CMS Annual Meeting

June 2010

Fredericton, Nova Scotia

Talk by M. Makkai

(in the session:

Tensor Categories)

$\boxed{\text{Semi-strict omega-categories}}$

$\boxed{\omega^{\cdot}\text{-category}}$ ← MM: "The word-problem for computads", 2005

$\mathbb{X}$: ω-graph

$\mathbb{X}_0$: n-cells for all $n \in \mathbb{N}$

with domain (d) and codomain (c)

assignments

$$\sim \quad \mathbb{X}_0 \underset{c}{\overset{d}{\leftleftarrows}} \mathbb{X}_1 \underset{c}{\overset{d}{\leftleftarrows}} \mathbb{X}_2 \cdots \underset{c}{\overset{d}{\leftleftarrows}} \mathbb{X}_n \underset{c}{\overset{d}{\leftleftarrows}} \cdots$$

$$dd = dc$$

Usual ω-category : compositions & identity arrows

compositions: for $k < n$:

$$\mathbb{X}_n \underset{\mathbb{X}_k}{\times} \mathbb{X}_n \longrightarrow \mathbb{X}_n$$

$$(a, b) \longmapsto a \cdot_k b$$

$$\underbrace{c^{(k)}}_{\text{dimension} = k}(a) = d^{(k)}(b)$$

'geometric' notation

$$a \cdot_k b = b \circ_k a$$

← "usual"

Special case:

Let: $a, b \in |\mathbb{X}| = \coprod_{n \in \mathbb{N}} \mathbb{X}_n$

$\dim(a) = m \geq 1, \quad \dim(b) = n \geq 1$

$$k = \min(m, n) - 1 \quad (\geq 0)$$

('permanent notation')

$$N = \max(m, n)$$

Assume: $c^{(k)}(a) = d^{(k)}(b)$.

WRITE

$$a \cdot b \overset{\text{def}}{=} \underbrace{1_a^{(N)} \cdot_k 1_b^{(N)}}_{\text{dimension: } = N}$$

For ω'-category – equivalent definition of ω-category –,

$a \cdot b$:

the primitive operations (with expected domain/codomain relations; identity cells: ignored here)

Axioms (equalities):

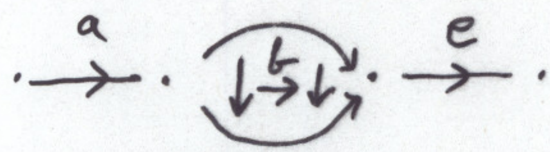
0) for identities (unit laws) : ...

1) <u>Associations</u> :

$$a(be) = (ab)e$$

"every time this makes sense"

Examples:



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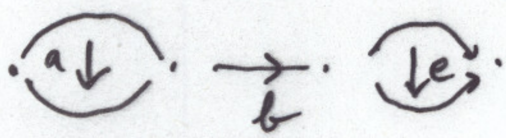

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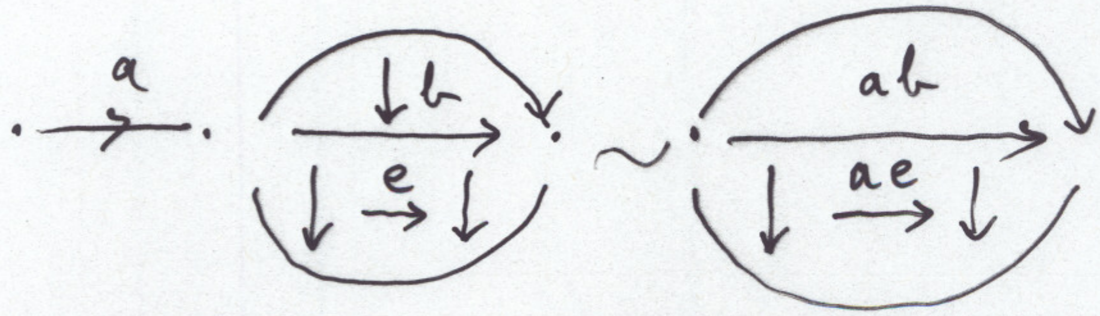
2) <u>Distributions</u> :

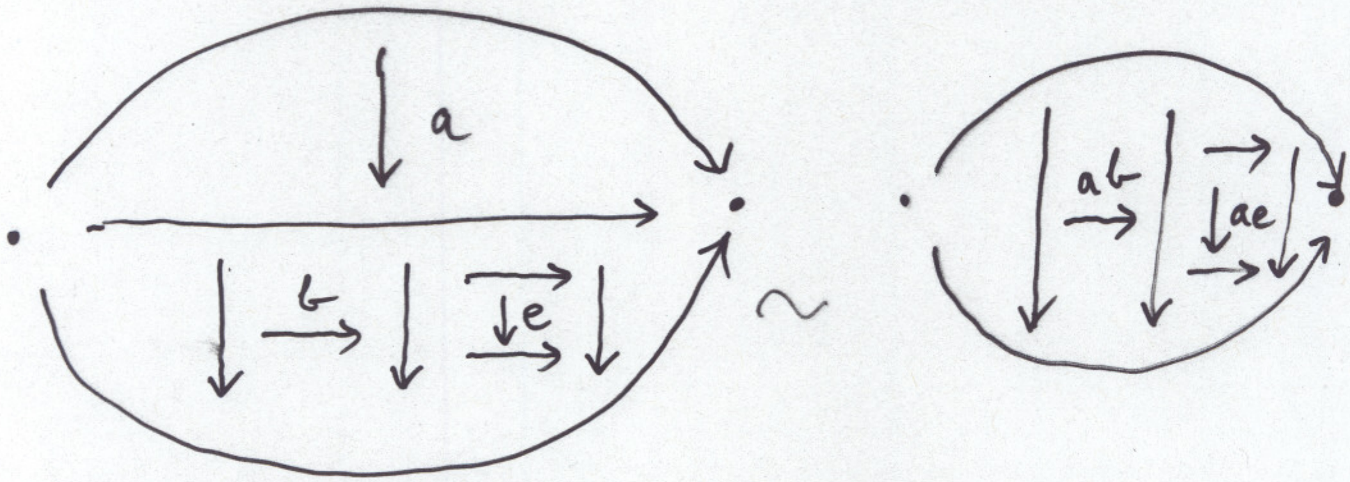
$$a(be) = (ab)(ae)$$

$$(ab)e = (ae)(be)$$

"whenever meaningful."

Examples:





3) <u>Commutations</u> :

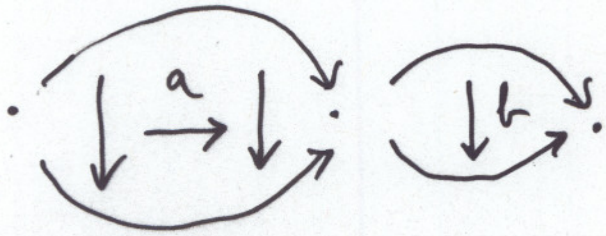
Let: a, b as before;

assume $m, n \geq 2$; $k \geq 1$;

$$c^{(k-1)}(a) = d^{(k-1)}(b)$$

Example:

Example:



$m = 3$ $\quad n = 2$

$k = 1$

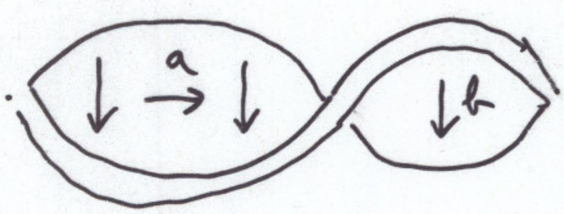
$k - 1 = 0$

Define:

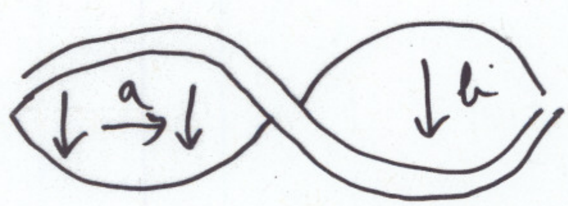
$$\boxed{\begin{aligned} \binom{a}{b} &\overset{\text{def}}{=} (a(\bar{d}\,b))(b(\bar{c}\,a)) \\ \binom{b}{a} &\overset{\text{def}}{=} ((\bar{d}\,a)\,b)((\bar{c}\,b)\,a) \end{aligned}}$$

$(\bar{d} = d^{(k)}, \quad \text{etc.})$

$\binom{a}{b}$:



$\binom{b}{a}$:



'commutation' is the axiom

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

end of def'n of ω'-cat

Theorem (MM, 2005)

$\omega^{\bullet}$-category $=$ ω-category

(in the strongest possible sense).

$\boxed{\omega^{-}\text{-category}}$ $=$ ω'-category

take away

the commutation laws

(proposed) semistrict-ω-category :

(Crans-ω-category) :

ω^{-}-category, plus new operations

Operations: for all pairs a, b as in the commutation law above

a cell $\langle a, b\rangle$:

$$\dim(\langle a, b\rangle) = \max(m, n) + 1$$

Domain / codomain of $\langle a, b\rangle$:

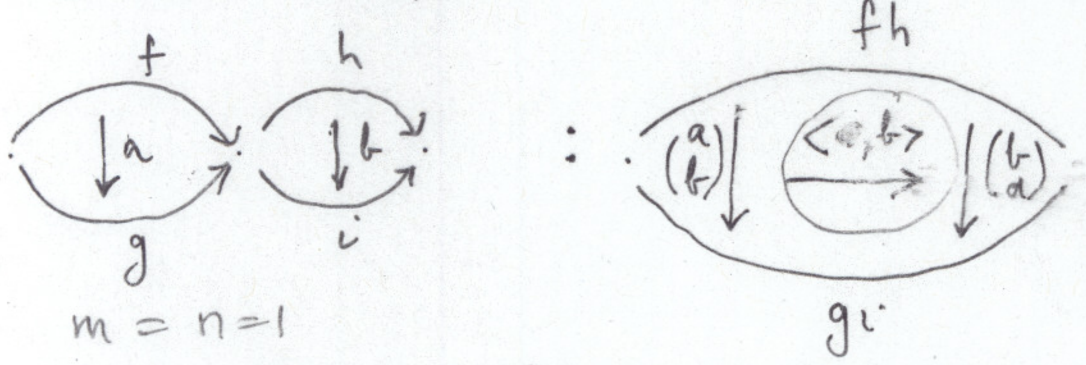
let $l = |m - n|$.

By symmetry, may assume: $m \geq n$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\dim(a)$ $\quad$ $\dim(b)$

Proceed by induction on l:

$l = 0$: Example $\quad\quad\quad\quad\quad\quad$ $\dim\langle a, b\rangle = 2$



$m = n = 1$

Now, $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} b \\ a \end{pmatrix}$ are parallel; I can say:

$$\langle a, b\rangle : \begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow[\cong]{} \begin{pmatrix} b \\ a \end{pmatrix}$$

(typical, modulo suspension!)

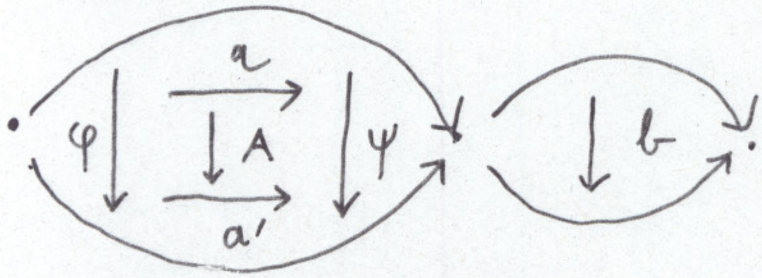
$\ell = 1$:

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\varphi\Downarrow \xrightarrow{a} \Downarrow\psi"] & \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow b"] & \bullet \end{tikzcd} \quad :$$

$$\begin{tikzcd}[column sep=huge, row sep=huge] \begin{pmatrix} \varphi \\ b \end{pmatrix} \arrow[r, "\langle \varphi, b\rangle"] \arrow[d, "\begin{pmatrix} a \\ b \end{pmatrix}"'] \arrow[dr, phantom, "\overset{\langle a, b\rangle}{\underset{\cong}{\Rrightarrow}}"] & \begin{pmatrix} b \\ \varphi \end{pmatrix} \arrow[d, "\begin{pmatrix} b \\ a \end{pmatrix}"] \\ \begin{pmatrix} \psi \\ b \end{pmatrix} \arrow[r, "\langle \psi, b\rangle"'] & \begin{pmatrix} b \\ \psi \end{pmatrix} \end{tikzcd}$$

$$\boxed{\langle a, b\rangle : \begin{pmatrix} a \\ b \end{pmatrix} \langle ca, b\rangle \xrightarrow[\cong]{} \langle da, b\rangle \begin{pmatrix} b \\ a \end{pmatrix}}$$

$l = 2$:



:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\binom{\varphi}{b} \arrow[r, "\langle\varphi, b\rangle"] \arrow[d, bend right=60, "\binom{a}{b}"'] \arrow[d, bend left=60, "\binom{a'}{b}"] \arrow[d, phantom, "\overset{\binom{A}{b}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\langle a', b\rangle}{\Rightarrow}"] & \binom{b}{\varphi} \arrow[d, "\binom{b}{a'}"] \\
\binom{\psi}{b} \arrow[r, "\langle\psi, b\rangle"'] & \binom{b}{\psi}
\end{tikzcd}$$

$$\langle A, b\rangle \; \Rrightarrow$$

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\binom{\varphi}{b} \arrow[r, "\langle\varphi, b\rangle"] \arrow[d, "\binom{a}{b}"'] \arrow[dr, phantom, "\overset{\langle a, b\rangle}{\Rightarrow}"] & \binom{b}{\varphi} \arrow[d, bend right=60, "\binom{b}{a}"'] \arrow[d, bend left=60, "\binom{b}{a'}"] \arrow[d, phantom, "\overset{\binom{b}{A}}{\Rightarrow}"] \\
\binom{\psi}{b} \arrow[r, "\langle\psi, b\rangle"'] & \binom{b}{\psi}
\end{tikzcd}$$

$$\boxed{\binom{A}{b}\langle c^{(2)}A, b\rangle\langle c^{(3)}A, b\rangle \xrightarrow{\langle A, b\rangle} \langle d^{(3)}A, b\rangle\langle d^{(2)}A, b\rangle\binom{b}{A}}$$

$\boxed{\text{Back to : “}\omega'\text{-cat} = \omega\text{-cat”}}$

Given cells a, b

$\dim a = m \qquad \dim b = n \qquad$ (as before)

$$k = \min(m, n) \geq 1$$

also : $j \leq k-1$

Assume $c^{(j)}(a) = d^{(j)}(a)$

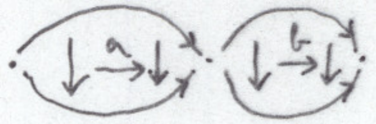
"level-j commutation situation"

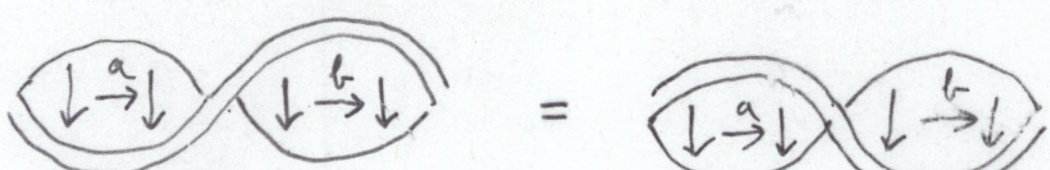
Define

$$\binom{a}{b}_{\ell} = (a(\bar{d}b))(b(\bar{c}a))$$

$$\binom{b}{a}_{\ell} = ((\bar{d}a)b)((\bar{c}b)a)$$

where $\bar{d} = d^{(j+1)}$, $\bar{c} = c^{(j+1)}$

Example :  $j=0\,;\ m=n=3$, $k=2$



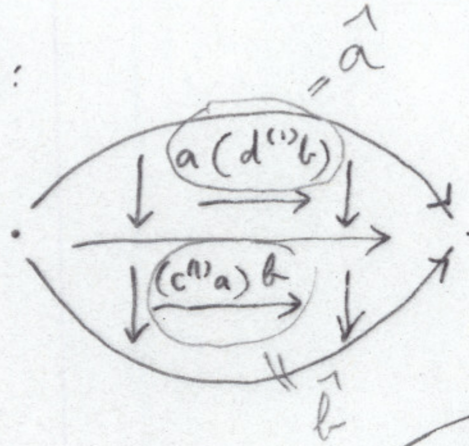
$$\binom{a}{b}_{j=0} \underset{\text{in an } \omega'\text{-cat}}{=} \binom{b}{a}_{j=0}$$

<u>Note</u>: this is the reason why <u>we do</u> <u>not</u> need an operation like

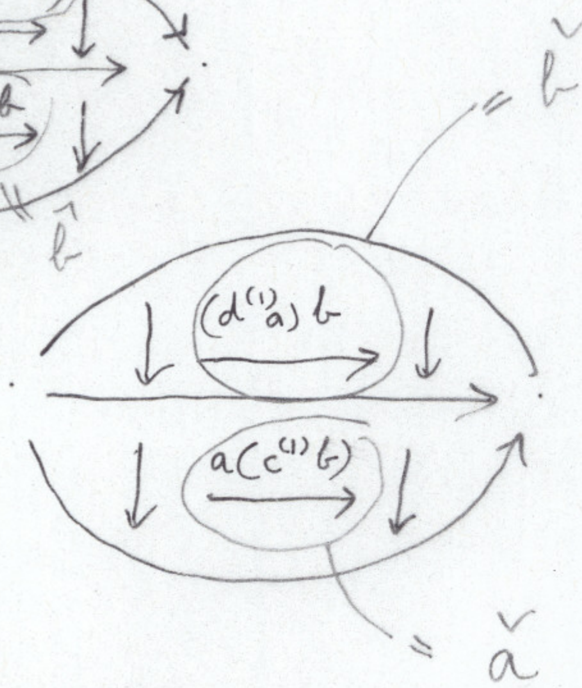
$$\begin{pmatrix} a \\ b \end{pmatrix}_j \xrightarrow{\langle a, b\rangle_j} \begin{pmatrix} b \\ a \end{pmatrix}_j$$

In the example:

$$\begin{pmatrix} a \\ b \end{pmatrix}_{j=0} =$$



$$\begin{pmatrix} b \\ a \end{pmatrix}_0 =$$



$$\left.\begin{matrix} \hat{a}, \hat{b} \\ \check{b}, \check{a} \end{matrix}\right\} : \text{ ordinary (level-1) commutation situation}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}_0 = \boxed{\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}} = \begin{pmatrix} \hat{b} \\ \hat{a} \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix}_0 = \begin{pmatrix} \check{b} \\ \check{a} \end{pmatrix} = \boxed{\begin{pmatrix} \check{a} \\ \check{b} \end{pmatrix}}$$

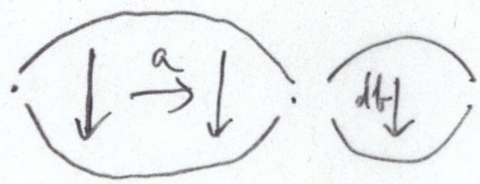
But:

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \Big(\underbrace{\hat{a}(d\hat{b})}\Big)\Big(\underbrace{(c\hat{a})}\,\hat{b}\Big)$$

$$\begin{pmatrix} \check{a} \\ \check{b} \end{pmatrix} = \Big(\overbrace{(d\check{b})\,\check{a}}^{=}\Big)\Big(\overbrace{\check{b}\,(c\check{a})}^{=}\Big)$$

because $\quad \hat{a}(d\hat{b}) = \begin{pmatrix} a \\ db \end{pmatrix}$

$$(d\check{b})\,\check{a} = \begin{pmatrix} db \\ a \end{pmatrix}$$



(3, 2) commutation