7. The category of multitopes

Suppose $S=(C_n, \boldsymbol{C}_n, \boldsymbol{D}_n, d_n)_{n\in\mathbb{N}}$, $\bar{S}=(\bar{C}_n, \bar{\boldsymbol{C}}_n, \bar{\boldsymbol{D}}_n, \bar{d}_n)_{n\in\mathbb{N}}$ are multitopic sets. A *morphism* $\Phi:S\to\bar{S}$ *of multitopic sets* consists of maps $\Phi_n:C_n\to\bar{C}_n$ such that

Φ_n and Φ_{n+1} combine to induce a, necessarily unique, strict morphism of multicategories $\boldsymbol{C}_{n+1}\longrightarrow\bar{\boldsymbol{C}}_{n+1}$;

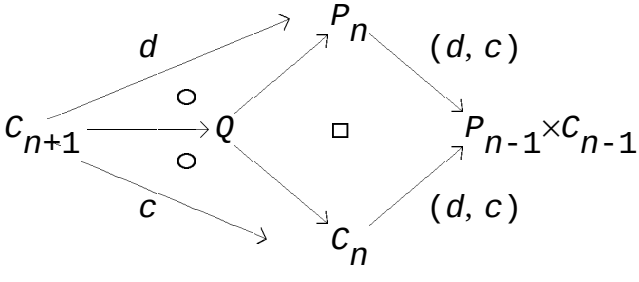
and

the Φ_n are compatible with the $\mathtt{d}$'s: $\Phi_{n-1}\circ\mathtt{d}_n=\mathtt{d}_n\circ\Phi_n$.

Because of the definition of $\boldsymbol{D}_n$ in terms of $\boldsymbol{C}_n$ and the $\mathtt{d}$'s, it follows that the Φ_n induce a strict morphism $\boldsymbol{D}_n\longrightarrow\bar{\boldsymbol{D}}_n$.

Under this definition of morphism, we have a category $\mathtt{MSet}$ of multitopic sets, with obvious composition and identities.

We are particularly interested in the terminal multitopic set $\mathcal{T}$. This is obtained if we stipulate that for each n, there be exactly one n-cell of any possible type; that is, exactly one $\mathtt{0}$-cell altogether; exactly one $\mathtt{1}$-cell altogether; and for each $n\geq\mathtt{1}$, for each pair $(\beta, b)\in P_n\times C_n$ such that $\mathtt{d}\beta=\mathtt{d}b$, $\mathtt{c}\beta=\mathtt{c}b$, there be exactly one $a\in C_{n+1}$ such that $da=\beta$, $ca=b$; in other words that for each $n\geq\mathtt{1}$, the mapping $C_{n+1}\longrightarrow Q$ in the diagram



induced by the pullback $Q=P_n\times_{P_{n-1}\times C_{n-1}}C_n$, be an isomorphism. It is easy to see, by going through the recursive buildup of any given multitopic set S as given in the previous

section, that if $\mathcal{T}$ is such as described, then there is a unique map $S\to\mathcal{T}:\mathcal{T}$ is terminal.

On the other hand, the *existence* of $\mathcal{T}$ as described is also assured by the same recursive procedure. Note that in this case, the non-deterministic element of the recursion, the arbitrary choice of the $(c_{n+1}, d_{n+1}, c_{n+1})$, is eliminated by the condition given.

Let us use the notations $\mathbf{C}_n=\boldsymbol{C}_n(\mathcal{T})$, $\mathbf{D}_n=\boldsymbol{D}_n(\mathcal{T})$, $\mathrm{C}_n=C_n(\mathcal{T})$, $\mathrm{P}_n=P_n(\mathcal{T})$; we continue to use d, c and i without further specifying tags both for $\mathcal{T}$ and for other multitopic sets as they might come up. It will be convenient to have a new element $*$, and set $\mathrm{P}_{-1}=\{*\}$.

Given an arbitrary multitopic set S, we use the notation $\Phi:S\to\mathcal{T}$ for the terminal map, as well as for its various components. For any entity a in S, $\Phi(a)$ is its *type*.

The first remark is that $\mathrm{d}:\mathrm{C}_n\xrightarrow{\cong}\mathrm{P}_{n-1}$ $(n\geq 0)$. This is true when $n=0$ and $n=1$. Let $n\geq 2$. We know that

$$\begin{array}{ccc}\mathrm{C}_n & \xrightarrow{\cong} & \mathrm{P}_{n-1}\times\mathrm{P}_{n-2}\times\mathrm{C}_{n-2}\,\mathrm{C}_{n-1}\\ a & \longmapsto & (\mathrm{d}a,\,\mathrm{c}a)\end{array}.$$

Therefore, it suffices to show that

$$\begin{array}{ccc}\mathrm{P}_{n-1}\times\mathrm{P}_{n-2}\times\mathrm{C}_{n-2}\,\mathrm{C}_{n-1} & \xrightarrow{\cong} & \mathrm{P}_{n-1}\\ (\beta,\,b) & \longmapsto & \beta\end{array}.$$

But for (β, b) as shown,

$$\mathrm{d}b=\mathrm{d}\beta,\ \mathrm{c}b=\mathrm{c}\beta;\qquad(1)$$

and, by the definition of $\mathcal{T}$, with any given $\beta\in\mathrm{P}_{n-1}$, there is exactly one $b\in\mathrm{C}_{n-1}$ satisfying (1).

What the last-shown fact signifies is that the sets C_n may be dropped, the sets P_n may be used for them as well. However, to avoid confusion, we continue to use the C_n. When

$\rho\in \mathtt{P}_{n-1}$, $\bar{\rho}$ is $\mathtt{d}^{-1}(\rho)\in \mathtt{C}_n$.

Our aim here is to show that there is a specific category $\mathtt{Multitope}$, the *category of multitopes*, such that the multitopic sets are the same as set-valued functors on $\mathtt{Multitope}$: $\mathtt{MSet}\simeq \mathtt{Set}^{\mathtt{Multitope}}$. The objects of $\mathtt{Multitope}$ are the elements of the $\mathtt{P}_n$ for all $n\in\mathbb{N}\cup\{-1\}$ (we write $\mathtt{P}_{-1}=\{*\}$); $\mathtt{Ob(Multitope)}=\dot{\bigcup}_{n\in\mathbb{N}\dot{\cup}\{*\}}\mathtt{P}_n$.

In what follows, S denotes an (arbitrary) multitopic set, with the notation for its ingredients we used before; $\Phi: S\to\mathcal{T}$, $\Phi(a)$ is the *type* of a.

Given any $\rho\in\mathtt{P}_{n-1}$, including the possibility $\rho=*\in\mathtt{P}_{-1}$, $C_n(\rho)$ is the set of n-cells of type $\bar{\rho}$: $C_n(\rho)=\{a\in C_n:\Phi(a)=\bar{\rho}\}$ for $n\geq 1$, and $C_0(*)=C_0$ for $n=-1$. Similarly, for $\rho\in\mathtt{P}_n$, let $P_n(\rho)=(\alpha\in P_n:\Phi(\alpha)=\rho\}$.

For emphasis, let us write $[\rho]$, or even $[\rho]_n$ (the subscript is n, not $n-1$, since $[\rho]_n$ is the "sort" for n-cells, not $(n-1)$-cells), for the object of $\mathtt{Multitope}$ corresponding to $\rho\in\mathtt{P}_{n-1}$ ($n\geq 0$). For the $\mathtt{Set}$-valued functor $\bar{S}:\mathtt{Multitope}\longrightarrow\mathtt{Set}$ corresponding to S, we will have $\bar{S}([\rho]_n)=C_n(\rho)$; $\bar{S}(*)=C_0$.

To identify the arrows of $\mathtt{Multitope}$, we have to do more work.

For $n\geq 1$, let us call $\alpha\in P_n$ *proper* if $\alpha\neq 1_b^{(c_n)}$ for any $b\in C_{n-1}$, i.e., if $|\langle\alpha\rangle|\neq\emptyset$; *improper* otherwise. For $n=0$, all $\alpha\in P_0=C_0$ are *proper*. We use "proper", "improper" for elements of $\mathtt{P}_n$ in a similar way.

Note the following fact. To know a *proper* pasting diagram $\alpha\in P_n$, it suffices to know its type $\rho=\Phi(\alpha)$ and the n-cells filling its places; in other words,

(1') supposing that $\alpha, \beta\in P_n(\rho)$ have the same *proper* type ρ, and for all $p\in |\langle\alpha\rangle| = |\langle\beta\rangle| = |\langle\rho\rangle|$, we have $\langle\alpha\rangle(p)=\langle\beta\rangle(p)$, then $\alpha=\beta$.

This is intuitively clear, and seen very easily by an induction on $\ell h(\langle\rho\rangle)$. Note, however, that the assignment $p\mapsto\langle\alpha\rangle(p)$ satisfies conditions, due to "links" within α; therefore, to determine α we need, beyond its type ρ, a suitably "linked" filling out its places with n-cells. Next, we give the description of these links.

Let $n\geq 1$, and $\rho\in \mathsf{P}_n$. Recall that $\mathsf{s}_{\mathbf{D}_n}(\rho)=\langle\rho\rangle$ and $\mathsf{s}_{\mathbf{C}_n}(\rho)=\langle \mathrm{d}\rho\rangle$; $p\in|\langle\rho\rangle|$ is a place where a "function-symbol", $\langle\rho\rangle(p)$, occurs in ρ; $r\in|\langle \mathrm{d}\rho\rangle|$ is a place where a "variable", $\langle \mathrm{d}\rho\rangle(r)$, occurs in ρ. If $\rho=f(\rho_1,\ldots\rho_m)$, $\langle\rho\rangle$ is the concatenation of $\langle f\rangle$ and the $\langle\rho_i\rangle$; we have the injections $\mu_i=\mu_i[f,\rho_1,\ldots\rho_m]:\langle\rho_i\rangle\longrightarrow\langle\rho\rangle$; $\mu_i(j)=1+\sum_{h<i}\ell h(\langle\rho_h\rangle)+j$: $\langle\rho\rangle$ is the coproduct of $\langle f\rangle$ and the $\langle\rho_i\rangle$ via the coprojections $\nu:|\langle f\rangle|\longrightarrow\rho:1\mapsto 1$ and the μ_i.

$\mathsf{Link}_1(\rho)$ is a set, specified below, of certain triples (p, q, s), the so-called 1-*links* of ρ, such that $p, q\in|\langle\rho\rangle|$, and for

$$\dot{p}=\langle\rho\rangle(p),\ \dot{q}=\langle\rho\rangle(q), \tag{2}$$

we have, in particular, $s\in|\langle \mathrm{d}\dot{p}\rangle|$, and

$$\langle \mathrm{d}\dot{p}\rangle(s) = \mathrm{c}\dot{q}\ . \tag{3}$$

Intuitively, $(p, q, s)\in\mathsf{Link}_1(\rho)$ means that the the function-symbol occurrence of $\langle\rho\rangle(q)$ at q in ρ plugs directly into the occurrence of $\langle\rho\rangle(p)$ at p in ρ at the place s of the function-symbol $\langle\rho\rangle(p)$. To see an exmaple, let $X, Y\in\mathsf{C}_{n-1}$, $a, b, c\in\mathsf{C}_n$, $a:\langle Y, Y\rangle\to X$, $b:\langle X, Y\rangle\to Y$, $c:\langle X, Y, Y\rangle\to Y$ in the multicategory $\mathbf{C}_n$, and let

$$\rho = a(c(X, b(X, Y)), Y), b(a(Y, Y), Y)) \in \mathsf{P}_n\ .$$

Then $\langle\rho\rangle=\langle a, c, b, b, a\rangle$, and

$$\mathsf{Link}_1(\rho)=\{(1,2,1),(2,3,2),(1,4,2),(4,5,1)\}\ ;$$

for instance, $(4,5,1)\in\mathsf{Link}_1(\rho)$ since the second occurrence of a, which is at place 5

in ρ, plugs directly into the second occurrence of b, which is at place 4 in ρ, and this takes place at the 1st place of the function-symbol b.

Here is the formal, recursive, definition. When ρ is improper, $\mathrm{Link}_1(\rho)=\emptyset$. Let $\rho=f(\rho_1, \ldots \rho_m)$. We define

$$\mathrm{Link}_1(\rho) = \bigcup_{i=1}^{m} \{(\mu_i(p), \mu_i(q), s): (p, q, s)\in \mathrm{Link}_1(\rho_i)\} \cup \{(1, \mu_i(1), i): i\in[1, m], |\langle \rho_i \rangle| \neq \emptyset\}.$$

The terms of the first union are there because all the 1-links in the ρ_i give rise to 1-links in ρ via the maps μ_i; the final term says that the head-occurrence (if any) of a function-symbol in ρ_i, which occurs at $\mu_i(1)$ in ρ, is plugged directly into the head-occurrence of f in ρ, at the argument-place i of f.

(3) (under (2)) is seen immediately for $(p, q, s)\in \mathrm{Link}_1(\rho)$.

Given ρ, let $\alpha\in P_n(\rho)$. For $p\in |\langle\alpha\rangle| = |\langle\rho\rangle|$, let us write $\hat{p}$ for $\langle\alpha\rangle(p)$; then $\dot{p}=\langle\rho\rangle(p)=\Phi(\hat{p})$; we have $\hat{p}\in C_n(\mathrm{d}\dot{p})$. Given $(p, q, s)\in \mathrm{Link}_1(\rho)$, we have

$$\langle \mathrm{d}\hat{p}\rangle(s)=\mathrm{c}\hat{q}\,; \qquad (3')$$

this is a consequence of (3). We claim that, conversely,

(4) given $\rho\in \mathrm{P}_n$ *proper*, and for each $p\in |\langle\rho\rangle|$, a cell $a_p\in C_n(\mathrm{d}\dot{p})$ such that for every $(p, q, s)\in \mathrm{Link}_1(\rho)$,

$$\langle \mathrm{d}a_p\rangle(s)=\mathrm{c}a_q \quad (\in C_{n-1}),$$

then there is a unique $\alpha\in P_n(\rho)$ such that $\hat{p}_{\mathrm{d\bar{e}f}}\langle\alpha\rangle(p)=a_p$ ($p\in |\langle\rho\rangle|$).

The proof is a relatively straightforward induction.

Next, $\mathrm{Link}_2(\rho)$ will denote a set of certain triples (p, r, s), the 2-*links* in ρ, such that, amomg others, $p\in|\langle\rho\rangle|$, $r\in|\langle d\rho\rangle|$ and, for $\dot{p}=\langle\rho\rangle(p)$, $s\in|\langle \mathrm{d}\dot{p}\rangle|$,

$$\langle \mathrm{d}\dot{p}\rangle(s) = \langle \mathrm{d}\rho\rangle(r) . \qquad (5)$$

Intuitively, the 2-links (p, r, s) are those for which the variable-occurrence in ρ at r plugs directly into the function-symbol occurrence in ρ at p, at the place s of the function-symbol $\langle\rho\rangle(p)$. For instance, let us consider the ρ taken as an example above, and assume that as far as ρ and its subterms are concerned, the multicategory $\mathbf{C}_n$ has standard amalgamation; in particular, $\mathrm{s}_{\mathbf{C}_n}(\rho)=\langle X, X, Y, Y, Y, Y, Y\rangle$, and in fact each i of the seven places 1 to 7 refers to the ith occurrence from the left of an element of C_{n-1} in ρ. (This assumption is, of course, not automatically true.) In this case, we have

$$\mathrm{Link}_2(\rho) = \{((2,1,1),(3,2,1),(3,3,2),(1,4,2),(5,5,1),(5,6,2),(4,7,2)\}$$

Formally, we define $\mathrm{Link}_2(\rho)$ as follows. When ρ is improper, $\mathrm{Link}_2(\rho)=\emptyset$. Let $\rho=f(\rho_1, \ldots, \rho_m)$. We have the amalgamating functions $\varphi_i:\mathrm{s}_{\mathbf{C}_n}(\rho_i)\longrightarrow\mathrm{s}_{\mathbf{C}_n}(\rho)$, that is, $\varphi_i:\langle \mathrm{d}\rho_i\rangle\longrightarrow\langle \mathrm{d}\rho\rangle$;

$$\mathrm{Link}_2(\rho) = \bigcup_{i=1}^{m}\{(\mu_i(p), \varphi_i(r), s): (p, r, s)\in\mathrm{Link}_2(\rho_i)\} \cup \{(1, \mu_i(1), i): i\in[1, m], |\langle\rho_i\rangle|=\emptyset\} .$$

(5) is immediately seen.

It is easy to see that

(5') if ρ is proper, then for every $r\in|\langle d\rho\rangle|$, there is a unique $(p, r, s)\in\mathrm{Link}_2(\rho)$ with second component the given r.

Supposing that $\alpha\in P_n(\rho)$, $p\in|\langle\rho\rangle|$, $\hat{p}=\langle\rho\rangle(p)=\Phi(\dot{p})$, and $(p, r, s)\in\mathrm{Link}_2(\rho)$, we have

$$\langle \mathrm{d}\hat{p}\rangle(s) = \langle \mathrm{d}\alpha\rangle(r)\,, \tag{6}$$

as a consequence of (5).

We are ready to define the category $\mathtt{Multitope}$. As we said, its objects are $[\rho]_n$, one for each $n\in\mathbb{N}$, $\rho\in \mathrm{P}_{n-1}$. Next, we give *generating arrows* for $\mathtt{Multitope}$. They are

Multitope[1] $\qquad [\rho]_{n+1} \xrightarrow{\ \mathrm{d}_{\rho,\,p}\ } [\mathrm{d}\dot{p}]_n\,,$

one for each $n\geq -1$, $\rho\in \mathrm{P}_n$ and $\mathrm{d}\dot{p}=\mathrm{d}(\langle\rho\rangle(p))\in \mathrm{P}_{n-1}$;

Multitope[2] $\qquad [\rho]_{n+1} \xrightarrow{\ \mathrm{c}_{\rho}\ } [\mathrm{d}\rho]_n\,,$

one for each $n\geq -1$ and $\rho\in \mathrm{P}_n$.

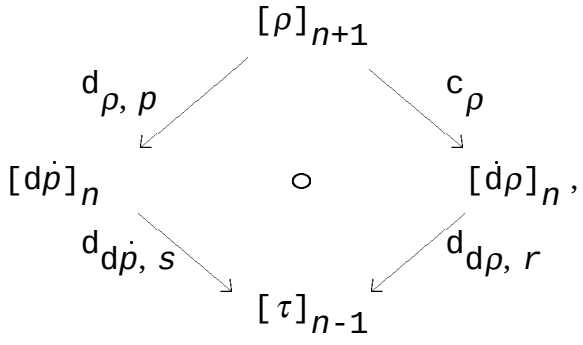
Finally, we give *defining relations* that the generating arrows are to satisfy. These come in four groups $\mathtt{Multitope[3]}$ to $\mathtt{Multitope[6]}$.

Multitope[3]:

$$\begin{array}{ccccc} & & [\rho]_{n+1} & & \\ & \swarrow^{\mathrm{d}_{\rho,\,p}} & & \searrow^{\mathrm{d}_{\rho,\,q}} & \\ [\mathrm{d}\dot{p}]_n & & \circlearrowright & & [\mathrm{d}\dot{q}]_n\,, \\ & \searrow_{\mathrm{d}_{\mathrm{d}\dot{p},\,s}} & & \swarrow_{\mathrm{c}_{\mathrm{d}\dot{q}}} & \\ & & [\theta]_{n-1} & & \end{array} \tag{7}$$

one for each $n\geq 0$, $\rho\in \mathrm{P}_n$, and $(p, q, s)\in \mathtt{Link}_1(\rho)$. Note that the codomain of $\mathrm{d}_{\mathrm{d}\dot{p},\,s}$ is $[\theta_1]_{n-1}$ for $\theta_1=\mathrm{d}(\langle\mathrm{d}\dot{p}\rangle(s))$, and the codomain of $\mathrm{c}_{\mathrm{d}\dot{q}}$ is $[\theta_2]_{n-1}$ for $\theta_2=\mathrm{d}(\mathrm{d}\dot{q})$; by the fact that $\mathrm{d}(\mathrm{d}\dot{q})=\mathrm{d}(\mathrm{c}\dot{q})$, and by (3), we have $\theta_1=\theta_2$, and so, the diagram (7) makes sense.

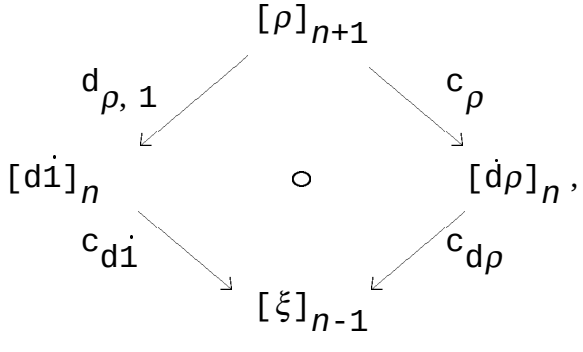
Multitope[4]:



(8)

one for each $n\geq 1$, $\rho\in \mathsf{P}_n$, and $(p, r, s)\in \mathtt{Link}_2(\rho)$. Note that the codomain of $\mathtt{d}_{\mathtt{d}\dot{p},\, s}$ is $[\tau_1]_{n-1}$ for $\tau_1=\mathtt{d}(\langle \mathtt{d}\dot{p}\rangle(s))$, and that of $\mathtt{d}_{\mathtt{d}\rho.r}$ is $[\tau_2]_{n-1}$ for $\tau_2=\mathtt{d}(\langle \mathtt{d}\rho\rangle(r))$, and by (5), $\tau_1=\tau_2$, thus, (8) is meaningful.

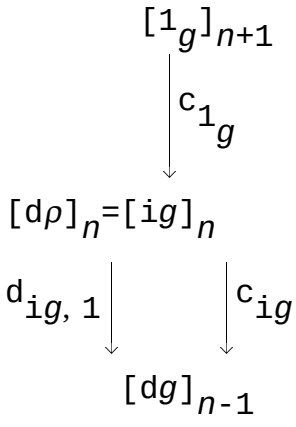
Multitope[5] :



(9)

one for each $n\geq 0$, *proper* $\rho\in \mathsf{P}_n$. Note that $1\in |\langle \rho\rangle|$; $\dot{1}=\langle p\rangle(1)$. The codomain of $\mathtt{c}_{\mathtt{d}\dot{1}}$ is $\mathtt{dd}\dot{1}$, that of $\mathtt{c}_{\mathtt{d}\rho}$ is $\mathtt{dd}\rho$. We have $\mathtt{c}\dot{1}=\mathtt{c}\rho$ as a general, and obvious, rule for all proper ρ . Therefore $\mathtt{dd}\dot{1}=\mathtt{dc}\dot{1}=\mathtt{dc}\rho=\mathtt{dd}\rho$, thus, (9) is meaningful.

Multitope[6] :



commutes: $d_{ig,\,1}\circ c_{1_g}=c_{ig}\circ c_{1_g}$,

whenever $g\in C_{n-1}$. Note that $1_g=1_g^{(\mathbf{c}_n)}\in P_n$, $ig=i_{n-1}(g)\in P_{n-1}$, $|\langle ig\rangle|=\{1\}$, $\langle ig\rangle(1)=g$, $dig=dg$.

The category $\mathtt{Multitope}$ is the one whose arrows are generated by the generating arrows, under the identification of arrows forced by the defining relations; briefly, the category whose presentation was given above.

Given any multitopic set S, we may define $\bar{S}:\mathtt{Multitope}\longrightarrow\mathtt{Set}$ as follows. We put $\bar{S}([\rho]_n)=C_n(\rho)$. For the generating arrow in $\mathtt{Multitope[1]}$, we note that $\bar{S}([d\dot{p}]_{n-1})=C_{n-1}(d\dot{p})$, and if $a\in\bar{S}([\rho]_n)=C_n(\rho)$, then $\alpha_{\bar{\bar{\mathtt{def}}}}da\in P_{n-1}(\rho)$, and for $\hat{p}=\langle d\alpha\rangle(p)$, $\Phi(\hat{p})=\dot{p}$, and thus $\hat{p}\in C_{n-1}(d\dot{p})$; this means that we can define

$$\bar{S}(d_{\rho,\,p}):C_n(\rho)\longrightarrow C_{n-1}(d\dot{p}) \;:\; a\longmapsto\hat{p}=\langle da\rangle(p)\,.$$

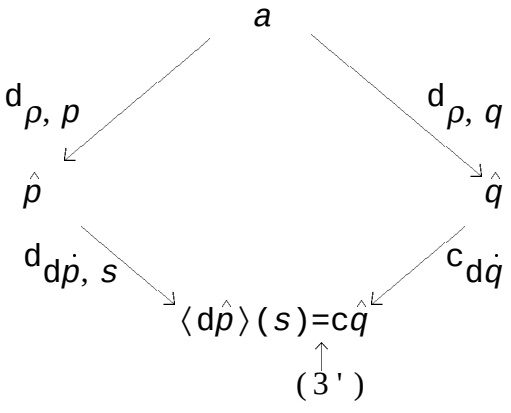
As for $\mathtt{Multitope[2]}$,

$$\bar{S}(c_\rho):C_n(\rho)\longrightarrow C_{n-1}(d\rho) \;:\; a\longmapsto ca\,;$$

note that since $d\Phi(ca)=dc\Phi(a)=dd\Phi(a)=d\Phi(da)=dd\rho$, we have $\Phi(ca)=d\rho$, and so $ca\in C_{n-1}(d\rho)$.

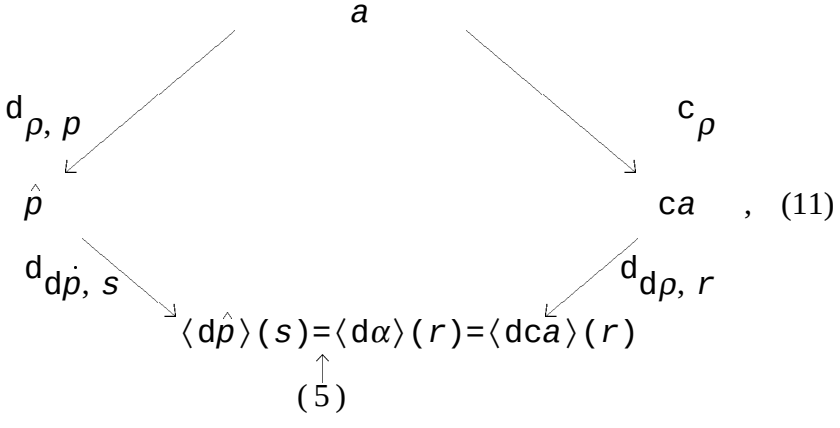
The diagrams which are the images under $\bar{S}$ of the ones in `Multitope[3]` to `Multitope[6]` commute:

`Multitope[3]`:



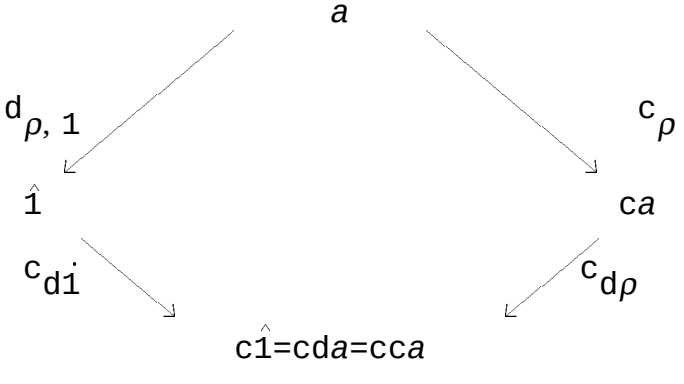
(10)

`Multitope[4]`:



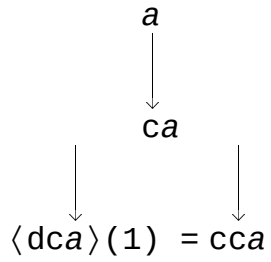
(11)

`Multitope[5]`:



(12)

Multitope[6]:



; (13)

the reason for the last equality is that now $da=\alpha=1_f$ for some $f\in C_{n-1}$, and so $dda=if\in P_{n-1}$, $cca=cda=f$, and so $\langle dca\rangle(1)=\langle dda\rangle(1)=f=cca$.

Therefore, the $\bar{S}:\mathtt{Multitope}\longrightarrow\mathtt{Set}$ is well-defined.

It is clear that any arrow $\Psi:S_1\to S_2$ in MSet gives rise to a natural transformation $\bar{\Psi}:\bar{S}_1\to\bar{S}_2$, whose components $C^1_n(\rho)\longrightarrow C^2_n(\rho)$ are the restrictions of $\Psi:C^1_n\longrightarrow C^2_n$. We have a functor $\Sigma:\mathtt{MSet}\longrightarrow\mathtt{Set}^{\mathtt{Multitope}}:S\mapsto\bar{S},\ \Psi\mapsto\bar{\Psi}$.

I omit the (easy) proof of the fact that Σ is full and faithful.

Let us show that Σ is surjective on objects. Let $T\in\mathtt{Set}^{\mathtt{Op}}$, to construct $S\in\mathtt{MSet}$ for which $\bar{S}=T$; we are going to use the standard notation for S; $C_k=C_k(S)$, etc.

We define $C_{0}\stackrel{=}{def}T([*]_0)$.

Let $n\geq 0$. Suppose we have constructed the n-truncation (see the previous section) $S{\restriction}n$ of S, so that

$$\overline{(S{\restriction}n)}=T{\restriction}n\ ; \tag{14}$$

here we are using a self-explanatory notation. $S{\restriction}n$ involves k-cells and k-pasting diagrams for all $k=-1, 0, \ldots, n$. $\mathtt{Multitope}{\restriction}n$ is the category whose objects are the $[\rho]_k$ for $\rho\in\mathtt{P}_k$, and for the same range of k, and whose arrows the arrows generated by the generators and relations in Multitope[1] to Multitope[6], with the value of n appearing in Multitope[1] to Multitope[6] restricted to the range from 0 the

present n (inclusive). $T\restriction n$ is the $\mathtt{Set}$-valued functor on $\mathtt{Multitope}\restriction n$ which is the restriction of T. For the canonical functor $\Xi:\mathtt{Multitope}\restriction n\longrightarrow\mathtt{Multitope}$, $T\restriction n$ is $T\circ\Xi$. (Incidentally, Ξ is full and faithful.)

Next, we define

(15) C_{n+1} and $\mathtt{d}_{n+1}:C_{n+1}\longrightarrow P_n$, $\mathtt{c}_{n+1}:C_{n+1}\longrightarrow C_n$ such that when $n\geq 1$, also $\mathtt{d}_n\mathtt{d}_{n+1}=\mathtt{d}_n\mathtt{c}_{n+1}$, $\mathtt{c}_n\mathtt{d}_{n+1}=\mathtt{c}_n\mathtt{c}_{n+1}$.

We put $C_{n+1}\ \mathtt{d\bar{e}f}\ \dot{\bigcup}_{\rho\in\mathtt{P}_n}T([\rho]_{n+1})$.

Let $\rho\in\mathtt{P}_n$, $a\in T([\rho]_{n+1})\subset C_{n+1}$.

We let $\mathtt{c}_{n+1}(a)=\mathtt{c}a\ _{\mathtt{d\bar{e}f}}T(\mathtt{c}_\rho)(a)$, for the arrow $\mathtt{c}_\rho$ from $\mathtt{Multitope[2]}$. Since $\mathtt{c}a\in T([\mathtt{d}\rho]_n)$, and (9), we have $\mathtt{c}a\in C_n(\mathtt{d}\rho)$.
To define $?:\alpha_{\mathtt{d\bar{e}f}}\mathtt{d}_{n+1}(a)=\mathtt{d}a\in P_n$, we distinguish two cases. First, assume ρ is improper; $\rho=1_g^{(\mathbf{c}_n)}$ for some $g\in\mathtt{C}_{n-1}$. The type of α has to be $\Phi\alpha=\rho$; thus, $\alpha=1_c^{(\mathbf{c}_n)}$ for a suitable $c\in C_{n-1}$ for which $\Phi(c)=g$; moreover, $c=\mathtt{c}\alpha=\mathtt{c}\mathtt{d}a=\mathtt{c}\mathtt{c}a$, and $\mathtt{c}a\in C_n(\mathtt{d}\rho)$ were determined above. So, let us define $\mathtt{d}a_{\mathtt{d\bar{e}f}}1_c^{(\mathbf{c}_n)}$ for $c_{\mathtt{d\bar{e}f}}\mathtt{c}\mathtt{c}a$.

For $n\geq 1$, we need $\mathtt{d}\mathtt{d}a=\mathtt{d}\mathtt{c}a$, $\mathtt{c}\mathtt{d}a=\mathtt{c}\mathtt{c}a$.

We have $\mathtt{d}\rho=\mathtt{d}(1_g)=\mathtt{i}g$; for $b=\mathtt{c}a$, we have $b\in C_n(\mathtt{i}g)$; so, $\mathtt{d}b=\mathtt{i}f$ for some $f\in C_{n-1}$; $f=\langle\mathtt{d}b\rangle(1)$. Applying $\mathtt{Multitope[6]}$ to T (and remembering (9)), we get that $\langle\mathtt{d}\mathtt{c}a\rangle(1)=\mathtt{c}\mathtt{c}a$: read (13) as to imply the equality stated in it. This means that $\mathtt{d}\mathtt{c}a=\mathtt{i}\mathtt{c}\mathtt{c}a$. Also, $\mathtt{d}\mathtt{d}a=\mathtt{d}(1_c)=\mathtt{i}c=\mathtt{i}\mathtt{c}\mathtt{c}a$. Therefore, $\mathtt{d}\mathtt{d}a=\mathtt{d}\mathtt{c}a$ is established.

Since $\mathtt{d}a=1_c$, $\mathtt{c}\mathtt{d}a=c$; $\mathtt{c}\mathtt{c}a=c$ by the definition of c; $\mathtt{c}\mathtt{d}a=\mathtt{c}\mathtt{c}a$ is established.

Second, assume ρ is proper. We now apply (4). We let $a_p\ _{\mathtt{d\bar{e}f}}T(\mathtt{d}_{\rho,\,p})(a)$ $(p\in|\langle\rho\rangle|)$. The fact that T respects the diagram $\mathtt{Multitope[3]}$ (compare (10)) gives that the condition in (4) is satisfied. Therefore, we have $\alpha\in P_n(\rho)$ such that

$\hat{p}=\langle\alpha\rangle(p)=T(\mathrm{d}_{\rho,\,p})(a)$ for all $p\in|\langle\rho\rangle|$. We let $\mathrm{d}a\stackrel{=}{\mathrm{def}}\alpha$.

To see that $\mathrm{cd}a=\mathrm{cc}a$, it suffices to invoke the fact that T "satisfies" Multitope[5]; see (12).

We have $\mathrm{dd}a$, $\mathrm{dc}a\in P_{n-1}(\mathrm{d}\rho)$. To show the equality $\mathrm{dd}a=\mathrm{dc}a$, we distinguish two cases: $\mathrm{d}\rho$ is proper (Case 1), $\mathrm{d}\rho$ is improper (Case 2).

Case 1. Let $\alpha=\mathrm{d}a$, $\beta=\mathrm{dc}a$. We have that $\beta\in P_{n-1}(\mathrm{d}\rho)$, and β satisfies the condition $\langle\mathrm{d}\hat{p}\rangle(s)=\langle\beta\rangle(r)$ for all $(p,r,s)\in\mathrm{Link}_2(\rho)$, by Multitope[4] applied to T . By (6), and (5'), thus $\langle\beta\rangle(r)=\langle\mathrm{d}\alpha\rangle(r)$ for all $r\in|\langle\mathrm{d}\rho\rangle|$, which means, by (1'), that $\beta=\mathrm{d}\alpha$; this is what we wanted.

Case 2. Now, $n-1\geq 1$. Any (improper) $\beta\in P_{n-1}(\mathrm{d}\rho)$ is determined by $\mathrm{c}\beta$; if $\beta_1,\beta_2\in P_{n-1}(\mathrm{d}\rho)$, and $\mathrm{c}\beta_1=\mathrm{c}\beta_2$, then $\beta_1=\beta_2$. For $\beta_1=\mathrm{dd}a$, $\beta_2=\mathrm{dc}a$,

$$\mathrm{c}\beta_1=\mathrm{cdd}a\ \underset{1}{=}\ \mathrm{ccd}a\ \underset{2}{=}\ \mathrm{ccc}a\ \underset{3}{=}\ \mathrm{cdc}a\,,$$

where 1 and 3 hold by the law "$cd=cc$" holding in $S\restriction n$, 2 by the fact that $\mathrm{cd}a=\mathrm{cc}a$.

We have completed (15).

By what we have done in the previous section, we now have $\hat{S}=S\restriction(n+1)$, the $(n+1)$-truncation of the desired S ; by recursion, we have the full S .

The construction of S clearly ensures that $\bar{S}=T$, for the effect on both objects and arrows of Multitope .

This completes the proof that $\mathrm{MSet}\simeq\mathrm{Set}^{\mathrm{Multitope}}$.

We develop a notation for multitopes.

In what follows, we work in $\mathcal{T}$, the terminal multitopic set. To have a notation for cells and pd's, we make the following decisions:

(i) We make each of the bijections $d_n : C_n \xrightarrow{\cong} P_{n-1}$ ($n \geq 0$) equal the identity map.

(ii) For elements of the free multicategory $\mathcal{F}(\mathcal{L})$ we may use Polish notation. That is, instead of writing $f(\alpha_1, \ldots, \alpha_n)$ ($f \in L(\mathcal{L})$, $\alpha_i \in A(\mathcal{F}(\mathcal{L}))$), we may write $f\alpha_1 \ldots \alpha_n$. In particular, if $n=0$, then the term is just f . (It is known that there is no loss of unique readability when we so omit all parentheses and commas. In a complex expression, the key to decoding is to use the arity of each operation symbol. In our notation, the arity will be contained in the notation of the operation symbol itself.) Alternatively, we may use the original parenthesis/comma notation as well.

(iii) The ingredients of the notation of members of $A(\mathcal{F}(\mathcal{L}))$ are the elements of $O(\mathcal{L})$ and those of $L(\mathcal{L})$. Thus, the ingredients for the elements of $P_n = A(\mathcal{F}(\mathcal{L}_n))$ ($n \geq 1$) are the elements of $C_{n-1} = P_{n-2}$, and those of $C_n = P_{n-1}$. When $\beta \in C_{n-1} = P_{n-2}$ is used in P_n , we put β in brackets [ ] ; when $\alpha \in C_n = P_{n-1}$ is used in P_n , we put α in curly brackets { } . When $n=0$, only the second part applies.

(iv) We denote the single element of $C_0 = P_{-1}$ by * (as we already did above).

The above fixes the notation, with a choice of using Polish notation, or the parenthesis/comma notation. The Polish normal form for the generating arrow $\beta \in C_n = L(\mathcal{L}_n) = P_{n-1}$ $(n \geq 1)$ as it appears in $P_n = A(\boldsymbol{C}_n)$ is $\{\beta\}[\gamma_1] \ldots [\gamma_\ell]$ where $\langle \beta \rangle = \langle \gamma_1, \ldots, \gamma_\ell \rangle \in P_{n-2}$. As an abbreviation, we will write $\{\beta\}$ for $\{\beta\}[\gamma_1] \ldots [\gamma_\ell]$. On the other hand, for $\gamma \in P_{n-2}$, we have $[\gamma] \in P_n$ in proper notation.

The single element of $C_1 = P_0$ is $\{*\}$. The elements of $C_2 = P_1$ are the expressions $\{\{*\}\}\{\{*\}\} \ldots \{\{*\}\}[*]$, with zero or more parts of the form $\{\{*\}\}$. If $\alpha = \{\{*\}\}\{\{*\}\} \ldots \{\{*\}\}[*] \in P_1 = C_2 = L(\mathcal{L}_2) \subset A(\boldsymbol{C}_2)$ has ℓ occurrences of $\{\{*\}\}$, then $s_{\mathcal{L}_2}(\alpha) = s_{\boldsymbol{C}_2}(\alpha) = \langle d\alpha \rangle = \langle \alpha \rangle = \langle \{*\}, \{*\}, \ldots, \{*\} \rangle \in C_1^*$ (ℓ occurrences), and $t_{\mathcal{L}_2}(\alpha) = t_{\boldsymbol{C}_2}(\alpha) = \{*\} \in C_1$. Consequently, an inductive definition of $C_3 = P_2$ is as follows:

(a) $[\{*\}] \in P_2$;

(b) if $\{\{*\}\}\{\{*\}\} \ldots \{\{*\}\}[*] \in P_1$ has ℓ occurrences of $\{*\}$, and $\alpha_1, \ldots, \alpha_\ell \in P_2$, then $\{\{\{*\}\}\{\{*\}\} \ldots \{\{*\}\}[*]\}\alpha_1, \ldots, \alpha_\ell \in P_2$.

Note that, for $\ell=0$, (b) gives $\{[*]\}$ as an example of a member of P_2; of course, it is different from the one in (a).

To have an example, consider the 2-pd γ considered in section 1; $\gamma\in P_2(S)$ in some multitopic set S that accommodates the cells involved. The type of γ, $\Phi(\gamma)\in P_2$, is as follows. Let us write upper bar for "type", $\bar{x}$ for $\Phi(x)$. Then $\bar{a}=\bar{c}=\bar{d}=\bar{e}=\{\{*\}\}\{\{*\}\}[*]=\varphi\in C_2=P_1$, and $\bar{b}=\{\{*\}\}\{\{*\}\}\{\{*\}\}[*]=\eta\in C_2=P_1$. The type of all of the f_i is $\{*\}\in C_1=P_0$. Thus, abbreviating $\#=[\{*\}]$, $\hat{\varphi}=\{\varphi\}$, $\hat{\eta}=\{\eta\}$, we have

$$\bar{\gamma} = \hat{\varphi}(\hat{\varphi}(\#, \hat{\varphi}(\hat{\eta}(\#, \#, \#), \hat{\varphi}(\#, \#))), \#)$$

or, in Polish notation,

$$\bar{\gamma} = \hat{\varphi}\hat{\varphi}\#\hat{\varphi}\hat{\eta}\#\#\#\hat{\varphi}\#\#\#$$

or, without abbreviations,

$$\begin{aligned}\bar{\gamma} = {} & \{\{\{*\}\}\{\{*\}\}[*]\}\{\{\{*\}\}\{\{*\}\}[*]\}[\{*\}]\{\{\{*\}\}\{\{*\}\}[*]\} \\ & \{\{\{*\}\}\{\{*\}\}\{\{*\}\}[*]\}[\{*\}][\{*\}][\{*\}] \\ & \{\{\{*\}\}\{\{*\}\}[*]\}[\{*\}][\{*\}][\{*\}] .\end{aligned}$$

This is not meant as a particularly intuitive representation; it is a systematic notation well-suited for mechanical manipulation.

The inductive definition of P_n and $c(\alpha)$ for $\alpha\in P_n$ is this:

(a) for any $\gamma\in P_{n-2}$, $[\gamma]\in P_n$; and $c[\gamma]=\{\gamma\}\in P_{n-1}$;

(b) whenever $\beta\in P_{n-1}$, $\langle\beta\rangle=\langle\gamma_1,\ldots,\gamma_\ell\rangle\in P^*_{n-2}$, $\alpha_i\in P_n$, $c\alpha_i=\gamma_i$, we have $\{\beta\}\alpha_1\ldots\alpha_\ell\in P_n$ and $c(\{\beta\}\alpha_1\ldots\alpha_\ell)=d\beta$.

For d on P_n, we have

$$d(\alpha_1 \circ_p \alpha_2) = (d\alpha_1) \square_p (d\alpha_2)$$

whenever $p \in |\langle d\alpha_1 \rangle|$. Also, $d(\{\beta\}) = \beta$ and $d([\gamma]) = \{\gamma\}$ for $\beta \in P_{n-1}$, $\gamma \in P_{n-2}$.

We look at the types of the 3-cells considered in subsection 1.6 of section 1. Let's use φ, $\hat{\varphi}$, $\#$ as above; $\sigma = \{\{*\}\}[*]$, $\hat{\sigma} = \{\sigma\}$. We have $\bar{f} = \bar{h} = \varphi$, $\bar{i} = \bar{g} = \sigma$, $\bar{\alpha} = \hat{\varphi}\#\hat{\sigma}\#$, $\bar{\beta} = \hat{\varphi}\hat{\sigma}\#\#$; also, $\overline{\beta \square_1 \alpha} = \bar{\beta} \square_1 \bar{\alpha} = \hat{\varphi}\hat{\sigma}\#\hat{\sigma}\#$; $\bar{\alpha}, \bar{\beta}, \bar{\beta} \square_1 \bar{\alpha} \in P_2$. For the 3-cells u, v, we have $\bar{u} = \bar{\alpha}$, $\bar{v} = \bar{\beta}$, both elements of $C_3 = P_2$. The 3-pd $\psi = v \circ_1 u = v(u(h, i), g)$ (from 1.6 too),

$$\bar{\psi} = \{\bar{\beta}\}\{\bar{\alpha}\}[\varphi][\sigma][\sigma] = \{\hat{\varphi}\hat{\sigma}\#\#\}\{\hat{\varphi}\#\hat{\sigma}\#\}[\varphi][\sigma][\sigma] ;$$

$$d\bar{\psi} = \bar{\beta} \square_1 \bar{\alpha} = \hat{\varphi}\hat{\sigma}\#\hat{\sigma}\# \quad .$$

Also, $\bar{\beta}' = \hat{\varphi}\#\hat{\sigma}\# = \bar{\alpha}$, $\bar{v}' = \bar{\beta}'$, $\psi' = v' \circ_1 u = v'(u(h, i), g)$, and

$$\bar{\psi}' = \{\bar{\alpha}\}\{\bar{\alpha}\}[\varphi][\sigma][\sigma] = \{\hat{\varphi}\#\hat{\sigma}\#\}\{\hat{\varphi}\#\hat{\sigma}\#\}[\varphi][\sigma][\sigma] ;$$

$$d\bar{\psi} = \bar{\beta}' \square_1 \bar{\alpha} = \hat{\varphi}\#\hat{\sigma}\hat{\sigma}\# \quad .$$