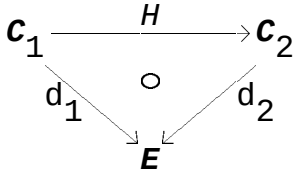
## 5. A 2-level multicategory with non-standard amalgamation

Let $\mathcal{L}$ be a language, and $\boldsymbol{C}$ a (not necessarily standard, but 1-level) multicategory free over $\mathcal{L}$ (see section 4). For $\alpha\in A=\mathrm{A}(\boldsymbol{C})$, we define $\langle\alpha\rangle$ to be "the tuple of occurrences of operation symbols in $\alpha$, listed from the left to the right". For a formal definition, we use unique redability (4.(1)), which enables us to employ a recursion. For $X\in O$, $\langle 1_X\rangle=\perp$, the empty tuple. For $f\in L(\mathcal{L})$, $m=\ell\mathrm{h}(\mathrm{s}(f))$, $\alpha_i\in A$, $n_i=\ell\mathrm{h}(\langle\alpha_i\rangle)$, $\alpha=f(\alpha_1, \alpha_2, \ldots, \alpha_m)\in A$, we put $\ell\mathrm{h}(\langle\alpha\rangle)=1+\sum_{i=1}^{m} n_i$, $\langle\alpha\rangle(1)=f$, and for $i\in[1, m]$, $k\in[1, n_i]$, $j=1+\sum_{h<i} n_h+k$, we define $\langle\alpha\rangle(j)=\langle\alpha_i\rangle(k)$. This is the same as saying that

$$\langle f(\alpha_1, \alpha_2, \ldots, \alpha_m)\rangle = \langle f\rangle^\wedge\langle\alpha_1\rangle^\wedge\langle\alpha_2\rangle^\wedge\ldots^\wedge\langle\alpha_m\rangle\,, \qquad (1)$$

where we used the well-known operation of concatenation of tuples; $\langle f\rangle$ means the one-term tuple whose only term is $f$; of course, it is also the same as $\langle\alpha\rangle$ in the sense being defined now, for $\alpha=f$.

Let us fix a multicategory $\boldsymbol{E}$, not necessarily 1-level, or with standard amalgamation. A *free multicategory over* $\boldsymbol{E}$ is a system $(\mathcal{L}, \boldsymbol{C}, \mathrm{d}:\boldsymbol{C}\to\boldsymbol{E})$ where $\mathcal{L}$ is a language, $\boldsymbol{C}$ is a (1-level) multicategory free over $\mathcal{L}$ such that $(\mathrm{O}(\mathcal{L})=)\mathrm{O}(\boldsymbol{C})=\mathrm{O}(\boldsymbol{E})$, the morphism $\mathrm{d}:\boldsymbol{C}\to\boldsymbol{E}$ of multicategories is the identity on upper-level objects, and it is strict. A *morphism* $H:(\mathcal{L}_1, \boldsymbol{C}_1, \mathrm{d}_1)\to(\mathcal{L}_2, \boldsymbol{C}_2, \mathrm{d}_2)$ of free multicategories over $\boldsymbol{E}$ is a mapping $H:\mathrm{A}(\mathcal{L}_1)\to\mathrm{A}(\mathcal{L}_2)$ such that for any $f\in\mathrm{A}(\mathcal{L}_1)$, we have $\mathrm{s}_{\mathcal{L}_2}(H(f))=\mathrm{s}_{\mathcal{L}_1}(f)$, $\mathrm{t}_{\mathcal{L}_2}(H(f))=\mathrm{t}_{\mathcal{L}_1}(f)$, and for the induced strict morphism $H:\boldsymbol{C}_1\to\boldsymbol{C}_2$ (which is the identity on objects) we have the commutative diagram




.

We also say that $H$ is a *morphism of languages*, and write $H:\mathcal{L}_1\to\mathcal{L}_2$ (note that by a "morphism of languages" one might *a priori* mean something more general; we do not need

the more general concept).

**Theorem.** There is a uniquely determined assignment of a multicategory $\boldsymbol{D}=\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ to any free multicategory $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ over $\boldsymbol{E}$ such that conditions (i) to (vi) hold.

**(i)** $\mathrm{O}(\boldsymbol{D})=\mathrm{L}(\mathcal{L})$ , $\dot{\mathrm{O}}(\boldsymbol{D})=\mathrm{A}(\boldsymbol{E})\times\mathrm{O}(\boldsymbol{E})$ , $\mathrm{A}(\boldsymbol{D})=\mathrm{A}(\boldsymbol{C})$ .

**(ii)** Using the abbreviation $\mathrm{T}(\alpha)=(\mathrm{d}(\alpha),\mathrm{t}_{\boldsymbol{C}}(\alpha))$ ( $\alpha\in A=\mathrm{A}(\boldsymbol{D})=\mathrm{A}(\boldsymbol{C})$ ) the mapping $\mathrm{O}(\boldsymbol{D})\longrightarrow\dot{\mathrm{O}}(\boldsymbol{D})$ is $s\mapsto\langle\mathrm{T}(s(i))\rangle_{i\in|s|}$ .

**(iii)** For $\alpha\in A$ , we have $\mathrm{t}_{\boldsymbol{D}}(\alpha)=\mathrm{T}(\alpha)$ and $\mathrm{s}_{\boldsymbol{D}}(\alpha)=\langle\alpha\rangle$ .

**(iv)** For $f\in\mathrm{L}=\mathrm{O}(\boldsymbol{D})$ , $1_f^{(\boldsymbol{D})}=f$ .

**(v)** Let us write $\square$ for $\circ^{(\boldsymbol{D})}$ , and $\circ$ for $\circ^{(\boldsymbol{C})}$ . Whenever $\alpha\square_p\beta$ is well-defined, we have that

$$\alpha=\alpha'\circ_q f(\alpha_1,\ldots,\alpha_n) \tag{2}$$

and

$$\alpha\square_p\beta \;\bar{\mathrm{def}}\; \alpha'\circ_q\beta(\alpha_1,\ldots,\alpha_n) \tag{3}$$

for $f=\langle\alpha\rangle(p)$ , and for some $\alpha',\alpha_1,\ldots,\alpha_n\in A$ and $q\in|\mathrm{s}_{\boldsymbol{C}}(\alpha)|$ (we are referring here to simultaneous composition in $\boldsymbol{C}$ , discussed in the last section).

**(vi)** Whenever $H:(\mathcal{L}_1,\boldsymbol{C}_1,d_1)\to(\mathcal{L}_2,\boldsymbol{C}_2,d_2)$ is a morphism of free multicategories over $\boldsymbol{E}$ , $\boldsymbol{D}_j=\boldsymbol{D}[\mathcal{L}_j,\boldsymbol{C}_j,d_j]$ , the mappings

$$\begin{aligned}
&H:\mathrm{O}(\boldsymbol{D}_1)=\mathrm{L}(\mathcal{L}_1)\longrightarrow\mathrm{O}(\boldsymbol{D}_2)=\mathrm{L}(\mathcal{L}_2)\\
&\mathrm{id}:\dot{\mathrm{O}}(\boldsymbol{D}_1)=\mathrm{A}(\boldsymbol{E})\times\mathrm{O}(\boldsymbol{E})\longrightarrow\dot{\mathrm{O}}(\boldsymbol{D}_2)=\mathrm{A}(\boldsymbol{E})\times\mathrm{O}(\boldsymbol{E})\\
&H:\mathrm{A}(\boldsymbol{D}_1)=\mathrm{A}(\boldsymbol{C}_1)\longrightarrow\mathrm{A}(\boldsymbol{D}_2)=\mathrm{A}(\boldsymbol{C}_2)
\end{aligned}$$

constitute a strict morphism $H:\boldsymbol{D}_1\longrightarrow\boldsymbol{D}_2$ of multicategories.

The multicategory $\boldsymbol{D}=\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ is called the *multicategory of function-replacement associated with* $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ . The name derives from the main clause, (v). This clause *tries to*

*say* that

" $\alpha\Box_p\beta$ results by replacing the "function-symbol" $f=\langle\alpha\rangle(p)$ at the place $p$ in $\alpha$ by the arrow $\beta$ ";

however, it actually says less, namely that

" $\alpha\Box_p\beta$ results by replacing the "function-symbol" $f=\langle\alpha\rangle(p)$ at *some* place in $\alpha$ by the arrow $\beta$ ".

Note the difficulty of saying the first of these two statements mathematically; this difficulty comes from the fact that $f$ may occur at more than one place in $\alpha$. The theorem avoids specifying the *particular* decomposition (2) that "belongs to" the place p , and still manages to give the complete definition of the concept of the multicategory of function replacement. The price we pay is that we do not have the definition of $\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ for any *particular* $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ spelled out in detail; rather, we have the complete definition of the *global* assignment $(\mathcal{L},\boldsymbol{C},\mathrm{d})\longmapsto\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ .

The rest of this section is devoted to the proof of the theorem; certain technical details will be relegated to the Appendix.

We prove the theorem in two stages. In the first, we fix $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ , a free multicategory over $\boldsymbol{E}$ , and define the operations for $\boldsymbol{D}=\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ partially, for certain combinations of arguments only, ones that we will call "separated". The second stage will involve the use of morphisms of free multicategories over $\boldsymbol{E}$ to complete the definition.

With the fixed $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ , we have $O=\mathrm{O}(\mathcal{L})=\mathrm{O}(\boldsymbol{C})$ , $L=\mathrm{L}(\mathcal{L})$ , $A=\mathrm{A}(\boldsymbol{C})$ . $\alpha,\beta,\gamma$ denote elements of $A$ , $f$, $g$ elements of $L$ .

The construction of $\boldsymbol{D}$ takes place in $\boldsymbol{C}$ . The role $\boldsymbol{E}$ and $\mathrm{d}$ have in the construction is summarized in the following lemma:

**(4) Lemma**.

**(i)** $\mathrm{d}(\alpha)=\mathrm{d}(\beta)$ implies that $\mathrm{s}(\alpha)=\mathrm{s}(\beta)$ .

**(ii)** $d(\alpha \circ_r \beta) = (d\alpha) \circ_r^{(\boldsymbol{E})} (d\beta)$ ,

and more generally

$$d(\alpha(\alpha_1, \ldots, \alpha_n)) = (d\alpha)(d\alpha_1, \ldots, d\alpha_n)$$

whenever $\alpha \circ_r \beta$ , $\alpha(\alpha_1, \ldots, \alpha_n)$ are well-defined; here, we refer to simultaneous composition, in $\boldsymbol{C}$ on the left and in $\boldsymbol{E}$ on the right.

**(iii)** Suppose that $d(\alpha_1)=d(\alpha_2)$ , $d(\beta_1)=d(\beta_2)$ and $c(\beta_1)=c(\beta_2)$ . Suppose $\alpha_1 \circ_r \beta_1$ is well-defined. Then also,

**(a)** $\alpha_2 \circ_r \beta_2$ is well-defined;

**(b)** $d(\alpha_1 \circ_r \beta_1) = d(\alpha_2 \circ_r \beta_2)$ ;

as a consequence,

**(c)** $s(\alpha_1 \circ_r \beta_1) = s(\alpha_2 \circ_r \beta_2)$ ;

and we have, for the amalgamating functions for $\boldsymbol{C}$ , that

**(d)** $\varphi[\alpha_1, \beta_1, r]=\varphi[\alpha_2, \beta_2, r]$ , $\psi[\alpha_1, \beta_1, r]=\psi[\alpha_2, \beta_2, r]$ .

This is essentially immediate from the fact that $d:\boldsymbol{C} \to \boldsymbol{E}$ is a strict morphism which is the identity on objects; here are some details.

For (i): since $d:\boldsymbol{C} \to \boldsymbol{E}$ is a strict morphism which is the identity on objects, $s_{\boldsymbol{E}}(d(\alpha))=s_{\boldsymbol{C}}(\alpha)=s(\alpha)$ . Therefore, $d(\alpha)=d(\beta)$ implies $s(\alpha)=s(\beta)$ .

For (ii): this is a consequence of the fact that $d$ is a strict morphism $d:\boldsymbol{C} \to \boldsymbol{E}$ of multicategories.

For (iii):

Remember that $\alpha \circ_r \beta$ is well-defined iff $r \in |s(\alpha)|$ and $s(\alpha)(r)=c(\beta)$ . Thus, (a) is clear.

Writing $\alpha,\beta$ for $\alpha_i, \beta_i$ , for either $i=1$ or $i=2$ , we have $d(\alpha \circ_r \beta) = d(\alpha) \circ_r^{\boldsymbol{E}} d(\beta)$ , implying (b). By the strictness of $d$ , $\varphi[\alpha,\beta,r]=\varphi^{\boldsymbol{E}}[d(\alpha),d(\beta),r]$ , and similarly for $\psi$ . The equalities $d(\alpha_1)=d(\alpha_2)$ , $d(\beta_1)=d(\beta_2)$ now clearly imply (d).

Let us write $\|\alpha\|$ for the range of the function $\langle\alpha\rangle$ , the set of function-symbols occurring in $\alpha$ . The definition of $\langle\alpha\rangle$ gives that

$$\|f(\alpha_1, \ldots, \alpha_n)\| = \{f\} \cup \|\alpha_1\| \cup \ldots \cup \|\alpha_n\|$$

and by induction on $\alpha$, we see that

$$\|\alpha \circ_q \beta\| = \|\alpha\| \cup \|\beta\| \ . \tag{5}$$

Note the obvious fact that, for $\alpha, \beta \in A$, the existence of at least one map $\langle \beta \rangle \longrightarrow \langle \alpha \rangle$ in $L^{\#}$ is equivalent to the condition $\|\beta\| \subset \|\alpha\|$.

Let $\alpha \in A$ and let $f \in \|\alpha\|$. Any representation of $\alpha$ in the form of (2), with suitable $\alpha'$, etc., is called a *decomposition of* $\alpha$ *at* $f$.

**(6) Lemma.** Assume $f \in \|\alpha\|$. There is at least one decomposition of $\alpha$ at $f$.

**Proof:** see Appendix.

$\alpha$ is *separated* if $\langle \alpha \rangle$ is a repetition-free tuple: the function $\langle \alpha \rangle : |\langle \alpha \rangle| \longrightarrow L$ is one-to-one. A system $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of terms is *separated* if each $\alpha_i$ is separated, and the ranges $\|\alpha_i\|$ are pairwise disjoint sets. Note the obvious fact that if $\alpha$ is separated, $\beta$ is any term, then there is at most one arrow $\langle \beta \rangle \to \langle \alpha \rangle$ in $L^{\#}$. For $f \in L$, the well-defined term $f(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is separated iff the system $(f, \alpha_1, \alpha_2, \ldots, \alpha_n)$ is separated. By induction on $\alpha$, we see that $\alpha \circ_q \beta$ is separated iff $(\alpha, \beta)$ is a separated system; in particular, if $\alpha \circ_q \beta$ is separated, then $\|\alpha\| \cap \|\beta\| = \emptyset$.

The following lemma is intuitively obvious.

**(7) Lemma.** For a separated term $\alpha$, the decomposition at $f \in \|\alpha\|$ (see (2)) is unique.

**Proof:** see Appendix.

Let $\alpha, \beta \in A$, and $f \in L$. We declare $\alpha \Box_f \beta$ to be *well-defined* if and only if $\alpha$ is separated, $f \in \|\alpha\|$, and $T(f)=T(\beta)$ (for $T$, see (ii) of the theorem); if so, $\alpha \Box_f \beta$ is given by the expression (3), that is,

$$\alpha \Box_f \beta = \alpha' \circ_q \beta(\alpha_1, \ldots, \alpha_n),$$

where we refer to (2), the (unique) decomposition of $\alpha$ at $f$ (by (6) and (7)). Note that, instead of a "place" $p$, we now have a function-symbol $f$ in the subscript position. We still have to see that the expression defining $\alpha \Box_f \beta$ is well-defined.

Note that $T(f)=T(\beta)$ implies that $s(f)=s(\beta)$ and $t(f)=t(\beta)$. The simultaneous composition $\beta(\alpha_1, \ldots, \alpha_n)$ is well-defined since $(s(\beta)(i))=(s(f)(i))=t(\alpha_i)$, the second equality from the fact that $f(\alpha_1, \ldots, \alpha_n)$ is well-defined. The composition at $q$ is well-defined since

$$(s(\alpha')(q)) = t(f) = t(\beta) = t(\beta(\alpha_1, \ldots, \alpha_n)) ;$$

the first equality holding since (2) is well-defined.

The first thing we check is that

$$T(\alpha \Box_f \beta)=T(\alpha) . \qquad (8)$$

provided $\alpha \Box_f \beta$ is well-defined. Applying $d$ to the expressions (2) and (3), and applying (4) repeatedly, we get that

$$d(\alpha) = d(\alpha') \circ_q d(f)(d(\alpha_1), \ldots, d(\alpha_n)),$$

and

$$d(\alpha \Box_f \beta) = d(\alpha') \circ_q d(\beta)(d(\alpha_1), \ldots, d(\alpha_n)) ;$$

on the right-hand side, we have simultaneous composition in $\boldsymbol{E}$. The equality $d(f)=d(\beta)$ ensures that $d(\alpha \Box_f \beta)=d(\alpha)$. We also have that $c(\alpha \Box_f \beta)=c(\alpha')=c(\alpha)$. By (4), it follows that (8) holds.

The definition of $\alpha\Box_f\beta$ gives immediately that we have

$$\|\alpha\Box_f\beta\| = (\|\alpha\| - \{f\}) \cup \|\beta\| \ . \tag{9}$$

Next, we claim

**(10) Lemma.** Assuming the pair $(\alpha, \beta)$ is separated and $f \in \|\alpha \circ_r \beta\|$,

$$(\alpha \circ_r \beta)\Box_f \gamma = \begin{cases} (\alpha\Box_f\gamma) \circ_r \beta & \text{if } f \in \|\alpha\| \\ \alpha \circ_r (\beta\Box_f\gamma) & \text{if } f \in \|\beta\| \end{cases}$$

(Note that the same place $r$ appears on the two sides. The upper right-hand occurrence of the composition $\circ_r$ at $r$ is meaningful, since by (8) and (4), $s(\alpha\Box_f\gamma) = s(\alpha)$ .)

**Proof:** see Appendix.

Note that, together with the equality $f\Box_f\beta = \beta$ (provided $f\Box_f\beta$ is well-defined), (10) determines the value of $\alpha\Box_f\gamma$ in all cases, since the generating arrows $f \in L$, together with the identities generate $A$. Of course, (10) cannot be used to define $\Box$, directly at least, since terms can, in general, be written in the form $\alpha \circ_r \beta$ in more than one way.

The definition of $\alpha\Box_f\beta$ in terms of $\circ_C$, and what we know about separatedness and $\circ_C$, makes it clear that if the pair $(\alpha, \beta)$ is separated, then so is the term $\alpha\Box_f\beta$ (a little less would in fact suffice).

We are ready to state and prove the associative and commutative laws for the separated case; the proof uses (10).

**(11) Lemma.** Assume that the triple $(\alpha, \beta, \gamma)$ is separated, $f \in \|\alpha\|$, $g \in \|\alpha\Box_f\beta\| = (\|\alpha\| - \{f\}) \cup \|\beta\|$, $T(\beta) = T(f)$ and $T(\gamma) = T(g)$. Then

$$(\alpha\Box_f\beta)\Box_g\gamma \;=\; \begin{array}{ll}(\alpha\Box_g\gamma)\Box_f\beta & \text{if } g\in\|\alpha\| \\ \alpha\Box_f(\beta\Box_g\gamma) & \text{if } g\in\|\beta\|\end{array} \quad .$$

(Note that the assumption implies that the left-hand side is well-defined. Note also that the right-hand expressions are well-defined too; if $g\in\|\beta\|$, we have, by (8), $T(\beta\Box_g\gamma)=T(\beta)=T(f)$, making $\alpha\Box_f(\beta\Box_g\gamma)$ well-defined.)

**Proof:** see Appendix.

Assume again that $(\alpha, \beta)$ is separated, and $f\in\|\alpha\|$, so $\alpha\Box_f\beta$ is well-defined. Let $p\in|\langle\alpha\rangle|$ such that $\langle\alpha\rangle(p)=f$. As we have noted, the separatedness of $\alpha\Box_f\beta$ ensures there is at most one morphism $\langle\alpha\rangle\backslash p\longrightarrow\langle\alpha\Box_f\beta\rangle$, and at most one $\langle\beta\rangle\longrightarrow\langle\alpha\Box_f\beta\rangle$. But also, since $\|\alpha\Box_f\beta\|=(\|\alpha\|-\{f\})\cup\|\beta\|$, there are such morphisms

$$\langle\alpha\rangle\backslash p\xrightarrow{\ \psi\ }\langle\alpha\Box_f\beta\rangle\xleftarrow{\ \varphi\ }\langle\beta\rangle\ , \qquad (15)$$
$$\psi=\psi_{\Box}(\alpha,\beta,f)\ ,\quad \varphi=\varphi_{\Box}(\alpha,\beta,f)\ ;$$

we have defined the amalgamating maps for the composition $\circ_{\boldsymbol{D}}=\Box$, partially, for the "separated case". Finally, we note that, provided the triple $(\alpha, \beta, \gamma)$ is separated, each one of the diagrams made up of amalgamating maps for the composition $\circ_{\boldsymbol{D}}=\Box$, associated with either the associative law $\delta_{\mathrm{d\bar{e}f}}(\alpha\Box_f\beta)\Box_g\gamma=\alpha\Box_f(\beta\Box_g\gamma)$ in 1.(xi) or the commutative law $\delta_{\mathrm{d\bar{e}f}}(\alpha\Box_f\beta)\Box_g\gamma=(\alpha\Box_f\beta)\Box_g\gamma$ in 1.(xii) as the case may be, is automatically commutative, by the separatedness of the term $\delta$, which implies that into $\langle\delta\rangle$ from any other object of $\boldsymbol{L}^{\#}$ there is at most one morphism.

This completes the work of establishing the multicategory structure in the restricted sense of applying to the "sufficiently separated" arguments. We now enter the second stage of the proof of the theorem.

Let $F:(\hat{\mathcal{L}},\hat{\boldsymbol{C}},\hat{\mathrm{d}})\longrightarrow(\mathcal{L},\boldsymbol{C},\mathrm{d})$ be a morphism of free multicategories over $\boldsymbol{E}$. We say that $F$ is *ample*, or that $(\hat{\mathcal{L}},\hat{\boldsymbol{C}},\hat{\mathrm{d}})$ is an *ample expansion of* $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ *via* $F$, if for each $f\in\boldsymbol{L}$ there are infinitely many distinct $\hat{f}\in\hat{\boldsymbol{L}}$ such that $F(\hat{f})=f$.

It is almost obvious that we have

**(12) Lemma.** Any free multicategory $(\mathcal{L},\boldsymbol{C},\mathrm{d})$ over $\boldsymbol{E}$ has ample expansions.

**Proof:** see Appendix.

Let $F:\hat{\boldsymbol{C}}\to\boldsymbol{C}$ be an ample expansion. Since $F$ is strict, for any $\beta\in\hat{A}=\mathrm{A}(\hat{\boldsymbol{C}})$, $\mathrm{s}_{\hat{\boldsymbol{C}}}(\beta)=\mathrm{s}_{\boldsymbol{C}}(F(\beta))$. We will write $\mathrm{s}(\beta)$, $\mathrm{t}(\beta)$ for $\mathrm{s}_{\hat{\boldsymbol{C}}}(\beta)$, $\mathrm{t}_{\hat{\boldsymbol{C}}}(\beta)$, respectively, just like in $\boldsymbol{C}$. We have, for any $\hat{\alpha}\in\hat{A}$, that

$$\langle F(\hat{\alpha})\rangle=F\circ\langle\hat{\alpha}\rangle\ :$$

$$\begin{array}{ccc} \langle F(\hat{\alpha})\rangle\,| = |\langle\hat{\alpha}\rangle| & \xrightarrow{\quad\langle\hat{\alpha}\rangle\quad} & \hat{L} \\ & \searrow^{\langle F(\hat{\alpha})\rangle} \quad \circlearrowright \quad \swarrow_{F} & \\ & L & \end{array} \tag{13}$$

Of course, "separated" terms in $\hat{\boldsymbol{C}}$ are meant as they were in $\boldsymbol{C}$.

Here is another "obvious" lemma.

**(14) Lemma.** $F:\hat{\boldsymbol{C}}\longrightarrow\boldsymbol{C}$ is surjective: For any $\alpha\in L$, there is at least one separated $\beta\in\hat{L}$ such that $F(\beta)=\alpha$. In fact, for any $\alpha\in L$, and any finite set $I$, there is at least one separated $\beta\in\hat{L}$ such that $F(\beta)=\alpha$ and $\|\beta\|\cap I=\emptyset$.

**Proof:** see Appendix.

We use the notations $\hat{\mathrm{c}}(\hat{\alpha})=\mathrm{t}_{\hat{\boldsymbol{C}}}(\hat{\alpha})$, $\hat{\mathrm{T}}(\hat{\alpha})=(\hat{\mathrm{d}}(\hat{\alpha}),\hat{\mathrm{c}}(\hat{\alpha}))$ for $\hat{\alpha}\in\hat{A}$. Since $\hat{\mathrm{d}}(\hat{\alpha})=\mathrm{d}(F\hat{\alpha})$, $\hat{\mathrm{c}}(\hat{\alpha})=\mathrm{c}(F\hat{\alpha})$, we have that $\hat{\mathrm{T}}(\hat{\alpha})=\mathrm{T}(\alpha)$ for $\alpha=F(\hat{\alpha})$.

Now, we can define $\alpha\square_p\beta$ for any $\alpha, \beta\in A=\mathsf{A}(\boldsymbol{C})$, and $p\in |\langle\alpha\rangle|$ such that, for $f=\langle\alpha\rangle(p)$, $\mathsf{T}(\beta)=\mathsf{T}(f)$. Let $\hat{\alpha}\in\hat{A}$ be separated such that $F(\hat{\alpha})=\alpha$ by (14); next, let $\hat{\beta}\in\hat{A}$ be separated such that $F(\hat{\beta})=\beta$ and $\|\beta\|\cap\|\alpha\|=\varnothing$ by (14) again. Let $\hat{f}=\langle\hat{\alpha}\rangle(p)$. By (13), $F(\hat{f})=f$; we have $\hat{\mathsf{T}}(\hat{\beta})=\hat{\mathsf{T}}(\hat{f})$, and $\hat{\alpha}\square_p\hat{\beta}=\hat{\alpha}\square_{\hat{f}}\hat{\beta}$ is well-defined. We define

$$\alpha\square_p\beta \ \overline{\text{def}} \ F(\hat{\alpha}\square_{\hat{f}}\hat{\beta}) .$$

Further, we define the amalgamating maps

$$\langle\alpha\rangle\backslash p \xrightarrow{\ \psi\ } \langle\alpha\square_p\beta\rangle \xleftarrow{\ \varphi\ } \langle\beta\rangle , \qquad (15)$$
$$\psi=\psi_{\square}(\alpha, \beta, p) , \quad \varphi=\varphi_{\square}(\alpha, \beta, p)$$

as the $F$-images of the maps

$$\langle\hat{\alpha}\rangle\backslash p \xrightarrow{\ \hat{\psi}\ } \langle\hat{\alpha}\square_{\hat{f}}\hat{\beta}\rangle \xleftarrow{\ \hat{\varphi}\ } \langle\hat{\beta}\rangle ,$$
$$\hat{\psi}=\psi_{\square}(\hat{\alpha}, \hat{\beta}, \hat{f}) , \quad \varphi=\varphi_{\square}(\hat{\alpha}, \hat{\beta}, \hat{f}) .$$

As maps of sets, $\psi$ and $\varphi$ are the same as $\hat{\psi}$ and $\hat{\varphi}$, respectively; by (13), e.g., $|\langle\alpha\rangle\backslash p| = |\langle\hat{\alpha}\rangle\backslash p|$, $|\langle\alpha\square_p\beta\rangle| = |\langle\hat{\alpha}\square_{\hat{f}}\hat{\beta}\rangle|$, thus, we can define $\psi: |\langle\alpha\rangle\backslash p| \longrightarrow |\langle\alpha\square_p\beta\rangle|$ as $\psi=\hat{\psi}$; similarly, $\varphi=\hat{\varphi}$; then by (13) again, $\varphi$ and $\psi$ are maps as in (15). It is also obvious that $(\varphi, \psi)$ are the coprojections of a coproduct in $L^{\#}$.

Let us show that this definition is legitimate: that is, the result does not depend on the choice of the ample expansion $F:\hat{\boldsymbol{C}}\to\boldsymbol{C}$, and the choice of $\hat{\alpha},\hat{\beta}$.

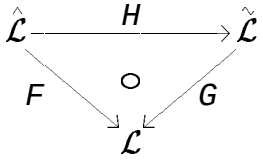
Let $\alpha,\beta\in\mathsf{A}(\boldsymbol{C})$, $p\in |\langle\alpha\rangle|$, $f=\langle\alpha\rangle(p)$. Let $\hat{\boldsymbol{C}},\hat{\alpha},\hat{\beta}$ and $\hat{f}$ be as above. Assume $G:\tilde{\boldsymbol{C}}=\mathcal{F}(\tilde{\mathcal{L}})\to\boldsymbol{C}$ is another ample expansion, $(\tilde{\alpha}, \tilde{\beta})$ a separated pair of terms in $\tilde{A}=\mathsf{A}(\tilde{\boldsymbol{C}})$

such that $G(\hat{\alpha})=\alpha$, $G(\hat{\beta})=\beta$; let $\tilde{f}=\langle\tilde{\alpha}\rangle(p)$; we want to show that

$$F(\hat{\alpha}\Box_{\hat{f}}\hat{\beta})=G(\tilde{\alpha}\Box_{\tilde{f}}\tilde{\beta}), \tag{16}$$

and that the amalgamating maps $\varphi,\psi$ also come out to be the same when we use the new data.

We claim that there is a morphism $H:\hat{\mathcal{L}}\to\tilde{\mathcal{L}}$ of languages (see above) such that




(17)

and such that $H\circ\langle\hat{\alpha}\rangle=\langle\tilde{\alpha}\rangle$, $H\circ\langle\hat{\beta}\rangle=\langle\tilde{\beta}\rangle$ for the particular $\alpha$ and $\beta$ given to us. The latter two conditions determine the effect of $H$ on the subset $\|\hat{\alpha}\|\dot{\cup}\|\hat{\beta}\|$ of $\hat{L}$; these conditions are possible to fulfill since $\langle\hat{\alpha}\rangle$, $\langle\hat{\beta}\rangle$ are injective functions, and $\|\hat{\alpha}\|\cap\|\hat{\beta}\|=\emptyset$. The restriction of $H$ to $\|\hat{\alpha}\|\dot{\cup}\|\hat{\beta}\|$ so determined will satisfy what it has for (17) to hold. On the rest of the set $\hat{L}$, $H$ can be defined arbitrarily, except for being subject to (17); the ampleness of $G$ ensures that for every $\hat{g}\in\hat{L}$ there is $\tilde{g}\in\tilde{L}$ such that $G(\tilde{g})=F(\hat{g})$; we may put $H(\hat{g})=\tilde{g}$.

$H$ gives rise to a morphism $H:\hat{\mathbf{C}}\longrightarrow\tilde{\mathbf{C}}$ for which the transition isomorphisms $\theta_{\hat{g}}$ ($\hat{g}\in\hat{L}$) are all identities. $H$ is the identity on objects. It follows from the strictness of $F$ and $G$, and (17) that $H$ is strict.

We claim that $H(\hat{\alpha})=\tilde{\alpha}$, $H(\hat{\beta})=\tilde{\beta}$. Note that for $\tilde{\alpha}_1=H(\hat{\alpha})$, $\tilde{\beta}_1=H(\hat{\beta})$, we have $G(\tilde{\alpha}_1)=G(\tilde{\alpha})$ and $G(\tilde{\beta}_1)=G(\tilde{\beta})$; also, $\langle\tilde{\alpha}_1\rangle=\langle\tilde{\alpha}\rangle=H\circ\langle\hat{\alpha}\rangle$, $\langle\tilde{\beta}_1\rangle=\langle\tilde{\beta}\rangle$. The assertion then follows from the following observation: if $\tilde{\gamma},\tilde{\delta}\in\tilde{L}$, $G(\tilde{\gamma})=G(\tilde{\delta})$, $\langle\tilde{\gamma}\rangle=\langle\tilde{\delta}\rangle$ then $\tilde{\gamma}=\tilde{\delta}$; this is proved by an induction on the length of $G(\tilde{\gamma})=G(\tilde{\delta})$.

The definition of $\hat{\alpha}\square_{\hat{f}}\hat{\beta}$ via the structure of $\hat{\boldsymbol{C}}$, and that of $\tilde{\alpha}\square_{\tilde{f}}\tilde{\beta}$ via the structure of $\tilde{\boldsymbol{C}}$ tell us that $H(\hat{\alpha}\square_{\hat{f}}\hat{\beta})=\tilde{\alpha}\square_{\tilde{f}}\tilde{\beta}$ holds as a consequence of $H(\hat{\alpha})=\tilde{\alpha}$, $H(\hat{\beta})=\tilde{\beta}$, the facts that $H$ is a strict morphism, and that $H$ is the identity on objects. For this, one notes, in the first place, that the decomposition of $\hat{\alpha}$ at $\hat{f}$ is carried by $H$ into the decomposition of $\tilde{\alpha}$ at $\tilde{f}$. Now, (16) follows by (17). The assertion on the amalgamating maps is also clear.

The condition (v) is clearly met by the construction.

We now prove assertion (vi) of the Theorem, even though we have not yet proved that $\boldsymbol{D}[\mathcal{L},\boldsymbol{C},\mathrm{d}]$ is a multicategory. Assume the data for (vi) as shown. The claim is that in this case we have

$$H(1_f^{(\boldsymbol{D}_1)}) = 1_{H(f)}^{(\boldsymbol{D}_2)} \tag{18}$$

$$H(\alpha\square_p^1\beta) = (H\alpha)\square_p^2(H\beta) \tag{19}$$

and

$$\varphi_{\square^1}[\alpha,\beta,p] = \varphi_{\square^1}[H\alpha,H\beta,p]\ ,\quad \psi_{\square^1}[\alpha,\beta,p] = \psi_{\square^1}[H\alpha,H\beta,p]\ , \tag{20}$$

every time $\alpha\square_p^1\beta$ is well-defined ( $\square^i$ is the composition in $\boldsymbol{D}_i$ ).

(18) is immediate. The proofs of (19) and (20) are also easy: having set up $F:\hat{\boldsymbol{C}}\to\boldsymbol{C}_1$, $\hat{\alpha}\overset{F}{\longmapsto}\alpha$, $\hat{\beta}\overset{F}{\longmapsto}\beta$ as is needed for the definition of $\alpha\square_p^1\beta$ as $\alpha\square_p^1\beta=F(\hat{\alpha}\square_{\hat{f}}^1\hat{\beta})$, we have a valid set-up $HF:\hat{\boldsymbol{C}}\to\boldsymbol{C}_2$, $\hat{\alpha}\overset{HF}{\longmapsto}H\alpha$, $\hat{\beta}\overset{HF}{\longmapsto}H\beta$ for the disambiguated definition of $(H\alpha)\square_p^2(H\beta)$ as $(H\alpha)\square_p^2(H\beta)=(HF)(\hat{\alpha}\square_{\hat{f}}^1\hat{\beta})=(H(F(\hat{\alpha}\square_{\hat{f}}^1\hat{\beta}))$, which shows (19); (20) is similarly seen.

The relation

$$T(\alpha \square_p \beta) = T(\alpha)$$

follows instantly from the definition of $\alpha \square_p \beta$, the variant (8) of the same law established in the "separated case", and applied in $\hat{\boldsymbol{C}}$, and by the fact $\hat{T}(\hat{\alpha}) = T(\alpha)$ when $F(\hat{\alpha}) = \alpha$.

The rest of the required laws for $\square$ are also easily established. We treat the associative law.

Suppose $\alpha, \beta, \gamma \in A$, $\alpha \square_p \beta$ and $\beta \square_q \gamma$ are well-defined, and $\bar{q} = \varphi_{\square}[\alpha, \beta, p](q)$, to see that $(\alpha \square_p \beta) \square_{\bar{q}} \gamma = \alpha \square_p (\beta \square_q \gamma)$. Choose an ample $F : (\hat{\mathcal{L}}, \hat{\boldsymbol{C}}, \hat{d}) \to (\mathcal{L}, \boldsymbol{C}, d)$ and by (14), choose arrows $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \in A(\hat{\boldsymbol{C}})$ such that $\hat{\alpha}$ is separated, $F(\hat{\alpha}) = \alpha$, $\hat{\beta}$ is separated, $\|\hat{\beta}\| \cap \|\hat{\alpha}\| = \emptyset$, $F(\hat{\beta}) = \beta$, $\hat{\gamma}$ is separated, $\|\hat{\gamma}\| \cap (\|\hat{\alpha}\| \cup \|\hat{\beta}\|) = \emptyset$, and $F(\hat{\gamma}) = \gamma$. Then, with $\hat{f} = \langle \hat{\alpha} \rangle(p)$, $\hat{g} = \langle \hat{\beta} \rangle(p)$, we have $(\hat{\alpha} \square_p \hat{\beta}) \square_{\bar{q}} \hat{\gamma} = (\hat{\alpha} \square_{\hat{f}} \hat{\beta}) \square_g \hat{\gamma}$, $\hat{\alpha} \square_p (\hat{\beta} \square_q \hat{\gamma}) = \hat{\alpha} \square_{\hat{f}} (\hat{\beta} \square_g \hat{\gamma})$, and $(\hat{\alpha} \square_{\hat{f}} \hat{\beta}) \square_g \hat{\gamma} = \hat{\alpha} \square_{\hat{f}} (\hat{\beta} \square_g \hat{\gamma})$ as $(\alpha, \beta, \gamma)$ is a separated triple, and in the separated case, we know that the associative law holds (see (11)). By (vi) (already proved), $F((\hat{\alpha} \square_p \hat{\beta}) \square_{\bar{q}} \hat{\gamma}) = (\alpha \square_p \beta) \square_{\bar{q}} \gamma$ and $F(\hat{\alpha} \square_p (\hat{\beta} \square_q \hat{\gamma})) = \alpha \square_p (\beta \square_q \gamma)$. It follows that $(\alpha \square_p \beta) \square_{\bar{q}} \gamma = \alpha \square_p (\beta \square_q \gamma)$.

The commutative law and the commutative diagrams involving the amalgamating maps are established similarly.

We have left the treatment of the identity arrows for the end.

For $f \in L = O(\boldsymbol{D})$, $1_f^{(\boldsymbol{D})}$ is defined to be $f \in A$ itself; since $s_{\boldsymbol{D}}(f) = \langle f \rangle$, and $t_{\boldsymbol{D}}(f) = \dot{f} = T(f)$, the source and the target of $1_f^{(\boldsymbol{D})}$ are as they should be.

To see that the first unit law holds, let us assume first that $\beta \in A$ is separated, $p \in |\langle \beta \rangle|$, $f = \langle \beta \rangle(p)$, to see that $\beta \square_p f = \beta$ and for $\varphi = \varphi_{\square}[\beta, f, p]$, $\psi = \psi_{\square}[\beta, f, p]$, we have $\varphi(1) = p$ and $\psi(i) = i$ for $i \in [1, lh(\beta)] - \{p\}$. By condition (v) in the Theorem (which we have proved), it is clear that $\beta \square_p f = \beta$. The separatedness of $\beta$ ensures that $\varphi$ and $\psi$ cannot be anything else but the ones described. Turning to the general case, let $F : \hat{\boldsymbol{C}} \to \boldsymbol{C}$ be an ample expansion, and $\hat{\beta} \in \hat{A}$ such that $F(\hat{\beta}) = \beta$. Let $\hat{f} = \langle \hat{\beta} \rangle(p)$. Then, by what we

just saw, $\hat{\beta}\Box_p\hat{f}=\hat{\beta}$ and for $\hat{\varphi}=\varphi[\hat{\beta}, \hat{f}, p]$, $\hat{\psi}=\psi[\hat{\beta}, \hat{f}, p]$, we have $\hat{\varphi}(1)=p$ and $\hat{\psi}(i)=i$ for $i\in[1, \ell h(\hat{\beta})]-\{p\}$. Since $\Box$ is preserved by $F$ (condition (vi), already proved), the desired result follows.

The other unit law is similarly seen.

The uniqueness assertion in the Theorem is clear from what we have gone through.