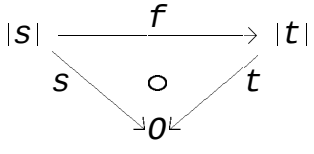
2. Multicategories

For $\ell\in\mathbb{N}=\{0, 1, 2, \ldots\}$, we write $[1, \ell]$ for the set $\{1, 2, \ldots, \ell\}$; $[1, \ell]=\emptyset$ when $\ell=0$.

Let O be a set. A *tuple* (string) of elements of O is a function of the form $\varphi:[1, \ell]\to O$, for some $\ell\in\mathbb{N}$. We write $|\varphi|$ for the set $[1, \ell]$, and $\ell h(\varphi)$ for ℓ. O^* is the set of all tuples of elements of O. $\perp\in O^*$ is the empty tuple ($|\perp|=\emptyset$). For $X\in O$, $\langle X\rangle\in O^*$ is the one-term tuple whose only term is X; $|\langle X\rangle|=[1, 1]=\{1\}$, $\langle X\rangle(1)=X$.

It will be convenient to work with the following category $O^{\#}$. Its objects are each a function s whose domain $|s|$ is a finite set (possibly empty) of positive integers, and whose range is a subset of O; $s:|s|\to O$. An arrow $s\to t$ is a function $f:|s|\to|t|$ such that



(the circle in a diagram denotes the assertion that the diagram commutes).

A *multicategory* $\boldsymbol{C}$ is given by data (i) to (vii) and conditions (viii) to (xi) as follows.

(i) A set $O=O(\boldsymbol{C})$ of *upper level objects*, or simply, *objects*.

(ii) A set $\dot{O}=\dot{O}(\boldsymbol{C})$ of *lower level objects.*

(iii) A map $O\longrightarrow\dot{O}:X\mapsto\dot{X}$, assigning a lower-level object $\dot{X}$ to every object X.

(iv) A set $A=A(\boldsymbol{C})$ of *arrows.*

(v) To each arrow f a *source* $\mathrm{s}(f)=\mathrm{s}f=\mathrm{s}_{\boldsymbol{C}}(f)\in O^{*}$, and a *target* $\mathrm{t}(f)=\mathrm{t}_{\boldsymbol{C}}(f)\in\dot{O}$ is assigned; we write $\vec{X}\xrightarrow{f}A$ if $\mathrm{s}(f)=\vec{X}$, $\mathrm{t}(f)=A$; here, $\vec{X}\in O^{*}$, $A\in\dot{O}$.

(vi) Given $\mathrm{s}(f)\xrightarrow{f}\mathrm{t}(f)$, $\mathrm{s}(g)\xrightarrow{g}\mathrm{t}(g)$, and $p\in|\mathrm{s}(g)|$ such that $(\mathrm{s}(g)(p))^{\cdot}=\mathrm{t}(f)$, which situation we indicate by the notation

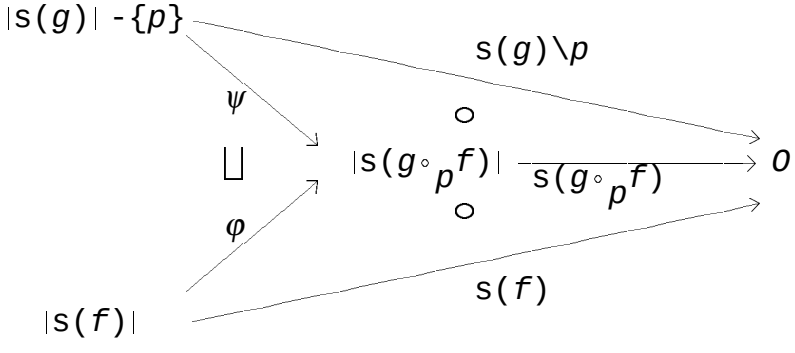
$$\mathrm{s}(f)\xrightarrow[p]{f}\mathrm{s}(g)\xrightarrow{g}\mathrm{t}(g),$$

a *composite* $g\circ_{p}f$ is defined; it is an arrow; we have $\mathrm{t}(g\circ_{p}f)=\mathrm{t}(g)$; furthermore, we have specified *amalgamating maps*

$$\psi=\psi[g,f,p]:\mathrm{s}(g)\backslash p\longrightarrow\mathrm{s}(g\circ_{p}f),$$

$$\varphi=\varphi[g,f,p]:\mathrm{s}(f)\longrightarrow\mathrm{s}(g\circ_{p}f)$$

(morphisms in $O^{\#}$), forming the coprojections of a coproduct in $O^{\#}$. ($\mathrm{s}(g)\backslash p$ means the restricted function $\mathrm{s}(g)\restriction(|\mathrm{s}(g)|-\{p\})$; also, for a subset $P\subset|\mathrm{s}(g)|$, we use the notation $\mathrm{s}(g)\backslash P$ in a similar sense.) In plain words, the set $|\mathrm{s}(g\circ_{p}f)|$ is the disjoint sum of the sets $|\mathrm{s}(g)|-\{p\}$ and $|\mathrm{s}(f)|$, with injections ψ and φ; and these injections are morphisms of the functions ($O^{\#}$-objects) $\mathrm{s}(g)\backslash p$, $\mathrm{s}(f)$; that is, we have the commutative diagram



It follows that $|\mathrm{s}(g\circ_{p}f)|$ is given as $[1,\ \ell+m-1]$ where $|\mathrm{s}(f)|=[1,\ \ell]$, $|\mathrm{s}(g)|=[1,\ m]$; however, this fact leaves open multiple possibilities for the amalgamating

maps φ and ψ. Let me emphasize that in general, $\psi=\psi[g, f, p]$, $\varphi=\varphi[g, f, p]$ depend in an essential way on all three arguments g, f, p; in particular, it is possible that $s(g')=s(g)$, $s(f')=s(f)$, but $s(g'\circ_p f')\neq s(g\circ_p f)$.

(vii) For each $Y\in O$, an identity map $\langle Y\rangle \xrightarrow{1_Y} \dot{Y}$.

For the data listed, we require the following laws to be obeyed.

(viii) (**unit law 1**) Whenever $g\in A$, $p\in |s(g)|$, and $Y=s(g)(p)$, which, in particular, implies $\langle Y\rangle \xrightarrow[p]{1_Y} s(g) \xrightarrow{g} t(g)$ (although the latter only says that $\dot{Y}=\dot{Y}_p$, which is weaker than what we are assuming now), we require that

$$g\circ_p 1_Y = g\ .$$

Moreover, we require that

$$\psi=\psi[g, 1_Y, p]=\mathrm{incl.}:(|s(g)|-\{p\})\to |s(g)|\ ,$$
$$\varphi=\varphi[g, 1_Y, p]=(1\mapsto p)$$

(which imply that $(\varphi, \psi\backslash p)$ is a coproduct pair in $O^{\#}$).

(ix) (**unit law 2**) Under the assumption that $s(f) \xrightarrow[1]{f} \langle Y\rangle \xrightarrow{1_Y} Y$ (that is, $t(f)=\dot{Y}$), we require that

$$1_Y\circ_1 f = f\ ,$$

and $\varphi=\varphi[1_Y, f, p]=\mathrm{id}$ (making $(\varphi, \psi\backslash 1=\perp)$ a coproduct pair in $O^{\#}$).

(x) (**associative law**) In the situation

$$s(f) \xrightarrow[p]{f} s(g) \xrightarrow[q]{g} s(h) \xrightarrow{h} t(h)\ ,$$

we require that

$$(h\circ_q g)\circ_{\bar{p}} f \;=\; h\circ_q (g\circ_p f)\,;$$

here, $\bar{p}=\varphi[h, g, q](p)$. Let us refer to the four compositions by the numbers as in

$$(h\overset{3}{\circ}_q g)\overset{4}{\circ}_{\bar{p}} f \;,\quad h\overset{2}{\circ}_q (g\overset{1}{\circ}_p f)\;.$$

Note that the compositions 1 and 3 are well-defined by the assumptions. 2 is meaningful since we have $\mathrm{t}(g\circ_p f)=\mathrm{t}(g)$. 4 is meaningful since, for $\varphi=\varphi[g, h, q]$,

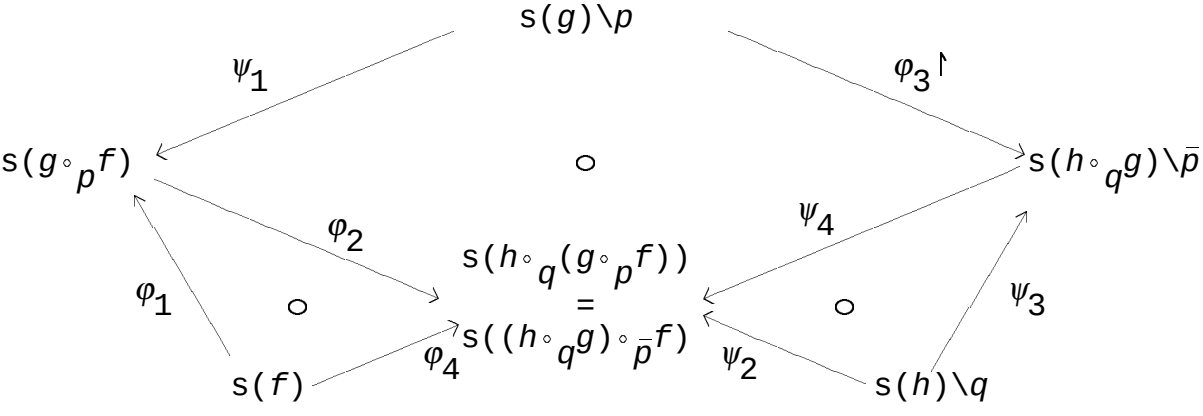
$$\mathrm{s}(h\circ_q g)(\bar{p}) = s(h\circ_q g)(\varphi(p)) = s(g)(p) = \mathrm{t}(f)\,;$$

the second equality because we have $\varphi:\mathrm{s}(g)\longrightarrow \mathrm{s}(h\circ_q g)$ in $O^{\#}$.

We abbreviate

$$\varphi_1=\varphi[g, f, p]\;,\ \psi_1=\psi[g, f, p]\;,\ \varphi_2=\varphi[h, g, q]\;,\ \psi_2=\psi[h, g, q]\;,$$
$$\varphi_3=\varphi[h, g, q]\;,\ \psi_3=\psi[h, g, q]\;,\ \varphi_4=\varphi[h\circ_q g, f, p]\;,\ \psi_4=\psi[h\circ_q g, f, p]\;.$$

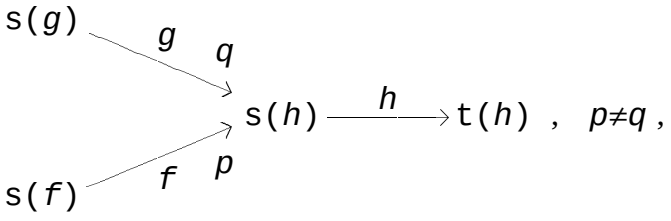
We require that the following diagram be commutative:



Here, $\varphi_3{\restriction}$ is the restriction of φ_3 to the appropriate domain. Since (φ_3, ψ_3) is a

coproduct, $\bar{p}=\varphi_3(p)\notin \mathrm{Im}(\psi_3)$, thus the use of ψ_3 in the diagram is legitimate.

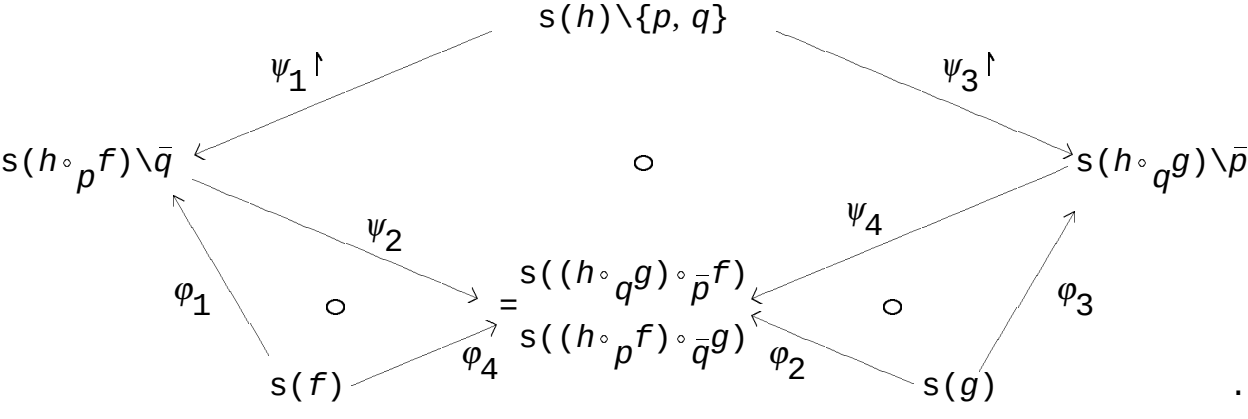
(xi) (**commutative law**) In the situation



for $\bar{q}=\psi[h, f, p](q)=\psi_1(q)$ (since $p\neq q$, $q\in \mathrm{dom}(s(h)\backslash p)$, so $\psi_1(q)$ is defined), $\bar{p}=\psi[h, g, q](p)=\psi_3(p)$, we require

$$(h \overset{1}{\circ}_p f) \overset{2}{\circ}_{\bar{q}} g \;=\; (h \overset{3}{\circ}_q g) \overset{4}{\circ}_{\bar{p}} f \;.$$

The fact that the composites 2, 4 are well-defined is seen as in the previous case. With φ_i, ψ_i similarly as above, we require the commutativities as in



Since $\bar{q}\in \mathrm{Im}(\psi_1)$, we have $\bar{q}\notin \mathrm{Im}(\varphi_1)$, so the use of φ_1 is justified; similarly for φ_3. The map ψ_1 is injective; so, $\bar{q}\notin \mathrm{Im}(\psi_1{\restriction}(|s(h)|-\{p, q\}))$, and the use of the restricted $\psi_1{\restriction}$ is justified. Similarly for $\psi_3{\restriction}$

(**end** of definition of "multicategory").

The standard definition of multicategory (see [L1], [L2]) is the special case in which (i) $\dot{O}=O$, $\dot{X}=X$ for all $X\in O$ (a *one-level* multicategory, as opposed to the general *two-level* notion), and (ii) we make the *standard* choice for the amalgamating maps as explained now.

For $\vec{X}=\langle X_i\rangle_{i\in[1,\,\ell]}$, $\vec{Y}=\langle Y_j\rangle_{j\in[1,\,n]}$, both in O^*, and for $p\in[1,m]$, a particular index, $\vec{Y}\square_p\vec{X}$ denotes the result of *inserting* $\vec{X}$ *into* $\vec{Y}$ *at the place* p; this in effect replaces Y_p by $\vec{X}$. This means that $\vec{Y}\square_p\vec{X}=\vec{Z}=\langle Z_k\rangle_{k\in[1,\,n]}$, where $n=\ell+m-1$, $Z_k=Y_k$ when $1\le k<p$, $Z_k=X_{k-p+1}$ when $p\le k<p+\ell$, and $Z_k=Y_{k-\ell+1}$ when $p+\ell\le k\le n$. Define

$$\varphi=\varphi[\vec{Y},\vec{X},p]:|\vec{X}|\to|\vec{Z}|\ ,\quad \psi=\psi[\vec{Y},\vec{X},p]:|\vec{Y}|-\{p\}\to|\vec{Z}|$$

by $\varphi(i)=p+i-1$; $\psi(j)=j$ when $1\le j<p$, and $\varphi(j)=p-j+1$ when $p<j\le n$; we have the coproduct diagram

$$\vec{X}\xrightarrow{\ \varphi\ }\vec{Z}\xleftarrow{\ \psi\ }\vec{Y}\backslash p$$

in $O^{\#}$. When the multicategory has the just specified connecting maps: for the composition $g\circ_p f$, $\varphi[g,f,p]=\varphi[s(g),s(f),p]$, $\psi[g,f,p]=\psi[s(g),s(f),p]$, we talk about a multicategory *with standard amalgamation*. In particular, the source of $g\circ_p f$ depends, in the standard case, only on the sources of the factors, and the place p; not necessarily so in the general case.

Note that in the standard case, the commutativities required for associativity and commutativity ((ix) and (x)) are automatic.

There is a further remark to be made about the commutative diagrams in the laws of associativity and commutativity, to the effect that they are, to a large extent, automatically true. Referring to the notation in (x), suppose that the functions $s(f)$, $s(g)$, $s(h)$ are one-to-one (non-repeating tuples), and their ranges are pairwise disjoint. I claim that, as a consequence of the preceding conditions, the commutativities required in (x) are now true. First of all, since $s_{\overline{\text{def}}}s(h\circ_q(g\circ_p f))$ is a coproduct (in $O^{\#}$) of non-repeating

(generalized) tuples, it is itself non-repeating. But then for any $t \in O^{\#}$, there can be at most one morphism $t \to s$ in $O^{\#}$. This implies each of the three commutativities in (x). The same can be said about (xi). We will exploit this fact in section 5.

As a consequence of the definition, in any multicategory, we have a concept of *simultaneous composition*. Assume $g \in A$, $p_i \in |\mathrm{s}(g)|$ for $i=1, \ldots, m$, $p_i \neq p_j$ when $i \neq j$. Assume that $f_i \in A$ for $i=1, \ldots, m$, such that $\mathrm{t}(f_i)=(\mathrm{s}(g)(i))$ for all $i \in [1, m]$. Then we define

$$h \underset{\mathrm{def}}{=} g(f_1/p_1, f_2/p_2, \ldots, f_m/p_m), \tag{1}$$

and with $P=\{p_1, \ldots, p_m\}$, the amalgamating functions

$$\psi : \mathrm{s}(g) \backslash P \longrightarrow \mathrm{s}(h), \qquad \psi = \psi[g, \langle f_j \rangle_{j \in [1, m]}, \langle p_j \rangle_{j \in [1, m]}]$$

and

$$\varphi_i : \mathrm{s}(f_i) \longrightarrow \mathrm{s}(h), \qquad \varphi_i = \varphi_i[g, \langle f_j \rangle_{j \in [1, m]}, \langle p_j \rangle_{j \in [1, m]}]$$

such that $\mathrm{s}(h)$ is the coproduct of the $O^{\#}$-objects $\mathrm{s}(g) \backslash P$, $\mathrm{s}(f_i)$ $(i \in [1, m])$ via the coprojections ψ, φ_i $(i \in [1, m])$. The definition is by recursion on m. When $m=0$, $h=g$, we have $\psi = \mathrm{id}_{\mathrm{s}(g)}$. Suppose $m \geq 1$, and assume that $\bar{h} = g(f_1/p_1, f_2/p_2, \ldots, f_{m-1}/p_{m-1})$ has been defined, with corresponding amalgamating functions

$$\bar{\psi} : \mathrm{s}(g) \backslash P^{-} \longrightarrow \mathrm{s}(\bar{h}) \quad (P^{-}=\{p_i : i \in [1, m-1]\}),$$
$$\bar{\varphi}_i : \mathrm{s}(f_i) \longrightarrow \mathrm{s}(\bar{h}) \quad (i \in [1, m-1]).$$

We put

$$h = \bar{h} \circ_{\bar{p}_m} f_m,$$

where $\bar{p}_m = \bar{\psi}(p_m)$, and, with $\hat{\varphi}=\varphi[\bar{h}, f_m, \bar{p}_m]$, $\hat{\psi}=\psi[\bar{h}, f_m, \bar{p}_m]$, we define the amalgamating functions for (1) as $\psi = \hat{\psi} \circ (\bar{\psi} \restriction P)$, $\varphi_i = \hat{\psi} \circ \bar{\varphi}_i$ $(i \in [1, m-1])$ and $\varphi_m = \hat{\varphi}$.

In the simultaneous composition, the order of the composed-in factors is immaterial. Precisely speaking, we have the *generalized commutative law*, which says the following:

$$g(f_1/p_1, f_2/p_2, \ldots, f_m/p_m) = g(\hat{f}_1/\hat{p}_1, \hat{f}_2/\hat{p}_2, \ldots, \hat{f}_m/\hat{p}_m)$$

provided for a permutation $\sigma:[1, m] \xrightarrow{\cong} [1, m]$, we have $\hat{p}_i=p_{\sigma i}$ and $\hat{f}_i=f_{\sigma i}$ ($i\in[1, m]$); moreover,

$$\hat{\psi}=\psi, \quad \hat{\varphi}_i=\varphi_{\sigma i}$$

where, of course, we have used the obvious notation for the corresponding amalgamating functions, that is,

$$\hat{\psi}=\psi[g, \langle \hat{f}_j \rangle_{j\in[1, m]}, \langle \hat{p}_j \rangle_{j\in[1, m]}],$$
$$\hat{\varphi}_i=\varphi_i[g, \langle \hat{f}_j \rangle_{j\in[1, m]}, \langle \hat{p}_j \rangle_{j\in[1, m]}].$$

For the case $m=2$, the generalized commutative law is identical to the original form of the commutative law (including the commutativity of the corresponding diagram). The general case be proved by using the commutative law alone, by representing the arbitrary permutation σ as a product of transpositions each of which exchanges two elements standing next to each other in the "previous" permutation.

Therefore, the best way of looking at simultaneous composition is that we have an arrow g, a set $P\subset|\mathrm{s}(g)|$, and a function $p\mapsto f_p:P\to A$, such that $\mathrm{t}(f_p)=\mathrm{s}(g)(p)$ $(p\in P)$, giving rise to the composite $h=g(\langle f_p/p \rangle_{p\in P})$, and to the amalgamating maps $\psi=\psi[g, \langle f_p \rangle_{p\in P}]:\mathrm{s}(g)\backslash P\to s(h)$, $\varphi_p=\varphi_p[g, \langle f_p \rangle_{p\in P}]:\mathrm{s}(f_p)\to \mathrm{s}(h)$ $(p\in P)$. In fact, we can define

$$h=g(\langle f_p/p \rangle_{p\in P}) = g(\bar{f}_1/p_1, \bar{f}_2/p_2, \ldots, \bar{f}_m/p_m)$$

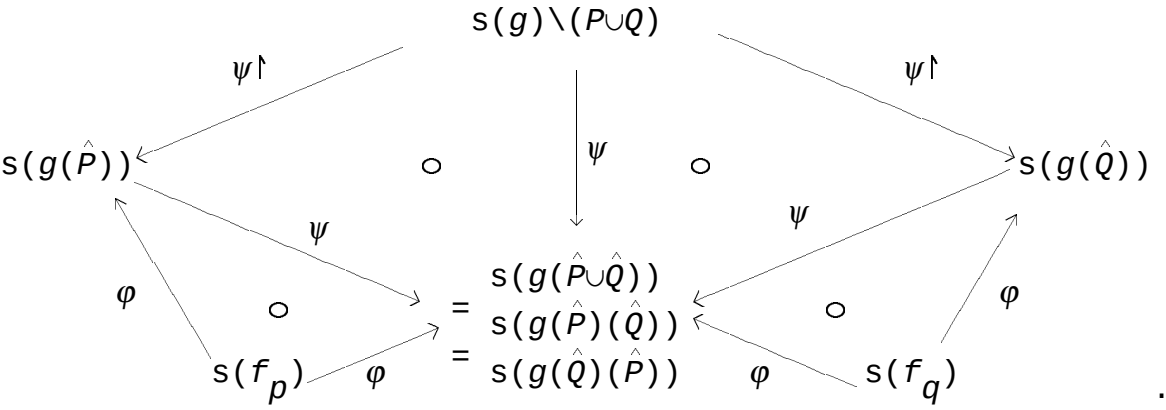
for an arbitrary repetition-free enumeration $\langle p_j \rangle_{j\in[1, m]}$ of P, and for $\bar{f}_i = f_{p_i}$; of course,

$$\psi[g, \langle f_p\rangle_{p\in P}]=\psi[g, \langle \bar{f}_j\rangle_{j\in[1,\,m]}, \langle p_j\rangle_{j\in[1,\,m]}]\ ,$$

and

$$\varphi_{p_i}[g, \langle f_p\rangle_{p\in P}]=\varphi_i[g, \langle \bar{f}_j\rangle_{j\in[1,\,m]}, \langle p_j\rangle_{j\in[1,\,m]}]\ .$$

Suppose $P, Q\subset |s(g)|$, $P\cap Q=\emptyset$; write $\hat{P}$ and $\hat{Q}$ for the assignments $\hat{P}=\langle f_p/p\rangle_{p\in P}$, $\hat{Q}=\langle f_q/q\rangle_{q\in Q}$. Suppose both $g(\hat{P}) = g(\langle f_p/p\rangle_{p\in P})$, $g(\hat{Q}) = g(\langle f_q/q\rangle_{q\in Q})$ are well-defined. We can consider $g(\hat{P}\cup\hat{Q}) = g(\langle f_r/r\rangle_{r\in P\cup Q})$, and we have $g(\hat{P}\cup\hat{Q})=g(\hat{P})(\hat{Q})=g(\hat{Q})(\hat{P})$, with the following diagram commuting:



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Here, the further specification of the maps is self-explanatory. It should be mentioned that the map $\varphi: s(f_p) \longrightarrow s(g(\hat{P}\cup\hat{Q}))=s(g(\hat{Q})(\hat{P}))$ has two meanings, which coincide: $\varphi=\varphi[g, \hat{P}\cup\hat{Q}, p]=\varphi[g(\hat{Q}), \hat{P}, p]$; similarly for q in place for p .

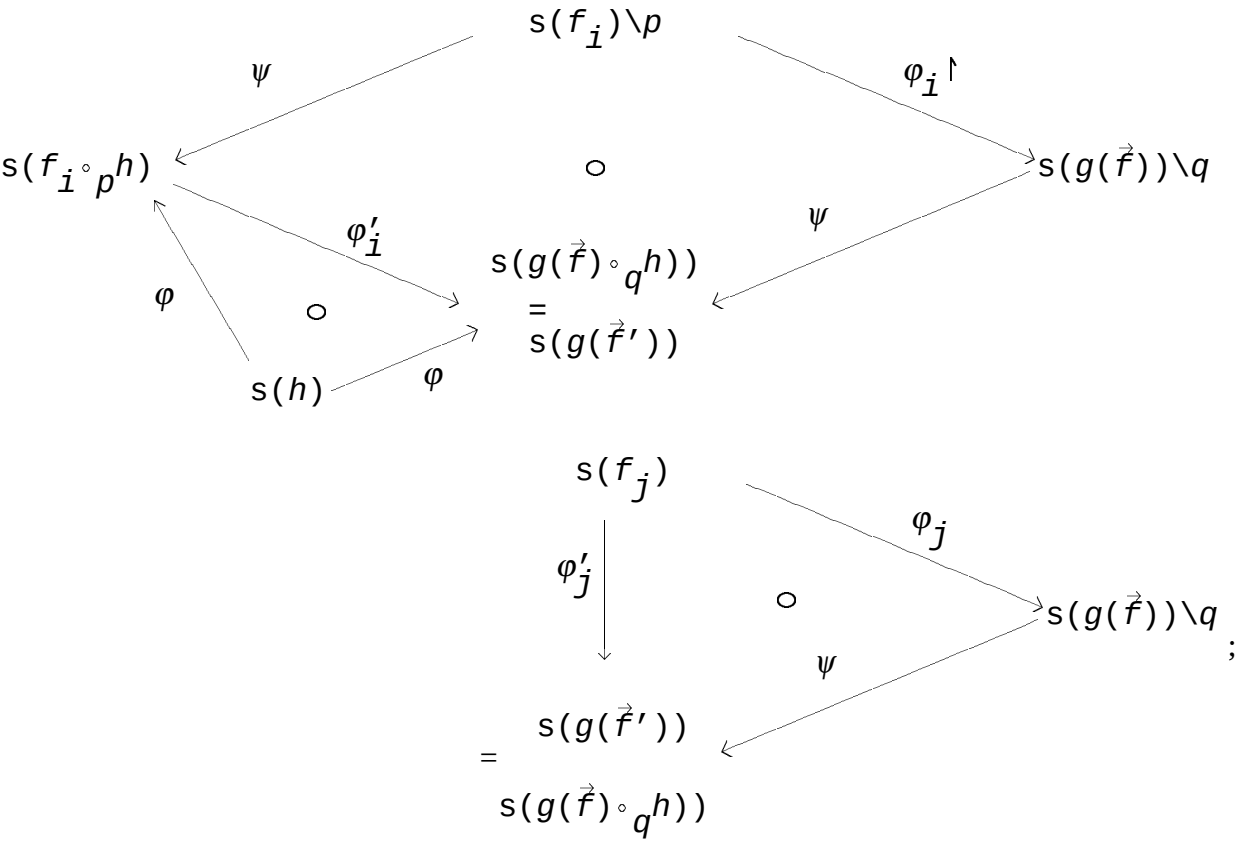
We write $g(f_1, f_2, \ldots, f_m)$, or $g(\langle f_i\rangle_{i=1}^{m})$, for $g(f_1/1, f_2/2, \ldots, f_m/m)$. The notation $g(f_1, f_2, \ldots, f_m)$ (in which there is no notation of the place where each f_i is being composed into g) will never be used unless *all* places of g are involved (that is, $m=\ell h(s(g))$), and f_i is composed into g at the place i . Now, $P=[1, m]$: the ψ-map for $g(f_1, f_2, \ldots, f_m)$ is empty: its domain is the empty set $|s(g)| - P=\emptyset$; $s(g(f_1, f_2, \ldots, f_m))$ is the coproduct of the $s(f_i)$ via the maps

$$\varphi_i=\varphi_i[g, \langle f_i\rangle_{i\in[1,\,m]}]: s(f_i)\to s(g(f_1, f_2, \ldots, f_m)) \quad (i\in[1, m])\ .$$

Let us formulate a version of the transitive law, involving a simultaneus composition. Using the notation of the previous paragraph, let $i\in[1,m]$, $p\in|s(f_i)|$, and suppose $f_i\circ_p h$ is well-defined. Then for $q=\varphi_i(p)$, we have

$$g(f_1, f_2, \ldots, f_m)\circ_q h = g(f_1, \ldots, f_{i-1}, f_i\circ_p h, f_{i+1}, \ldots, f_m)\ .$$

The coherence commutativities in this case are:



;

we have used the abbreviations $g(\vec{f})=g(f_1, f_2, \ldots, f_m)$,
$g(\vec{f}')=g(f_1, \ldots, f_{i-1}, f_i\circ_p h, f_{i+1}, \ldots, f_m)$; $j\neq i$.

3. Morphisms of multicategories

Given multicategories $\mathcal{C}$, $\mathcal{D}$, a morphism $F:\mathcal{C}\longrightarrow\mathcal{D}$ is given by data and conditions as follows.

(i) Maps $F:\mathrm{O}(\mathcal{C})\to\mathrm{O}(\mathcal{D})$, $\dot{F}:\dot{\mathrm{O}}(\mathcal{C})\to\dot{\mathrm{O}}(\mathcal{D})$ on objects such that

$$\begin{array}{ccc} \mathrm{O}(\mathcal{C}) & \xrightarrow{(\ )^{\cdot}} & \dot{\mathrm{O}}(\mathcal{C}) \\ {\scriptstyle F}\downarrow & \circ & \downarrow{\scriptstyle \dot{F}} \\ \mathrm{O}(\mathcal{D}) & \xrightarrow[(\ )^{\cdot}]{} & \dot{\mathrm{O}}(\mathcal{D}) \end{array} \quad .$$

(ii) A map $F:\mathrm{A}(\mathcal{C})\longrightarrow\mathrm{A}(\mathcal{D})$ on arrows; $\mathrm{t}_{\mathcal{D}}(Ff)=\dot{F}(\mathrm{t}_{\mathcal{C}}(f))$ is required for all $f\in\mathrm{A}(\mathcal{C})$.

(iii) For any $f\in\mathrm{A}(\mathcal{C})$, a *transition* bijection $\theta_f\colon |\mathrm{s}_{\mathcal{C}}(f)| \xrightarrow{\cong} |\mathrm{s}_{\mathcal{D}}(Ff)|$ such that

$$\begin{array}{ccc} |\mathrm{s}_{\mathcal{C}}(f)| & \xrightarrow{\theta_f} & |\mathrm{s}_{\mathcal{D}}(Ff)| \\ {\scriptstyle \mathrm{s}_{\mathcal{C}}(f)}\downarrow & \circ & \downarrow{\scriptstyle \mathrm{s}_{\mathcal{D}}(Ff)} \\ \mathrm{O}(\mathcal{C}) & \xrightarrow[F]{} & \mathrm{O}(\mathcal{D}) \end{array} \quad . \qquad (1)$$

Note that this is the same as to say that $\theta_f:F\circ\mathrm{s}_{\mathcal{C}}(f)\xrightarrow{\cong}\mathrm{s}_{\mathcal{D}}(Ff)$ in $\mathcal{O}(\mathcal{D})^{\#}$.

(iv) F preserves identities: $F(\langle Y\rangle \xrightarrow{1_Y} \dot{Y}) = \langle FY\rangle \xrightarrow{1_{FY}} (FY)^{\cdot}$.

(v) F preserves composition. Given $f, g\in\mathrm{A}(\mathcal{C})$, $p\in|\mathrm{s}_{\mathcal{C}}(g)|$, $\mathrm{t}_{\mathcal{C}}(f)=(\mathrm{s}_{\mathcal{C}}(g)(p))^{\cdot}$ (so that $g\circ_p f$ is well-defined), for $\hat{p}=\theta_g(p)\in|\mathrm{s}_{\mathcal{D}}(Fg)|$ we have that

$$t_{\boldsymbol{D}}(Ff) \underset{(ii)}{=} \dot{F}(t_{\boldsymbol{C}}(f)) = \dot{F}(s_{\boldsymbol{C}}(g)(p)\dot{}) \underset{(i)}{=} F(s_{\boldsymbol{C}}(g)(p)\dot{}) \underset{(iii)}{=} s_{\boldsymbol{D}}(Fg)(\hat{p})\dot{} \quad ,$$

thus $(Fg)\circ_{\hat{p}}(Ff)$ is well-defined. We require that

$$F(g\circ_p f) = (Fg)\circ_{\hat{p}}(Ff) \; ;$$

moreover,

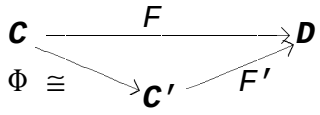
$$\begin{array}{ccc}
|s_{\boldsymbol{C}}(g)|\text{-}\{p\} & \xrightarrow{\theta_g\restriction} & |s_{\boldsymbol{D}}(Fg)|\text{-}\{\hat{p}\} \\
\psi[f,g,p]\downarrow & \circ & \downarrow\psi[Ff,Fg,\hat{p}] \\
|s_{\boldsymbol{C}}(g\circ_p f)| & \xrightarrow{\theta_{g\circ_p f}} & |s_{\boldsymbol{D}}(F(g\circ_p f))| = |s_{\boldsymbol{D}}((Fg)\circ_{\hat{p}}(Ff))| \\
\varphi[f,g,p]\uparrow & \circ & \uparrow\varphi[Ff,Fg,\hat{p}] \\
|s_{\boldsymbol{C}}(f)| & \xrightarrow[\theta_f]{} & |s_{\boldsymbol{D}}(Ff)|
\end{array} \quad .$$

There is a composition of morphisms of multicategories. Given $\boldsymbol{C}\xrightarrow{F}\boldsymbol{D}\xrightarrow{G}\boldsymbol{E}$, for $H{=}G\circ F{:}\boldsymbol{C}\longrightarrow\boldsymbol{E}$, we have H is the usual composite as far as the effect on objects and arrows is concerned, and $\theta_f{:}\ |s_{\boldsymbol{C}}(f)|\longrightarrow|s_{\boldsymbol{E}}(Hf)|$ is given as the composite $|s_{\boldsymbol{C}}(f)|\xrightarrow{\theta_f^{(F)}}|s_{\boldsymbol{D}}(Ff)|\xrightarrow{\theta_{Ff}^{(G)}}|s_{\boldsymbol{E}}(GFf)|$. It is fairly clear that H is so defined is indeed a morphism of multicategories. We also have the obvious *identity morphism* $\mathrm{Id}_{\boldsymbol{C}}{:}\boldsymbol{C}\to\boldsymbol{C}$.

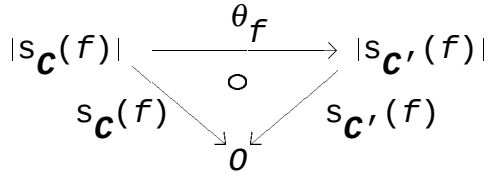
The said items form the category `Multicat` of (small) multicategories and their morphisms.

Let us emphasize that *multicategories* are treated here as 0-dimensional objects, that is, objects of a 1-dimensional, ordinary, category, in contrast to the fact that *categories* are usually treated as 1-dimensional objects in a 2-dimensional category. This fact is the key specific feature of our approach. There are *isomorphisms* of multicategories, but there are, at least for us, *no equivalences* of them.

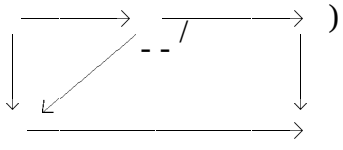
Let us note that every morphism $F:\mathbf{C}\to\mathbf{D}$ can be factored, in a unique manner, in the form



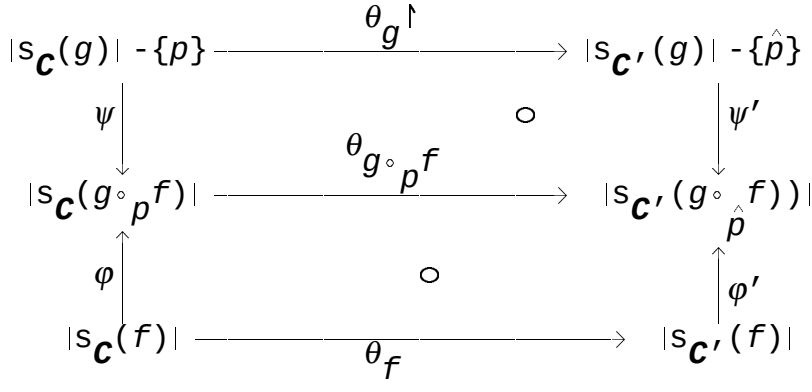
so that the isomorphism Φ is the identity on objects and arrows, and F' is *strict*, that is, all its transition maps are identities. To define $\mathbf{C}'$, we put $O(\mathbf{C}')=O(\mathbf{C})=O$, $A(\mathbf{C}')=A(\mathbf{C})=A$, and, for any $f\in A$, $t_{\mathbf{C}'}(f)=t_{\mathbf{C}}(f)=t(f)$. For any $f\in A$, $s_{\mathbf{C}'}(f)$ is defined by $|s_{\mathbf{C}'}(f)| = |s_{\mathbf{C}}(f)|$, and the commutative diagram



(this diagram is obtained by decomposing (1) in the form



using the transition map θ_f for F. Given $f, g\in A$, $\hat{p}\in |s_{\mathbf{C}'}(g)|$ such that $t(f)=s_{\mathbf{C}'}(f)(\hat{p})$, we put $g\circ_{\hat{p}}^{\mathbf{C}'} f = g\circ_{p}^{\mathbf{C}} f$ with $p=\theta_f^{-1}(\hat{p})$. To define the amalgamating functions $\varphi'=\varphi_{\mathbf{C}'}[f, g, \hat{p}]$, $\psi'=\psi_{\mathbf{C}'}[f, g, \hat{p}]$, we use the commutative squares in



.

where φ and ψ are the amalgamating functions given with $\boldsymbol{C}$. It is immediate that $\boldsymbol{C'}$ is well-defined. The transition maps for Φ are the given θ_f; the effect of F' on objects and arrows is that of F.

4. The free multicategory

We need only the free multicategory in the case there is only one level of objects; therefore, we restrict the definition to this case. However, note that the free one-level multicategory will be free with respect to the general, two-level, variety.

Suppose O is a set (of "objects"), L is a set (of "generating arrows"), and for each $f \in L$, we are given $s(f) \in O^*$, and $t(f) \in O$. Such data determine a *language* $\mathcal{L}$; we may write $O = O(\mathcal{L})$, $L = L(\mathcal{L})$, $s = s_{\mathcal{L}}$, $t = t_{\mathcal{L}}$. The free multicategory $\mathcal{F}(\mathcal{L}) = \boldsymbol{C}$ on the given language is defined by the universal property as follows. We have that $O(\mathcal{L}) = O(\boldsymbol{C})$, $L(\mathcal{L}) \subset A(\boldsymbol{C})$, $s_{\boldsymbol{C}}$, $t_{\boldsymbol{C}}$ extend the given maps $s_{\mathcal{L}}$ and $t_{\mathcal{L}}$; and every time $\boldsymbol{D}$ is a multicategory, and we are given $F(X) \in O(\boldsymbol{D})$, $F(f) \in O(\boldsymbol{D})$ for $X \in O$, $f \in L$ such that $t_{\boldsymbol{D}}(F(f)) = (F(t_{\mathcal{L}}(f)))^{\cdot}$, and we are also given $\theta_f : |s_{\mathcal{L}}(f)| \xrightarrow{\cong} |s_{\boldsymbol{D}}(Ff)|$ such that

$$\begin{array}{ccc} |s_{\mathcal{L}}(f)| & \xrightarrow{\theta_f} & |s_{\boldsymbol{D}}(Ff)| \\ \downarrow{\scriptstyle s_{\mathcal{L}}(f)} & \circ & \downarrow{\scriptstyle s_{\boldsymbol{D}}(Ff)} \\ O(\mathcal{L}) & \xrightarrow[F]{} & O(\boldsymbol{D}) \end{array}$$

(when θ_f is the identity, $s_{\boldsymbol{D}}(Ff) = F \circ s_{\mathcal{L}}(f)$) for all $f \in L(\mathcal{L})$, there is a unique morphism $F : \boldsymbol{C} \to \boldsymbol{D}$ of multicategories extending the given data F and $\theta_{(\ )}$. The uniqueness, up to isomorphism, of $\mathcal{F}(\mathcal{L})$ is clear; its existence could be proved routinely by the Adjoint Functor Theorem (or, Initial Object Theorem; see [M L]). Instead, we will find a direct description and proof of existence for $\mathcal{F}(\mathcal{L})$.

We first formulate a characterization.

(1) Let $\mathcal{L}$ be a language as above. Suppose $\boldsymbol{C}$ is a 1-level multicategory with $O(\boldsymbol{C}) = O(\mathcal{L})$, $L(\mathcal{L}) \subset A(\boldsymbol{C})$, and $s_{\boldsymbol{C}}$, $t_{\boldsymbol{C}}$ extend $s_{\mathcal{L}}$ and $t_{\mathcal{L}}$, respectively. Then $\boldsymbol{C}$ is free on $\mathcal{L}$ **if and only if** the following condition (2) holds:

(2) ***(Unique Readability)*** for every $\alpha \in \mathrm{A}(\boldsymbol{C})$,

either **(a)** $\alpha = 1_X$ for some $X \in \mathrm{O}(\mathcal{L})$,

or **(b)** $\alpha = f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ for some $f \in \mathrm{L}(\mathcal{L})$, $m = \ell\mathrm{h}(\mathrm{s}(f))$, $\alpha_i \in \mathrm{A}(\boldsymbol{C})$ such that $\mathrm{t}(\alpha_i) = \mathrm{s}(f)(i)$ $(i \in [1, m])$

($f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ refers to simultaneous composition; see the end of section 1.);

and furthermore,

exactly one of **(a), (b)** is the case; in case **(a)**, X is uniquely determined by α, and in case **(b)**, the items f, α_i are uniquely determined by α.

Note that there are no additional conditions put on the amalgamating functions.

Note that in case **(b)**, for each i, $\ell\mathrm{h}(\mathrm{s}(\alpha_i)) < \ell\mathrm{h}(\mathrm{s}(\alpha))$, which fact implies that under (2), $\mathrm{A}(\boldsymbol{C})$ is generated by $\mathrm{L}(\mathcal{L})$ in the obvious sense: $\mathrm{A}(\boldsymbol{C})$ is the least set $\mathcal{X}$ containing each 1_X $(X \in \mathrm{O}(\mathcal{L}))$ and such that if $f \in \mathrm{L}(\mathcal{L})$, $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathcal{X}$ and $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is well-defined, then $f(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{X}$. In fact, if the condition (2) holds, we may apply *structural induction*, respectively, *structural recursion*, to prove that a property holds for all arrows of $\boldsymbol{C}$, respectively, to define a function, say Φ, whose domain is $\mathrm{A}(\boldsymbol{C})$. In the latter case, we should have the definition of the function Φ at arguments 1_X, $X \in \mathrm{O}(\boldsymbol{C})$, and a way that determines the value of Φ at any argument of the form $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ $(f \in \mathrm{L}(\mathcal{L}))$ from the following data: $f, \alpha_1, \alpha_2, \ldots, \alpha_m$ and $\Phi(\alpha_1), \Phi(\alpha_2), \ldots, \Phi(\alpha_m)$; unique readability ensures that thereby Φ is uniquely determined.

The proof of the **if** part consists in verifying the universal property of $\boldsymbol{C}$ under the condition (2). Let us use the notation in the statement of the universal property. The effect of F on the arrows α of $\boldsymbol{C}$, including the connecting maps θ_α, is defined by structural recursion on α. Of course, the amalgamating maps for the composition in $\boldsymbol{C}$ and those for the composition in $\boldsymbol{D}$ are used in this definition. The details are put into the Appendix.

(Note that the **if** part of (1) is an important piece in the justification of the generalized notion

of multicategory introduced in this paper; the **if** part of (1) shows that the generalized notion is, after all, not so far from the standard concept of multicategory; in fact, the **if** part of (1) shows that, in a sense, the generalized notion is the algebraic essence of the standard notion.)

Next, for a given language $\mathcal{L}$, we exhibit a particular multicategory $\mathcal{F}(\mathcal{L})$ *with standard amalgamation* satisfying the condition (2). Then, by the **if** part already shown, $\mathcal{F}(\mathcal{L})$ is free on $\mathcal{L}$; and since any multicategory free on $\mathcal{L}$ is isomorphic to $\mathcal{L}$, and the condition (2) is clearly invariant under isomorphism, the "only if" part will follow.

$\boldsymbol{C}=\mathcal{F}(\mathcal{L})$, in the specific sense now to be adopted, is defined to have objects $\mathrm{O}(\boldsymbol{C})=\mathrm{O}(\mathcal{L})$. The arrows are defined inductively as follows:

(i) each $X\in S$ is an *arrow*; $\mathrm{s}_{\boldsymbol{C}}(X)=\langle X\rangle$, $\mathrm{t}_{\boldsymbol{C}}(f)=X$.

(ii) whenever $f\in L$ with $\ell\mathrm{h}(f)=n$, and, for each $i\in[1, n]$, α_i is an *arrow* such that $\mathrm{t}_{\boldsymbol{C}}(\alpha_i)=\mathrm{s}(f)(i)$, then

$$\alpha \;\bar{=}_{\mathrm{def}}\; f(\langle\alpha_i\rangle_{i\in[1,\,m]}) = f(\alpha_1, \alpha_2, \ldots, \alpha_m) \qquad (3)$$

is an *arrow*, and $\mathrm{s}_{\boldsymbol{C}}(\alpha)$ is the *concatenation* $\mathrm{s}_{\boldsymbol{C}}(\alpha_1)\hat{}\mathrm{s}_{\boldsymbol{C}}(\alpha_2)\hat{}\ldots\hat{}\mathrm{s}_{\boldsymbol{C}}(\alpha_m)$; that is, with $n=\ell\mathrm{h}(\mathrm{s}_{\boldsymbol{C}}(\alpha))$, $n_i=\ell\mathrm{h}(\mathrm{s}_{\boldsymbol{C}}(\alpha_i))$, we have $n=\sum_{i=1}^{n} n_i$, and for any $j\in[1, n]$, with $i\in[1, m]$ determined such that $j\in[(\sum_{h<i} n_h)+1, \sum_{h\leq i} n_h]$, we have $\mathrm{s}_{\boldsymbol{C}}(\alpha)(j)=\mathrm{s}_{\boldsymbol{C}}(\alpha_i)(j-\sum_{h<i} n_h)$.

In (3), $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ means something determined from f and the α_i so that, conversely, f and the α_i can be recovered from it. Thus, $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$ may be the concatenation of the strings $\langle f\rangle$, α_1, α_2, $\ldots$, α_n. This is all right if and only if the sets $\mathrm{O}(\mathcal{L})$ and $\mathrm{L}(\mathcal{L})$ are disjoint. In the general case, a set-theoretical construct such as $\langle 1, f, \alpha_1, \alpha_2, \ldots, \alpha_m\rangle$ (function on $[1, m+2]$ with values as listed) can be taken for $f(\alpha_1, \alpha_2, \ldots, \alpha_m)$; now, for clause (i), we take $\langle 0, X\rangle$ to be the arrow, rather than plain X.

The notation in (3) is in agreement with the notation for simultaneous composition, as will become clear when we have defined composition in $\mathcal{F}(\mathcal{L})$.

The composition in $\mathcal{F}(\mathcal{L})$ is defined by *substitution*. Given

$$s(\beta) \xrightarrow[p]{\beta} s(\alpha) \xrightarrow{\alpha} t(\alpha),$$

$\alpha \circ_p \beta$ is defined as $\alpha(\beta/p)$, the result of *substituting* β in α for $t(\beta)=s(\alpha)(p)$ at the place p. The value of the expression $\alpha(\beta/p)$ is defined by recursion on the complexity of α. If $\alpha=X\in O$, and thus $s(\alpha)(p)=X$, then $\alpha(\beta/p)=\beta$. If $\alpha=f(\langle \alpha_i \rangle_{i\in[1,m]})$, then, using the notation adopted under (ii), for a specific $j\in[1,m]$, we $p=\sum_{h<j} n_j+q$ with $q\in[1,n_j]$; and we put $\alpha(\beta/p)=f(\langle \hat{\alpha}_i \rangle_{i\in[1,m]})$, where $\hat{\alpha}_i=\alpha_i$ when $i\in[1,m]-\{j\}$, and $\hat{\alpha}_j=\alpha_j(b/q)$. It is left to the reader to verify that in this way we have defined a multicategory with standard amalgamation.