1. An informal description

1.1. *n*–graphs and multitopic sets.

In the classical, strict, concept of higher-dimensional category (HDC), an HDC $\boldsymbol{A}$ consists of k-*cells* in each of several *dimensions* k, where k ranges over a set $\{0, \ldots, n\}$ (n-*category*), or over all natural numbers (ω-*category*). Let us denote the class of all k-cells of $\boldsymbol{A}$ by C_k. For $k>0$, each k-cell a is "based on" two $(k-1)$-cells, the *domain* da and *codomain* ca of a; when $b=da$, $c=ca$, we write $a:b\longrightarrow c$; we have the assignments $d_k=d:C_k\longrightarrow C_{k-1}$, $c_k=c:C_k\longrightarrow C_{k-1}$ as part of the structure of the HDC $\boldsymbol{A}$. The part of the structure of $\boldsymbol{A}$ so far described is an n-*graph* in the case of "n-category", ω-*graph* in the case of "ω-category"; the data for an n-graph can be summarized in the diagram.

$$C_0 \underset{c_1}{\overset{d_1}{\leftarrow\!\!\leftarrow}} C_1 \underset{c_2}{\overset{d_2}{\leftarrow\!\!\leftarrow}} C_2 \ \ldots\ C_{n-1} \underset{c_n}{\overset{d_n}{\leftarrow\!\!\leftarrow}} C_n \qquad (1)$$

A feature of n-graphs, is *globularity*: for any $a\in C_k$, $k\geq 1$, $b=da$ and $c=ca$ must be *parallel*, that is, either $k-1=0$, or else $db=dc$, $cb=cc$:

$$e \underset{c}{\overset{b}{\rightrightarrows}} f$$

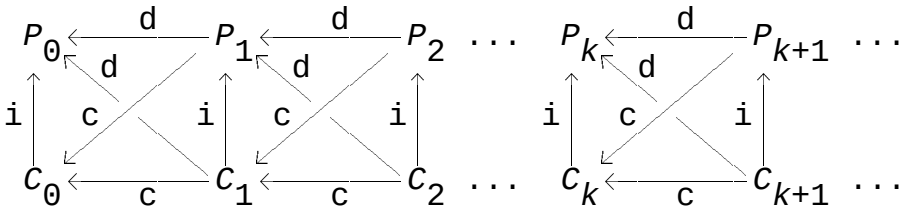
where $e=db=dc$, $f=cb=cc$. Put another way,

$$dd=dc\,,\ dc=cc\,, \qquad (2)$$

where d and c ambiguously denote any of the domain, respectively codomain maps $d_k:C_k\longrightarrow C_{k-1}$, $c_k:C_k\longrightarrow C_{k-1}$, with the restriction that the composites intended should be meaningful. n-graphs are defined by having data as in (1), the domain/codomain assignments satisfying globularity (2). An n-category (in the usual sense) has several additional operations of *composition*; see, e.g., [S2].

The notion of HDC of the present paper will retain the above general features, except for one thing: the *domain* of a cell is no longer a cell itself; rather, it is a *pasting diagram* (see below) of cells. Note the asymmetry: we only mentioned "domain", not "codomain"; codomains will remain single cells.

The role of n-graphs is taken up by (n-dimensional) *multitopic set*s; below, there will be an explanation for the choice of the name of the concept. The data for a multitopic set are summarized in the diagram



where C_k is the set of k-cells, P_k the set of k-dimensional pasting diagrams (k-pd's for short), each $\mathtt{i}$ is an inclusion map, and the $\mathtt{d}$ and $\mathtt{c}$ are domain and codomain maps. All meaningful instances of the globularity condition (2) will hold.

In the next subsection, we will explain the notion of pasting diagram; here, we note that they are not independent data governed by relations and properties; rather, they are defined explicitly in terms of cells. The most important point to keep in mind that there is an essential recursive character to the notion of multitopic set; this is because the notion of $(k+1)$-cell cannot be explained before we know what k-pd's are, and k-pd's, in turn, are defined in terms of k-cells.

The higher dimensional categories, *multitopic n-categories,* whose definition is the eventual goal of the present paper, are based on multitopic sets, just as n-categories are based on n-graphs. As a compensation for the increased complexity in multitopic sets in comparison to higher dimensional graphs, we have the fundamental fact that a multitopic n-category is an $(n+1)$-dimensional multitopic set *with additional properties* only; no additional data are required. (Note, however, the placing of the prefixes n and $n+1$ in this description.)

1.2. Pasting diagrams

The expression of "pasting diagram" refers to the idea of a *composable diagram*, one which, *if* a concept of composition of cells were available, would result in a single cell after all the meaningful compositions denoted in the diagram are performed. This is an approximate expression of an intuitive idea. It turns out that composability in higher dimensions is a difficult concept, and despite several contributions (e.g., [S1], [J], [P1], [P2]) it is not yet completely clarified. It is to be emphasized that the concept "composable diagram" is a geometric one in that it does not involve composition of cells in the algebraic sense. Composability is the *geometric precondition* of (iterated) composition.

An important point for this paper, inspired by the Baez/Dolan work, is the restriction of cells to the form $a:\alpha\to b$, where α is a pasting diagram (pd), but b is a single cell. The first consequence is that the notion of pd itself becomes simple, and abstractly manageable, in comparison with the (potential) more comprehensive concept that would allow both the domain and the codomain to be arbitrary pd's. The "Baez/Dolan restriction" (as we may call the above-mentioned restriction) is not a *necessary* feature of the intended notion of HDC; it is, rather, a simplifying idea; the thus simplified notion of pd turns out to be *sufficient* for carrying the intended structure of an HDC.

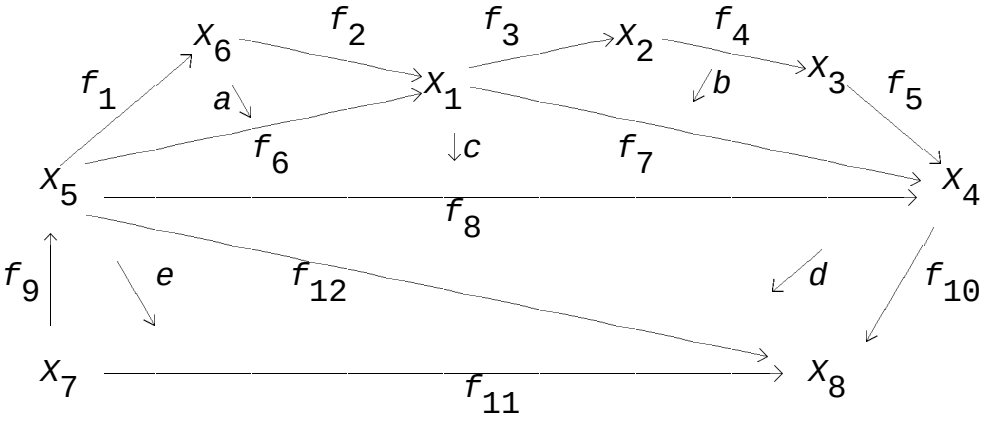
A 0-dimensional pd (or 0-pd) is just a 0-cell (object). A 1-dimensional pd (or 1-pd) is a composable string of 1-cells:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \ldots \xrightarrow{f_n} X_{n+1} \qquad (3)$$

where the X_i's are 0-cells, the f_i's are 1-cells. $n=0$ is allowed, in which case there are no arrows; but in this case, there is still an object, X_1, and we have the empty string of arrows starting and ending in X_1.

A 2-pd consists of 0-, 1- and 2-cells; each 2-cell in it is *from* a 1-pd, a string of one cells (possibly empty), *to* a single 1-cell; and the whole thing is *composable*. Here is an example of a 2-pd, which we denote by the single letter

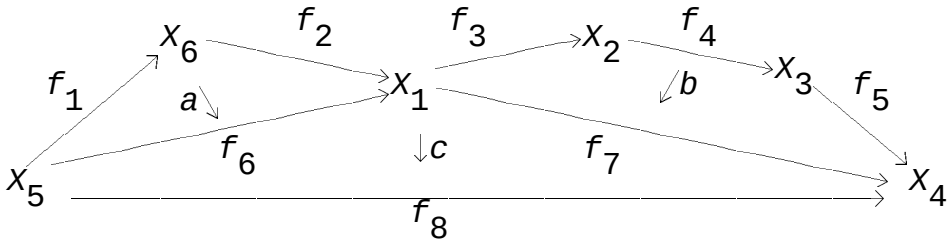
γ :



γ consists of the 0-cells x_i, $1 \le i \le 8$, 1-cells f_j, $1 \le j \le 11$ (numbered in no particular order), and the 2-cells a, b, c, d, e. The figure is supposed to make clear the domain/codomain relations among the cells and 1-pd's involved. Notice the constraint that each 2-cell targets a single 1-cell; in a 2-pd in a more general sense, both domains and codomains could be general 1-pd's. Perhaps it is superfluous to say that the 2-pd γ is the totality of the items listed; it is not the result of some kind of composition performed on those items. Of course, the relative position of its component 2-cells is part of the defining data of the 2-pd.

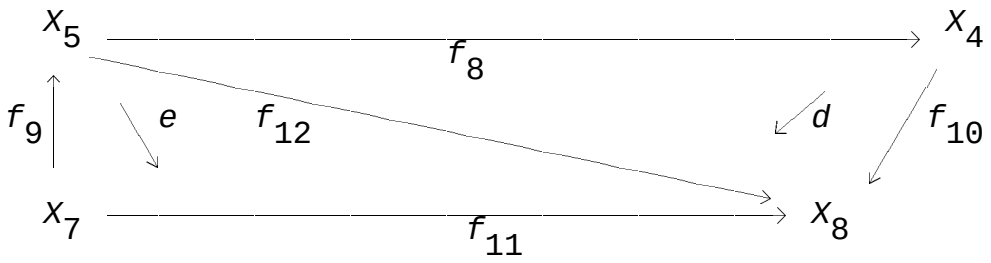
There are features of 2-pd's that become important elements of the general concept of a k-pd. The above 2-pd can be regarded as obtained by *composition*, in a new sense of "formal" composition, which applies to pd's rather than cells. This composition may also be called *grafting*. For instance, γ is obtained by grafting from the following two pd's α and β :

α :



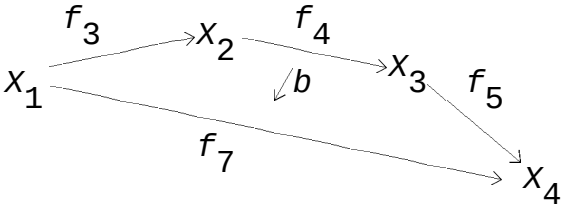
and

β :



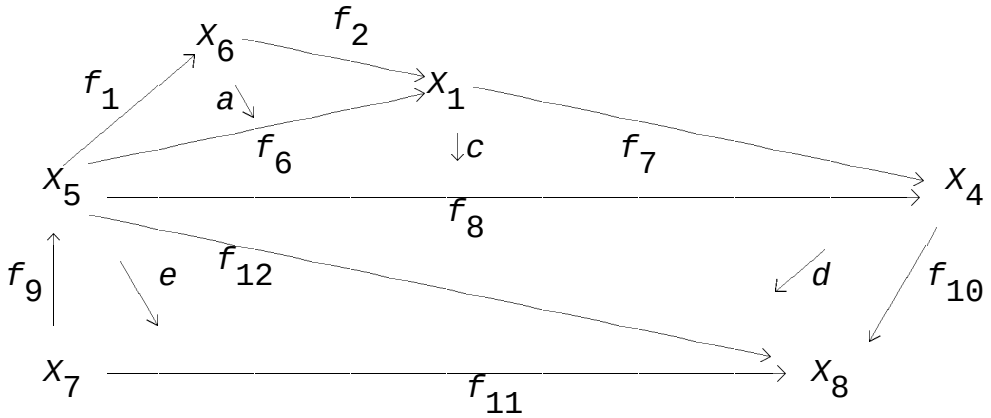
More precisely, we graft α into β *at* f_8, and obtain the original γ. Of course, the same pd γ can also be obtained in several other ways as the result of grafting, e.g. by grafting δ into ε, where

δ :



(which is a pd consisting of a single 2-cell), and

ε :

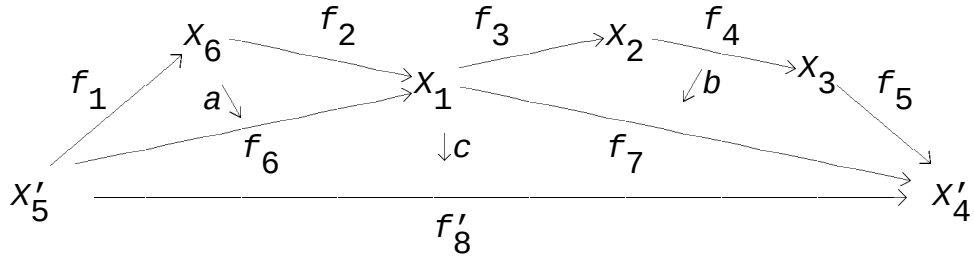


The grafting composition is a binary operation as far as the number of arguments that are pd's is concerned; but it also has a third argument, the *place* at which the grafting takes place. The

two grafting compositions displayed are denoted as $\beta \circ_{f_8} \alpha = \gamma$, and $\varepsilon \circ_{f_7} \delta = \gamma$; read e.g. the first as α composed (grafted) into β at f_8 is γ.

Given β as above, and, say,

α' :



where the primed items may or may not be equal to the corresponding non-primed items in β, the composite $\beta \circ_{f_8} \alpha'$ is meaningful if and only if $f'_8 = f_8$, and as a consequence, $x'_5 = x_5$, $x'_4 = x_4$. f'_8 is distinguished as the *target*-1-*cell* of α; $t(\alpha) \underline{\underline{\text{def}}} f'_8$. For the given β, and an undetermined α', the condition for $\beta \circ_{f_8} \alpha'$ to be well-defined is that $t(\alpha') = f_8$.

It is perfectly possible that several items in the above pd's that are now denoted by different symbols are actually the same. For instance, it is possible that all the 0-cells are the same, and all the 1-cells are the same. If so, the 2-cells a, c, d, e could all be the same, although b cannot be the same as those since its shape is different: its domain pd is a length-3 1-pd, whereas the domains of the others are of length 2. Assuming, e.g., that all the said coincidences actually take place, the subscript f_8 in $\beta \circ_{f_8} \alpha$ cannot refer to the f_8 simply as a 1-cell; it has to refer to *the place of* f_8; we have $f_8 = f_9 = f_{10}$, and we can just as well compose α into β at the *two other places*, now denoted f_9 and f_{10}, and the results of these compositions are all very different, distinguished already by their shapes. This tells us that in the concept of pd there has to be an essential element that we may call *place*; in a 2-pd, there are *places* for 1-cells, each of which carries the "occurrence" of a particular 1-cell.

Note that it does not make sense to compose anything into α at f_{11}, or into γ at places other than $f_9, f_1, f_2, f_3, f_4, f_5, f_{10}$; the result would not be a "composable diagram".

The listed places of γ, the ones at which it is legitimate to compose something into γ, are the *source* places of γ; they are, together with the *target* place f_{11}, "outer places"; the "inner places" are the rest, f_6, f_7, f_8, f_{12}. $s(\gamma)$ denotes the tuple $\langle f_{10}, f_5, f_4, f_3, f_2, f_1, f_9\rangle$, and it is called the *source* of γ; the reason for the order will be explained below. In the example, $s(\gamma)$ is a function on the set $[1, 6]=\{1, 2, 3, 4, 5, 6\}$, and its values are $s(\gamma)(1)=f_{10}$, *etc.* The source places themselves of γ are identified with the natural numbers $1, 2, 3, 4, 5, 6$; the place 1 carries an occurrence of f_{10}, the place 2 one of f_5, etc. Writing $|s(\gamma)|$ for the domain of the function $s(\gamma)$, the source-places of γ are the elements of $|s(\gamma)|$.

Similarly, $s(\beta)=\langle f_{10}, f_8, f_1\rangle$. Since the place of f_8 in β is 2, we will write $\beta\circ_2\alpha$ for $\beta\circ_{f_8}\alpha$; we have $\gamma=\beta\circ_2\alpha$.

For general 2-pd's α and β,

(3) $\beta\circ_p\alpha$ makes sense if and only if $p\in|s(\beta)|$ and $s(\beta)(p)=t(\alpha)$.

We have identified what we take to be the essential structure on pd's: the *placed composition* $\alpha\circ_p\beta$, a ternary operation as explained above.

1.3. Multicategories

The abstract concept of structure for the operation of placed composition is called *multicategory*. Multicategories were introduced by J. Lambek in 1969 [L1]; one of the uses he made of them was to define a multicategory of proofs in the Gentzen formal system for intuitionistic logic, where the placed composition corresponds to the Cut-rule. A *Lambek multicategory* $\boldsymbol{C}$ has a set $O=O(\boldsymbol{C})$ of objects, and a set $A=A(\boldsymbol{C})$ of arrows; each arrow α has a source $s(\alpha)$ which is a finite tuple of objects, and a target $t(\alpha)$ which is a single object; when $s(\alpha)=\vec{X}$, $t(\alpha)=Y$, we write $\alpha:\vec{X}\longrightarrow Y$; $\boldsymbol{C}$ has, for each object X, an identity arrow $1_X:\langle X\rangle\to X$; and $\boldsymbol{C}$ has a placed composition as in (3) above. These data are to satisfy certain laws, the first of which regulates the source and the target of a composite, with the remaining laws being two identity laws, an associativity law, and a commutativity

law. The definition will be given in section 2; the reader will notice that the definition in section 2 is, initially, something more general and more complicated than the one indicated here; later in that section, however, it is pointed out what exactly the Lambek concept is as a special case. Later in this introduction we will turn to the reasons why we need the more general concept of multicategory.

Thus, the 2-pd's (in a given HDC $\mathbf{A}$) form a Lambek multicategory (the 1-pd's also do, in fact, they form an ordinary category). More is true: the 2-pd's form a *free* multicategory, with objects the 1-cells, and generating arrows the 2-cells. Hence, all 2-pd's are generated by the 2-cells by using the operation of placed composition. This should be seen as an intuitively natural fact about pasting (composable) diagrams. (Let us remind ourselves that here we are in the business of *defining* what pasting diagrams are; the definition is constrained by intuitive ideas, which we are trying to make explicit.) Freeness is meant here in the sense of a strict universal property; it will be crucial later that the free Lambek multicategory maintains its universal property in the larger context of all (generalized) multicategories in the sense of Section 2.

For precise definitions concerning morphisms of multicategories, and free multicategories, see sections 3 and 4. Here we only give a brief idea.

Let O be a set of *objects*, L a set of *arrows*, with each $f \in L$ equipped with a source $s(f) \in O^*$, and a target $t(f) \in O$; data as described define a *language* $\mathcal{L}$. The terminology is natural, since $\mathcal{L}$ is exactly what is usually called a language (signature) for multi-sorted algebras; the elements of O are the sorts; the elements of L are the sorted operation symbols. The free multicategory, $\mathbf{C} = \mathcal{F}(\mathcal{L})$, on $\mathcal{L}$ is defined by the conditions that $O(\mathbf{C}) = O$, $L \subset A(\mathbf{C})$, and any "interpretation" (a rather obvious notion) $\mathcal{L} \longrightarrow \mathbf{D}$ to any multicategory $\mathbf{D}$ can be uniquely extended to a morphism $\mathbf{C} \longrightarrow \mathbf{D}$. It turns out that the concrete description of $\mathcal{F}(\mathcal{L})$ is very simple. Its arrows are the *terms*, in the sense used in describing the syntax of first order logic, built up from sorted variables and the operation symbols of L, with the further simplification that we use only a single variable for each sort X, which variable therefore may just as well be identified with X itself.

Thus, we now have a *term-representation* of 2-pd's. Turning to the examples above, we have the following:

$\gamma : e(d(f_{10}, c(b(f_5, f_4, f_3), a(f_2, f_1))), f_9)$,
$\alpha : \quad c(b(f_5, f_4, f_3), a(f_2, f_1))$,
$\beta : e(d(f_{10}, f_8), f_9)$,
$\delta : \quad b(f_5, f_4, f_3)$
$\varepsilon : e(d(f_{10}, c(f_7, a(f_2, f_1))), f_9)$

To understand these, consider the following. Any expression $x(y, z, \ldots)$ stands for a repeated composition; $x(y, z, \ldots) = \ldots(x \circ_1 y) \circ_{\bar{2}} z \ldots$; $\bar{2}$ is the place in $x \circ_1 y$ that "corresponds to" the place 2 in x . Each f_i stands for 1_{f_i} , the identity arrow $\langle f_i \rangle \xrightarrow{1_{f_i}} f_i$. Since $a(f_2, f_1)$ is a with identities composed into a , $a(f_2, f_1)$ equals a itself; we could write a in place of $a(f_2, f_1)$ above, except that in that case we would have not used the normal form which is intended by the term-representation. For $t_1 = b(f_5, f_4, f_3)$, $t_2 = a(f_2, f_1)$, the term $\alpha = c(t_1, t_2)$ is, really, the multicategory composite $(c \circ_1 t_1) \circ_{\bar{2}} t_2 = (c \circ_2 t_2) \circ_{\bar{1}} t_1$; the equality is the commutative law; $\bar{2} = 4$, $\bar{1} = 1$ (why?). We also see that placed composition corresponds to *substitution:* the fact that $\beta \circ_2 \alpha = \gamma$ is reflected in the fact that γ is the result of substituting α for f_8 in β .

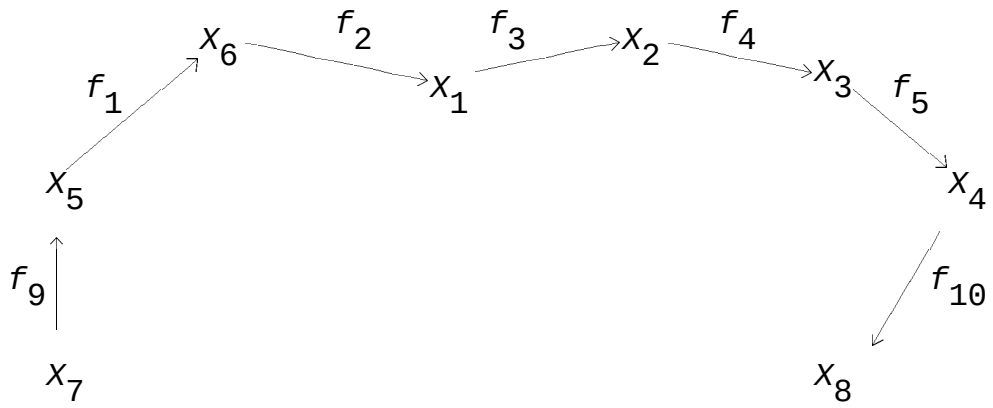
The term-representation is a simple linear way of writing down 2-pd's; in fact, it will also be available for k-pd's for any k . However, note that in this notation, several elements that are clear in the geometric picture are suppressed. All 0-cells, and all but the input 1-cells are suppressed, although they can be be recovered by the information concerning the *targets* of the 2-cells involved.

Let us note that the 1-pd's also admit a term representation, since they also form a multicategory, which in fact is an *ordinary category*, since only unary arrows appear. The 1-pd in (1) is represented by the term $f_n(\ldots(f_2(f_1(x_1))\ldots)$. The source-assignment to 2-cells above follows the left-to-right order in the term-representation; this is the reason why we used the "reverse" order for those sources above.

Let us move from dimension 2 to dimension 3 .

A 3-cell u is to have a 2-pd $\mathrm{d}u (= \mathrm{d}_3 u)$ as domain, and a 2-cell $\mathrm{c}u$ as codomain. Globularity requires that we should have $\mathrm{dd}u = \mathrm{dc}u$, $\mathrm{cd}u = \mathrm{cc}u$; however, we have not

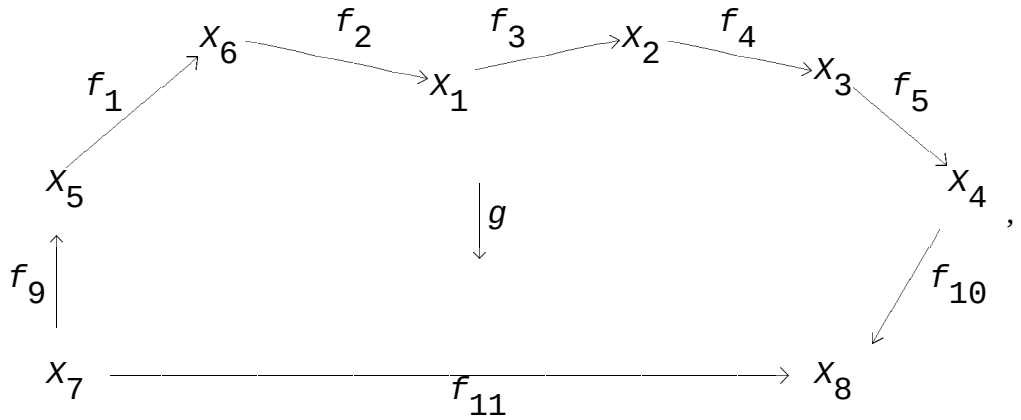
defined $\mathtt{d}\alpha$, $\mathtt{c}\alpha$ for 2-pd's α as yet, and we need them now for $\alpha=\mathtt{d}u$. The definition of the domain of a pd is a major issue in our enterprise; the codomain is easy. In the case of the example γ above, $\mathtt{d}\gamma$ is



that is,

$$x_7 \xrightarrow{f_9} x_5 \xrightarrow{f_1} x_6 \xrightarrow{f_2} x_1 \xrightarrow{f_3} x_2 \xrightarrow{f_4} x_3 \xrightarrow{f_5} x_4 \xrightarrow{f_{10}} x_8\,;$$

this is the "upper part of the contour (boundary) of γ". $\mathtt{c}\gamma$ is the 1-cell $x_7 \xrightarrow{f_{11}} x_8$, the "lower" part of the contour of γ, the cell that "closes off" $\mathtt{d}\gamma$. Thus, a 3-cell u for which $\mathtt{d}u=\gamma$, with γ as in the example, looks necessarily like $u:\gamma\to g$, where g is a 2-cell of the following "shape":



which means that $\mathtt{d}g=\mathtt{d}\gamma$, $\mathtt{c}g=\mathtt{c}\gamma$. One cannot faithfully represent u in a 2-dimensional drawing; but u has a good 3-dimensional geometric representation; in this, the 2-pd γ is

placed in the plane of the table, say; the 2-cell g is spanned out in a curved surface above the table, with its contour joining the contour of γ according to the the identifications inherent in the facts $\mathrm{d}g=\mathrm{d}\gamma$, $\mathrm{c}g=\mathrm{c}\gamma$; the 3-cell u "fills" the space between γ and g, "in the direction" from γ to g.

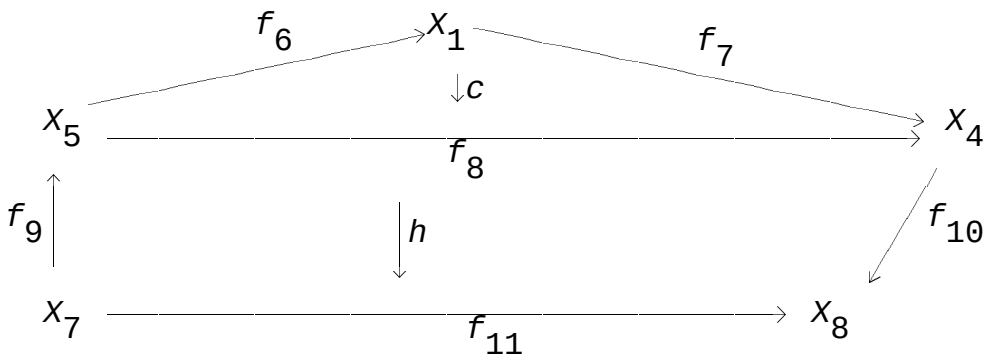
3-pd's will be construed as arrows in the free multicategory on the language whose objects (sorts) are the 2-cells, operation-symbols the 3-cells, and in which the sorting of the latter is given as follows. Every 3-cell u comes with $\mathrm{d}u$, a 2-pd; regard $\mathrm{d}u$ in the term-representation; look at all the operation-symbol occurrences in $\mathrm{d}u$, which are 2-cells; define $\mathrm{s}(u)$ to be the left-to-right tuple $\langle \mathrm{d}u \rangle$ of those occurrences; $\mathrm{s}(u) \in C_2^*$ as it should be. $\mathrm{t}(u)$ is defined to be $\mathrm{c}(u)$.

For instance, for $u : \gamma \to g$ considered above, $\mathrm{s}(u)=\langle e, d, c, b, a \rangle$.

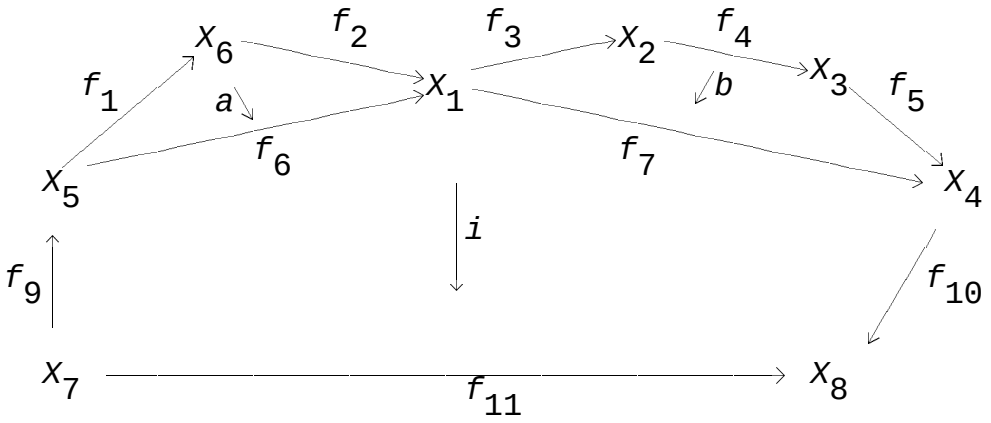
We will now describe a 3-pd φ which is *parallel* to the 3-cell u considered before. This involves the statement that $\mathrm{d}\varphi=\mathrm{d}u$, and therefore involves the determination of the domain $\mathrm{d}\varphi$ of a 3-pd φ. The systematic way of defining the domain of a pd is our main task.

Let us use the 2-pd's β and δ introduced above, as well as the following η and λ; we will use two new 2-cells, h and i:

$\eta = h(f_{10}, c(f_7, f_6), f_9)$:



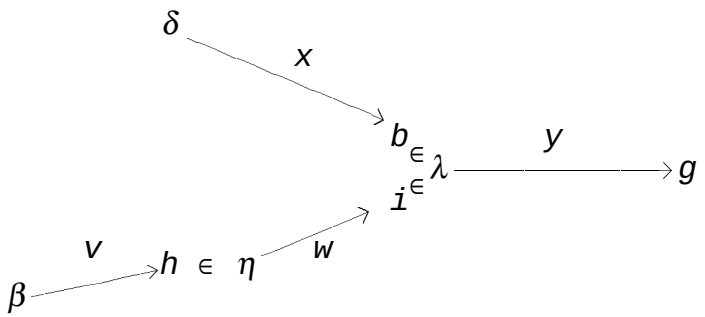
$\lambda = i(f_{10}, b(f_5, f_6, f_3), a(f_2, f_1), f_9)$:



Now we introduce the 3-cells:

$$\beta \xrightarrow{\ v\ } h\ ,\ \ \eta \xrightarrow{\ w\ } i\ ,\ \ \delta \xrightarrow{\ x\ } b\ ,\ \ \lambda \xrightarrow{\ y\ } g\ .$$

The first thing to check is that these are well-formed, that is, in each case the assigned domain (a 2-pd) and codomain (a 2-cell) are parallel; this is true. Now, notice that these four 3-cells "line up" as follows:



(3')

In fact, we have

$$s(v) = \langle e, d\rangle\ ,\ \ s(w)=\langle h, c\rangle\ ,\ \ s(x)=\langle b\rangle\ ,\ \ s(y)=\langle i, b, a\rangle\ ;$$

$$h=s(w)(1)\ ,\ \ i=s(y)(1)\ ,\ \ b=s(y)(2)\ ;$$

and

$$\varphi \stackrel{\mathrm{def}}{=} y(w(v(e,d),c),x(b),a)$$

is well-defined as a 3-pd. Note that, to an even larger extent than before, what φ really is cannot be directly seen on its defining expression; only by taking into account the descriptions of all the ingredients, which themselves were defined in similar ways, can we grasp what φ is. The faithful geometric representation of the 3-pd φ is a 3-dimensional object, obtained by joining the 3-dimensional cells v, w, x, y; the target 2-cell h of v is joined with the occurrence of h in η, similarly for i and b; we get a spherical (simply connected) 3-dimensional object subdivided appropriately. The full entity φ involves four levels of ingredients: k-cells for all of $k=0, 1, 2, 3$. The 2-dimensional boundary of this object consists of the 2-pd γ as domain, and the 2-cell g as codomain; we have $\mathrm{d}\varphi=\gamma$, $\mathrm{c}\varphi=g$. The 2-cells h, i and one of the occurrences of b are "inner" 2-cells in φ, not denoted in the term representation. φ is indeed parallel to the 3-cell $u:\gamma\to g$; as a consequence, a 4-cell of the shape $\varphi\longrightarrow g$ is possible.

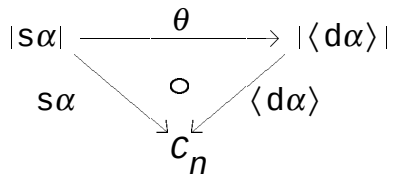
1.4. The domain of a pasting diagram

We turn to explaining how $\mathrm{d}\varphi$, and in general, the domain of an arbitrary pd, is determined algebraically.

As explained before for the cases $k=1$, 2 and 3, we construe the set P_k of k-pd's of the HDC $\boldsymbol{A}$ as the arrows of a free multicategory $\boldsymbol{C}_k^0$ whose objects are the elements of C_{k-1} ($(k-1)$-cells), and whose generating arrows are the elements of C_k. (We use the superscript 0 since there will be a modified ("twisted") variant $\boldsymbol{C}_k$ which will be the final version.) The k-cells $a\in C_k$ come with a domain $\mathrm{d}a\in P_{k-1}$ and a codomain $\mathrm{c}a\in C_{k-1}$. For the determination of $\boldsymbol{C}_k^0$, we also need $\mathrm{s}a$ and $\mathrm{t}a$ for $a\in C_k$; as done above for low values of k, we put $\mathrm{s}a=\langle \mathrm{d}a\rangle$, and $\mathrm{t}a=\mathrm{c}a$.

Let $k\geq 1$ be arbitrary, and let $\alpha\in P_{k+1}$. For any $\gamma\in P_k$, we let $\langle\gamma\rangle$ denote the left-to-right list of function-symbol occurrences in γ. Thus, $\mathrm{s}\alpha$ is a tuple of elements of C_k, and $\mathrm{d}\alpha$ is to be defined in such a way that $\langle \mathrm{d}\alpha\rangle$ is also a tuple of elements of C_k.

The first fact on how $\mathrm{d}\alpha$ is defined is that $\mathrm{s}\alpha$ and $\langle \mathrm{d}\alpha \rangle$ are *almost* equal; one is obtained from the other by a permutation. That is, $|\mathrm{s}\alpha| = |\langle \mathrm{d}\alpha \rangle|$, and there is a permutation $\theta_\alpha = \theta : |\mathrm{s}\alpha| \xrightarrow{\cong} |\langle \mathrm{d}\alpha \rangle|$ such that



Note that, by what was said above, for α a single cell, $\mathrm{d}\alpha$ is already defined, and $\mathrm{s}\alpha = \langle \mathrm{d}\alpha \rangle$; for such α, θ_α can be taken to be the identity.

The second, and main, fact about the way $\mathrm{d}\alpha$ is defined is that there is an operation assigning a new "composite" $\gamma \square_q \delta$ to any $\gamma, \delta \in P_k$ and $q \in |\langle \gamma \rangle|$ satisfying certain conditions of compatibility (that we will see below in detail) such that

$$\mathrm{d}(\alpha \circ_p \beta) = (\mathrm{d}\alpha) \square_{\theta(p)} (\mathrm{d}\beta) ; \qquad (4)$$

that is, the domain of the grafting composite of two $(k+1)$-pd's is the $\square$-composite of the domains of the $(k+1)$-pd's. This, together with knowing what $\mathrm{d}\alpha$ is for single-cell pd's α determines the operation d.

Let us describe the operation $\square$. In fact, this can be done on an arbitrary free multicategory.

Start with $\boldsymbol{C} = \mathcal{F}(\mathcal{L})$, the free Lambek multicategory on the arbitrary language $\mathcal{L}$; we use the notation we had before; $O = \mathrm{O}(\mathcal{L}) = \mathrm{O}(\boldsymbol{C})$ is the set of objects of $\boldsymbol{C}$; $A = \mathrm{A}(\boldsymbol{C})$ is the set of arrows of $\boldsymbol{C}$; we write $\mathrm{s}\alpha$ for $\mathrm{s}_{\boldsymbol{C}}(\alpha)$, $\mathrm{t}\alpha$ for $\mathrm{t}_{\boldsymbol{C}}(\alpha)$. For any $\alpha \in A$, we let $\langle \alpha \rangle$ denote the left-to-right list of function-symbol occurrences in α, as we did before. We let $\mathrm{T}(\alpha) \stackrel{\mathrm{def}}{=} (\mathrm{s}\alpha, \mathrm{t}\alpha)$. Note that $\mathrm{T}(\alpha) = \mathrm{T}(\beta)$ means that α and β are "parallel in the multicategory $\boldsymbol{C}$".

We are going to define a partial operation

$$(\alpha, \beta, p) \longmapsto \alpha \square_p \beta \qquad (\alpha, \beta \in A, p \in |\langle \alpha \rangle| ; \alpha \square_p \beta \in A) .$$

defined whenever