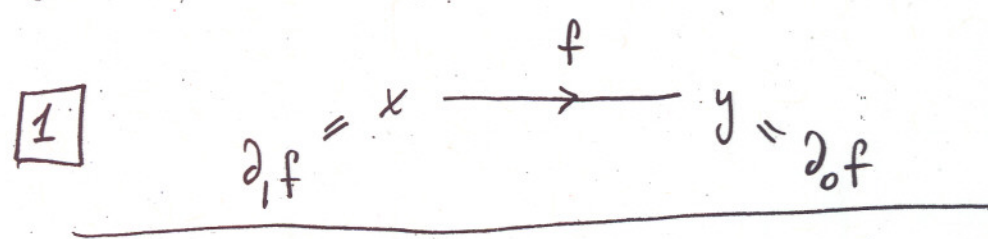
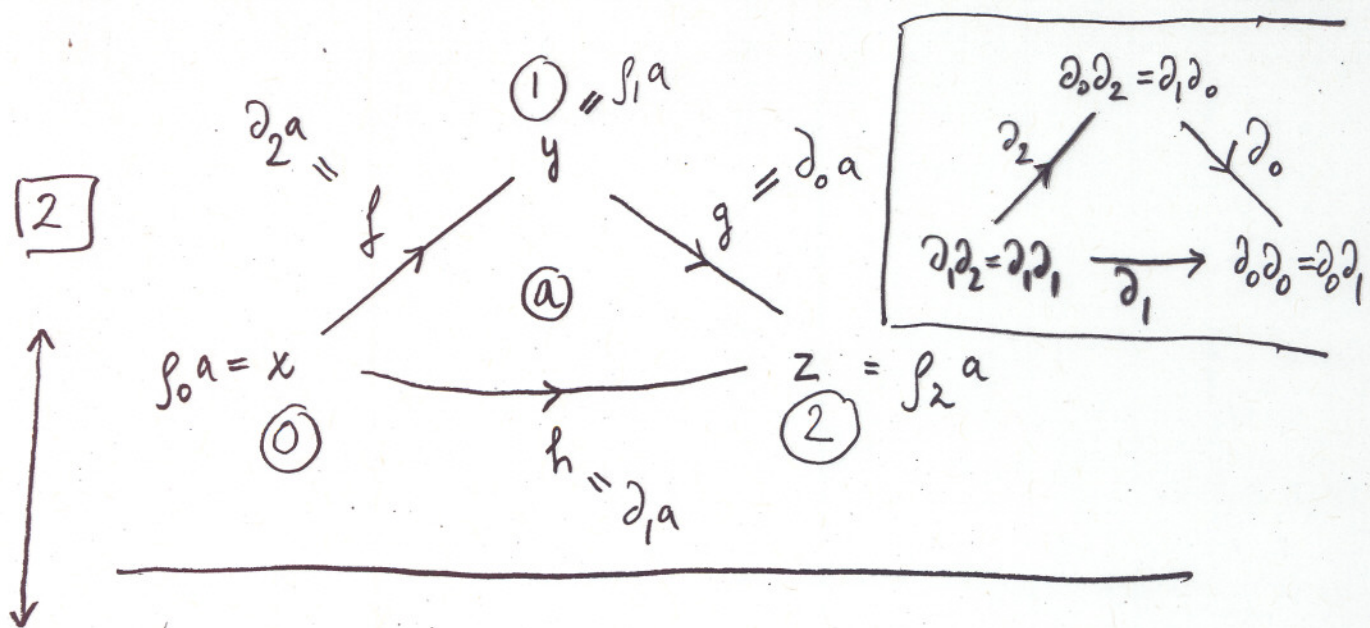
Let: M be an L_1 -structure; write S_0 for MS_0 , etc.

Use: $x, y, \dots \in S_0$; $f, g, \dots \in S_1$; $a, b, \dots \in S_2$.

Then: picture of two (M) -contexts

(elements of M):

[an element of S_i : i -simplex; $i=0,1,2$ for now]



Example for a (variable-) context: 'same' as [2] above:

$$\langle x: S_0, y: S_0, z: S_0, f: S_1(x, y), g: S_1(y, z), h: S_1(x, z), \\ a: S_2(x, y, z, f, g, h) \rangle$$

$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ f_0 & f_1 & f_2 & f_2 & f_0 & f_1 \end{matrix}$

Examples for sentences:

$\forall x \forall y \forall z \forall f \forall g \exists h \exists a. \#$	$(\wedge^1(2))$
$\forall x \forall y \forall z \forall f \forall h \exists g \exists a. \#$	$(\wedge^0(2))$
$\forall x \forall y \forall z \forall g \forall h \exists f \exists a. \#$	$(\wedge^2(2))$

non-FOLDS-formula:

$$\boxed{\forall x \exists a. \#}$$

in more detail:

$$\forall x: S_0. \exists a: S_2(x, y, z, f, g, h). \#$$

$f: \text{free (!) and depends on } x:$ } illegal
 $f: S_1(x, y)$ } (see 5) above)
 PB

The type $S_2(x, y, z, f, g, h)$ is well-formed
 only if the variables $x, \dots, h$ 'line up'

as in the picture ; i.e., given as in the first line of
 the context displayed.

Identification allowed :

$$\langle x: S_0, f: S_1(x, x), a: S_2(x, x, x, f, f, f) \rangle$$

is a legal context.

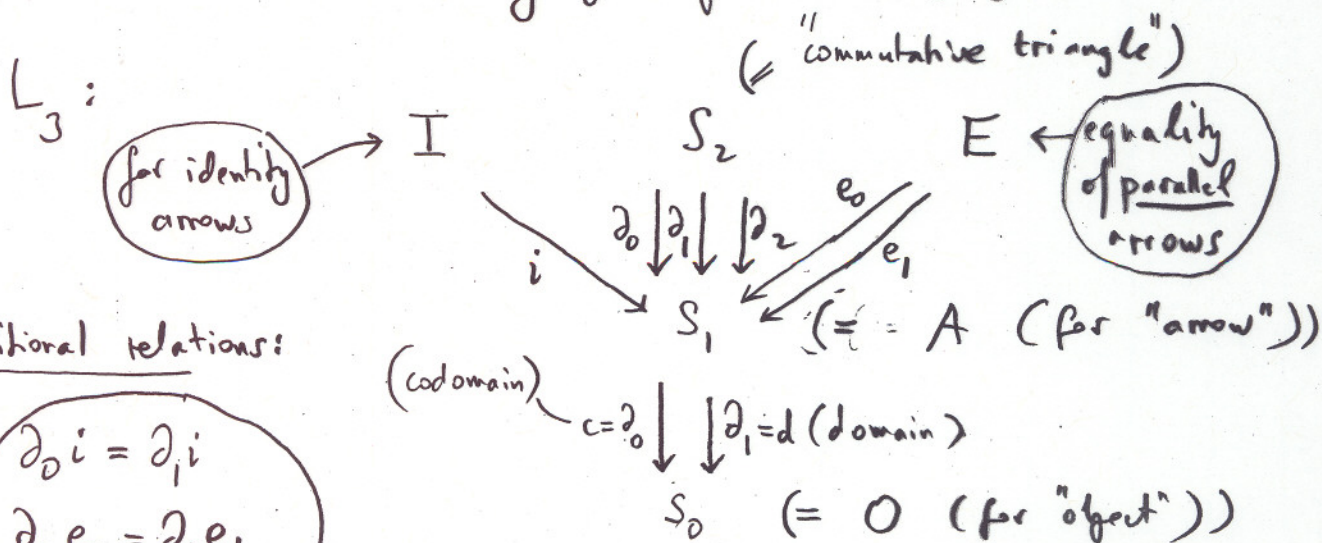
General remark: $\text{Var}(\Phi)$, the set of free variables
 of $\Phi \in \text{FOLDS}$ is always a (finite) context of variables:

$$x \in \text{Var}(\Phi) \ \& \ x \text{ depends on } y \Rightarrow y \in \text{Var}(\Phi)$$

$$\left[\text{for } x: K(\{y_p\}_{p \in |K|}), y = y_p \text{ for some } p \right]$$

Example 3 (extension of Ex. 2)!

FOLDS - language for "category":

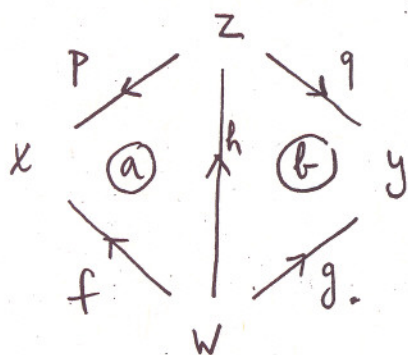


additional relations:

$$\begin{aligned} \partial_0 i &= \partial_1 i \\ \partial_0 e_0 &= \partial_0 e_1 \\ \partial_1 e_0 &= \partial_1 e_1 \end{aligned}$$

Example: (to illustrate the expressive power of FOLDS):
Sentence expressing existence of binary products:

" $z = x \times y$ "



$$\begin{aligned} &\langle x, y, z, w: O; \\ &\quad f: A(w, x), g: A(w, y), \dots \\ &\quad h, h_1, h_2: A(w, z); \\ &\quad a: S_2(w, z, x; h, p, f), \\ &\quad b: \dots, a_1: \dots, \dots \\ &\quad e: E(w, z; h_1, h_2) \rangle \end{aligned}$$

$$\begin{aligned} &\forall x \forall y \exists z \exists p \exists q \forall f \forall g [(\exists h \exists a \exists b. \#) \\ &\quad \wedge (\forall h_1 \forall h_2 \forall a_1 \forall a_2 \forall b_1 \forall b_2 \exists e. \#)] \end{aligned}$$

① L-structure : functor $M: L \rightarrow \text{Set}$

morphism of L-structures $P \xrightarrow{m} M$

natural transformation (= homomorphism):

$$L \begin{array}{c} \xrightarrow{P} \\ \downarrow m \\ \xrightarrow{M} \end{array} \text{Set}.$$

L-Str : category of L-structures

$$\text{L-Str} = \text{Set}^L$$

② Auxiliary notation & terminology:

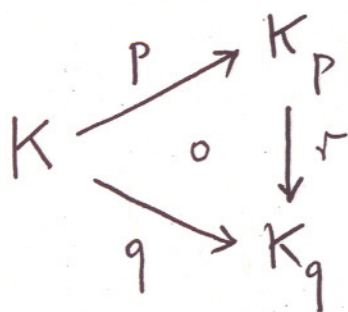
Fix FOLDS-signature L .

- $p, p', q, \dots$: proper arrows in L .
- $\underline{d}_p = \text{domain}(p)$; $K_p = \text{codomain}(p)$
- $|K| \stackrel{\text{def}}{=} \{p : \underline{d}_p = K\} = \text{set of arrows out of } K$
- $M, N, P, \dots$: L-structures
- $K, K', \dots$: objects of L
- A context for K in M :

$$\alpha = \langle \alpha(p) \rangle_{p \in |K|} \quad \text{such that}$$

$$\alpha(p) \in K_p \quad (p \in |K|)$$

and



$\Rightarrow$

$$\alpha(p) \in MK_p$$

$$\begin{array}{c}
 \alpha(p) \in MK_p \\
 \downarrow M(r) \\
 \alpha(q)
 \end{array}$$

$(p, q \in |K|;$
 $r \in \text{Arr}(L))$

$(\alpha: \text{'compatible' } |K| \text{-indexed family of elements of } M).$

- $M[K] \stackrel{\text{def}}{=} \text{set of all contexts for } K \text{ in } M.$

Note: for any $a \in MK$,

$$\begin{array}{l}
 \bullet [a] = \langle (M_p)(a) \rangle_{p \in |K|} \in M[K] \\
 \quad \quad \quad \uparrow \\
 \quad \quad \quad \text{context of } a
 \end{array}$$

- For $\alpha \in M[K]$:

[13]

$$MK(\alpha) \stackrel{\text{def}}{=} \{a \in MK : [a] = \alpha\}$$

= fiber of MK over α .

- Have: projection

$$\begin{array}{ccc} MK & & a \\ \pi_K \downarrow & : & \downarrow \\ M[K] & & [a] \end{array}$$

- $MK = \bigcup_{\alpha \in M[K]} MK(\alpha)$

- For $m : P \rightarrow M$, $K \in L$,

have: induced map $m_{[K]} : P[K] \rightarrow M[K]$

$$\begin{array}{ccc} PK & \xrightarrow{m_K} & MK \\ \pi_K \downarrow & \circ & \downarrow \pi_K \\ P[K] & \xrightarrow{m_{[K]}} & M[K] \end{array}$$

and:

— and, therefore, for any $\xi \in P[K]$

and for $\alpha \stackrel{\text{def}}{=} m_{[K]}(\xi) \in M[K]$

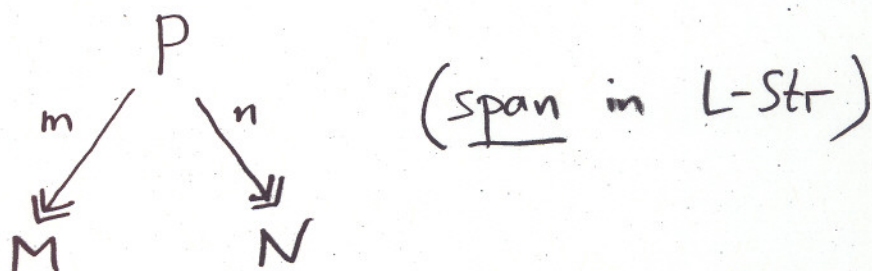
the induced map:

$$m_{\xi} : PK(\xi) \longrightarrow MK(\alpha)$$

• $P \xrightarrow{m} M$ is called fiberwise surjective (FS)

if m_{ξ} is surjective for all $\xi \in P[K]$
for all $K \in L$.

• An L-equivalence of L-structures M, N
is a triple $E = (P, m, n)$:



such that both m and n are FS.

Notation: $(E : M \simeq_L N)$;

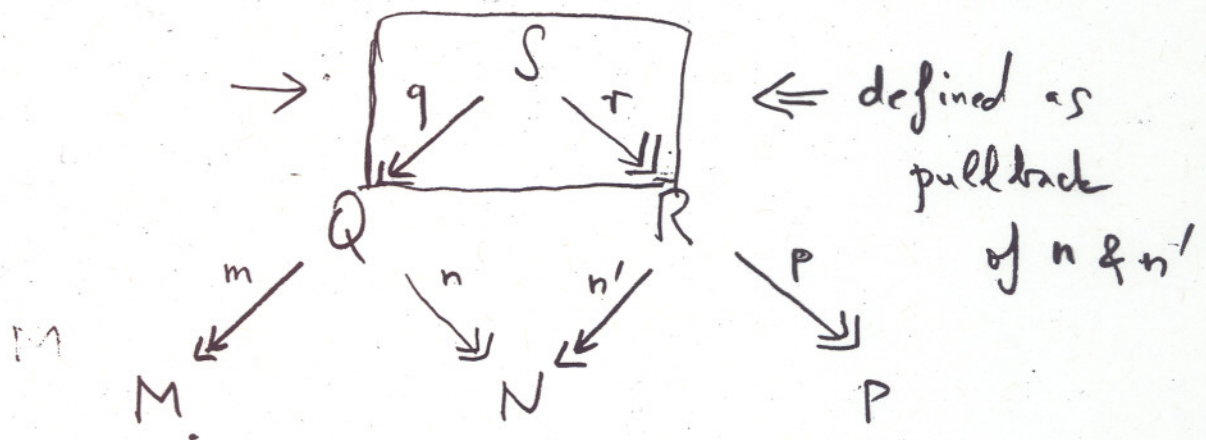
① Main concept: L-equivalence $\cong_L$:

$$M \cong_L N \quad \stackrel{\text{def}}{\iff} \quad \exists E \text{ s.t. } E: M \cong_L N.$$

Remark: sometimes $\cong$ is written for $\cong_L$.

① $\cong_L$ is an equivalence relation.

Transitivity: $(Q, m, n): M \cong N$ & $(R, n', p): N \cong P$
 $\Rightarrow$ $(?) : M \cong P$



Pullback of FS morphism is FS: $n' \text{ FS} \Rightarrow q \text{ FS}$
 $n \text{ FS} \Rightarrow r \text{ FS}$

Composite of FS morphisms is FS:

mq & pr are FS
 $\therefore (S, mq, pr): M \cong P.$

$$\odot \quad M \cong N \quad \Rightarrow \quad M \approx N$$

$\uparrow$
 ordinary
 isomorphism

(existence of

$$M \begin{matrix} \xleftarrow{f} \\ \xrightarrow{g} \end{matrix} N$$

$$gf = id_M, fg = id_N)$$

- Important general observation:

a context $\mathcal{X}$ of variables is essentially the same as an L-structure $\hat{\mathcal{X}} = \hat{C} : L \rightarrow \text{Set}$;

(for instance, in Example 2, when

$$\mathcal{X} = \langle x, y, z : S_0; f : S_1(x, y), g : S_2(y, z), h : S_3(x, z) \rangle$$

(pictorially:



then $\hat{\mathcal{X}}$ is the functor

$$\begin{matrix} \emptyset \\ \downarrow \partial_0 \downarrow \partial_1 \downarrow \partial_2 \end{matrix}$$

$$\{g, h, f\}$$

$$\downarrow \partial_0 \downarrow \partial_1$$

$$\{z, y, x\}$$

$$\bullet \quad \partial_0 g = z$$

$$\bullet \quad \partial_1 g = y$$

$$\partial_0 h = z$$

$$\partial_1 h = x$$

$$\partial_0 f = y$$

$$\partial_1 f = x$$

An evaluation of $\mathcal{X}$ in M is the same as a morphism $\hat{\mathcal{X}} \xrightarrow{\mu} M$.

(This "functorial language of contexts" is a great help ...)

Any $\hat{\mathcal{X}} \xrightarrow{\mu} M$ is a system of M -elements of the shape $\mathcal{X}$.

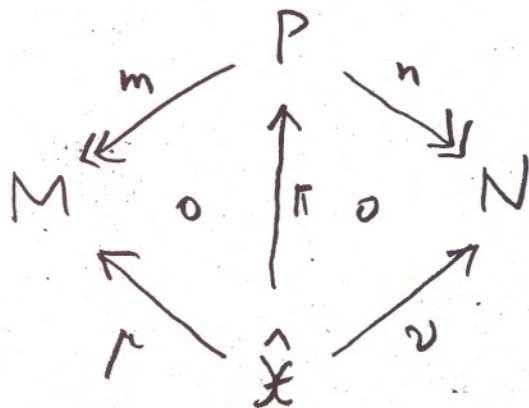
• Notation: for a context of variables $\mathcal{X}$, and $\mathcal{X}$ -systems $\hat{\mathcal{X}} \xrightarrow{\mu} M$, $\hat{\mathcal{X}} \xrightarrow{\nu} N$,

we write

$$(P, m, n) : (M, \mu) \simeq (N, \nu)$$

(read: "the equivalence (P, m, n) takes μ to ν ")

if there exists $\pi : \hat{\mathcal{X}} \rightarrow P$ such that



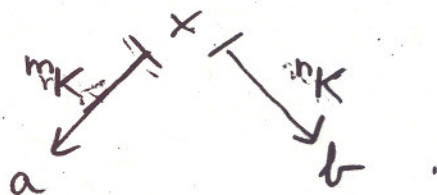
[15.3]

For single elements $a \in MK$, $b \in NK$

$$(P, m, n) : (M, a) \simeq (N, b)$$

means (by definition):

$$\left\{ \begin{array}{l} (P, m, n) : M \simeq N \\ \text{and there is } x \in PK \text{ such that} \end{array} \right.$$



This is equivalent to

$$(P, m, n) : (M, \langle a \rangle) \simeq (N, \langle b \rangle)$$

$$\text{for } \hat{\mathcal{X}} \xrightarrow{\langle a \rangle} M, \quad \hat{\mathcal{X}} \xrightarrow{\langle b \rangle} N$$

where $\hat{\mathcal{X}} = YK$, and $\langle a \rangle$ is $[a]$
↑
"Yoneda"

above, together with: $\langle a \rangle (\text{id}_K) = a$.

p.12, last line

① Fact: for any $\underbrace{L}_{!}(\text{FOLDS})$

formula Φ with free variables
in the context $\mathcal{X}$,

$$(M, \mu) \simeq (N, \nu) \Rightarrow [M \models \Phi[\mu] \Leftrightarrow N \models \Phi[\nu]]$$

(invariance under equivalence).

② Invariance theorems:

"essentially, only $\underbrace{L}(\text{FOLDS})$ -formulas
are preserved under L -equivalence".

See [M1] and [M2].

Examples :

① Example 1 continued (from p. 4):

(i) • Every $M \in L_0\text{-Str}$ is L_0 -equivalent to a normal one (for each (u,v) , discard all but one $r \in R(u,v)$).

(ii) For $u, v \in MU$, written just U , define

$$u \approx v \stackrel{\text{def}}{\iff} \forall w ((Ruw \leftrightarrow Rvw) \wedge (Rwu \leftrightarrow Rvw))$$

abbreviation
of $\exists r: R(u,v).$

M is skeletal if $u \approx v \iff u = v$ ($u, v \in U$)

Then:

- every $M \in L_0\text{-Str}$ is L_0 -equivalent to a normal and skeletal structure.

(iii) A normal $M \in L_0\text{-Str}$ is the same as an ordinary structure $(U; R \subseteq U \times U)$.

- For M, N normal and skeletal:

$$M \cong_{L_0} N \iff M \cong N$$

↑
ordinary isomorphism

Remark: for a (normal) $M = (U; R)$, which is a pre-order, $u \approx v \iff Ruv \wedge Rvu$, the usual equiv. rel'n induced by R . The skeletal version of M is the corresponding partial order.

① Example 2, continued (from p.6):

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Let: $P \xrightarrow{m} M$; $x, \dots \in M$; $\bar{x}, \dots \in P$.

"m is FS" means $\boxed{1} \& \boxed{2} \& \boxed{3}$:

$P \xrightarrow{m} M$

$\boxed{1}$ $\forall:$ x
For $K=S_0$: $\exists: \bar{x} \mapsto x$

$\boxed{2}$ $\forall:$ $\bar{x} \mapsto x$
For $K=S_1$: $\forall:$ $\begin{cases} \bar{x} \mapsto x \\ \bar{y} \mapsto y \end{cases}$
 $\exists:$ $\begin{cases} \bar{x} \mapsto x \\ \bar{f} \mapsto f \\ \bar{y} \mapsto y \end{cases}$

$\boxed{3}$ For $K=S_2$:
 $(K=S_2) \forall:$ $\begin{cases} \bar{x} \mapsto x \\ \bar{f} \mapsto f \\ \bar{y} \mapsto y \\ \bar{z} \mapsto z \end{cases} \mapsto \begin{matrix} f & y & g \\ x & \textcircled{a} & z \\ & h & \end{matrix} [a \in M_{S_2}(x, y, z, f, g, h)]$
 $= [a]$
 $\exists:$ $\begin{cases} \bar{x} \mapsto x \\ \bar{f} \mapsto f \\ \bar{y} \mapsto y \\ \bar{z} \mapsto z \end{cases} \mapsto a$

① Example 3 continued (from p. 9):

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- ① Every category $\mathcal{C}$ may be regarded,
in a canonical way, as an L_3 -structure $M = \tilde{\mathcal{C}}$.

$$\left(\begin{array}{l} \text{E.g., } M(I) = \{ \text{id}_X : X \in \text{ob}(\mathcal{C}) \}, \\ \text{id}_X \xrightarrow[M(i)]{} X. \end{array} \right)$$

- ② There is a finite set Σ_{cat} of L_3 -FOLDS sentences such that, for any L_3 -structure M

$$M \models \Sigma_{\text{cat}} \iff \text{there is a category } \mathcal{C} \text{ such that } M \cong_{L_3} \tilde{\mathcal{C}}.$$

$\therefore$ The concept of "category" is FOLDS-definable.

- ③ For categories $\mathcal{C}, \mathcal{D}$:

$$\tilde{\mathcal{C}} \cong_{L_3} \tilde{\mathcal{D}} \iff \mathcal{C} \cong \mathcal{D}$$

(ordinary) category equivalence

$$\left(\mathcal{C} \cong \mathcal{D} \iff_{\text{def}} \exists: \left(\begin{array}{c} \textcircled{GF \cong \text{id}_M G} \\ \mathcal{C} \xrightleftharpoons[F]{G} \mathcal{D}' \end{array} \right) \textcircled{\mathcal{D}' \ni \text{id}_{\mathcal{D}} \cong FG} \right)$$

Model theory for FOLDS

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$$L_{\infty\omega}(\text{FOLDS}) \subset L_{\infty\omega}(\text{full FOLW/})$$

Model theory for FOLDS uses only $L_{\infty\omega}(\text{FOLDS})$

— and replaces isomorphism by L-equivalence

No equality — for a replacement, see later!

In full FOL, it is trivial that every structure can be characterized up to isomorphism by a sentence in $L_{\infty\omega}$

— but this uses equality heavily — I don't know what happens in $L_{\infty\omega}(\text{FOLDS})$ & L-equivalence,

But the "Scott-sentence" constructions in $L_{\infty\omega}$ work.

DEF There is a concept:

$$M \overset{\text{weak}}{\underset{L}{\cong}} N$$

"back and forth equivalence"

whose definition is similar in style to that of

$\cong_L$ — but using functors

$$\mathbb{B} = \mathbb{B}[L] \longrightarrow \text{Set}$$

$$\text{for } \mathbb{B}[L] = ((\text{Set}^L)_{\text{finite}})^{\text{op}}$$

($\mathbb{B}[L]$ is the finite-limit completion of L .)

$$\text{so } \frac{M: L \longrightarrow \text{Set}}{M: \mathbb{B} \xrightarrow{\text{ex}} \text{Set}}$$

PROP

$$M \equiv_{\substack{\text{FOLDS} \\ \mathcal{L}_w}} N \Leftrightarrow M \overset{\text{weak}}{\underset{\mathcal{L}}{\sim}} N$$

$$M \models \Phi \Leftrightarrow N \models \Phi$$

for all $\mathcal{L}_w$ (FOLDS) sentences Φ

PROP

When L, M, N are countable,

$$M \stackrel{\text{weak}}{\underset{L}{\approx}} N \Rightarrow M \underset{L}{\approx} N;$$

hence,

$$M \underset{\omega\omega}{\equiv}^{\text{FOLDS}} N \Leftrightarrow M \underset{L}{\approx} N.$$

PROP

For L, M countable, $\alpha < \omega_1$, there is $\sigma_\alpha[M] \in L_{\omega\omega_1}^{\text{(FOLDS)}}$ of quantifier complexity α such that for any L -structure N ,
 $N \underset{\omega\omega}{\equiv}^{\alpha \text{ (FOLDS)}} M \Leftrightarrow N \models \sigma_\alpha[M].$

elem. equiv. for sentences of quant. compl. $\leq \alpha$

PROP

For L, M countable, there is $\alpha < \omega_1$ such that

$$N \underset{\omega\omega}{\equiv}^{\alpha \text{ (FOLDS)}} M \Rightarrow N \underset{\omega\omega}{\equiv}^{\text{(FOLDS)}} M$$

PROP

Hence, for L, M countable, there is $\sigma[M] \in L^{\text{(FOLDS)}}_{\omega\omega_1} ("Scott \text{ sentence}")$ such that, for cble N , $N \models \sigma[M] \Leftrightarrow N \underset{L}{\approx} M.$

PROP

Gödel-completeness for $L_{\omega\omega}$ (FOLDS)

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and $L_{\omega\omega_1}$ (FOLDS) work, by usual

Henkin-type arguments.

Remarks: Just like a variable, each individual constant c has its context $[c]$; the set of constants involved in $[c]$ is denoted by $|c|$.

In FOLDS, only (contextually) closed sets of individual constants are usable: C is such if for all $c \in C$, $|c| \subseteq C$.

Fact: Suppose $C_1 \subset C_2$, both closed.

Then there is a (possibly transfinite) sequence

$\langle C_\gamma \rangle_{\gamma < \alpha}$ such that $\{C_\gamma : \gamma < \alpha\} = C_2 - C_1$

and for all $\beta \leq \alpha$, $C_1 \cup \{C_\gamma : \gamma < \beta\}$ is closed.

"There are enough closed sets of ind. constants."

PROP

Compactness, Łoś ultraproduct theorem

for FOLDS are trivial consequences of the

same things for ordinary FOL

(unlike Gödel completeness).

[see top of p. 19]

DEF

FOLDS - formulas are closed under the Boolean operations. Therefore, for any given FOLDS theory T in $L_{\omega\omega}$ ($T = (L, \Sigma)$, Σ : set of axioms), and for any given (mostly, but not necessarily, finite) context X of variables, we have the Boolean algebra $F_T(X)$ of (equiv. classes of) formulas with free variables in X ,

$\therefore$ We have the notions of consistent and of complete types $p(X)$ in the variables X over T .

$\therefore$ We have types over a model as types over the complete (FOLDS-) diagram of the model

$\therefore$ We have the notions of saturated model,

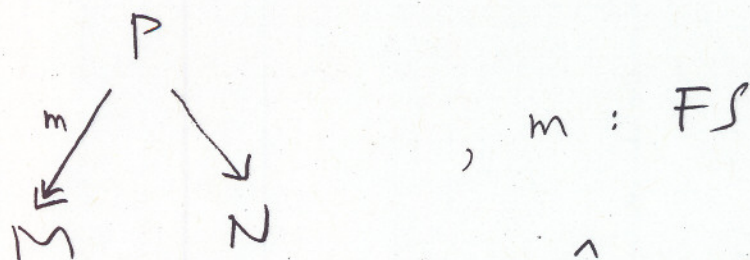
homogeneous model, ...
we have concepts of stability: stable, totally transcendental, ...
 M, N saturated of the same cardinality:

PROP

$$M \equiv_{\omega\omega}^{(FOLDS)} N \iff M \simeq N.$$

(This can be used to prove the
"invariance theorem")

DEF, Instead of "elementary embedding",
the useful concept is: span in $L\text{-Str}$



s.t. for all finite $\gamma : \hat{X} \rightarrow P$ &
all $\mathbb{I}$, $M \models \mathbb{I}[m \circ \gamma] \Leftrightarrow N \models \mathbb{I}[n \circ \gamma]$

(used in a proof of the intuitionistic
version of the "invariance theorem")

PROP Omitting Types Theorem : OK.

DEF Because of lack of equality, cardinality is
not an $\cong$ -invariant. But, for a FOLDS-theory

$T = (L, \Sigma)$, we may call it $\aleph_0$ -categorical

if $M, N \models T \Rightarrow M \equiv_{\omega\omega} N$ (right?)

PROPRyll-Nardzewski's thm holds:For a countable FOLDS theory $T = (L, \Sigma)$, T is $\aleph_0$ -categorical iff $F_T(\mathbb{X})$ is
finite for every finite context $\mathbb{X}$.

□ Intrinsic identity

Suppose: $E = (P, m, n) : M \cong M$

We say that E extends the identity if
there exists $M \xrightarrow{P} P$

Such that

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \text{id}_M & & \searrow \text{id}_M & \\ M & & P & & M \\ & \nwarrow & \downarrow P & \nearrow & \\ & M & & M & \end{array}$$

(This implies that E takes a to itself (too!):

$$E : (M, a) \cong (M, a)$$

for any $a \in |M|$.)

Definition. Suppose a and b are elements of
 M of the same kind: $a, b \in MK$.

We say: a & b are intrinsically identical:

$$a \approx_M b \iff \text{there exists } E : M \cong M$$

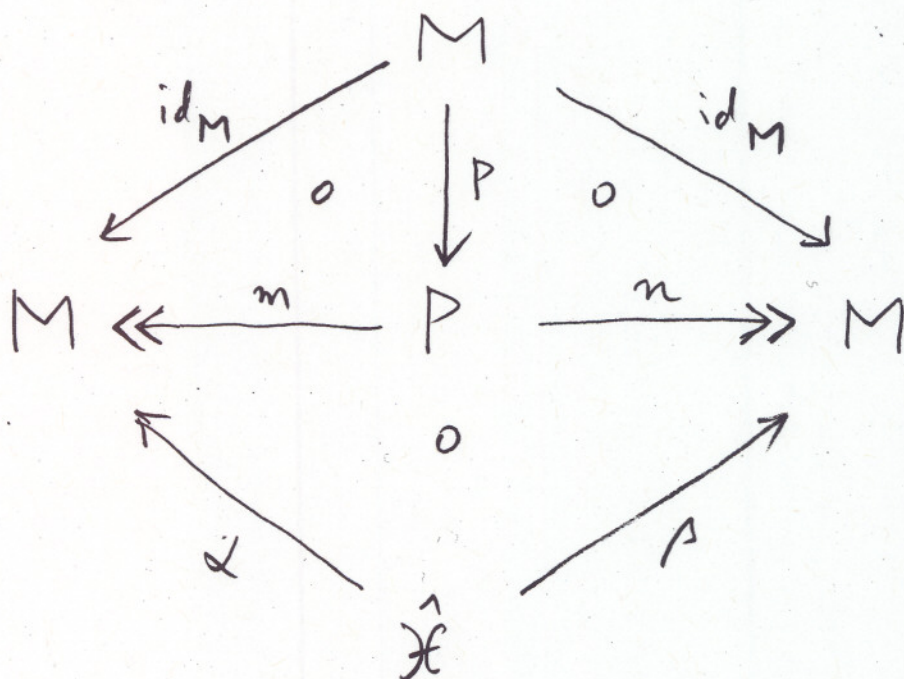
(i) extending the identity
and

(ii) taking a to b .

For systems $\hat{X} \xrightarrow[\beta]{\alpha} M$
 of the same shape $\hat{X}$, the definition
 of $\alpha \approx_M \beta$ is analogous.

The full diagram : $\alpha \approx_M \beta$ iff:

$\exists$:



PROP 1: Intrinsic identity is an equivalence relation.

PROP 2: Intrinsic identity is invariant under
 equivalence : if $E: M \approx N$, and
 E takes $\alpha \in M$ to $\alpha' \in N$, $\beta \in M$ to $\beta' \in N$,
 then $\alpha \approx_M \beta \Rightarrow \alpha' \approx_N \beta'$.

DEF $L' \underset{\text{convex}}{\subset} L$ means: L' is a

full subcategory of L , and

$$K \in L' \quad \& \quad K \xrightarrow{f} K' \Rightarrow K' \in L'$$

For any $n \in \mathbb{N}$, $L' \upharpoonright_{\leq n} \underset{\text{convex}}{\subset} L$.

objects: of $\dim \leq n$

$(P, m, n): M \cong N$ is L' -fixed

(for $L' \underset{\text{convex}}{\subset} L$) if $P \upharpoonright L' = M \upharpoonright L'$

and $m \upharpoonright L': P \upharpoonright L' \rightarrow M \upharpoonright L'$ is $\text{id}_{M \upharpoonright L'}$

- and similarly for n .

$\alpha \underset{L', M}{\approx} \beta$: defined similarly as $\alpha \underset{M}{\approx} \beta$

L' -fixed intrinsic identity

but with $(P, m, n): M \cong M$

being L' -fixed

PROP L' -fixed $\approx$ is an equivalence relation; it is invariant under L -equivalence.

Examples:

Example 1 continued:

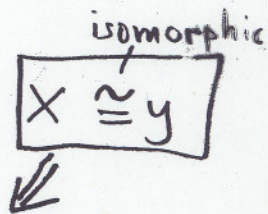
the relation $\approx$ defined on p. 16 is now intr. id. (for elements $u, v \in U$).

Example 3 continued:

for L_3 -structures in general: not much to say - but for models of (L_3, Σ_{cat}) ('categories, essentially):

For $x, y \in O$ (objects)

$$x \approx y \iff \exists : \left\{ \begin{array}{c} x \xrightarrow{f} y \\ y \xrightarrow{g} x \\ ((gf)E(id_x))^{\delta}, (fg)E(id_y) \end{array} \right.$$



$$\iff \exists f: A(x, y). \exists g: A(y, x). \exists i: I(x).$$

$$\exists j: I(y). \exists h: A(x, x). \exists a: S_2(x, y, x, f, g, h).$$

$$\exists k: A(y, y). \exists b: S_2(y, x, y, g, f, k) \exists e: E(x, x, h, i)$$

$$\exists e': E(y, y, k, j). \quad \mathbf{+}$$

For $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$,

$$f \approx f' \quad \text{iff} \quad \exists: \quad \begin{array}{ccc} x & \xrightarrow{f} & y \\ \cong \downarrow & \circ & \downarrow \cong \\ x' & \xrightarrow{f'} & y' \end{array}$$

and

$$f \approx_{L \uparrow \leq 0} f' \quad \text{iff} \quad x = x', \quad y = y' \\ \text{and} \quad f \in f'$$

DEF 1.0 Let: $s: M \xrightarrow{\cong} N$,

$$E = (P, m, n) : M \cong_L N.$$

E coincides with s if

$$E: (M, a) \cong (N, b) \Leftrightarrow b = s(a).$$

DEF M is skeletal if

for all $n \in \mathbb{N}$

$$K \in L, \dim K = n$$

$$a, b \in MK :$$

$$a \underset{L \upharpoonright K^n}{\approx} b \Rightarrow a = b$$

PROP Have: for M, N skeletal :

$$E: M \cong N \overset{\checkmark}{\Rightarrow} E \text{ coincides with an isomorphism } M \xrightarrow{\cong} N$$

DEF Let: $\mathcal{K}$ be a class of L -structures closed under L -equivalence. $\mathcal{K}$ is skeletalizable if for every $M \in \mathcal{K}$, there is skeletal $M^\# \in \mathcal{K}$ such that $M \cong M^\#$. [$M^\#$ is determined up to isomorphism]

Many (all?) interesting classes are skeletalizable.
So are: $\mathcal{K} = L_0\text{-Str}(\underline{\text{Ex 1}})$,

$$\mathcal{K} = \text{Mod}(L_0, \Sigma_{\text{cut}}) = \underline{\text{Cat}}(\underline{\text{Ex 3}})$$

DEF Class $\mathbb{K} \subset L\text{-Str}$ has
definable intrinsic identities if:

for each finite context X , there is
 formula $\Phi(X, X')$

$\uparrow$ disjoint copy of X

such that for $M \in \mathbb{K}$, $\alpha \in M^X$, $\beta \in M^X$:

$\alpha: X \rightarrow M$

$\alpha \approx_M \beta \iff M \models \Phi[\alpha, \beta].$

Ex's: (Example 1) $L_0\text{-Str}$
 and

(Example 3) $\text{Mod}(L_3, \Sigma_{\text{cat}})$

have definable intrinsic identities (see above!)

— . —

When $\mathbb{K}$ is skeletizable and

has definable intrinsic identities, then

the FOLDS -theory of $\mathbb{K}$ is reduced

to the ordinary model theory of its skeletal part.

PROP

Suppose $\mathcal{IK} \subseteq \mathcal{L}\text{-Str}$ is

axiomatizable, not necessarily in FOLDS, and possibly with additional predicates (PC_Δ class).

Suppose $\mathcal{IK}$ has definable intrinsic identities.

Then for any finite context $\mathcal{X}$, formula

$\mathcal{L}(\mathcal{X})$ in full first-order logic (with

equality) over $\mathcal{L}$, there is $\mathcal{L}$ -FOLDS formula

$\Phi(\mathcal{X})$ such that:

in every skeletal $M \in \mathcal{IK}$,

$$M \models \forall \mathcal{X} (\mathcal{L}(\mathcal{X}) \leftrightarrow \Phi(\mathcal{X})).$$

□ Simplicial sets

- Category Δ :

objects : $[n] = \{0, 1, \dots, n\} \quad (n \in \mathbb{N})$

arrows :

$[m] \xrightarrow{f} [n] : \leq\text{-preserving function}$

- $\Delta^\uparrow$: same objects as Δ

arrows : injective ($\leq$ -preserving)
arrows of Δ .

- $L \stackrel{\text{def}}{=} (\Delta^\uparrow)^{\text{op}}$: FOLDS-signature

Example 2 : L_2 is the ≤ 2 -dimensional part of L .

- $\underline{\text{SimpSet}} \stackrel{\text{def}}{=} \text{Set}^{\Delta^{\text{op}}}$

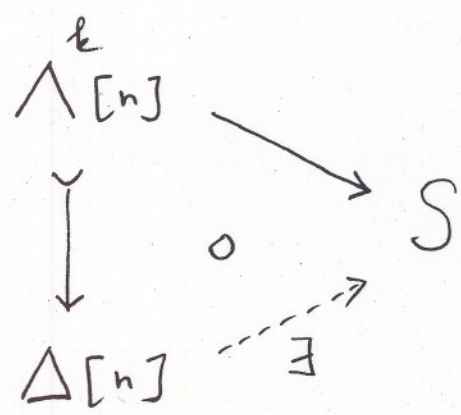
- For $S \in \text{simpSet}$, $S^\uparrow$ is its reduct to L .

• Homotopy for simplicial maps: $X \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} Y$ in simp Set:
 $f \sim g \iff \text{formally } \dots$ Same definition, with I
 replaced by $\Delta[1]$

$$= Y[1] = \text{hom}_{\Delta}[-, [1]].$$

— • —

• $S \in \text{simp Set}$ is a Kan-complex if



$\bigwedge^k [n]$: subfunctor of
 $\Delta[n] = \text{hom}_{\Delta}(-, [n])$

missing the two elements: $[n] \xrightarrow{\text{id}} [n]$;
 and $[n-1] \xrightarrow[\text{injective}]{\delta_k} [n]$, $k \notin \text{Im}(\delta_k)$
 ("omit k ")

Significance: For $T, S \in \underline{\text{simpSet}}$,

the realization functor $\underline{\text{simpSet}} \rightarrow \underline{\text{TopK}}$

induces

$$\text{hom}_{\underline{\text{simpSet}}}(T, S) \xrightarrow{\varphi} \text{hom}_{\underline{\text{TopK}}}(|T|, |S|)$$

If S is Kan, then φ induces

a bijection of the homotopy classes

of maps $T \rightarrow S$, and the homotopy classes of maps

$|T| \rightarrow |S|$.

"The homotopy category of Kan-complexes is the same as the homotopy category of spaces"

DEF Kan-complexes S and T are
homotopy-equivalent, $S \sim T$, if $\exists$:

$$\begin{array}{ccc} \textcircled{gf \sim id_S} \hookrightarrow S & \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{f} \end{array} & T \begin{array}{c} \xleftarrow{id_T \sim fg} \end{array} \end{array}$$

Proposition S, T : Kan-complexes

1. $S \sim T \iff S^\uparrow \cong_L T^\uparrow$

2. Intrinsic identity in $S^\uparrow$
 coincides with homotopy of maps into S .

3. For $M \in L\text{-Str}$,

$$M \cong_L S^\uparrow \implies \exists T \text{ Kan complex} \\ M = T^\uparrow$$

4. Any two degeneracy structures on $S^\uparrow$ are intrinsically identical (homotopic).