Makkai: Computads & weak ω-categories
Carvoeiro, June 2007

$\boxed{0}$ 2-categories and bifunctors:

(= homomorphisms of bicategories)

needed for Makkai - Paré: Accessible Categories

(2-functors are not enough;

but bicategories beyond 2-cats are

not needed)

$\underbrace{\text{2-cat's}}_{\text{0-cells}}$ and $\underbrace{\text{bifunctors}}_{\text{1-cells}}$, with

pseudo-natural transformations: 2-cells

&

modifications : 3-cells

form a proper tri-category:

not equivalent to a 3-category

∴ tri-categories

and

weak-n-categories

are needed.

$\boxed{1}$ In a multitopic category

(Baez - Dolan - Hermida - M - Power)

composition itself is defined by a universal property; in particular, not uniquely determined.

① Composition of 1-cells f & g:

$$\forall \quad \begin{tikzcd} & \cdot \arrow[r, "g"] & \cdot \\ \cdot \arrow[ur, "f"] & & \end{tikzcd} \qquad \exists : \quad \begin{tikzcd} & \cdot \arrow[r, "g"] \arrow[d, phantom, "\Downarrow u"] & \cdot \\ \cdot \arrow[ur, "f"] \arrow[urr, bend right, "k"'] & & \end{tikzcd} \qquad (1)$$

$\forall f, g$ as shown $\exists\, k\ \exists\, u$

○ : universal [see later]

□ : $\exists$

domain of u: (f, g)

not a composite of f and g

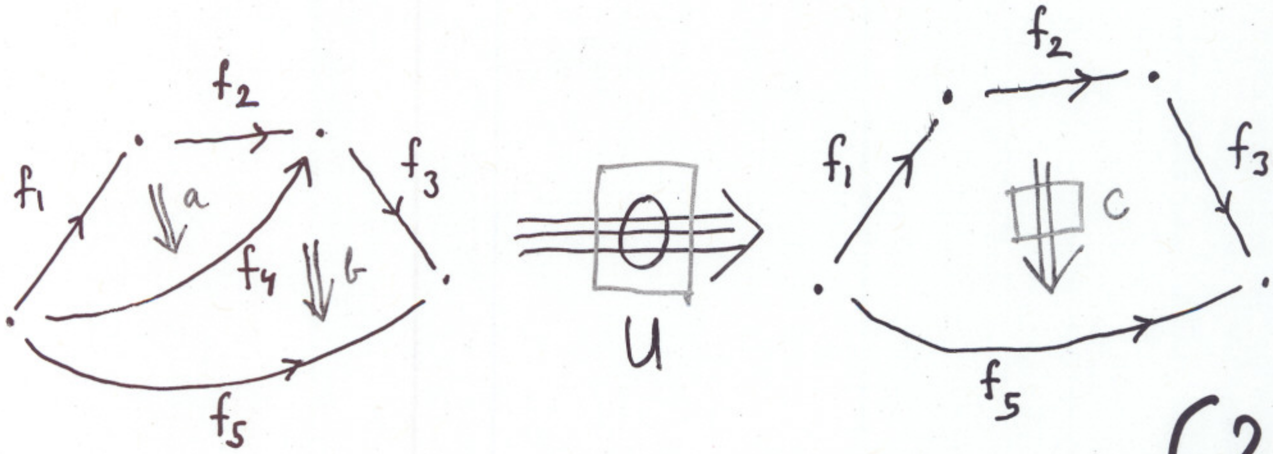
<u>Example:</u> tensor product of Abelian groups

$$k = f \otimes g$$

u: universal bilinear map

(more generally: tensor product of bimodules)

⊙ Composition of 2-cells a & b:

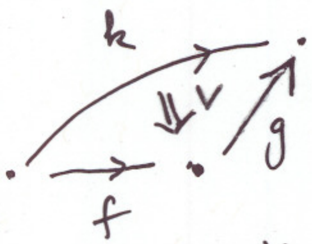


(2)

In (1), u universal

$\approx$ "invertible" (?)

which would mean: $\exists$ v as in



$c(v) = (f, g)$

plus further properties connecting u & v

but the shape

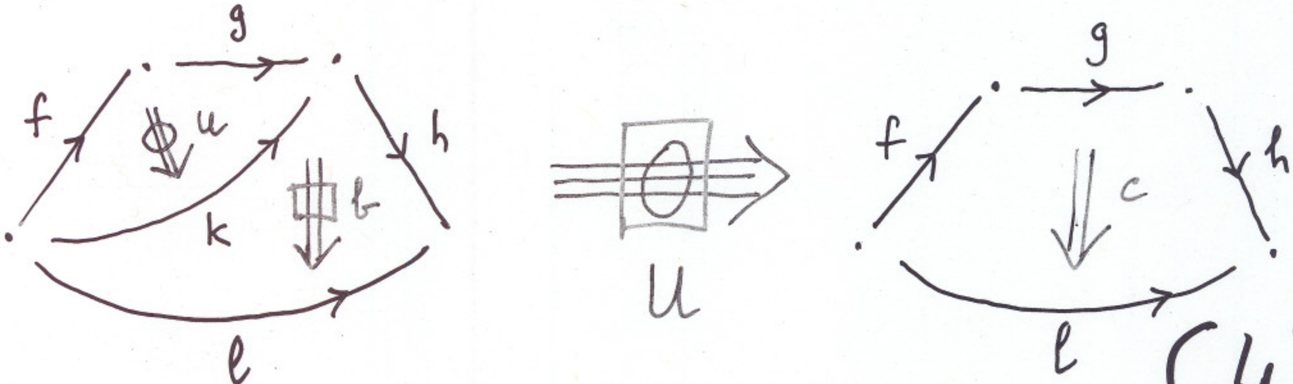
is not allowed

(3)

in a multitopic category!

For the definition of universal, we require consequences of the existence of the 'inverse' v:

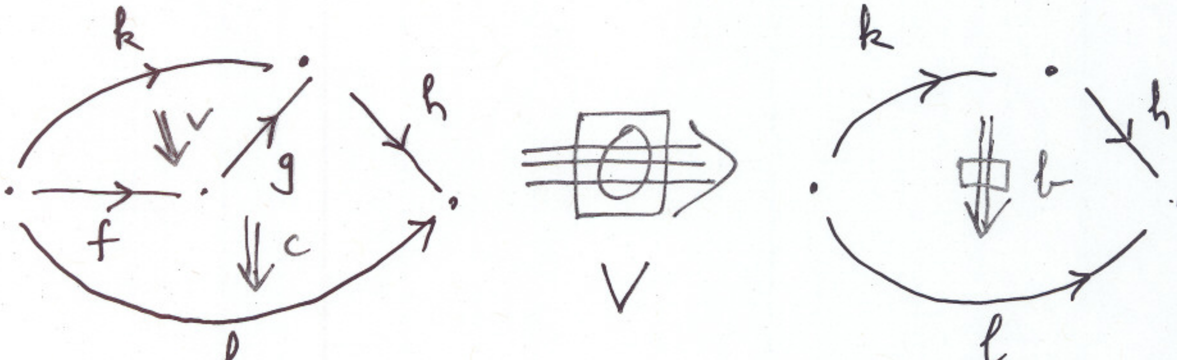
example:



(4)

$$\forall \underset{\text{univ}}{u}\ \forall c\ \exists b\ \exists \underset{\text{univ}}{U}$$

(With the forbidden v, this would be a composition:



)

— · —

Why don't we allow the shape (3) – and "all" shapes, in fact?

Answer: we get too-strong notion: essentially strict ω-category.

$\boxed{2}$ Computad (Ross Street)

$=$

polygraph (Albert Burroni)

Given: $\left\{\begin{array}{l}\omega\text{-category} \\ \infty\text{-category}\end{array}\right\}$ $\mathbb{X}$

$\mathcal{U}$: set of <u>indeterminates</u>

<u>attached</u> to $\mathbb{X}$:

$$u \in \mathcal{U} \longmapsto \left\{\begin{array}{l} du \\ \\ cu \end{array}\right. \in \|\mathbb{X}\| = \coprod_n \mathbb{X}_n$$

such that $du \parallel cu$

define **ω-cat**

$$\mathbb{X}[\mathcal{U}]$$

"freely adjoin each $u \in \mathcal{U}$ to $\mathbb{X}$":

$$\boxed{\begin{array}{c} \mathbb{X}[\mathcal{U}] \xrightarrow{F} \mathbb{Y} \\ \hline \mathbb{X} \xrightarrow{\Phi} \mathbb{Y} \;;\quad u \overset{\Psi}{\longmapsto} \hat{u} \in \|\mathbb{Y}\| \\ \text{subject to: } \left\{\begin{array}{l} du \overset{\Phi}{\longmapsto} d\hat{u} \\ cu \overset{\Phi}{\longmapsto} c\hat{u} \end{array}\right. \end{array}}$$

$$\boxed{F \longleftrightarrow (\Phi, \Psi)}$$

Definition ω-category $\mathbb{X}$ is a computad

if$_{def}$: $\forall n \in \mathbb{N} \cup \{-1\}$

$$\mathbb{X} \restriction (n+1) = \mathbb{X}[U_{n+1}]$$

for some set U_{n+1} of $(n+1)$-dim indets.

$|\mathbb{X}| = \dot{\bigcup}_{n \in \mathbb{N}} U_n$: indets in $\mathbb{X}$

can be recovered as the indecomposables:

a indecomposable $\Longleftrightarrow$

$\forall b \quad a \neq 1_b$

&

$\forall k \quad \forall b, c \quad a = b \cdot_k c \Longrightarrow$

$\exists d \in \mathbb{X}_k \quad b = 1_d^{(n)}$ or $c = 1_d^{(n)}$

Morphism of computads: $\mathbb{X} \xrightarrow{F} \mathbb{Y}$ ω-cat map

such that: indet $\xmapsto{F}$ indet

$\underline{\mathrm{Comp}}$: category of (small) computads

non-full inclusion

$$\underline{\mathrm{Comp}} \xrightarrow{\langle - \rangle} \underline{\omega\mathrm{Cat}}$$

$\underline{\mathrm{Comp}}$ is a locally finitely presentable cat

(computad $\mathbb{X}$ is finitely presentable

$\iff$ $|\mathbb{X}|$ is a finite set)

set of indets

But $\underline{\mathrm{Comp}}$ is not a presheaf category

and I think it is not a topos.

$\langle - \rangle$ **has a right adjoint** : $\langle - \rangle \dashv [-]$

$$\begin{tikzcd} \underline{\mathrm{Comp}} \arrow[r, shift right=1.5ex, "\langle - \rangle"'] \arrow[r, phantom, "\top"] & \omega\mathrm{Cat} \arrow[l, shift right=1.5ex, "{[-]}"'] \\ {[\mathbb{X}]} & \mathbb{X} \arrow[l, mapsto] \end{tikzcd}$$

nerve of $\mathbb{X}$

counit $\quad [\mathbb{X}] \xrightarrow{\varepsilon} \mathbb{X}$

defined recursively:

Suppose $[\mathbb{X}]\upharpoonright n$ and

$$[\mathbb{X}]\upharpoonright n \xrightarrow{\varepsilon} \mathbb{X}\upharpoonright n$$

has been def'd:

$\varepsilon = [\![-]\!]$
'evaluation'

$$[\mathbb{X}]\upharpoonright(n+1) \overset{DEF}{=} ([\mathbb{X}]\upharpoonright n)[U_{n+1}]$$

where

$$u \in U_{n+1} \iff u = (\delta, \gamma, a)$$

with: $\delta, \gamma \in [\mathbb{X}]_n$ (arbitrary n-cells in $[\mathbb{X}]$)

such that $\delta \parallel \gamma$

and

$a \in \mathbb{X}_{n+1}$ (arbitrary $n+1$-cell in $\mathbb{X}$)

such that

$$du = [\![\delta]\!] \quad (= \varepsilon(\delta))$$
$$cu = [\![\gamma]\!] .$$

$$du \overset{DEF}{=} \delta \; ; \quad cu \overset{DEF}{=} \gamma \; ; \quad \underset{\varepsilon u}{[\![u]\!]} \overset{DEF}{=} a$$

picture:

$$\begin{tikzcd}
\delta \arrow[rr, "u"] \arrow[d, "\varepsilon"'] & {} \arrow[d, "\varepsilon"] & \gamma \arrow[d, "\varepsilon"] \\
da \arrow[rr, "a"'] & {} & ca
\end{tikzcd}$$

$\boxed{3}$ **Many**-to-one computads:

underlying the multitopic categories

Computad $\mathbb{X}$ is many-to-one if:

codomain of any indet in $\mathbb{X}$ is an indet.

(of dim ≥ 1)

$\underline{\mathrm{Comp}}_{m/1} \subset \mathrm{Comp}$

full subcat

a presheaf category

A multitopic category is a many-to-one computad $\mathbb{X}$ with the additional <u>property</u>:

"every pd (pasting diagram) $\in \|\mathbb{X}\|$ has a composite":

$$\forall n \geq 1.\ \forall a \in \mathbb{X}_n\ \exists x \in |\mathbb{X}|_n\ \exists u \in |\mathbb{X}|_{n+1}$$

Such that $x \| a$, u is universal, and $du = a$, $cu = x$:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "a"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "u\Downarrow"] & \cdot \end{tikzcd}$$

Here: 'universal' has a "coinductive" definition, which is somewhat complicated

<u>Example</u> for a clause in this: display (4) on page 4:

$$\forall u \in \mathrm{Univ}\ \forall c\ \exists b\ \exists U \in \mathrm{Univ}\ \dots$$

— · —

Suggestion: use arbitrary computads

result: computad-weak-ω-category

The def'n becomes simplified because of the availability of "inverse" shapes.

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Fix ω-cat $\mathbb{X}$.

4 Equivalence cells

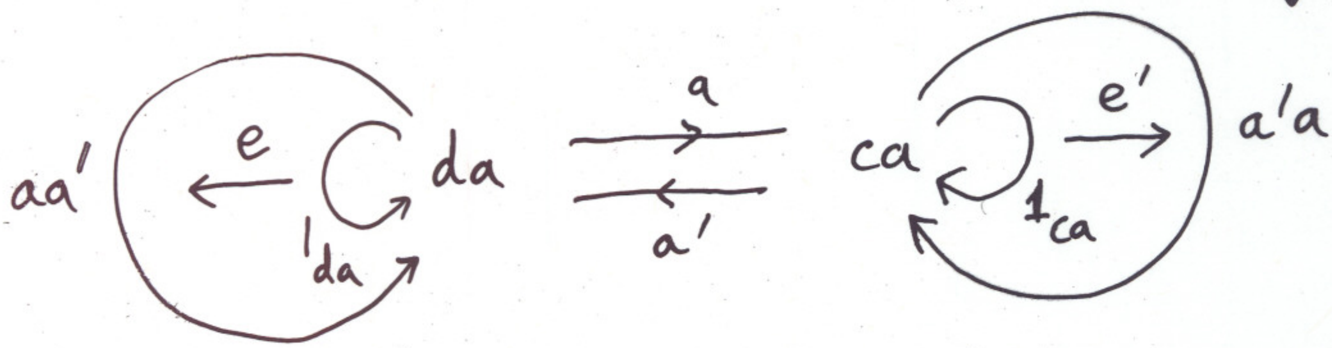
$\mathbb{X}$: ω-cat

Let: $E \subseteq \dot{\bigcup}_{n \geq 1} \mathbb{X}_n$

E is coinductive if_{def}

N.B.: geometric composition notation!

$a \in E \Rightarrow \exists\, a', e, e'$:



<u>and</u> $e \in E$, $e' \in E$.

Any union of coinductive sets is coinductive. The largest coinductive set: $E_{\mathbb{X}}$ = union of all coinductive sets is, by def'n, the set of all equivalence cells of $\mathbb{X}$.

Examples of properties of equivalence cells, proved by 'coinduction':

$$e, a \in E_{\mathbb{X}} \quad \Rightarrow \quad e \cdot_k a \in E_{\mathbb{X}}$$

$$e \in E_{\mathbb{X}} \quad \& \ \dim(e) > \dim(a)$$
$$\Rightarrow \quad e \cdot_k a \in E_{\mathbb{X}}$$

In particular, all 0-cells plus all equivalence cells form a sub-ω-cat.

Definition A computad-weak-ω-category is a computad $\mathbb{X}$ such that [see p. 9!]

$$\forall n \geq 1.\ \forall a \in \mathbb{X}_n\ \exists x \in |\mathbb{X}|_n\ \exists u \in |\mathbb{X}|_{n+1}$$

with:

$$x \| a\ , \quad u \in E_{\mathbb{X}}\ , \quad du = a,\ cu = x:$$

(!)

$$\begin{tikzcd} \cdot \arrow[r, bend left, "a"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \end{tikzcd}$$

Proposition (easy). For any ω-category $\mathbb{X}$, the nerve-computad $[\mathbb{X}]$ is a computad-weak-ω-category.

5 Equivalence of computads.

Definition 1) Computad map $P \xrightarrow{F} \mathbb{X}$ is fiberwise surjective (f.s.) if_{def}:

$$
\begin{tikzcd}
P: & \delta \arrow[d, mapsto, "F"'] & \parallel & \gamma \arrow[d, mapsto, "F"] \\
\mathbb{X}: & dx \arrow[rr] & & cx
\end{tikzcd}
\qquad \exists\, y: \qquad
\begin{tikzcd}
\delta \arrow[r, "y"] & \gamma \arrow[d, mapsto, "F"'] & : P \\
dx \arrow[r, "x"'] & cx & : \mathbb{X}
\end{tikzcd}
$$

$\forall$: $x \in |P|$ (indet)

2) $\mathbb{X} \underset{\text{equivalent}}{\simeq} \mathbb{Y} \overset{\text{DEF}}{\Longleftrightarrow} \exists$:

$$
\begin{tikzcd}
& P \arrow[dl, "F"'] \arrow[dr, "G"] & \\
\mathbb{X} & & \mathbb{Y}
\end{tikzcd}
$$

such that F and G are <u>f.s.</u>

Theorem Any computad-weak-ω-category is equivalent to the nerve of an ω-category:

if $\mathbb{X}$ is a comp-w-ω-cat

then $\mathbb{X} \simeq [\mathbb{X}]$.

In fact:

$$\exists: \quad \begin{tikzcd} {[\mathbb{X}]} \arrow[d, dotted, bend left, "\rho"] \\ \mathbb{X} \arrow[u, "\eta"] \arrow[loop below, "1_{\mathbb{X}} = \rho \circ \eta"] \end{tikzcd} \qquad \rho: \text{computad map}$$

For the proof, we need a construction: $\mathcal{H}(\mathbb{X})$ the ω-cat of homotopies in an ω-cat $\mathbb{X}$.

This follows / modifies Metayer's $H(\mathbb{X})$
(Metayer does not use equivalence cells)

$\boxed{6}$ The ω-cat of homotopies.

$\boxed{6.1}$ Lax ∞-natural transformations

Recall: lax 2-natural transformations:

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow H"] & \mathbb{X} \end{tikzcd}$$

($\mathbb{A}$, $\mathbb{X}$: **2**-cats; F, G: 2-functors)

Data for H:

$$\left.\begin{array}{l} A \in \mathbb{A}_0 \longmapsto HA \in \mathbb{X}_1 \\ f \in \mathbb{A}_1 \longmapsto Hf \in \mathbb{X}_2 \end{array}\right\} :$$

Such that

$$A \xrightarrow{f} B \;::\; \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "HA"'] \arrow[dr, phantom, "\Downarrow Hf"] & FB \arrow[d, "HB"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

$$\boxed{Hf : Ff \cdot HB \longrightarrow HA \cdot Gf} \quad (0)$$

[geom. notation for composition]

Conditions:

1) $\boxed{\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & B \end{tikzcd}}$ generates:

$$\begin{tikzcd}
FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] \arrow[d, "HA"'] & FB \arrow[d, "HB"] \\
GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GB
\end{tikzcd}
\quad\Bigg|\quad
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "HA"'] \arrow[dr, phantom, "Hf\Downarrow"] & FB \arrow[d, "HB"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}
\quad\Bigg|\quad
\begin{tikzcd}
FA \arrow[r, "Fg"'] \arrow[d, "HA"'] \arrow[dr, phantom, "\Downarrow Hg"] & FB \arrow[d, "HB"] \\
GA \arrow[r, "Gg"'] & GB
\end{tikzcd}$$

with two pastings:

$$F\alpha \,\square\, Hg \underset{\substack{\uparrow \\ \text{require}}}{=} Hf \,\square\, G\alpha \qquad (1)$$

2) $\boxed{A \xrightarrow{f} B \xrightarrow{g} C}$ generates:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "HA"'] \arrow[dr, phantom, "\Downarrow Hf"] & FB \arrow[r, "Fg"] \arrow[d, "HB"] \arrow[dr, phantom, "\Downarrow Hg"] & FC \arrow[d, "HC"] \\
GA \arrow[r, "Gf"'] & GB \arrow[r, "Gg"'] & GC
\end{tikzcd}
\quad\Bigg|\quad
\begin{tikzcd}
FA \arrow[r, "F(fg)"] \arrow[d, "HA"'] \arrow[dr, phantom, "\Downarrow H(fg)"] & FC \arrow[d, "HC"] \\
GA \arrow[r, "G(fg)"'] & GC
\end{tikzcd}$$

$$Hg \,\square\, Hf \underset{\substack{\uparrow \\ \text{require}}}{=} H(fg) \qquad (2)$$

3) $A \xrightarrow{1_A} A$ generates:

$$\begin{tikzcd} FA \arrow[r, "1_{FA}"] \arrow[d, "HA"'] \arrow[dr, phantom, "\Downarrow H(1_A)"] & FA \arrow[d, "HA"] \\ GA \arrow[r, "1_{GA}"'] & GA \end{tikzcd}$$

$$H(1_A) = 1_{HA} \qquad (3)$$

require

(end of def of lax 2-nat tr.)

—·—

In the higher-dimensional generalization, we turn equality (1) into an arrow

$$F\alpha \,\square\, Hg \xrightarrow[H\alpha]{} Hf \,\square\, G\alpha$$

but leave equalities (2) & (3) alone.

—·—

General definition:

ω-cats $\mathbb{A}, \mathbb{X}$; ω-functors F, G

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow H"] & \mathbb{X} \end{tikzcd} \qquad H: ?$$

$$H = \bigcup_{n=0}^{\infty} H_n \quad ; \quad H_n : \mathbb{A}_n \longrightarrow \mathbb{X}_{n+1}$$

write H for H_n

such that

1) $a \in \mathbb{A}_n$:

$$d(Ha) = ((Fa \cdot H(c^{(0)}a)) \cdot H(c^{(1)}a)) \cdots H(c^{(n-1)}a)$$
$$c(Ha) = H(d^{(n-1)}a) \cdots (H(d^{(1)}a)) \cdot ((H(d^{(0)}a) \cdot Ga))$$

[$\dim(c^{(i)}a) = i$;

$$c^{(i)}a = \underbrace{c}_{1} \cdots \underbrace{c}_{n-i} a$$

where $n = \dim(a)$

same for 'd']

2) $H(a \underset{k}{\cdot} b) =$

$$= H(b) \,\square\, H(a)$$

$$= (Fa \underset{k}{\cdot} H(b)) \underset{k+1}{\cdot} (H(a) \underset{k}{\cdot} G(b))$$

3) $H(1_a) = 1_{H(a)}$

(End of def)

$\boxed{6.2}$ Homotopy

$$\begin{tikzcd} \mathbb{A} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow H"] & & \mathbb{X} \end{tikzcd}$$

$$H : F \sim G$$

$$\overset{}{\underset{\text{DEF}}{\Longleftrightarrow}}$$

$H : F \xrightarrow{\square} G$ and

$$\forall a \in \|\mathbb{A}\| \quad Ha \in E_{\mathbb{X}}$$

$\boxed{6.3}$ The object of internal homotopies

Note:

$$\mathbb{B} \xrightarrow{K} \mathbb{A} \underset{G}{\overset{F}{\underset{\Downarrow H}{\rightrightarrows}}} \mathbb{X}$$

gives rise to

$$\mathbb{B} \underset{G\circ K}{\overset{F\circ K}{\underset{\Downarrow H\circ K}{\rightrightarrows}}} \mathbb{X}$$

by $(H \circ K)(b) \underset{\text{DEF}}{=} H(K(b))$

We have, for given $H : F \twoheadrightarrow G$:

$$\hom(\mathbb{B}, \mathbb{A}) \xrightarrow[H^*]{} \mathrm{Lax}(\mathbb{B}, \mathbb{X})$$

and for $H : F \sim G$

$$\hom(\mathbb{B}, \mathbb{A}) \xrightarrow[H^*]{} \underset{\text{homotopies}}{\mathrm{Ht}(\mathbb{B}, \mathbb{X})}$$

<u>Proposition</u> For any ω-cat $\mathbb{X}$

there are

$$\begin{tikzcd} H(\mathbb{X}) \arrow[r, shift left=2, "b"] \arrow[r, shift right=2, "l"'] \arrow[r, phantom, "\Downarrow m"] & \mathbb{X} \end{tikzcd} \leftarrow \text{(Metayer) (essentially)}$$

and

$$\begin{tikzcd} \mathcal{H}(\mathbb{X}) \arrow[r, shift left=2, "b"] \arrow[r, shift right=2, "l"'] \arrow[r, phantom, "\sim\downarrow m"] & \mathbb{X} \end{tikzcd}$$

$$m : b \sim l$$

Such that

$$\hom(\mathbb{B}, H(\mathbb{X})) \xrightarrow[m^*]{\cong} \mathrm{Lax}(\mathbb{B}, \mathbb{X})$$

bijection

$$\hom(\mathbb{B}, \mathcal{H}(\mathbb{X})) \xrightarrow[m^*]{\cong} Ht(\mathbb{B}, \mathbb{X})$$

bijection

$\mathcal{H}(\mathbb{X})$: "ω-category of homotopies in $\mathbb{X}$"

Metayer:

$H(\mathbb{X})$: "ω-cat of paths in $\mathbb{X}$"

$\boxed{6.3}$ Using the object of homotopies

Construction of ρ of the Theorem :

by dimensional recursion , we construct the diagram

$$\nu = [-] \text{ (nerve)}$$

$$\begin{tikzcd}
{[\mathbb{X}]} \arrow[r, "h"] \arrow[rr, bend left=50, "\nu"] \arrow[rr, bend right=50, "\rho"'] & \mathcal{H}(\mathbb{X}) \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\sim\downarrow m"] & \mathbb{X}
\end{tikzcd}$$

$$m \circ h = \varphi : \nu \xrightarrow{\sim} \rho$$