6. Multitopic sets

An *multitopic set* S, by definition, consists of data (i) to (iii), subject to conditions (iv) to (viii):

(i) a sequence $\langle C_k \rangle_{k\in\mathbb{N}}$ of sets [to indicate dependence on S, we may write $C_k(S)$ for C_k, and similarly for the other ingredients to follow],

(ii) sequences $\langle \boldsymbol{C}_k \rangle_{k\in\mathbb{N}}$, $\langle \boldsymbol{D}_k \rangle_{k\in\mathbb{N}}$ of multicategories,

and

(iii) morphisms $\mathrm{d}_{k+1}:\boldsymbol{C}_{k+1}\longrightarrow\boldsymbol{D}_k$ $(k\in\mathbb{N})$ of multicategories

such that

(iv) $\boldsymbol{C}_0$ has only identity arrows, and $\mathrm{O}(\boldsymbol{C}_0)=C_0$;

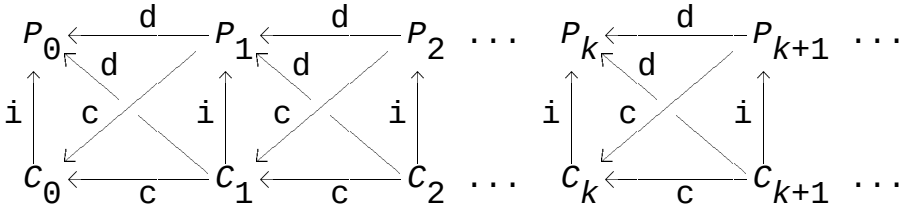
(v) for $k\geq 1$, $\boldsymbol{C}_k$ is free on a language $\mathcal{L}_k$ for which $\mathrm{O}(\mathcal{L}_k)=C_{k-1}$, $\mathrm{L}(\mathcal{L}_k)=C_k$;

(vi) $\boldsymbol{D}_0=\boldsymbol{C}_0$;

(vii) $\mathrm{O}(\boldsymbol{D}_k)=C_k=\mathrm{O}(\boldsymbol{C}_{k+1})$, and $\mathrm{d}_{k+1}:\boldsymbol{C}_{k+1}\longrightarrow\boldsymbol{D}_k$ is a strict morphism which is the identity on upper level objects;

(viii) for $k\geq 1$, $\boldsymbol{D}_k$ is the multicategory of function-replacement associated with $\boldsymbol{C}_k \xrightarrow{d_k} \boldsymbol{D}_{k-1}$ (see the previous section).

The multitopic set S gives rise to the following diagram of sets and functions:



Here, C_k is the set of *k-cells*, and it is given in (i) in the data for S. $P_k=A(\boldsymbol{C}_k)=A(\boldsymbol{D}_k)$; its elements are called the *k-pasting diagrams* of S. We have omitted subscripts from the maps; each should be understood with the same subscript as its domain; e.g., $\mathrm{d}_{k+1}:P_{k+1}\to P_k$, which is the effect on arrows of the morphism $\mathrm{d}_{k+1}:\boldsymbol{C}_{k+1}\to\boldsymbol{D}_k$.

$c_{k+1}=t\boldsymbol{C}_{k+1}$.

In general, $d(a)$ is the *domain*, $c(a)$ is the *codomain* of a whatever a is to which the d or c in question applies. In particular, we talk about domain and codomain of both cells and pasting diagrams.

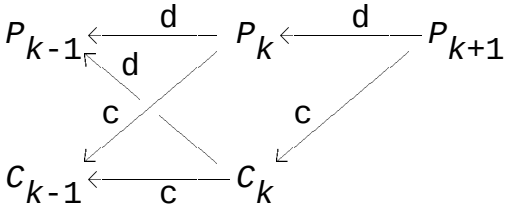
$i_k:C_k\to P_k$ is the inclusion of the generating arrows into the free $\boldsymbol{C}_k=\mathcal{F}(\mathcal{L}_k)$. d_k and c_k with domain C_k are the composites $d_k i_k$, $c_k i_k$, respectively.

We have

$$dd=dc\ ,\ cd=cc\ ;$$

in more detail,

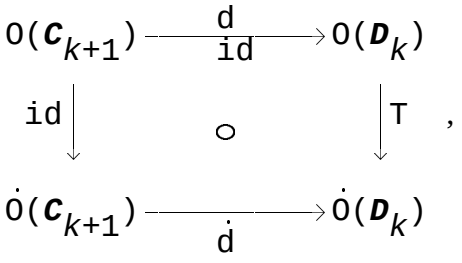
$$d_k d_{k+1}=d_k c_{k+1}\ ,\ c_k d_{k+1}=c_k c_{k+1}\ : \qquad (1)$$



for all $k\in\mathbb{N}-\{0\}$. This is the familiar "globular" aspect of higher-dimensional category theory: it says that the domain and the codomain of a cell of dimension $k+1$ greater than 1 are *parallel* to each other, that is, they agree as far as their domains and codomains are concerned. Note however that domains and codomains here are very different things; the domains are pasting diagrams, the codomains are individual cells.

To see (1), let us abbreviate d_{k+1} and c_{k+1} by d and c , respectively. d on upper level objects is the identity id_{C_k} . Recall that, by the definition of $\boldsymbol{D}_k$ as in (viii) , we have that $\dot{a}$, the lower level object corresponding to the upper level object $a\in C_k$, is $\dot{a}=T(a)=(d_k(a), c_k(a))$. Remember also that $t_{\boldsymbol{D}_k}(\beta)=T(\beta)=(d_k(\beta), c_k(\beta))$ ($\beta\in P_k$).

On lower level objects, the effect $\dot{d}$ of d is forced by the required commutativity of



to be $\dot{d}(a)=T(a)$. Since $d:\boldsymbol{C}_{k+1}\longrightarrow\boldsymbol{D}_k$ preserves "target", for any $\alpha\in P_{k+1}$, $t_{\boldsymbol{D}_k}(d\alpha))=\dot{d}(t_{\boldsymbol{C}_{k+1}}(\alpha))$, that is $T(d\alpha)=T(c\alpha)$, which is (1).

Note that $O(\boldsymbol{D}_k)=L(\mathcal{L}_k)=C_k$, $\dot{O}(\boldsymbol{D}_k)=A(\boldsymbol{D}_{k-1})\times O(\boldsymbol{C}_k)=P_{k-1}\times C_{k-1}$, and we have $A(\boldsymbol{D}_k)=A(\boldsymbol{C}_k)$.

The fact that $d_{k+1}:\boldsymbol{C}_{k+1}\longrightarrow\boldsymbol{D}_k$ is a strict morphism which is the identity on objects implies that $s_{\boldsymbol{C}_{k+1}}(\alpha)=s_{\boldsymbol{D}_k}(\alpha)=\langle\alpha\rangle$, where $\langle\alpha\rangle$ is defined as the left-to-right tuple of function-symbol occurrences in α ; see the previous section.

The fundamental equality is

$$d(\alpha\circ_p\beta) = (d\alpha)\square_p(d\beta) \qquad (p\in|\langle\alpha\rangle| , c\beta=\langle\alpha\rangle(p)),$$

signifying part of the fact that $d=d_{k+1}$ is a strict morphism of multicategories; here $\alpha, \beta\in C_{k+1}$, $\circ$ is the composition in $\boldsymbol{C}_{k+1}$, $\square$ is the composition in $\boldsymbol{D}_k$.

It is possible to build an multitopic set recursively. An *n-truncated multitopic set* is given by data $\langle C_k\rangle_{k\in[0,\,n]}$, $\langle\boldsymbol{C}_k\rangle_{k\in[0,\,n]}$, $\langle\boldsymbol{D}_k\rangle_{k\in[0,\,n-1]}$ as above in (i), (ii), (iii), but with the index k ranging over the integers 0 to n inclusive in the first two sequences, and up to $n-1$ in the last; the conditions (iv) to (viii) are assumed. Every (full) multitopic set S gives rise to its *n-truncation* $S\restriction n$, an n-truncated multitopic set; also, if $n\geq m$, an n-truncated one, S , gives rise to $S\restriction m$, an m-truncated one. On the other hand, if for each n , there is given S_n , an n-truncated multitopic set , and $S_{n+1}\restriction n=S_n$ for all n , then

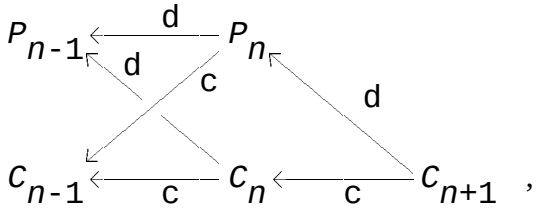
there is a unique multitopic set S for which $S\restriction n = S_n$ for all n.

With C_0 an arbitrary set (of 0-cells), we let $\boldsymbol{C}_0 = \mathcal{F}(\mathcal{L}_0)$ be the (free) multicategory whose objects are the 0-cells, and whose only arrows are identities ($\mathrm{O}(\mathcal{L}_0)=C_0$, $\mathrm{L}(\mathcal{L}_0)=\emptyset$). We have $P_0 = \mathrm{A}(C_0) \cong C_0$. This is all there is to a 0-truncated multitopic set.

After having determined C_0 , we let C_1 , the set of 1-cells, be any set, and we let $\mathrm{d}_1 : C_1 \to P_0 = C_0$, $\mathrm{c}_1 : C_1 \to C_0$ be arbitrary functions (the domain and codomain assignments for 1-cells). We let $\boldsymbol{D}_0 = \boldsymbol{C}_0$. The language $\mathcal{L}_1$ has $\mathrm{O}(\mathcal{L}_1) = C_0$, and $\mathrm{L}(\mathcal{L}_1) = C_1$; $\mathrm{s}_{\mathcal{L}_1}(f) = \langle \mathrm{d}_1(f) \rangle \in C_0^*$ (singleton tuple) , $\mathrm{t}_{\mathcal{L}_1}(f) = \mathrm{c}_1(f) \in C_0$. There is nothing to say about the amalgamating functions for $\boldsymbol{C}_1$.

Given an

n-truncated multitopic set S , with notation used above,
an arbitrary set C_{n+1} (of $(n+1)$-cells);
functions $\mathrm{c}_{n+1} : C_{n+1} \to C_n$, $\mathrm{d}_{n+1} : C_{n+1} \to P_n$ such that
$\mathrm{d}_n \mathrm{d}_{n+1} = \mathrm{d}_n \mathrm{c}_{n+1}$, $\mathrm{c}_n \mathrm{d}_{n+1} = \mathrm{c}_n \mathrm{c}_{n+1}$:



,

we have a uniquely determined $(n+1)$-truncated multitopic set $\hat{S}$ which extends the given data. To define $\hat{S}$, we first let $\boldsymbol{D}_n$ be the multicategory of function-replacement associated with $d_n : \boldsymbol{C}_n \to \boldsymbol{D}_{n-1}$.

Next, we define the language $\mathcal{L}_{n+1}$ to have $\mathrm{O}(\mathcal{L}_{n+1}) = C_n$, $\mathrm{L}(\mathcal{L}_{n+1}) = C_{n+1}$. For $f \in C_{n+1}$, $\mathrm{s}_{\mathcal{L}_{n+1}}(f) \ d\bar{\bar{e}}f \ \langle \mathrm{d}_{n+1}(f) \rangle \in C_n^*$; We let $t_{\mathcal{L}_{n+1}}(f) \ \mathrm{d}\bar{\bar{e}}\mathrm{f} \ \mathrm{c}_{n+1}(f) \in C_n$.

We define $\boldsymbol{C}^0_{n+1}$ (not yet $\boldsymbol{C}_{n+1}$) as the free multicategory on $\mathcal{L}_{n+1}$ with standard amalgamation.

To define $d^0 = d^0_{n+1} : \mathcal{C}^0_{n+1} \longrightarrow \mathcal{D}_n$, a morphism of multicategories, we use the freeness of $\mathcal{C}^0_{n+1}$. d^0 is defined on upper level objects as the identity: $d^0 : O(\mathcal{C}^0_{n+1}) = C_n \longrightarrow O(\mathcal{D}_n) = C_n$ is id_{C_n}. On lower level objects, the effect of d^0 is forced by the required commutativity of

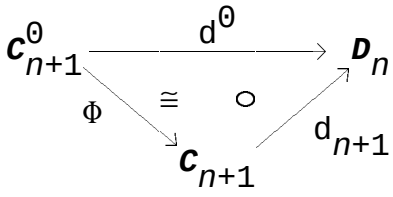
$$\begin{array}{ccc} O(\mathcal{C}^0_{n+1}) & \xrightarrow[id]{d^0} & O(\mathcal{D}_n) \\ \Big\downarrow id & \circlearrowleft & \Big\downarrow T \\ \dot{O}(\mathcal{C}^0_{n+1}) & \xrightarrow[\dot{d}^0]{} & \dot{O}(\mathcal{D}_n) \end{array} \quad ;$$

to be $\dot{d}^0 = T$, we put, for $a \in \dot{O}(\mathcal{C}^0_{n+1}) = C_n$, $\dot{d}^0(a) = T(a) = (d_n(a), c_n(a))$.

On a generating arrow $f \in L(\mathcal{L}_{n+1}) = C_{n+1}$, we put $d^0(f) = d_{n+1}(f)$. We need to have that $t_{\mathcal{D}_n}(d^0(f)) = \dot{d}^0(t_{\mathcal{L}_{n+1}}(f))$; but this means $T(d_{n+1}(f)) = T(c_{n+1}(f))$, which reduces to $d_n(d_{n+1}(f)) = d_n(c_{n+1}(f))$ and $c_n(d_{n+1}(f)) = c_n(c_{n+1}(f))$, which are true by the assumptions we have made on d_{n+1}, c_{n+1}.

For $f \in C_{n+1}$, $s_{\mathcal{L}_{n+1}}(f) = \langle d_{n+1}(f) \rangle$, and $s_{\mathcal{D}_n}(d^0(f)) = \langle d_{n+1}(f) \rangle$; also, the effect of d^0 on the upper level objects is the identity; therefore, it is legitimate to define the transition isomorphism $\theta_f : |s_{\mathcal{L}_{n+1}}(f)| \xrightarrow{\cong} |s_{\mathcal{D}_n}(d^0(f))|$ (see section 3.) to be the identity.

The freeness of $\mathcal{C}^0_{n+1}$ on $\mathcal{L}_{n+1}$ ensures the existence and uniqueness of $d^0 : \mathcal{C}^0_{n+1} \longrightarrow \mathcal{D}_n$, a morphism of multicategories, extending the determination of d^0 given on $\mathcal{L}_{n+1}$. In particular, d^0 is the identity on the upper level objects. However, d^0 is not, in general, a strict morphism, since $\mathcal{D}_n$ may have nonstandard amalgamation. We factor d^0 as in



so that Φ is an isomorphism which acts as the identity on objects and arrows (but may be non-strict), and d_{n+1} is strict (see the end of section 2.). This is the definition of the desired $d_{n+1}: C_{n+1} \longrightarrow D_n$. Since $C_{n+1} \cong C^0_{n+1}$, by an isomorphism which is the identity on objects and arrows, C_{n+1} is also free on $\mathcal{L}_{n+1}$, with possibly nonstandard amalgamation. We have completed the definition of $\hat{S}$.

Putting $S_{n+1}=\hat{S}_n$, we produce a sequence of truncated multitopic sets S_n which togerther define a full multitopic set S . The definition of the S_n and S is by a non-deterministic recursion, with the data on the n-cells and their domains and codomains being parameters that are to some extent arbitrary.