Appendix

We prove the **if** direction of 3.(1). We use the notation introduced in and before the statement of 3.(1). On the basis of the data $\mathcal{L}$, $\boldsymbol{C}$, F defined on $\mathcal{L}$, and $\langle \theta_f \rangle_{f\in L}$, satisfying the conditions in the definition of the free multicategory, we have to define $F:\boldsymbol{C}\to\boldsymbol{D}$, including $\langle \theta_f \rangle_{f\in A}$.

We write s, t, O, L, A for $s_{\boldsymbol{C}}=s_{\mathcal{L}}$, $t_{\boldsymbol{C}}=t_{\mathcal{L}}$, $O(\boldsymbol{C})=O(\mathcal{L})$, $L(\mathcal{L})$, $A(\boldsymbol{C})$, respectively. Let's write $\bar{O}$ for $O(\boldsymbol{D})$, $\bar{A}$ for $A(\boldsymbol{D})$, $\bar{X}$ for $F(X)$ ($X\in O$), $\bar{f}$ for $F(f)$ ($f\in A$). We will write $s(\beta)$, $t(\beta)$ for $s_{\boldsymbol{D}}(\beta)$, $t_{\boldsymbol{D}}(\beta)$, respectively, when $\beta\in\bar{A}$.

We define $\bar{\alpha}=F(\alpha)$ by structural recursion.

If $\alpha=1_X:\langle X\rangle\to\dot{X}$, we let $\bar{1}_X:\langle \bar{X}\rangle\to\dot{\bar{X}}$.

Assume that $f\in L$, $|s(f)|=[1,n]$, $\alpha_1,\ldots,\alpha_n\in A$, $t(\alpha_i)=(s(f)(i))\cdot$; thus, $\alpha=f(\vec{\alpha})=f(\alpha_1,\ldots,\alpha_n)$ is well-defined. Consider

$$\theta_f:F\circ s(f)\xrightarrow{\cong} s(\bar{f}), \tag{1}$$

and write $\bar{i}$ for $\theta_f(i)$, and $\tilde{j}$ for $\theta_f^{-1}(j)$ ($i,j\in[1,n]$). Assume that $\bar{\alpha}_i$ is defined for all $i\in[1,n]$, and $t(\bar{\alpha}_i)=(\overline{t(\alpha_i)})\cdot$ (induction hypothesis). We let

$$\bar{\alpha} = \bar{f}(\underset{1}{\bar{\alpha}_{\tilde{1}}},\underset{2}{\bar{\alpha}_{\tilde{2}}},\ldots,\underset{n}{\bar{\alpha}_{\tilde{n}}}) = \bar{f}(\underset{1}{\bar{\alpha}_{\tilde{1}}}/1,\underset{2}{\bar{\alpha}_{\tilde{2}}}/2,\ldots,\underset{n}{\bar{\alpha}_{\tilde{n}}}/n).$$

We observe that this is well-defined:

$$t(\underset{j}{\bar{\alpha}_{\tilde{j}}})=(\overline{t(\underset{j}{\alpha_{\tilde{j}}})})\cdot=(\overline{\overline{s(f)(\tilde{j})}})\cdot=s(\bar{f})(\theta_f(\tilde{j}))\cdot=s(\bar{f})(j)\cdot;$$

$\updownarrow$ $*$

here, the equality marked * is the content of (1). Also, $t(\bar{\alpha})=t(\bar{f})=(\overline{t(f)})^{\cdot}=(\overline{t(\alpha)})^{\cdot}$. This completes the definition of the mapping $\alpha\mapsto\bar{\alpha}=F(\alpha)$.

Next, we define $\theta_\alpha: F\circ s(\alpha)\xrightarrow{\cong} s(\bar{\alpha})$. For $\alpha=1_X$, there is no choice. Consider $\alpha=f(\vec{\alpha})$ as above. Let $\vec{\bar{\alpha}}=\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_n\rangle$, and

$$\varphi_i = \varphi_i^{(\boldsymbol{C})}[f, \vec{\alpha}]: s(\alpha_i)\xrightarrow{\cong} s(\alpha),$$
$$\bar{\varphi}_j = \varphi_j^{(\boldsymbol{D})}[\bar{f}, \vec{\bar{\alpha}}]: s(\bar{\alpha}_j)\xrightarrow{\cong} s(\bar{\alpha}),$$

arrows in $O^{\#}$, $\bar{O}^{\#}$, respectively. We know that the morphisms $\bar{\varphi}_j$ are coprojections making $s(\bar{\alpha})$ a coproduct of the $s(\bar{\alpha}_j)$ in $\bar{O}^{\#}$. Therefore, there is a uniquely determined morphism $\theta_\alpha: F\circ s(\alpha)\longrightarrow s(\bar{\alpha})$ for which the following diagram in $\bar{O}^{\#}$ commutes for each $j\in[1, n]$:

$$\begin{array}{ccc} F\circ s(\alpha) & \xrightarrow{\quad\theta_\alpha\quad} & s(\bar{\alpha}) \\ \uparrow{\scriptstyle \varphi_{\tilde{j}}} & \circ & \uparrow{\scriptstyle \bar{\varphi}_j} \\ F\circ s(\alpha_{\tilde{j}}) & \xrightarrow[\theta_{\alpha_{\tilde{j}}}]{} & s(\bar{\alpha}_{\tilde{j}}) \end{array} \quad ; \tag{2}$$

it is also clear that θ_α is an isomorphism. This completes the definition of the θ_α.

We need to verify the requisite properties of $F: \boldsymbol{C}\to\boldsymbol{D}$ defined by the above specifications. First, we show

$$F(\alpha\circ_p\beta)=\overline{\alpha\circ_p\beta}=\bar{\alpha}\circ_{\bar{p}}\bar{\beta}\,; \tag{3}$$

here, $\bar{p}=\theta_\alpha(p)$.

We employ a structural induction on α. When $\alpha=1_X$, the assertion is obviously true. Let $\alpha=f(\vec{\alpha})$, with all the accompanying notation used above. Assume that $\alpha\circ_p\beta$ is well-defined. There are uniquely determined $i\in[1,n]$ and $q\in|s(\alpha_i)|$ for which $p=\varphi_i(q)$; the associative law, in the form applying to a simultaneous composition, gives that

$$\alpha\circ_p\beta = f(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i\circ_q\beta, \alpha_{i+1}, \ldots, \alpha_n) \ . \tag{4}$$

Let $j=\bar{i}$; $\tilde{j}=i$. According to the definition of $\gamma\longmapsto\bar{\gamma}$, we have

$$\overline{\alpha\circ_p\beta}=\bar{f}(\underset{1}{\bar{\alpha}_{\sim}}/1, \ldots, \underset{(j-1)}{\bar{\alpha}_{\sim}}/(j-1), \overline{\alpha_i\circ_q\beta}/j, \underset{(j+1)}{\bar{\alpha}_{\sim}}/j+1, \ldots, \underset{n}{\bar{\alpha}_{\sim}}/n) \tag{5}$$

Let $\bar{q}=\theta_{\alpha_i}(q)$; by the induction hypothesis,

$$\overline{\alpha_i\circ_q\beta} = \bar{\alpha}_i\circ_{\bar{q}}\bar{\beta} \ . \tag{6}$$

Consider the diagram (2), and chase the element q in the lower left corner. We obtain that $\bar{\varphi}_j(\bar{q})=\bar{p}$. Since we have

$$\bar{\alpha} = \bar{f}(\underset{1}{\bar{\alpha}_{\sim}}/1, \underset{2}{\bar{\alpha}_{\sim}}/2, \ldots, \underset{n}{\bar{\alpha}_{\sim}}/n) \ ,$$

we get, by associativity in $\boldsymbol{D}$, that

$$\bar{\alpha}\circ_{\bar{p}}\bar{\beta}=\bar{f}(\underset{1}{\bar{\alpha}_{\sim}}/1, \ldots, \underset{(j-1)}{\bar{\alpha}_{\sim}}/(j-1), \bar{\alpha}_i\circ_{\bar{q}}\bar{\beta}/j, \underset{(j+1)}{\bar{\alpha}_{\sim}}/j+1, \ldots, \underset{n}{\bar{\alpha}_{\sim}}/n) \tag{7}$$

which, after a comparison with (5) and (6), gives (3).

It remains to show that the θ_α satisfy the coherence condition in 2.(v).

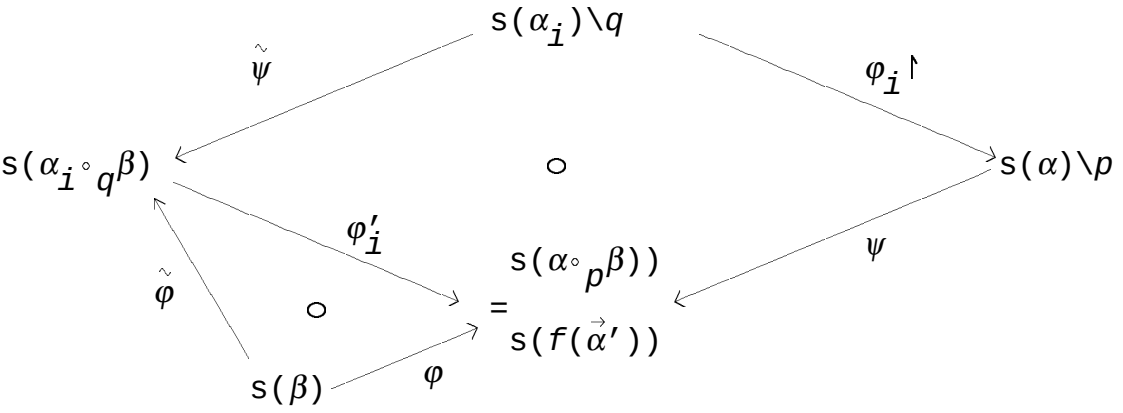
We consider $\alpha=f(\vec{\alpha})$ and $\alpha\circ_p\beta$ as before; we write $f(\vec{\alpha}')$ to abbreviate the right-hand side of (4), $\bar{f}(\vec{\bar{\alpha}}')$ for the right-hand side of (5) (equivalently, (7)). We want to prove the commutativities in

$$\begin{array}{ccc}
(F\circ s(\alpha))\backslash p & \xrightarrow{\theta_\alpha\ \vdash} & s(\bar{\alpha})\backslash\bar{p} \\
\downarrow{\scriptstyle \psi} & \circlearrowright ? & \downarrow{\scriptstyle \bar{\psi}} \\
F\circ s(\alpha\circ_p\beta) & \xrightarrow{\theta_{\alpha\circ_p\beta}} & s(\overline{\alpha\circ_p\beta})=s(\bar{\alpha}\circ_{\bar{p}}\bar{\beta}) \\
\uparrow{\scriptstyle \varphi} & \circlearrowright ? & \uparrow{\scriptstyle \bar{\varphi}} \\
F\circ s(\beta) & \xrightarrow[\theta_\beta]{} & s(\bar{\beta})
\end{array} \qquad (8)$$

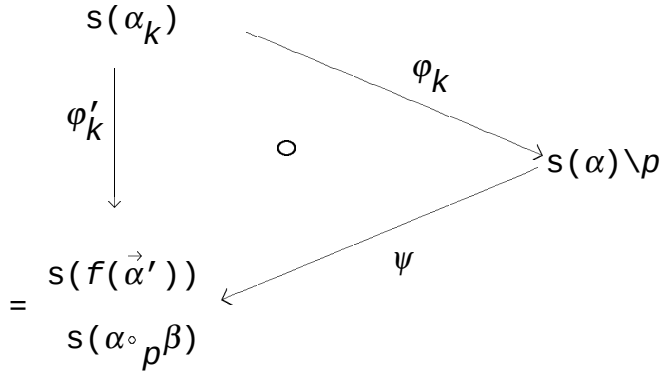
What we have to go on are the following facts. First, as the induction hypothesis, the commutativities

$$\begin{array}{ccc}
(F\circ s(\alpha_i))\backslash q & \xrightarrow{\theta_{\alpha_i}\ \vdash} & s(\bar{\alpha}_i)\backslash\bar{q} \\
\downarrow{\scriptstyle \hat{\psi}} & \circlearrowright & \downarrow{\scriptstyle \hat{\psi}} \\
F\circ s(\alpha_i\circ_p\beta) & \xrightarrow{\theta_{\alpha_i\circ_p\beta}} & s(\overline{\alpha_i\circ_q\beta})=s(\bar{\alpha}_i\circ_{\bar{q}}\bar{\beta}) \\
\uparrow{\scriptstyle \hat{\varphi}} & \circlearrowright & \uparrow{\scriptstyle \hat{\varphi}} \\
F\circ s(\beta) & \xrightarrow[\theta_\beta]{} & s(\bar{\beta})
\end{array} \qquad (9)$$

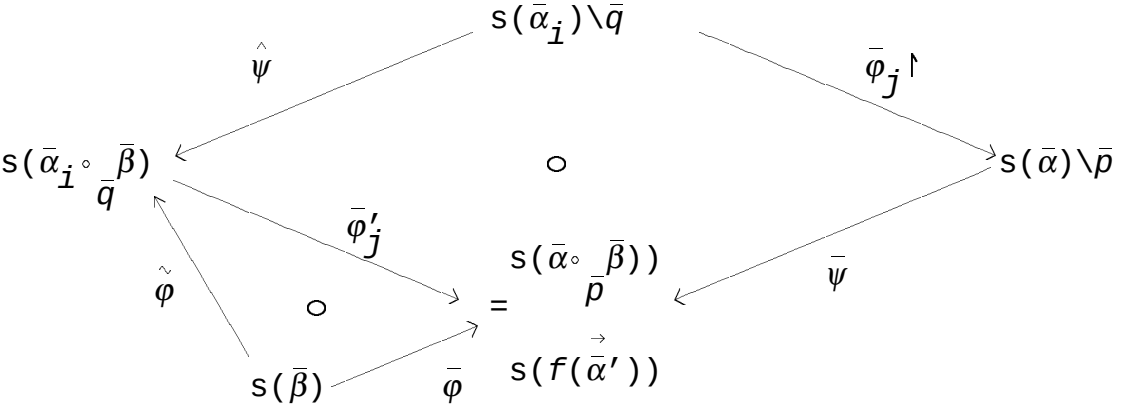
where we used the notations $\hat{\psi},\hat{\varphi},\hat{\psi},\hat{\varphi}$ in the obvious senses. Second, we have the coherence conditions associated with associativity, both in $\boldsymbol{C}$ and $\boldsymbol{D}$, to wit



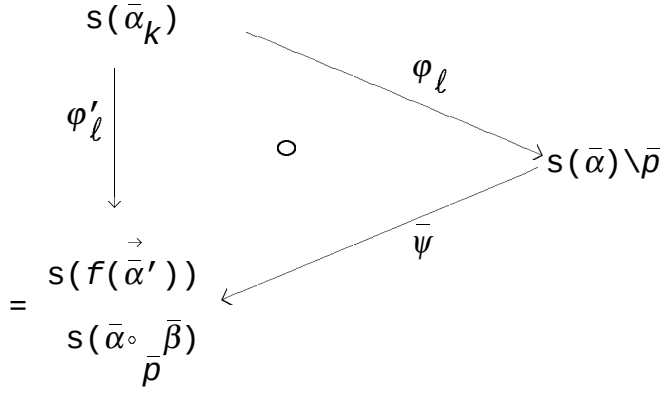
(10)



(k≠i) (11)

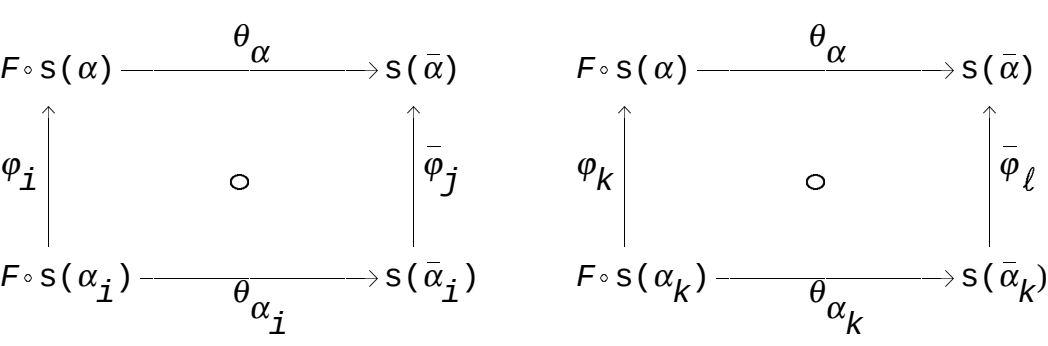


(12)



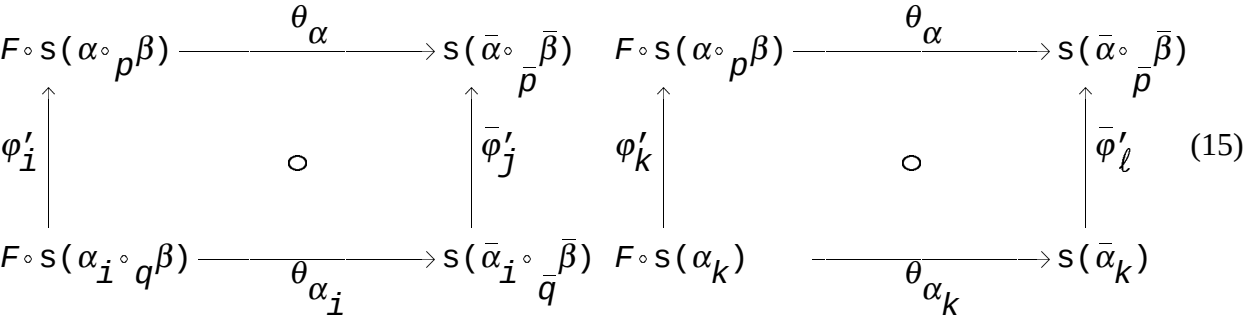
(k≠i, ℓ=θ_f(k)) (13)

Further, we have the definition of θ_α in terms of the diagrams (2), that is



(14)

and the definition of $\theta_{\alpha\circ_p\beta}=\theta_{f(\vec{\alpha}')}$, that is

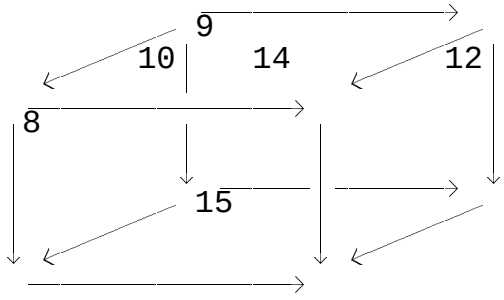


(15)

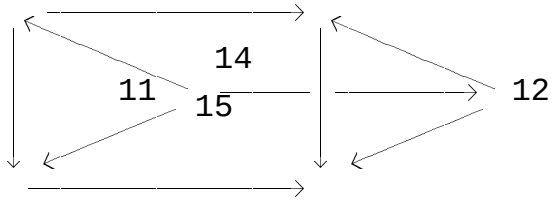
Let us remark that in each of these diagram, we actually have functions on sets, and as functions of sets, e.g. $\varphi_i : s(\alpha_i) \longrightarrow s(\alpha)$ and $\varphi_i : F\circ s(\alpha_i) \longrightarrow F\circ s(\alpha)$ are the same, namely $\varphi_i : |s(\alpha_i)| \longmapsto |s(\alpha)|$.

Consider the upper commutativity in (8). $s(\alpha)\backslash p$ is the coproduct of $s(\alpha_i)\backslash q$ and $s(\alpha_k)$ $(k\neq i)$ via the coprojections φ_i , φ_k ; therefore, the required commutativity is proved if we can show that it holds when we precompose it with the said coprojections.

We observe (not without a certain amount of experimentation) that using some of the diagrams above, we can fit together the cube



in which the front face is (8) upper (the one we want to see commute), the left face is (10) upper, the right face is (12) upper, the bottom face is (15) left, the top face is (14) left, and the back face is (9) upper. All of these, except the front face, commute. It follows that the front face commutes when precomposed by the left upper edge, which is $\varphi_i \restriction$; this is the first thing we want. As far as precomposing with φ_k $(k \neq i)$ is concerned, the back face of the cube collapses, and we get



here again, the front face is (8) upper, the left face is (11), the right face is (13), the bottom face is (15) right, the top face is (14) right; thus, we again have the desired conclusion. As we said, this shows the commutativity of (8) upper.

The proof for (8) lower is left to the reader.

This completes the proof of the existence of the morphism $F : \mathbf{C} \to \mathbf{D}$; the uniqueness of F is clear from what we have seen.

Next, the proofs of some lemmas in section 4 are provided.

Proof of 4.(6) Lemma:

We employ induction according to 3.(2).

If $\alpha=1_X$, then $|\langle\alpha\rangle|$ is empty; the assertion is vacuously true.

Let $\alpha=g(\beta_1, \ldots, \beta_m)$, with $g\in L$; $m_i=\ell h(s(\beta_i))$. There are two cases: $p=1$ (Case 1), $p\neq 1$ (Case 2).

In Case 1, we have $f=\langle\alpha\rangle(1)=g$ and $n=m$; we put $\alpha'=1_X$ where $X=t(g)$, and $\alpha_i=\beta_i$ ($i\in[1, m]$); (1) is clear.

In Case 2, by (1'), there is $i\in[1, m]$ such that $p\in|\langle\beta_i\rangle|$. By assumption, we have

$$\beta_i=\gamma\circ_r f(\alpha_1, \ldots \alpha_n)$$

for suitable $\gamma, r, \alpha_1, \ldots \alpha_n$. Then for

$$\delta \overset{=}{def} g(\beta_1, \ldots \beta_{i-1}, 1_Y, \beta_{i+1}, \ldots, \beta_m),$$
$$\varepsilon \overset{=}{def} f(\alpha_1, \ldots \alpha_n),$$
$$s \overset{=}{def} \varphi_i[g; \beta_1, \ldots \beta_{i-1}, 1_Y, \beta_{i+1}, \ldots, \beta_m](1)\in|s(\delta)|,$$
$$\alpha' \overset{=}{def} \delta\circ_s\gamma = g(\beta_1, \ldots \beta_{i-1}, \gamma, \beta_{i+1}, \ldots, \beta_m)$$

[α' is well-defined since $t(\gamma)=t(\beta)$],

$$q \overset{=}{def} \varphi[\delta, \gamma, r](r),$$

we have 4.(2):

$$\alpha'\circ_q f(\alpha_1, \ldots \alpha_n) = (\delta\circ_s\gamma)\circ_q\varepsilon = \delta\circ_s(\gamma\circ_q\varepsilon) = \delta\circ_s\beta_i = \alpha.$$

This completes the proof.

Proof of 4.(7) Lemma:

For the proof, let us first make a general statement. Given an identity

$$g(\beta_1, \ldots, \beta_m) = \eta \circ_s \xi$$

with $g \in L$, $\beta_1, \ldots, \beta_m, \eta, \xi \in A$, it follows that

either $\eta = 1_X$ and $g(\beta_1, \ldots, \beta_m) = \xi$,
or for unique $i \in [1, m]$, $\hat{\beta}$ and $t \in |s(\hat{\beta})|$, we have

$$\beta_i = \hat{\beta} \circ_t \xi \text{ and } \eta = g(\beta_1, \ldots, \beta_{i-1}, \hat{\beta}, \beta_{i+1}, \ldots, \beta_m),$$

and for $\varphi_i = \varphi_i[g; \beta_1, \ldots, \beta_{i-1}, \hat{\beta}, \beta_{i+1}, \ldots, \beta_m]$, we have $s = \varphi_i(t)$.

This is proved by an induction on η, according to 3.(2); no separatedness is involved.

Let $f \in L$, $n = \ell h(s(f))$, $\vec{\beta} = (\beta_1, \ldots, \beta_n)$, $\vec{\delta} = (\delta_1, \ldots, \delta_n)$, $f(\vec{\beta})$, $f(\vec{\delta})$ well-defined. Assume

$$\varepsilon = \alpha \circ_p f(\vec{\beta}) = \gamma \circ_q f(\hat{\delta})$$

and ε is separated, to show that $\alpha = \gamma$, $\vec{\beta} = \vec{\delta}$ and $p = q$. We do an induction on α.

Let first $\alpha = 1_X$; $X = t(f)$ necessarily. Then $\varepsilon = f(\vec{\beta})$. If $\gamma = 1_Y$, then $Y = X = t(f)$, and $\alpha = \gamma$. Otherwise, $\|\gamma\| \neq \emptyset$, and $\|\gamma\| \cap \|f(\hat{\delta})\| = \emptyset$, and so $f \notin \|\gamma\|$; but $\gamma = g(\vec{\gamma})$ for a suitable $g \in L$, $\vec{\gamma} \in A^*$, and so $\gamma \circ_q f(\hat{\delta}) = g(\vec{\eta})$ for a suitable $\vec{\eta} \in A^*$; $\varepsilon = f(\vec{\beta}) = \gamma \circ_q f(\hat{\delta}) = g(\vec{\eta})$ is impossible; contradiction.

Next, $\alpha = g(\vec{\alpha})$, $g \in L$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in L^*$. Then for a unique $i \in [1, m]$, for $\varphi_i = \varphi_i[g, \vec{\alpha}]$, and for a unique $r \in |s(\alpha_i)|$, we have $p = \varphi_i(r)$, and for

$\tilde{\alpha}=\alpha_i \circ_r f(\vec{\beta})$, we have

$$\varepsilon=\alpha\circ_p f(\vec{\beta})=g(\alpha_1, \ldots \alpha_{i-1}, \tilde{\alpha}, \alpha_{i+1}, \ldots, \alpha_m) .$$

Since ε is separated, we have that $g\neq f$.

$$\varepsilon=\gamma\circ_q f(\hat{\delta}) = g(\alpha_1, \ldots \alpha_{i-1}, \tilde{\alpha}, \alpha_{i+1}, \ldots, \alpha_m)$$

implies that

either $\gamma=1_X$ and $\varepsilon=f(\hat{\delta})$, a case excluded by $f\neq g$;

or for some $j\in[1, m]-\{i\}$, $\hat{\alpha}$ and $s\in |s(\hat{\alpha})|$, we have $\alpha_j=\hat{\alpha}\circ_s f(\hat{\delta})$, a case again excluded since $\|\alpha_j\|\cap\|\tilde{\alpha}\|=\emptyset$ by the separatedness of ε ;

or for some $\hat{\alpha}$ and $s\in |s(\hat{\alpha})|$, we have $\tilde{\alpha}=\hat{\alpha}\circ_s f(\hat{\delta})$ and

$$\gamma=g(\alpha_1, \ldots \alpha_{i-1}, \hat{\alpha}, \alpha_{i+1}, \ldots, \alpha_m) ,$$

and for $\hat{\varphi}_i=\varphi_i[g; \alpha_1, \ldots \alpha_{i-1}, \hat{\alpha}, \alpha_{i+1}, \ldots, \alpha_m]$, we have $\hat{\varphi}_i(s)=q$.

In this case, we have $\tilde{\alpha}=\alpha_i \circ_r f(\vec{\beta})=\hat{\alpha}\circ_s f(\hat{\delta})$. By the induction hypothesis applied to α_i , we get $\alpha_i=\hat{\alpha}$, $r=s$ and $\vec{\beta}=\vec{\delta}$. But then

$$\begin{aligned}\hat{\varphi}_i&=\varphi_i[g; \alpha_1, \ldots \alpha_{i-1}, \hat{\alpha}, \alpha_{i+1}, \ldots, \alpha_m]\\ &=\varphi_i[g; \alpha_1, \ldots \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_m]=\varphi_i ,\end{aligned}$$

$$p = \varphi_i(r)= \hat{\varphi}_i(s) = q ,$$

and

$$\alpha=g(\alpha_1, \ldots \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_m)=g(\alpha_1, \ldots \alpha_{i-1}, \hat{\alpha}, \alpha_{i+1}, \ldots, \alpha_m)=\gamma ;$$

and the proof is complete.

Proof of 4.(10) Lemma:

Assume first that $f \in \|\alpha\|$. Write α in the form 4.(2), with $\vec{\alpha}$ abbreviating $\alpha_1, \ldots, \alpha_n$:

$$\alpha = \alpha' \circ_q f(\vec{\alpha}) ,$$

and

$$\alpha \square_f \gamma = \alpha' \circ_q \gamma(\vec{\alpha}) \quad .$$

We have that $s(\alpha)$ is the coproduct of $s(\alpha')$ and the $s(\alpha_i)$ $(i \in [1, n])$, via the amalgamating maps

$$\psi \underset{\text{def}}{=} \psi[\alpha', f(\vec{\alpha}), q] : s(\alpha') \to s(\alpha) ,$$
$$\varphi \underset{\text{def}}{=} \varphi[\alpha', f(\vec{\alpha}), q] : s(f(\vec{\alpha})) \to s(\alpha) .$$

Therefore, we have the two mutually exclusive cases $r \in \text{Im}(\psi)$ (Case 1) , and $r \in \text{Im}(\varphi)$ (Case 2). Leaving aside the easier Case 1, we take up Case 2. $s(f(\vec{\alpha}))$ is the coproduct of the $s(\alpha_i)$ $(i \in [1, n])$, via the maps

$$\varphi_i = \varphi_i[f, \vec{\alpha}] : s(\alpha_i) \longrightarrow s(f(\vec{\alpha})) .$$

We have $r = \varphi(s)$ for a uniquely determined s ; and $s = \varphi_i(t)$ for uniquely determined i and t . Two applications of the associative law (one of the original form, the other of the form related to simultaneous composition) give that

$$\alpha \circ_r \beta = (\alpha' \circ_q f(\vec{\alpha})) \circ_r \beta = \alpha' \circ_q (f(\vec{\alpha}) \circ_s \beta) =$$
$$\alpha' \circ_q f(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \circ_t \beta, \alpha_{i+1}, \ldots, \alpha_n) \quad .$$

Therefore,

$$(\alpha \circ_r \beta) \square_f \gamma = \alpha' \circ_q \gamma(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \circ_t \beta, \alpha_{i+1}, \ldots, \alpha_n) \ . \qquad (1)$$

On the other hand,

$$\alpha \square_f \gamma = \alpha' \circ_q \gamma(\vec{\alpha}) \ .$$

Comparing $d(\alpha' \circ_q \gamma(\vec{\alpha}))$ and $d(\alpha' \circ_q f(\vec{\alpha}))$, by 4.(4) we see that they are equal, because $d(\gamma)=d(f)$; also, $\varphi[\alpha', \gamma(\vec{\alpha}), q]=\varphi[\alpha', f(\vec{\alpha}), q]=\varphi$, and similarly for the ψ's. Therefore,

$$(\alpha \square_f \gamma) \circ_r \beta = (\alpha' \circ_q \gamma(\vec{\alpha})) \circ_r \beta = \alpha' \circ_q (\gamma(\vec{\alpha}) \circ_s \beta) \ ,$$

with the same s as the one determined above. For similar reasons, we have, for the same i and t as above,

$$\gamma(\vec{\alpha}) \circ_s \beta \ = \ \gamma(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \circ_t \beta, \alpha_{i+1}, \ldots, \alpha_n) \ ,$$

and so,

$$(\alpha \square_f \gamma) \circ_r \beta \ = \ \alpha' \circ_q \gamma(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i \circ_t \beta, \alpha_{i+1}, \ldots, \alpha_n) \qquad (2)$$

(1) and (2) confirm the desired equality $(\alpha \circ_r \beta) \square_f \gamma = (\alpha \square_f \gamma) \circ_r \beta$.

We will not consider Case 1, neither the second case in the lemma, $f \in \|\beta\|$, and take 4.(10) as having been proved.

Proof of 4.(11) Lemma:

In the possession of 4.(10), we can perform an induction on α, in the sense of showing the equalities for $\alpha=1_X$ and $\alpha\in L$, and assuming them for δ and ε in place of α, with all other parameters unconstrained, we prove it for $\alpha=\delta\circ_r\varepsilon$. Since A is the least set containing all 1_X, L and closed under the operations $(\delta,\varepsilon)\longmapsto\delta\circ_r\varepsilon$, this is a valid procedure.

The basis case $\alpha=1_X$ is vacuous. Let $\alpha=h\in L$. Then $g\in\|\alpha\|$ would mean $g=h$, and since $f\in\|\alpha\|$, $f=h=g$; however, $f\notin\|\alpha\square_f\beta\|$ and $g\in\|\alpha\square_f\beta\|$, contradiction. It remains to consider the case $g\in\|\beta\|$. Since $\alpha\square_f\beta$ is well-defined, $h=f$, and $\alpha\square_f\beta=\beta$. Both $(\alpha\square_f\beta)\square_g\gamma$ and $\alpha\square_f(\beta\square_g\gamma)$ are equal to $\beta\square_g\gamma$.

For the induction step, we let $\alpha=\delta\circ_r\varepsilon$, and distinguish, because of $\|\alpha\|=\|\delta\|\dot\cup\|\varepsilon\|$, the following six mutually exclusive and jointly exhaustive cases:

[1]: $f\in\|\delta\|$, $g\in\|\delta\|$;
[2]: $f\in\|\delta\|$, $g\in\|\varepsilon\|$;
[3]: $f\in\|\varepsilon\|$, $g\in\|\delta\|$;
[4]: $f\in\|\varepsilon\|$, $g\in\|\varepsilon\|$;
[5]: $f\in\|\delta\|$, $g\in\|\beta\|$;
[6]: $f\in\|\varepsilon\|$, $g\in\|\beta\|$.

[1] : we want to show: $((\delta\circ_r\varepsilon)\square_f\beta)\square_g\gamma\stackrel{?}{=}((\delta\circ_r\varepsilon)\square_g\gamma)\square_f\beta$;

we have:

$$((\delta\circ_r\varepsilon)\square_f\beta)\square_g\gamma=((\delta\square_f\beta)\circ_r\varepsilon)\square_g\gamma=((\delta\square_f\beta)\square_g\gamma)\circ_r\varepsilon$$
$$\underset{*}{=}((\delta\square_g\gamma)\square_f\beta)\circ_r\varepsilon=((\delta\square_g\gamma)\circ_r\varepsilon)\square_f\beta=((\delta\circ_r\varepsilon)\square_g\gamma)\square_f\beta;$$

all equalities except the one marked $(*)$ which is the induction hypothesis for δ, are valid

instances of 4.(10), sometimes used "backwards".

[2] : $((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma \stackrel{?}{=} ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta$:

$$((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma = ((\delta \square_f \beta) \circ_r \varepsilon) \square_g \gamma = (\delta \square_f \beta) \circ_r (\varepsilon \square_g \gamma)$$
$$= (\delta \circ_r (\varepsilon \square_g \gamma)) \square_f \beta = ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta \ ;$$

in this case, the induction hypothesis is not used.

[3]: $((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma \stackrel{?}{=} ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta$:

$$((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma = (\delta \circ_r (\varepsilon \square_f \beta)) \square_g \gamma = (\delta \square_g \gamma) \circ_r (\varepsilon \square_f \beta)$$
$$= ((\delta \square_g \gamma) \circ_r \varepsilon) \square_f \beta = ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta \ .$$

[4]: $((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma \stackrel{?}{=} ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta$:

$$((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma = (\delta \circ_r (\varepsilon \square_f \beta)) \square_g \gamma = \delta \circ_r ((\varepsilon \square_f \beta) \square_g \gamma)$$
$$\underset{*}{=} \delta \circ_r ((\varepsilon \square_g \gamma) \square_f \beta) = (\delta \circ_r (\varepsilon \square_g \gamma)) \square_f \beta = ((\delta \circ_r \varepsilon) \square_g \gamma) \square_f \beta \ .$$

[5]: $((\delta \circ_r \varepsilon) \square_f \beta) \square_g \gamma \stackrel{?}{=} (\delta \circ_r \varepsilon) \square_f (\beta \square_g \gamma)$:

$$((\overset{\downarrow}{\delta}\circ_r\varepsilon)\square_f\beta)\square_g\gamma \;=\; ((\delta\square_f\beta)\circ_r\varepsilon)\square_g\gamma \;=\; ((\delta\square_f\beta)\square_g\gamma)\circ_r\varepsilon$$

$$\underset{*}{=}\; (\delta\square_f(\beta\square_g\gamma))\circ_r\varepsilon \;=\; (\delta\circ_r\varepsilon)\square_f(\beta\square_g\gamma)\ .$$

[6]: $$((\delta\circ_r\overset{\downarrow}{\varepsilon})\square_f\beta)\square_g\gamma \overset{?}{=} (\delta\circ_r\varepsilon)\square_f(\beta\square_g\gamma)\ :$$

$$((\delta\circ_r\overset{\downarrow}{\varepsilon})\square_f\beta)\square_g\gamma \;=\; (\delta\circ_r(\varepsilon\square_f\beta))\square_g\gamma \;=\; \delta\circ_r((\varepsilon\square_f\beta)\square_g\gamma)$$

$$\underset{*}{=}\; \delta\circ_r(\varepsilon\square_f(\beta\square_g\gamma)) \;=\; (\delta\circ_r\varepsilon)\square_f(\beta\square_g\gamma)\ .$$

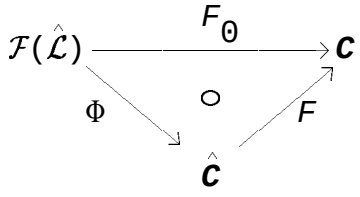
This completes the proof of 4.(11).

Proof of 4.(12) Lemma:

Assume $\mathcal{L}_1$, $\mathcal{L}_2$ are languages with $O(\mathcal{L}_1)=O(\mathcal{L}_2)$. Temporarily, we will mean by an morphism $H:\mathcal{L}_1\to\mathcal{L}_2$ of languages a mapping $H:A(\mathcal{L}_1)\to A(\mathcal{L}_2)$ such that $s_{\mathcal{L}_2}(H(f))=s_{\mathcal{L}_1}(f)$ and $t_{\mathcal{L}_2}(H(f))=t_{\mathcal{L}_1}(f)$ for all $f\in L(\mathcal{L}_1)$ (thus, we do not consider any action on the objects themselves now). Note that an "ample expansion" $F:\hat{\boldsymbol{C}}\to\boldsymbol{C}$ restricts to a morphism $\hat{\mathcal{L}}\to\mathcal{L}$.

We first choose a language $\hat{\mathcal{L}}$ with a morphism $F:\hat{\mathcal{L}}\to\mathcal{L}$ (as described) such that for every $f\in L$ there are infinitely many $\hat{f}\in\hat{L}=L(\hat{\mathcal{L}})$ with $F(\hat{f})=f$. Take $\hat{\boldsymbol{C}}=\mathcal{F}(\hat{\mathcal{L}})$ to be the free multicategory on $\hat{\mathcal{L}}$, say, with standard amalgamation. Using $\theta_g=\mathrm{id}_{s(g)}$ for all $g\in\hat{L}$, we have a uniquely determined morphism $F_0:\mathcal{F}(\hat{\mathcal{L}})\longrightarrow\boldsymbol{C}$ of multicategories for which $F_0(g)=F(g)$ as given to begin with, for each $g\in\hat{L}$, and whose transition isomorphisms, at

each $g\in\hat{L}$, are identities. "Twist" $\mathcal{F}(\hat{\mathcal{L}})$ by using the transition isomorphisms of F_0; that is, use the factorization of F_0 into an isomorphism Φ which is the identity on objects and arrows, and a strict morphism as in 2.:



$\hat{\boldsymbol{C}}$ is free as well, since, by the characterization 3.(1), being free is invariant under twisting (or, because $\hat{\boldsymbol{C}}\cong\tilde{\boldsymbol{C}}$). $F:\hat{\boldsymbol{C}}\to\boldsymbol{C}$ is the desired ample expansion. We define $\hat{d}:\hat{\boldsymbol{C}}\to\boldsymbol{E}$ as the composite $\hat{d}=d\circ F$; $\hat{d}$ is a strict morphism which is the identity on objects;

Proof of 4.(13) Lemma:

The proof is by induction on α.

If $\alpha=1_X$, $\beta=1_X$ is the necessary choice. Let $\alpha=f(\alpha_1,\ldots\alpha_n)$. We apply the induction hypothesis successively to $\alpha_1,\ldots,\alpha_n$; there is separated $\beta_1\in\hat{A}$ with $F(\beta_1)=\alpha_1$ such that $I\cap\|\beta_1\|\neq\emptyset$; there is $\beta_2\in\hat{A}$ with $F(\beta_2)=\alpha_2$ and $\|\beta_2\|\cap(I\cup\|\beta_1\|)=\emptyset$; ...; there is $\beta_n\in\hat{A}$ with $F(\beta_n)=\alpha_n$ and $\|\beta_n\|\cap(I\cup\|\beta_1\|\cup\|\beta_2\|\cup\ldots\cup\|\beta_{n-1}\|)=\emptyset$. Now, let $J=I\cup\|\beta_1\|\cup\|\beta_2\|\cup\ldots\cup\|\beta_n\|$. By assumption, there is $g\in\hat{L}$ such that $g\notin\psi$, and $F(g)=F(f)$. In particular, $s(g)=s(f)$. Thus, $\beta=g(\beta_1,\ldots,\beta_n)\in\hat{A}$ is well-defined, and since the sets $\{g\}$, $\|\beta_1\|$, $\|\beta_2\|$, ..., $\|\beta_n\|$ are pairwise disjoint, β is separated. Also, the construction ensures that $\|\beta\|\cap I=\emptyset$. Since F is a morphism, $F(\beta)=F(g)(F(\beta_1),\ldots,F(\beta_n))=\alpha$. The proof is complete.