A new set theory

1. Abstract sets

Naive set theory seems, to "most of us mathematicians", to be a natural, indeed indispensable, basis of mathematics. It is another matter when we come to formalized forms of set theory. The most common reaction of the practicing mathematician to formalized set theory is that it can safely be left to the logicians to worry about. This indifference to (any) formalized set theory is based, in the first place, on a firm trust in a common, unanalyzed, naive intuition of what is all right to do with sets and what is, exceptionally, not all right to do with them [?0 exhibited, for example, by van den Dries, at the start of the FOM exchange]. In the second place, however, there is here a further element of *refusal*; namely, the refusal to go along with the *restrictions* implied by any given formalism for set theory. Naive set theory is an unprincipled assemblage of intuitions some of which are seen quite irreconcilable with each other, once we approach the matter with the "pedantry" of the logician. The mathematician, on his part, will not give up any of these intuitions, and finds it unproblematic how to resolve the apparent contradictions, namely, mainly by ignoring them: saying that they occur in only outlandish extremes that are never to be faced in (real) mathematics.

On the one hand, we have the basic and very clear intuition of the Cantorian universe (perhaps more appropriately called the von Neumann universe [1? look up Hallett?]), the hierarchy of well-founded regular (pure) sets. A very nice exposition and defense of this intuition is given by Adrian Mathias in [2?], "defense" indeed against the category theorists' attack. [3? Did Mathias emphasize the fact that all set-theoretical constructions of concrete mathematical entities, such as real numbers, complex numbers, functions on the reals, etc., fall completely naturally into the Cantorian universe, with the possible exception of ordered pairs?]. The standard formal systems, the main one being ZFC, formalize the language to reason in about the Cantorian universe.

On the other hand, consider the following fragment of an imaginary mathematical text:

"Let G and H be arbitrary groups. We are going to describe the construction of a third group out of G and H; this group will be denoted by $G\Box H$. The underlying set of $G\Box H$ is the intersection $|G|\cap|H|$ of the underlying sets of the groups if this set is non-empty, and it is $\{1\}$ otherwise. The group operation on $G\Box H$ is defined as follows: ..."

I don't have to finish the example (of course, this can be done in more than one formal-set-theoretically rigorous way) to make the point that this construction is felt by "many of us" to be an improper definition in group theory. The most immediate explanation of this feeling is that, in a proper construction starting with two arbitrary groups, we cannot rely on the relationships such as the equality or the lack of it of the elements of one of the groups to the elements of the other. The two "arbitrary" groups are given "abstractly"; we know nothing, and should not assume anything, about their elements, and by implication, about what the overlap of their underlying sets is.

A more mathematical explanation is that the construction as described is not (cannot be) invariant under *isomorphism*, a requirement that seems natural. $G \cong G'$ and $H \cong H'$ should imply $G \square H \cong G' \square H'$ which requirement clearly fails (must fail) in the above example ($G \cong G'$ denotes: G is *isomorphic* to G' ; see more on this below).

The connection between the two explanations is at the heart of what I have to say today. To anticipate the main point: it is possible to set up a formal system of a "set theory" in which (1) it is *impossible* to define an operation $G, H \longmapsto G \square H$ of groups which is not invariant under isomorphisms in the sense above, and yet, (2) all the usual isomorphism-invariant such operations *are* definable.

Underlying modern-day abstract mathematics, such as group theory, there is an intuition of "abstract set" as opposed to the "concrete sets" of the Cantorian universe. The elements of an abstract set are featureless points. The idea is familiar as "sets with urelements". An ambitious modern-day treatment of set-theory that mixes urelements with the Cantorian hierarchy is [?3 Barwise].

My aim today is to describe a formal theory of abstract sets. I will again act as a pedantic logician, demanding severe restrictions on mathematical practice, which is based on naive set theory. These restrictions will be quite different compared to, and probably even more distasteful to the mathematician than, those presented by formal ZFC, say. I have no apologies to offer other than to say that I hope the system reflects a reasonable, even if one-sided, view, of the stuff of mathematics. To put it more positively, I think that the system exhibits an objectively present and significant *structure* in mathematical reasoning.

The ultimate goal of the system is provide a genuinely non-Cantorian way, that is, a way that has nothing to do with the "doctrine of size" [?4 Hallett], out of the impasse reached by

Gottlob Frege as pointed out by Bertrand Russell.

Imagine a "set theory" (it could also be called a "class theory", or a "theory of totalities"), the set theory I am trying to explain here, in which *all sets* are abstract: only urelements appear as elements of sets. On the other hand, there are things, in fact a whole hierarchy of things, other than sets. You are apt to say: this is cheating; I just call those other things other names, but in fact I will smuggle in non-abstract sets as well. The reply to this objection is this: in the theory in question, all groups without exception, all topological spaces, and all other "such things" (note: a *category* is not such a thing!; see later) will have *abstract* sets as underlying sets. To put it in another way: the grammar of the system will prevent contemplating a group whose underlying "set" is anything other than an "abstract set", the only kind of totality that is called *set* in the system. On the other hand, other kinds of totalities, in fact many kinds of totalities other than sets, make an appearance in the system.

2. Equality and identity

I will use "equality" in the specific sense we are accustomed to in mathematics, when we say things like "such and such natural number is equal to 2 ", "two such and such sets of reals are equal to each other, that is, are the same", etc. On the other hand, I will have to talk about *identity* as a relation, as in "categories $\mathbf{X}$ and $\mathbf{A}$ are identical", in a more general sense, the clarification of which sense being one of the main tasks at hand. This way of talking has its difficulties. The phrase "categories $\mathbf{X}$ and $\mathbf{A}$ are identical" will actually mean something *weaker* than "categories $\mathbf{X}$ and $\mathbf{A}$ are equal", a fact one has to get used to. In each of many specific cases, (abstract) mathematical practice has its established way of saying what I want: in the example given, this is " $\mathbf{X}$ and $\mathbf{A}$ are equivalent categories". I need the single term "identity" to encompass several different specific meanings (further example: "bicategories $\mathbf{X}$ and $\mathbf{A}$ are biequivalent", whatever that means). [?5 Am I now doing what Wittgenstein advises against: violating the ordinary use of a word?]. The word "equivalent" feels too neutral for the purpose at hand.

From now, by 'set' I mean "abstract set".

In the present conception to be explicated now (this qualification is to be understood throughout: I do not want to exclude the possibility of someone else using abstract sets in a different way), in the realm of (abstract) sets, equality is restricted to one set at a time. Each set A comes with its own equality predicate $=_A$, a binary relation *defined for elements of* A *only*. It is meaningless, rather than false, to say that $a=_Ab$, for instance, for an element b of an unspecified set, say, and an element a of A . It is also meaningless to ask, with a and b as before, if $a=b$, with $=$ appearing without a subscript, simply because there is no such primitive $=$ in the system. In short: $a=_Ab$ is meaningful (and has a truth-value such as true or false) if and only if a and b stand for elements of A .

For each set A , $=_A$ is assumed to be an equivalence relation, or, in an alternative formulation, a partial equivalence relation (PER) (familiar from Boolean valued sets [?6] and "realizability universes" [?7]) (the latter misses the reflexive condition).

We reject (once again, within the present conception) the view that the totality of mathematical objects has a meaningful global equality relation on it. Equality is "expensive"; we can afford it in small bites only, on each set separately.

In type theory, it is customary to restrict meaningfulness of relations to entities that are assumed to be of definite types. For instance, in second order number theory, writing $a \in A$ assumes that a is of the type "integer", A is of the type "set of integers". Here, we treat equality in a similar, strictly typed manner.

We enforce said restrictions through an imposed grammar. This has the general character of a theory of types. *In one version of the System* (I will allow myself to write "the System" for the thing I want to explain), there is a type called **SET** ; we have variables A , B , ... of type **SET** ; intuitively, A , B , ... range over (abstract) sets. We write $A \in \mathbf{SET}$ to *declare* that the variable A is of type SET . (Within the system, the symbol $\in$ will only be used for variable declaration.) Then (and this is the essential "new" [?8 suitably understood: refer to Martin-Lof] grammatical feature in the System), we have the *dependent* type $\mathbf{E}(A)$ of elements of A : the variables of type $\mathbf{E}(A)$ are meant to range over the elements of A . Thus, $a \in \mathbf{E}(A)$ is to mean that a belongs to the set A. (There is no harm in abbreviating $a \in \mathbf{E}(A)$ to $a \in A$).

What is important here is that $a \in \mathbf{E}(A)$ is a *variable declaration*, rather than a *proposition*, in the System. For instance, one cannot *deny* (grammatically, that is) that $a \in \mathbf{E}(A)$; $\neg(a \in \mathbf{E}(A))$ is ungrammatical. In contrast,

$$\forall A \in \mathbf{SET}.\ \forall x \in \mathbf{E}(A).\ \forall y \in \mathbf{E}(A).\ (x=_Ay \longrightarrow y=_Ax)$$

is a grammatical, in fact, a "true", proposition.

The last displayed formula revals much of the basic *syntax* of *First Order Logic with Dependent Sorts* (FOLDS), the *restricted* version of first order logic underlying the System. FOLDS is a general-purpose language, much like ordinary first order logic. Just as in the case of the latter, we have a variable *signature*, which is, however, something slightly more structured than an ordinary first-order signature: it is a special kind of category. The main point of the *semantics* of FOLDS is the concept of *identity* associated with every signature; more on this later.

I talked about FOLDS at the 1995 Haifa meeting of the ASL. The paper in the proceedings of the meeting, published in 1998, gives a reasonably complete description of FOLDS. There is also a manuscript of a monograph containing detailed work on FOLDS. It is available from my website, which was accepted for publication at one point, but which then I withheld because I thought I could make it better-looking.

There is precious little one can say about abstract sets without things connecting them. These things will be the *functions*, a new primitive of the System. (Of course, we are now taking the hint from Category Theory, in which sets form a *category*; but we do not have to formally bring in categories at this point). Having in mind the word "arrow", abbreviated as **A** in the notation, for "function", we display the start of the grammar of functions by the variable declaration

$$A, B \in \mathbf{SET}\ ;\ f \in \mathbf{A}(A, B)$$

which declares f to be a function from the set A to the set B. This features a *dependent type* $\mathbf{A}(A, B)$ *depending* on two variables A and B.

One way of proceeding is to use the *kind* (type-heading) **Apply**, subject to the grammar

$$A, B \in \mathbf{SET}\ ;\ f \in \mathbf{A}(A, B)\ ;\ a \in \mathbf{E}(A)\ ;\ b \in \mathbf{E}(B)\ ;\ \tau \in \mathbf{Apply}(f, a, b)$$

which is to read as:

" τ is a witness to the fact that, applying the function $f: A \to B$ to the element a of A, we obtain the element b of B ".

For $A \xrightarrow{f} B$, that is, for $\langle A, B \in \mathbf{SET}\ ;\ f \in \mathbf{A}(A, B) \rangle$, we use $f'a=b$ as an abbreviation for $\exists \tau \in \mathbf{Apply}(f, a, b).\top$ ($\top$ is "true", the nullary propositional connective).

(This example also shows how FOLDS handles relations and operations in the theory via types. Strictly speaking, the relation $=_A$ should be, and in fact is, treated similarly.)

Then, as an example, the fact of the *existence* of the composite $A \xrightarrow{h=gf} C$ of the functions $A \xrightarrow{f} B \xrightarrow{g} C$ will be expressed by the sentence

$$\forall A, B, C \in \mathbf{SET}.\forall f \in \mathbf{A}(A, B).\forall g \in \mathbf{A}(B, C).\exists h \in \mathbf{A}(A, C).$$
$$\forall a \in A).\forall c \in B).(h'a=c \longleftrightarrow \exists b \in \mathbf{E}(B)(f'a=b\ \&\ g'b=c)) \quad .$$

Since equality of *parallel* functions $A \underset{g}{\overset{f}{\rightrightarrows}} B$ can be *defined* by

$$f=g \underset{\text{def}}{\Longleftrightarrow} \forall a \in A.\forall b \in B(f'a=b \longleftrightarrow g'a=b) \quad ,$$

we can easily state (as an axiom) the fact that the composite is uniquely defined.

I do not need to go on like this. Although not formulated in the language used above, *topos theory*, based on the notion of *elementary topos* [?9], shows that a considerable amount of set theory, including nth order number theory for any finite n, can be developed [?10] in the above described context. In fact, a good intuitive basis for understanding what happens in topos theory is to pose oneself the problem of developing the theory of abstract sets and functions within the constraints outlined above. [?11: Lawvere's "abstract sets"]

For instance, consider the fact that the Cartesian product $A \times B$ of sets A and B *cannot* be defined as the set of the ordered pairs (a, b) with $a \in A$, $b \in B$ simply because, whatever they are, ordered pairs are *not* urelements. What one *can* do is say that, whatever $C=A \times B$ is, it comes with two projections $\pi_A : C \to A$, $\pi_B : C \to B$, the maps $(a, b) \mapsto a$, $(a, b) \mapsto b$, which together are recognized to have something called a universal property in Category Theory, a property that can be written down in our language.

We have arrived at an interesting point. Even without having fully written down the "category theoretical" definition of (Cartesian) product, it seems clear that the product $A \times B$ can never by such means be defined *uniquely*, since whichever set C, with appropriate π_A, π_B, qualifies, if another set D *isomorphic* to C, in notation $D \cong C$ (meaning that there exist arrows

$$D \underset{g}{\overset{f}{\rightleftarrows}} C$$

such that the diagrams

$$\begin{array}{ccccc} & & C & & \\ & f \nearrow & \circ & \searrow g & \\ D & & \xrightarrow[\mathrm{id}_D]{} & & D \end{array} \qquad \begin{array}{ccccc} & & D & & \\ & g \nearrow & \circ & \searrow f & \\ C & & \xrightarrow[\mathrm{id}_C]{} & & C \end{array}$$

commute), then D will also qualify as the product (with $\pi_A \circ f$, $\pi_B \circ f$ as the new projections). In general, a "category theoretical" definition of a set, in terms of some given sets, and possibly also some given functions, will define its object *at most* up to isomorphism. On the other hand, the product *is*, as a matter of fact, defined uniquely up to isomorphism.

Which brings us to the recognition that not only *we did not need* "the" usual equality of sets in our theory, but in fact, we discovered another notion of identity, that of isomorphism, derived from, in fact, *defined* in terms of, the primitives, which is the operative concept of *identity* for sets.

It is instructive to go through the steps to recognizing that "the usual" definition of equality of sets, the one which defines equality in terms of extensionality, cannot be grammatically stated in the System.

3. Totalities of sets: categories.

We want to go beyond sets: I want to consider various totalities of sets. The question is: "what kind of totality do (all) sets form?"

One important element in this quest is that *structured sets* are also to be included here as "sets": we want to consider totalities of sets structured in any one of many specific ways: we want to consider the totality of groups, that of topological spaces, etc. This requirement calls for a suitably general concept of totality.

Secondly: the totality of sets is not a set. Not, in this way of thinking, for a reason like "there are too many sets for them to form a set", but rather because there isn't, or does not seem to be, a reasonable notion of *equality* of sets.

Finally: the kind of totality sets form is essentially revealed by the way we talk about sets, samples of this talk having been shown above. It is now a matter of a good act of abstraction to find the right concept.

The concept of *category* is the answer: sets form a *category*.

The concept of category was introduced by S. Mac Lane and S. Eilenberg in 1945 [?12], and its use has become widespread in "abstract" mathematics.

(Some remarks on the notion of category compared to the language exhibited above for sets and functions. Talking about a category, we do not use elements of an object (set in the category of sets) as a primitive as we did above, and we do not use application of functions to elements. Instead, we use *composition* of functions (arrows) as primitive. In the category of sets, elements of as set A can be recovered as functions $a{:}1 \to A$, where 1 is the "terminal" object (one-element set), and application reduces to a special case of composition;

$$1 \xrightarrow{a} A \xrightarrow{f} B : \qquad f'a = f \circ a \ .)$$

As a first element of the "foundational reform"(ulation) I am carrying out, the definition of the concept of "category" is rewritten, in the form of a structure for a particular FOLDS-signature denoted $\mathcal{L}_{cat}$, with the intention that talk about a category should be framed within FOLDS over the signature $\mathcal{L}_{cat}$. Section 8 of my above-mentioned paper gives $\mathcal{L}_{cat}$ (see top of page 172; $\mathcal{L}_{cat}$ is a very small, five-object category), along with examples of statements of the language of FOLDS. All the axioms for "category" are (can be) stated in FOLDS over $\mathcal{L}_{cat}$. In particular, *one never uses equality of objects, and one uses equality on arrows only if those arrows are assumed parallel.*

Let me mention that the notion of $\mathcal{L}_{cat}$-equivalence, the canonical concept of identity associated with the signature $\mathcal{L}_{cat}$, coincides with the established notion of equivalence of categories for the case when the $\mathcal{L}_{cat}$-structures are in fact categories. This is a first instance in a series of facts that show that FOLDS-equivalence works. This is a welcome circumdtance since

The notion of *isomorphism* of objects is defined in any category.

The foundational reform continues with increased radicalism in regard to the notion of *functor*, the straighforward notion of structure-preserving map of categories. Because of the fact that isomorphism is the now-accepted notion of identity for objects in a category, any operation that results in an object should determine that object *only* up to isomorphism (as the operation of (categorical) product does) (this is an instance of the Leibniz rule of the indiscernibility of identicals). This makes it necessary to change the usual notion of functor, to something that determines its value-object up to isomorphism only. It is not entirely obvious how to do this (although it is not really difficult either) so that one retains all the essential features and uses of functors. This is described in section 5 of *loc.cit.* The full theory, set in the context of set theory without choice, is in the earlier paper [?M] whose title talks about avoiding the axiom of choice. Indeed, the main point is that, using anafunctors (as the new concept is called), one avoids the otherwise ubiquitous use of the Axiom of Choice in category theory, a use that may be said to vitiate some of claims as to the canonical nature of categorical constructions. It is to be noted that the axiom of choice, in a form in which we are supposed to pick a single object out of a totality of objects, simultaneously for all totalities from a given family, cannot even be stated in the System; thus, we'd better be able to do what we do in [?] if we wish to entertain any hope to do category theory in the new System.

Faithful to the spirit of formal uniformity, anafunctors too are (can be) presented as structures for a suitable FOLDS signature: see section 10, p.183, of *loc.cit.*