Introduction

In [B/D2] and [B/D3], John C. Baez and James Dolan have introduced a concept of weak higher dimensional category. The present paper is inspired by the work of Baez and Dolan. It is the first of two papers in which a modification of the Baez/Dolan proposal is offered and described in detail.

There are other proposals for related concepts; see [Ba], [T].

The problem of the identification of the weak higher dimensional categories has been recognized for some time; see e.g. [S1], [S2], [S3], [S4] . The motivations for the Baez/Dolan work were described in [B/D1]. In [M2], the second author of this paper describes another motivation, one that relates higher dimensional categories to the foundations of mathematics. In [M2], a program for a new type-theoretical foundation, termed *structuralist*, is described in which there is a hierarchy of totalities of higher and higher dimensions, starting with sets. In this framework, sets are taken to be totalities with an equality predicate. However, no equality is assumed between elements of different sets, and, essentially as a consequence, no equality of sets is contemplated. Because of this, sets do not form a set, or even a set-like totality like a class. Instead, sets form a *category*, the category of sets; and the role of equality as principle of identity is taken over by *isomorphism*, a concept derived from the structure of category. When we say that equality of objects is not part of the structure of the category, we have in mind a notion of category that is not the same as the one we deal with on the basis of the standard set-theoretical foundation. The negative statement of the denial of equality can be given objective content only by specifying a suitably constrained language to be adopted as the formal language of the structuralist foundation. The work [M3] proposes First Order Logic with Dependent Sorts (FOLDS) as the basis for such a language.

Classically, categories form 2-categories; the latter concept can already be found in [M L]. The structuralist foundation involves the program of revising category-theoretical concepts in which *equality of objects* of a category is used by replacing that equality by *specified isomorphisms of objects*. As a matter of fact, it has been widely accepted among category theorists that equality of objects should be avoided; the tendency to replace equality of objects by isomorphism is a common one in category theory. Jean Benabou's notion [Be] of bicategory is an instance of this tendency. In the case of a 2-category, the 1-arrows from a fixed 0-cell to another 0-cell form a(n ordinary) category. Applying the "isomorphisms-for-equality"

treatment to the part of the definition of 2-category which explicitly refers to equality of 1-arrows (e.g., the the associative law of composition of 1-arrows) results in the concept of bicategory. We do not simply require the existence of certain isomorphism-2-arrows, but introduce specified ones (*coherence isomorphisms*), and we attach them to the structure. Furthermore, certain natural *coherence conditions* are imposed on the coherence isomorphisms (the Mac Lane pentagon is an example; see [M L], p. 158, formulated for monoidal categories, that is, bicategories with a single 0-cell). It should be emphasized that the concept of bicategory was motivated in the first place by more mathematical considerations than the ones connected to the structuralist foundation. Bicategories have turned out to be extremely useful, and a great deal more flexible than 2-categories.

The paper [M1] deals with a more elementary instance of replacing equality of objects by isomorphism; the notion of (*saturated*) *anafunctor* is introduced, in which the value-object of a(n ana)functor at any given argument-object is determined (strictly) up to isomorphism. Anafunctors are "mathematically equivalent" to functors, but only at the cost of an application of the Axiom of Choice. The replacement of the composition-functors in the definition of a bicategory by anafunctors results in *anabicategories*, which are held, in [M1] and [M3], to be the right concept for totalities of categories, at least from the point of view of the structuralist foundation. Saturated anabicategories are equivalent to bicategories, again via Choice. Saturated anabicategories are equivalent in a canonical manner, without the use of Choice, to the Baez/Dolan weak 2-categories, and the *multitopic 2-categories* that the sequel to this paper will describe.

Besides being the first answer to a long-standing problem, the Baez/Dolan proposal has several remarkable features. The main one is a complete elimination of explicit lists of coherence structure and conditions. This feature is already fully apparent when one looks at the case $n=2$, a Baez/Dolan weak 2-category. It is related to a bicategory as a fibration is related to a pseudo-functor [G]. The coherence isomorphisms and conditions present in the definition of pseudo-functor are, in the corresponding fibration, eliminated in favor of a structure defined by a universal property, that of Cartesian arrows. Such an elimination of coherence takes place in a Baez/Dolan (B/D) weak n-category as well, for all n. For $n=2$, the composition of 1-cells is defined by a universal property, and accordingly, its result, the composite, is not a uniquely defined thing, but one which is determined up to isomorphism; recall that the last feature is present also in anabicategories. There are no coherence isomorphisms (such as the associativity isomorphism), no coherence conditions (such as Mac Lane's pentagon). The way this is achieved is similar to the case of fibrations inasmuch one adds *more entities* to the

original (pseudo-functor, respectively, bicategory) to get the new structure (fibration, respectively, B/D weak 2-category). In the case of a fibration, the arrows between objects in *different fibers* of the total category are new with respect to the data of the pseudo-functor. In the case of the B/D weak 2-category, we have 2-cells whose domain is a composable string of 1-cells, of arbitrary finite lengths in fact, instead of just a single 1-cell. These "multi-arrows" are new entities with respect to the corresponding (ana)bicategory, and they are taken away when one passes from the B/D 2-category to the corresponding bicategory; of course, before being taken away, they are used to define the data for the bicategory.

Multitopic higher dimensional categories, as we will call the objects that we intend to introduce, will share the above general aspects of the Baez/Dolan weak higher dimensional categories.

Although the proposal to be explained here was directly inspired by the B/D proposal, its exposition will not make this fact clear. In fact, at the present time, we do not see the *precise* equivalence of the two proposals. A conspicuous difference is the absence here, and the presence in [B/D3], of actions of permutation groups. It is possible to introduce an "up to isomorphism" variant of the basic notion of *multicategory* used in this paper (more on this will follow soon); this higher-dimensional variant of "multicategory" (in which, for instance, isomorphisms between arrows in a multicategory would appear) seems more directly related to [B/D3] than what is found here.

On the other hand, even if there are close ties between the proposal of [B/D3] and that of this paper, their mathematical forms are entirely different. The [B/D3] concept is abstract and conceptual; ours here is concrete and geometric.

The above description concerning the 2-dimensional case already indicates the starting point of the approach of the present paper. We define a concept of *k-dimensional cell*, or k-cell, for all $k=0, 1, 2, 3, \ldots$, in an inductive way. For $k>0$, a k-cell has a *domain* and a *codomain*; the codomain is an $(k-1)$-cell, but the domain is a *pasting diagram* of $(k-1)$-cells. The inductive character of the definition lies in the definition of pasting diagrams. These are related to what go under the same name in the literature (see e.g. [P1], [P2]), but are greatly simplified by the fact that the codomains of cells is always a single cell. Despite the fact that the Baez/Dolan concept is not explained in terms of cells whose domains are pasting diagrams of lower cells, the crucial restriction to single-cell codomains also originates in [B/D3].

The present paper's approach is consciously geometrical. At the same time, great care is taken to express everything in algebraic terms. The main algebraic tool we use is the concept of *multicategory*, a modified form of the same-named notion introduced by Joachim Lambek in 1969; see [L1] and [L2]. It is worth remarking that one of the first uses Lambek made of multicategories was to proof-theory, for an algebraic formulation of Gentzen's proof-system for intuitionistic propositional logic.

Lambek's concept is closely related to monoidal categories. A multicategory may be said to be mathematically equivalent to a strictly associative monoidal category in which the monoid of the objects under the tensor-product is a free monoid (on the objects of the multicategory as generators). In a multicategory, we have objects and arrows; each arrow has a *source* which is a finite tuple of objects, and a *target*, a single object. The main distinguishing point about the notion of multicategory is that it is phrased in terms of a *composition*, a ternary operation, two of whose arguments are arrows, the third being the *place* where the target of one the arrows is to fit into the source of the other; of course, the result of composition is an arrow. From the point of view of the arrows, we have a system of binary compositions. Two of the laws are an associative law and a commutative law of composition as in the ordinary binary case, but suitably decorated with places.

We generalize Lambek's notion in two steps, one major and a minor. The major step is to make explicit and generalize the *amalgamation* that takes place in composition. When two arrows are composed, the source of the composite results by *amalgamating* the sources of the original arrows in a certain way. In the Lambek case, this amalgamation is the standard one of *inserting* the source of one of the arrows into the source of the other at the given place. In the generalized concept, the amalgamation is made arbitrary, subject to certain laws. It should be noted that for the precise statement of the laws of multicategory, one has to make an explicit reference to this amalgamation already in Lambek's case. Lambek does not make the amalgamation explicit, but there is an acknowledgement of the resulting incompleteness of the formulation in lines 12 and 11 from the bottom on p.222 of [L2].

It does not seem possible to relate the general concept of multicategory with that of monoidal category as closely as in the case of the Lambek multicategory. The new concept is "essentially geometric"; it has geometric instances (see below), but it does not seem to have "semantical" instances, apart from the standard Lambek case, which does have many "semantical" examples.

On the other hand, the generalized concept is a *mild* generalization. This is witnessed by the fact that the free multicategory in the Lambek sense on a set of objects and generating arrows is also the free multicategory on the same generating data in the generalized sense.

The main point of the new notion is that multicategories with non-standard amalgamation appear in nature. The *multicategory of function-replacement* derived from a free multicategory plays a central role in our work; it is needed for the definition of the domain, a $(k\text{-}1)$-pasting diagram, of a k-pasting diagram.

The first section of the paper is an extended informal introduction. After the next three sections on multicategories, on morphisms of multicategories, and free multicategories, respectively, section 5 gives the construction of the multicategory of function-replacement.

Section 6 uses the preceding machinery to put together the definition of *multitopic set*, the main notion arrived at in this paper. A *multitopic n-category*, the main object we want, will, in the sequel to this paper, be defined as an $(n+1)$-dimensional multitopic set *with additional properties*; no new data are needed. Baez and Dolan used *opetopic sets* instead; the name of their notion is derived from *operads*, the abstract algebraic concept at the basis of their work. Let us note that by a multitopic set. we mean what also could be called an ω-dimensional multitopic set; an n-dimensional one is in fact a truncated one.

Section 7 identifies a particular category, the category $\mathtt{Multitope}$ of *multitopes*, and identifies multitopic sets defined in the section 6 as set-valued functors on the category of multitopes. More precisely, we prove that $\mathtt{MSet}$, the naturally defined category of multitopic sets, is equivalent to the category of functors from $\mathtt{Multitope}$ to $\mathtt{Set}$. $\mathtt{Multitope}$ is related to the terminal object $\mathcal{T}$ of $\mathtt{MSet}$. The objects of $\mathtt{Multitope}$ are identical to the pasting diagrams of the multitopic set $\mathcal{T}$; on the other hand, the identification of the arrows of $\mathtt{Multitope}$ takes additional work. It should be emphasized that all the complexity involved in the definition of multitopic sets in general is already present in the definition of the terminal one, $\mathcal{T}$, despite the fact that this object is absolutely uniquely given.

The category $\mathtt{Multitope}$ and, for any $\mathtt{n=0,1,2,...,}$ its truncation $\mathtt{Multitope[}n\mathtt{]}$ to include k-pasting diagrams of $\mathcal{T}$ for $k=0, \ldots, n$, are fundamental from the point of view taken in this paper. In [M3], a concept of L-equivalence, for variable signatures L for FOLDS, is introduced, and it is shown that, when used in conjunction with the ana-concepts of

[M1], L-equivalence becomes identified with categorical equivalence in many cases, for instance in the case of biequivalence for bicategories. `Multitope[`n`]` is the FOLDS-signature for multitopic n-sets. In view of the fact that multitopic n-categories are multitopic n-sets with additional properties *formulated in FOLDS*, `Multitope[`n`]` is the FOLDS-signature also for multitopic n-categories. Thus, now, even before we have given the further details of the definition of multitopic n-category, we have a notion of equivalence of multitopic n-categories. In Baez's and Dolan's work, we also find a notion of equivalence for weak n-categories. The comparison awaits further work.

The Appendix contains some details of proofs for sections 4 and 5.

It should be emphasized that this paper is only a part, in fact, just a beginning, of the work of establishing the concept of weak higher dimensional category. Even when we have the full definition (which is given by [B/D3], and promised, in a modified form, to be given by the sequel to this paper), the accompanying structures are still to be provided.

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