The multitopic ω-category of all multitopic ω-categories

by M. Makkai (McGill University)

September 2, 1999; corrected: June 5, 2004

Introduction. Despite its considerable length, this paper is only an announcement. It gives two definitions: that of "multitopic ω-category", and that of "the (large) multitopic set of all (small) multitopic ω-categories". It also announces the theorem that the latter is a multitopic ω-category.

The work has two direct sources. One is the paper [H/M/P] in which, among others, the concept of "multitopic set" was introduced. The other is the author's work on FOLDS, first order logic with dependent sorts. The latter was reported on in [M2]. A detailed account of the work on FOLDS is in [M3]. For the understanding of the present paper, what is contained in [M2] suffices. In fact, section 1 of this paper gives the definitions of all that's needed in this paper; so, probably, there won't be even a need to consult [M2].

The concept of multitopic set, the main contribution of [H/M/P], was, in turn, inspired by the work of J. Baez and J. Dolan [B/D]. Multitopic sets are a variant of opetopic sets of *loc. cit.* The name "multitopic set" refers to multicategories, a concept originally due to J. Lambek [L], and given an only moderately generalized formulation in [H/M/P]. The earlier "opetopic set" is based on a concept of operad; see [B/D]. I should say that the exact relationship of the two concepts ("multitopic set" and "opetopic set") is still not clarified. The main aspect in which the theory of multitopic sets is in a more advanced state than that of opetopic sets is that, in [H/M/P], there is an explicitly defined *category* `Mlt` *of multitopes*, with the property that the category of multitopic sets is equivalent to the category of Set-valued functors on `Mlt`, a result given a detailed proof in [H/M/P]. The corresponding statement on opetopic sets and opetopes is asserted in [B/D], but the category of opetopes is not described. In this paper, the category of multitopes plays a basic role.

Multitopic sets and multitopes are described in section 2 of this paper; for a complete treatment, the paper [H/M/P] should be consulted.

The author's research is supported by NSERC Canada and FCAR Quebec

The indebtedness of the present work to the work of Baez and Dolan goes further than that of [H/M/P]. The second ingredient of the Baez/Dolan definition, after that of "opetopic set", is the concept of "universal cell". The Baez/Dolan definition of weak n-category achieves the remarkable feat of specifying the composition structure by universal properties taking place in an opetopic set. In particular, a (weak) opetopic (higher dimensional) category is an opetopic set with additional *properties* (but with no additional data): the *existence* of "sufficiently many" of the various types of universal cells. In [H/M/P], no universal cells were defined, although it was mentioned that their definition could be supplied without much difficulty by imitating [B/D]. In this paper, the "universal structure" is supplied, albeit in a somewhat unexpected manner, by using the concept of FOLDS-equivalence introduced in [M2].

In [M2], the concepts of "(FOLDS-)signature" and "(FOLDS-)equivalence" are introduced. A signature is a category with certain special properties; for a signature L, an L-structure is a $\mathtt{Set}$-valued functor on L. To each signature L, a particular relation $\simeq_L$, called L-equivalence, between two variable L-structures is defined. The slogan of the work on FOLDS is that *all meaningful properties of L-structures should be invariant under L-equivalence*. The main effort in [M3] goes into specifying a language, First Order Logic with Dependent Sorts, and showing that the *first order* properties invariant under L-equivalence are precisely the ones that can be defined in FOLDS. It is a basic realization of [M3] that the usual concepts of "equivalence" in category theory, including the higher dimensional variants such as "biequivalence", are all special cases of L-equivalence, upon suitable, and natural, choices of the signature L.

In this paper, the language of FOLDS plays no role. The concepts of "signature" and "equivalence" are fully described in section 1 of this paper.

The definition of multitopic ω-category goes, in outline, as follows. For an arbitrary multitope σ of dimension ≥ 2, for a multitopic set $\mathcal{S}$, for a pasting diagram α in $\mathcal{S}$ of shape the domain of σ and a cell a in $\mathcal{S}$ of the shape the codomain of σ such that a and α are parallel, we define what it means to say that a is a *composite* of α. First, we define a FOLDS signature $\mathrm{L}\langle\sigma\rangle$ extending $\mathtt{Mlt}$, the signature of multitopic sets. Next, we define structures $\mathcal{S}\langle a\rangle$ and $\mathrm{S}\langle\alpha\rangle$, both of signature $\mathrm{L}\langle\sigma\rangle$, the first constructed from the data $\mathcal{S}$ and a, the second from $\mathcal{S}$ and α, both structures extending $\mathcal{S}$. a *is a composite* of α if there is a FOLDS-equivalence between $\mathcal{S}\langle a\rangle$ and $\mathrm{S}\langle\alpha\rangle$ that restricts to the identity equivalence from $\mathcal{S}$ to $\mathcal{S}$; below, I'll refer to this FOLDS equivalence as an equipment for a being a composite of α.

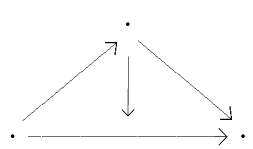
. A multitopic set $\mathcal{S}$ is *a multitopic* ω*-category* if every pasting diagram in it has a composite.

The analog of the universal arrows in the Baez/Dolan style definition is as follows. A *universal arrow* is defined to be an arrow of the form $b:\alpha\to a$ where a is a composite of α via an equipment that relates b with the identity arrow on a; in turn, the identity arrow on a is a composite of the empty pasting diagram of dimension $\mathrm{dim}(a)+1$ on a.

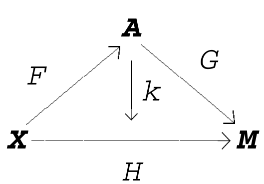
A new feature in the present treatment is that what are being defined directly are ω-dimensional weak categories; the finite dimensional ones are obtained as truncated versions of the ω-dimensional concept. It is important to emphasize that a multitopic ω-category is still just a multitopic set with additional properties, with no additional data.

The definition of "multitopic ω-category" is given in section 5. It uses sections 1, 2 and 4, but not 3.

The second main thing done in the paper is the definition of $\mathtt{Mlt}\omega\mathtt{Cat}$. This is a particular large multitopic set. Its definition is completed only by the end of the paper. The 0-cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ are the small multitopic ω-categories, defined in section 5. Its 1-cells, which we call *1-transfors* (thereby borrowing, and altering the meaning of, a term used by Sjoerd Crans [Cr]) are what stand for "morphisms", or "functors", of multitopic ω-categories. There are n-dimensional transfors for each $n\in\mathbb{N}$. For each multitope (that is, "shape" of a higher dimensional cell) π, we have the π*-transfors*, the cells of shape π in $\mathtt{Mlt}\omega\mathtt{Cat}$. For instance, when π is the 2-dimensional multitope depicted as


,

the π-transfors are the "binary" "natural transformations". In the following diagram:



,

representing a π-transfor for the given π, $\boldsymbol{X}$, $\boldsymbol{A}$, $\boldsymbol{M}$ denote multitopic ω-categories, F, G, H are "functors" between them, and $k:(F, G)\to H$ is a "bi-" (as in "bilinear") "natural transformation".

For each fixed multitope π, a π-transfor is what we call a *π-colored multitopic set,* with additional properties. There is a particular multitopic set $\mathcal{T}[\pi]$ the cells of which are the *π-colored multitopes*. As $\mathcal{T}[\pi]$ is a Set-valued functor on Mlt, we can take $\mathrm{El}(\mathcal{T}[\pi])$, the category of elements of $\mathcal{T}[\pi]$; this is the (FOLDS-)signature for the π-colored multitopic sets.

Section 3 describes $\mathcal{T}[\pi]$, and related "syntactical" concepts.

Section 6 gives the definition of "π-transfor", by supplying the properties a π-colored multitopic set is to satisfy to qualify being a π-transfor. Just as in the case of "multitopic ω-category" (= " 0-transfor"), these properties consist in the *existence* of items satisfying certain universal properties with respect to specific data, the universal properties being defined via FOLDS equivalence. However, there is one new element. For $\dim(\pi)\geq 2$, the concept of π-transfor involves a universal property which is an "omega-dimensional, FOLDS-style" generalization of the concept of Kan-extension (right lifting, in the terminology used by Ross Street). This is a "right-adjoint" type universal property, contrasted with the "left-adjoint" type involved in the concept of composite.

The main theorem announced, but not proved, in this paper is that the large multitopic set MltωCat is a multitopic ω-category. The proof of the theorem will be given in a sequel to this paper.

The material in this paper has been applied to give formulations of ω-dimensional versions of various concepts of homotopy theory; details will appear elsewhere.

I thank Victor Harnik and Marek Zawadowski for many stimulating conversations and helpful suggestions. I thank the members of the Montreal Category Seminar for their interest in the subject of this paper, which made the exposition of the material when it was still in an unfinished state very enjoyable for me.

1. FOLDS–signatures and FOLDS–equivalence

FOLDS is an acronym for "first order logic with dependent sorts"; see [M2]. FOLDS has a syntax and a semantics. Its syntax will not concern us here; but it is in fact quite completely described by the name. "Dependent sorts" are meant in a sense similar to the dependent types of [M-L] and [Ca].

FOLDS is based on (FOLDS-)*signatures* that are categories of a special kind. A *signature* is a category that satisfies the following two conditions:

(1) it is reverse well-founded: there are no infinite chains

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{k-1}} X_k \xrightarrow{f_k} X_{k+1} \xrightarrow{f_{k+1}} \ldots$$

of *non-identity* arrows f_k ($k \in \mathbb{N}$);

(2) it has finite fan-out: for every object X, the set of arrows with domain equal to X is finite.

Let L be a signature.

L is a skeletal category; also, every endomorphism $X \to X$ is the identity. Moreover, $\mathrm{Ob}(L)$ is the disjoint union $\dot{\bigcup}_{n\in\mathbb{N}} L_n$ of *levels* L_n, so that for every non-identity arrow $X \to Y$, we have $X \in L_n$, $Y \in L_m$ with $n > m$ (and if $n > 0$, and $X \in L_n$, then there is at least one arrow $X \to Y$ with $Y \in L_{n-1}$).

An L-*structure* M is a functor $M: L \to \mathbf{Set}$.

The semantics of FOLDS includes a theory of *identity* for structures: there is a separate concept of identity for each signature L, L-*equivalence*, $\simeq_L$, a relation between L-structures. The main part of the general theory of FOLDS deals with the exact matching of the syntax of FOLDS with the concept of FOLDS-equivalence. Roughly speaking, for any given FOLDS-signature L, the properties of L-structures formulated in a first order language possibly extending L that are invariant under L-equivalence are exactly the ones that can be expressed in FOLDS over the given L .

There is a *higher-order logic with dependent sorts over* L . It involves using signatures extending L ; properties formulated in it are also invariant under L-equivalence. In fact, our definition of "multitopic ω-category" is given in terms of an existential second-order property of multitopic sets, expressible in higher-order logic with dependent sorts over L , and thus invariant under L-equivalence; here, L is $\mathtt{Mlt}$, the category of multitopes, the signature for multitopic sets (see below).

In [M3], it is shown (it is also sketched in [M2]) how FOLDS equivalence specializes to the various notions of "equivalence" used in category theory: equivalence of categories, biequivalence of bicategories, and so on. The said concepts turn out to be cases of L-equivalence with suitable choices of the signature L . In this paper, L-equivalence, for various specially constructed signatures L , is used for the definition of the concepts of multitopic ω-category, and other related structures.

To define L-equivalence, I introduce some auxiliary concepts. Let K be a *kind*, that is an object of L . By $|K|$, we mean the set of all arrows in L with domain equal to K ; and we let $K|L=|K|-\{\mathrm{id}_K\}$. The set $K|L$ is the *arity* of K ; when K is used to form a dependent sort, one uses one variable x_p of the appropriate sort for each $p\in K|L$ to fill in the "places" of K . We write K_p for the codomain of the arrow p .

Consider the elements of $K|L$ to be the objects of a category, also denoted by $K|L$, in which an arrow $r:p\to q$ is any $r:K_p\to K_q$ such that $rp=q$; $K|L$ is a full subcategory of the slice-category $K\backslash L$. Let $\varphi_K:K|L\to L$ be the forgetful functor. Given any L-structure M , the limit of the composite $M\circ\varphi_K$, a set, is denoted by $M[K]$, and its elements are called the *K-contexts* in M . A K-context α is a "compatible" family $\alpha=\langle\alpha(p)\rangle_{p\in K|L}$ of elements $\alpha(p)\in M(K_p)$ indexed by the arity $K|L$ of K . "Compatibility" means that if $p, q\in K|L$ and $r\circ p=q$, then $M(r)(\alpha(p))=\alpha(q)$.

$M(K)$ is fibered over $M[K]$ by the canonical mapping $\lambda_K:M(K)\longrightarrow M[K]$; for $a\in M(K)$, $\lambda_K(a)\in M[K]$ is the *K-context of* a . The fiber $\lambda_K^{-1}(\alpha)$ of $\alpha\in M[K]$ is denoted as $M(K)(\alpha)$.

When $a \in M(K)(\alpha)$, we also say is that a is *of sort* $K(\alpha)$; to be sure, the sort $K(\alpha)$ is an *interpreted* dependent sort; the places of the kind K heading the sort are not filled by variables, but by elements $\alpha(p)$ interpreting such variables.

Let $\mu : P \to M$ be a morphism of L-structures: a natural transformation of functors $L \to \mathrm{Set}$. For any kind K, we have the induced map $\mu_{[K]} : P[K] \longrightarrow M[K]$, and also, for any $\xi \in P[K]$ and $\alpha \overset{=}{\mathrm{def}} \mu_{[K]}(\xi) \in M[K]$, the induced map $\mu_{K(\xi)} : P(K)(\xi) \longrightarrow M(K)(\alpha)$, which is a restriction of $\mu_K : P(K) \longrightarrow M(K)$. We say that μ is *fiberwise surjective* if $\mu_{K(\xi)}$ is a surjective function for all choices of $K \in L$ and $\xi \in P[K]$.

In order to formulate a property of fiberwise surjective maps, we introduce some auxiliary concepts.

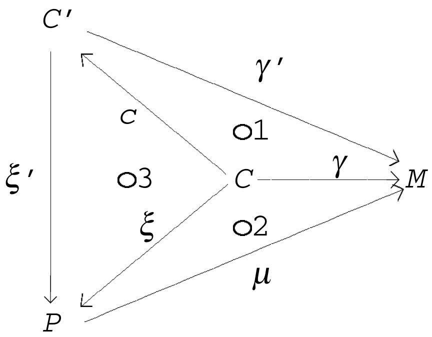
Let us define the cardinality $\# C$ of a Set-valued functor $C : L \to \mathrm{Set}$ to be $\# C = \#(\bigsqcup_{K \in \mathrm{Ob}(L)} C(K)) = \sum_{K \in \mathrm{Ob}(L)} \# C(K)$. A functor $C : L \to \mathrm{Set}$ is said to be *finite* if $\# C$ is finite; this is to say that $C(K)$ is empty for all but finitely many kinds K, and $C(K)$ is finite for all K. Finite functors $L \to \mathrm{Set}$ will now be called *formal contexts*. For a formal context C, and an L-structure M, a C-*context* in M is a natural transformation $\gamma : C \to M$. We write $M(C)$ or just MC for $\mathrm{hom}(C, M)$, the set of C-contexts in M. Sometimes we call C-contexts in M collectively for all C $\mathit{concrete}$ contexts in M.

The representable functor $\hat{K} = \mathrm{hom}(K, -)$ is a formal context, and so is the subfunctor $[K]$ of $\hat{K}$, defined by the condition that for every $U \in L$, the component $\iota_U : [K](U) \to \hat{K}(U)$ of the inclusion $\iota : [K] \to \hat{K}$ is an identity function, except for $U = K$ itself, when it is the inclusion of the empty set into the singleton $\{\mathrm{id}_K\}$. These facts are consequences of the L being a FOLDS signature. What we called a K-context above is the same as a $[K]$-context: $M[K] \cong \mathrm{hom}([K], M)$.

$[K]$ is a functor of K: $[-] : L^{\mathrm{op}} \to \mathrm{Set}^L$, since if $p : K_1 \to K_2$, then $\mathrm{hom}(K, p) : \hat{K}_2 \to \hat{K}_1$ restricts to a map $[p] : [K_2] \to [K_1]$; this, again, is something that depends on the special nature of the category L.

It is immediate now that $M[K]$ is a functor of K: $M[\ ] : L \to \mathrm{Set}$.

Suppose $\mu: P \to M$ is fiberwise surjective. Let $c: C \to C'$ be a map (natural transformation) of formal contexts *which is a monomorphism* (in Set^{L}); let γ be a C-context, γ' a C'-context in M, γ' extending γ via c as shown by commutativity ○1 in the next diagram:



Suppose further that ξ is a C-context in P that lifts γ via μ (commutativity □2). Then, the assertion is, that there exists (at least one) C'-context ξ' in P lifting ξ (see ○3).

This last property of fiberwise surjective maps is a generalization of the definition; the defining property is obtained when $c: C \to C'$ is taken to be $\iota: [K] \to \hat{K}$. L being a FOLDS-signature is needed to have the generalized property follow from the definition.

Given L-structures M and N, an *L-equivalence-span* E *between* M *and* N (in notation: $E: M \simeq_L N$) is a span

$$E = (P, \mu, \nu) = (M \xleftarrow{\mu} P \xrightarrow{\nu} N)$$

of natural transformations in which μ and ν are fiberwise surjective. We write $M \simeq_L N$, and say that M and N are *L-equivalent* if there exists E such that $E: M \simeq_L N$.

We may omit the subscript L, and write $E: M \simeq N$, and similarly in other similar notations.

Suppose $E = (P, \mu, \nu): M \simeq_L N$. For a formal context C, and C-contexts α, β and ξ in M, N and P, respectively, we write

$$(E, \xi) : (M, \alpha) \simeq_L (N, \beta) \qquad (3)$$

if ξ is a lifting of α via μ, and also a lifting of β via ν. We may omit ξ, and write

$$E: (M, \alpha) \simeq_L (N, \beta) ; \qquad (4)$$

this, of course, means that there exists ξ such that (3) holds. When (4) holds, we say that α and β are *related* by E.

For $K \in L$, and elements $a \in M(K)$, $b \in N(K)$, $x \in P(K)$, we write $(E, x) : (M, a) \simeq_L (N, b)$ to mean that $\mu_K(x) = a$, $\nu_K(x) = b$. This last notation is in fact a special case of the previous one involving contexts: take $C = \hat{K}$, and $\alpha : \hat{K} \to M$ the unique natural transformation for which $\alpha_K(\mathrm{id}_K) = a$, and similarly for β and ξ; then $(E, x) : (M, a) \simeq_L (N, b)$ iff (3).

A notation of the form

$$E: (M, \vec{a}) \simeq_L (N, \vec{b})$$

where $\vec{a}$ and $\vec{b}$ are tuples of elements in M and N, respectively, will be used as an abbreviation for a relation (4); $\vec{a}$ lists the values of a suitable context α, and similarly for $\vec{b}$ and β.

The defining property of L-equivalence-span can be expressed, in a slightly incomplete way, thus. Let $K \in L$, $\alpha \in M[K]$, $\beta \in N[K]$, and assume that

$$E: (M, \alpha) \simeq_L (N, \beta)$$

Then for any $a \in M(K)(\alpha)$, there is $b \in N(K)(\beta)$ such that

$$E: (M, \alpha, a) \simeq_L (N, \beta, b) .$$

and *vice versa*.

We note that the concept of L-equivalence naturally extends to " L-structures" with value-categories other than the category of sets. The extended concept is important in formulating when two different notions of higher dimensional category are essentially equivalent.

We discuss a general construction of new signatures out of an old one.

Suppose L is a signature, $M:L\to\mathrm{Set}$ an L-structure. Then the category $\mathrm{El}(M)$ of elements is a signature too. $\mathrm{El}(M)$ has as objects pairs (K, x) where $K\in\mathrm{Ob}(L)$ and $x\in M(K)$; an arrow $(K, x)\longrightarrow(H, y)$ is an arrow $f:K\to H$ such that $M(f)(x)=y$. There is an obvious forgetful functor $\Sigma:\mathrm{El}(M)\to L$. Verifying that $\mathrm{El}(M)$ satisfies the conditions for "signature" if L does is immediate; in fact, the "fan-out" of the object (K, x) has the same cardinality as that of K .

Let us remind the reader of a simple general fact. This is the equivalence of categories

$$\Pi\ :\ \mathrm{Hom}(L,\mathrm{Set})/M \xrightarrow{\ \simeq\ } \mathrm{Hom}(\mathrm{El}(M),\mathrm{Set})$$

$$(L\xrightarrow{C}\mathrm{Set},\gamma:C\to M)\longmapsto\left[\begin{array}{c}(K,x)\\ f\big\downarrow\\ (H,y)\end{array}\longmapsto\begin{array}{c}\{a\in C(K):\gamma_K(a)=x\}\\ \Big\downarrow\quad \begin{array}{c}a\\ \downarrow\\ C(f)(a)\end{array}\\ \{b\in C(H):\gamma_H(b)=y\}\end{array}\right]$$

holding for an arbitrary category L and functor $M:L\longrightarrow\mathrm{Set}$. On arrows $(C, \gamma)\xrightarrow{\varphi}(D, \delta)$, Π acts as follows:

$$\Pi(\varphi)_{(K,\ x)}(a)=\varphi_K(a)\ .$$

The quasi-inverse Π^{-1} of Π has, for $\Gamma:\mathrm{El}(M)\to\mathrm{Set}$, $\Pi^{-1}(\Gamma)=(C, \gamma)$ where, for $f:K\to H$ in L ,

$$\begin{array}{ccccc}
C(K) & = & \coprod_{x\in M(K)} \Gamma((K,x)) & = & \{(x,s): x\in M(K),\ s\in\Gamma((K,x))\} \\
{\scriptstyle C(f)}\downarrow & = & \downarrow & : & (x,s) \mapsto (M(f)(x), \Gamma(f)(s)) \\
C(H) & = & \coprod_{y\in M(H)} \Gamma((H,y)) & = & \{(y,t): y\in M(H),\ t\in\Gamma((H,y))\}
\end{array}$$

and

$$\gamma_K((x,s)) = x .$$

It is immediate that if $\Pi((C,\gamma)) = \Gamma$, then $\#\Gamma = \#C$. We conclude that, in case L is a signature, and we write $L[M]$ for $\mathrm{El}(M)$, then

the category of formal contexts of $L[M]$ *is canonically equivalent to the category of concrete contexts in* M,

where (of course) we mean by the latter category the corresponding subcategory of $\mathrm{Hom}(L,\mathrm{Set})/M$.

The same equivalence Π expresses also that

an $L[M]$*-structure is essentially the same thing as an* L*-structure* N *together with a map* $N\to M$.

2. Multitopes and multitopic sets

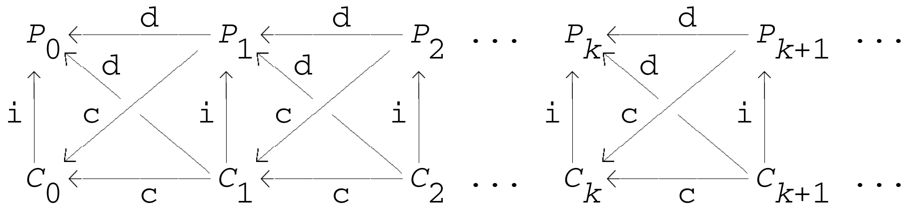
Here, I will recall the notions in the title; they are dealt with in [H/M/P] in complete detail. The present outline cannot quite eliminate the need for a detailed reference to the paper [H/M/P]. However, I hope that for a first orientation, it is of some use. In this section, references to sections are to [H/M/P].

A multitopic set $\mathcal{S}$ consists of *cell*s, each of a definite *dimension* that can be any non-negative integer; if the dimension of the cell a is n, a is said to be an n-*cell*. The set of k-cells is denoted by C_k, or, if $\mathcal{S}$ needs to be explicitly referred to, by $C_k(\mathcal{S})$.

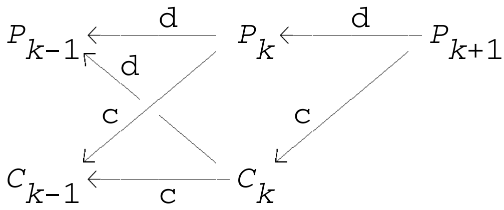
For every positive n, each n-cell a has a *domain* da and a *codomain* ca; ca is an $(n-1)$-cell; da is a *pasting diagram of dimension* $(n-1)$, briefly, an $(n-1)$-*pd*. In general, a k-pd is a "meaningful formal composite" of k-cells, just what you would expect from the term "pasting diagram". Note that the idea of a pasting diagram (composable system of higher dimensional cells) involved here is essentially simpler than the general one, because of the fact that the cells making up the pd are all "many-to-one" (their codomain is a single cell), in contrast to the general "many-to-many" concept. The set of k-pd's is denoted P_k (or $P_k(\mathcal{S})$).

For given $\beta\in P_{n-1}$, $b\in C_{n-1}$, a necessary condition for the existence of $a:\beta\to b$ (which abbreviates $da=\beta$, $ca=b$) is that β and b be *parallel*, that is, either $n-1=0$, or $n-1>0$, and $d\beta=db$, $c\beta=cb$ (*globularity*). Note that we now used the *domain* $d\beta$ and the *codomain* $c\beta$ of a k-pd β, for positive $k\in\mathbb{N}$; $d\beta$ is a $(k-1)$-pd, $c\beta$ is a $(k-1)$-cell.

The items mentioned so far are organized into the following diagram of sets:



Here, we used some abbreviations. Besides omitting subscripts to distinguish the various maps denoted by the same symbol (for instance, we have, in more detail, $d_{k+1}: P_{k+1} \to P_k$ for the arrow denoted d in the right-most position), we also have the *inclusions*, denoted i, of cells in pd's, and written d, c for the composites di, ci, respectively. The globularity condition is the commutativity of each instance of



Now, the basic fact about the concept of multitopic set is that globularity is the only limiting condition that regulates the existence of cells. Given any multitopic set $\mathcal{S}$, an $(n-1)$-pd β and an $(n-1)$-cell b in it, one can *adjoin* an n-cell $a:\beta \to b$ to $\mathcal{S}$ provided β and b are parallel; this adjunction will not effect anything else, that is, the cells in the new multitopic set will be the old ones plus the single new one a. Similarly, one can, for a fixed n, simultaneously adjoin any number of n-cells, each with prescribed domain and codomain, provided the latter are parallel.

An immediate consequence of this circumstance is the possibility of a recursive construction of multitopic sets. Assuming that we have all the cells of dimensions less than n, together with their domains and codomains, we now can introduce the n-cells by the process of simultaneous adjunction.

(Small) multitopic sets form a category $\mathtt{MltSet}$; the morphisms are the natural structure preserving maps.

It turns out (see Section 7) that $\mathtt{MltSet}$ is *presheaf-category*: there is a particular category, called the category of *multitopes*, and denoted by $\mathtt{Mlt}$, such that $\mathtt{MltSet}$ is equivalent to the category of Set-valued functors on $\mathtt{Mlt}$:

$$\mathtt{MltSet} \simeq \mathtt{Set}^{\mathtt{Mlt}} .$$

The situation is interesting from a logical point of view: it is *not* the case that the definition of "multitopic set" as " $\mathtt{Set}$-valued functor on $\mathtt{Mlt}$ " is easier than the original, somewhat complex, one (see Section 6). The reason is that the definition of the category $\mathtt{Mlt}$ is not direct. In fact, the objects of $\mathtt{Mlt}$, the *multitopes*, are defined to be the cells of the terminal multitopic set (the terminal object of $\mathtt{MltSet}$); the arrows of $\mathtt{Mlt}$ are then defined by an additional, geometrically straightforward, but algebraically rather tedious, manner (see Section 7). Moreover, the terminal multitopic set $\mathcal{T}$ (before we know that the category $\mathtt{MltSet}$ is a presheaf category!) is given by the recursive method mentioned above. The principle is simple: having introduced all $<n$-multitopes, that is, $<n$-cells in $\mathcal{T}$, we adjoin exactly one n-multitope a for each possible type (β, b), a parallel pair of an $(n-1)$-pd and an $(n-1)$-cell in $\mathcal{T}$, such that $da=\beta$, $ca=b$. Moreover, one notes the fact that, by the uniqueness of the cells involved, here the cell b is determined by β by the requirement that it is to be parallel to β; thus, in fact, each multitope is uniquely determined by its domain, a pd of multitopes of one lower dimension.

Multitopes are the *shapes* of cells in multitopic sets. Indeed, for any multitopic set $\mathcal{S}$, the unique map $\mathcal{S}\to\mathcal{T}$ in $\mathtt{MltSet}$ assigns to each cell in $\mathcal{S}$ its *shape*. For ρ a multitope, and the multitopic set $\mathcal{S}$ construed as a functor $\mathcal{S}:\mathtt{Mlt}\to\mathtt{Set}$, $\mathcal{S}(\rho)$ is the set of cells whose shape is ρ.

To eliminate ambiguity, we adopt the concept of "multitopic set" meaning "$\mathtt{Set}$-valued functor on $\mathtt{Mlt}$" as the official definition. The *total set* $|\mathcal{S}|$ of the multitopic set $\mathcal{S}$ is the sum

$$|\mathcal{S}| = \coprod_{\rho\in \mathrm{Ob}(\mathtt{Mlt})} \mathcal{S}(\rho) .$$

However, often we write $a\in\mathcal{S}$ for $a\in|\mathcal{S}|$.

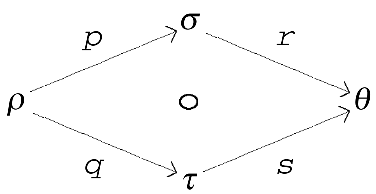
Throughout the paper, the Greek letters $\kappa, \pi, \rho, \sigma, \tau, \theta$ will denote multitopes.

For a multitope ρ, we write $|\rho|$ for the set of all arrows in $\mathtt{Mlt}$ whose domain is ρ; the elements of $|\rho|$ are called the *faces* of ρ. For $p\in|\rho|$, we write K_p (as in the previous section) for the codomain of p.

Let $\mathcal{S}$ be a multitopic set, ρ a multitope, $a\in\mathcal{S}(\rho)$. For $p\in|\rho|$, we have that $a\hat{\ }p \underset{\mathrm{def}}{=} \mathcal{S}(p)(a) \in \mathcal{S}(\mathrm{K}_p)$. The pair $(p, a\hat{\ }p)$, and sometimes, more sloppily, the cell $a\hat{\ }p$ by itself also, is called the *p-face* of a. Intuitively speaking, the faces of a are the same as the *occurrences* of cells "involved" in a: these include a itself, the one-lower-dimensional cell-occurrence that is the codomain of a, and the ones that are in the domain of a, and (inductively) all still lower dimensional cells involved in the ones previously mentioned.

Talking about faces of a multitope and talking about faces of a cell in a multitopic set are compatible; the former is a special case of the latter, done in the terminal multitopic set $\mathcal{T}$.

The composition structure of $\mathtt{Mlt}$ represents the geometry of the cells. For instance, if we have the commutative diagram



in $\mathtt{Mlt}$, a cell a in $\mathcal{S}$ of shape ρ, and $b=a\hat{\ }p$, $c=a\hat{\ }q$, then $d=b\hat{\ }r=c\hat{\ }s$: the cells b and c are located in a at the places p and q, respectively, in such a way that they share an occurrence of d, namely the one at the place $rp=sq$; in other words, they overlap in such a way that d is in the overlap.

The category $\mathtt{Mlt}$ is generated by arrows each of which is of the form $\rho\to\tau$ where $\mathrm{dim}(\rho)=\mathrm{dim}(\tau)+1$. Understanding by the *length* of an arrow $\rho\to\tau$ the difference $\mathrm{dim}(\rho)-\mathrm{dim}(\tau)$, we have that, for any given multitope ρ of positive dimension, the length-1 elements of $|\rho|$ are in two disjoint classes $|\rho|_{\mathrm{c}}$ and $|\rho|_{\mathrm{d}}$. One of these is a singleton, $|\rho|_{\mathrm{c}}=\{\mathrm{c}_{\rho}:\rho\to\mathrm{c}\rho\}$; the other is $|\rho|_{\mathrm{d}}=\{\mathrm{d}_{\rho,\,p}:p\in|\langle\rho\rangle|\}$, where we used the notation of Section 7, items Mlt[2] and Mlt[1] [in that order; here we use a different notation for the codomain of c_{ρ}. Here, I am not going to explain the notation $p\in|\langle\rho\rangle|$. Also, I use the letter p for denoting an element of $|\rho|$, and not for what it is used in [H/M/P] .) Elements of $|\rho|_{\mathrm{c}}$ are called *c-arrows*, those of $|\rho|_{\mathrm{d}}$ *d-arrows*. The cardinality of $|\rho|_{d}$ can be any natural number, including zero. Let a be a cell of shape ρ, $\mathrm{dim}(\rho)=n\geq 1$. For $p\in|\rho|_{c}$, the p-face of a is $\mathrm{c}a$, the codomain of a; and for $p\in|\rho|_{\mathrm{d}}$, the p-face of a is an $(n-1)$-cell in the domain $\mathrm{d}a$ of a.

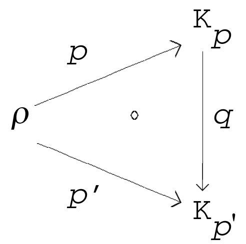
The geometry of a single multitope ρ is encapsulated in the structure which is the slice-category $\mathtt{Mlt}/\rho$ (with objects the elements of $|\rho|$), endowed with a *sign-structure* which is the label "c" or "d" on each of the length-1 arrows of $\mathtt{Mlt}/\rho$ (which are also arrows of $\mathtt{Mlt}$). The fact that a multitope is "no more than its geometric representation" is expressed precisely in the statement that if $\mathtt{Mlt}/\rho$ and $\mathtt{Mlt}/\tau$ are isomorphic by an isomorphism (of categories) respecting the sign-structure, then $\rho=\tau$. We also have that the only sign-respecting automorphism of $\mathtt{Mlt}/\rho$ is the identity.

$\mathtt{Mlt}$ is a FOLDS signature; the multitopic sets are the $\mathtt{Mlt}$-structures. Multitopic ω-categories (to be defined below) are multitopic sets with additional properties invariant under $\mathtt{Mlt}$-equivalence. For a finite n , the concept of multitopic n-category (see below) is a first-order concept, which can be formulated in the language of FOLDS over a truncation of $\mathtt{Mlt}$.

In what follows, we reformulate the ingredients of multitopic sets we started with in this section in terms of the definition of "multitopic set" as " $\mathtt{Set}$-valued functor on $\mathtt{Mlt}$ ". The results of this exercise will help us to define, in section 6, the generalizations of the relevant concepts to "colored multitopic sets".

Let us write $\rho\|$ for the set of all arrows p with domain equal to ρ such that $p\neq\mathrm{id}_\rho$ and $p\neq\mathrm{c}_\rho$.

Let $k\geq 1$. A $(k-1)$-pd β in $\mathcal{S}$ is given by a k-dimensional multitope ρ , and a function β such that $\mathrm{dom}(\beta)=\rho\|$; $\beta(p)\in\mathcal{S}(\mathrm{K}_p)$ whenever $p\in\rho\|\mathrm{Mlt}$; and if



then $\mathcal{S}(q)(\beta(p))=\beta(p')$. The pd β is then of *shape* ρ , or, β is a *ρ-pd.*

In particular, it is clear that for any $a\in\mathcal{S}(\tau)$, the family $\langle b_p\rangle_{p\in\rho\|}$, where $b_p=\mathcal{S}(p)(a)$, defines a pd in the sense just described; this pd is what we denoted by $\mathrm{d}a$. We write $\mathcal{S}[\rho^{-}]$ for the set of pd's of shape ρ .

A useful way of looking at ρ-pd's is that they are $[\rho^-]$-contexts in $\mathcal{S}$, where $[\rho^-]$ is the subfunctor of $\hat{\rho}=\mathrm{hom}(\rho,-)$ in which $[\rho^-](\rho)=\emptyset$, $[\rho^-](c\rho)=[\rho](c\rho)-\{c_\rho\}$, and $[\rho^-](\kappa)=[\rho](\kappa)$ for all $\kappa\in\mathrm{Ob}(\mathrm{Mlt})-\{\rho,c\rho\}$. The domain $\mathrm{d}a$ of a cell $a\in\mathcal{S}(\rho)$ is the composite $[\rho^-]\xrightarrow{\mathrm{incl}}\hat{\rho}\xrightarrow{\dot{a}}\mathcal{S}$, where $\dot{a}$ is the arrow corresponding to a by Yoneda: $\dot{a}_\rho(\mathrm{id}_\rho)=a$.

Note that the formal context $[\rho^-]$, and hence the set $\mathcal{S}[\rho^-]$ too, are (contravariant) functors of $\rho\in\mathrm{Mlt}$.

For a ρ-pd $\beta:[\rho^-]\to\mathcal{S}$, $\mathrm{d}\beta$, a pd of dimension one less than that of β, is obtained as the composite

$$\mathrm{d}\beta : [(c\rho)^-]\xrightarrow{\ \iota\ }[\rho^-]\xrightarrow{\beta}\mathcal{S},$$

where ι is an arrow determined by the commutative diagram

$$\begin{array}{ccc} [\rho^-] & \xrightarrow{\mathrm{incl}} & \hat{\rho} \\ \iota\uparrow & \circ & \uparrow(c_\rho)^* \\ [(c\rho)^-] & \xrightarrow[\mathrm{incl}]{} & \widehat{(c\rho)} \end{array}$$

Let θ be an arbitrary multitope; $\dim(\theta)=k-1$. There is a particular multitope θ^* of dimension k, uniquely determined by θ by the facts that $\theta=c\theta^*$, there are exactly two distinct non-identity arrows with domain θ^* and codomain of dimension $k-1$, and both of them are of the form $\theta^*\to\theta$; one of them, of course, is c_{θ^*}; we denote the other arrow $\theta^*\to\theta$ by $\mathrm{d}_{\theta^*}:\theta^*\to\theta$. θ^* is the shape of a k-cell both of whose domain and codomain are of shape θ. A special feature of the pair

$$\theta^* \overset{\mathrm{d}_{\theta^*}}{\underset{\mathrm{c}_{\theta^*}}{\rightrightarrows}} \theta$$

of arrows is that they are equalized by any non-identity arrow out of their codomain.

Given any cell $a\in\mathcal{S}(\rho)$, with $\mathrm{dim}(\rho)=k$, we have the ρ^*-pd $\langle a\rangle\in\mathcal{S}[\rho^*]$ for which $\langle a\rangle_\rho(\mathrm{c}_{\rho^*})=a$. $\langle a\rangle$ is the same as $\mathrm{i}(a)$ for the "inclusion" $\mathrm{i}:\mathrm{C}_k(\mathcal{S})\to\mathrm{P}_k(\mathcal{S})$ at the beginning of this section.

The codomain $\mathrm{c}\beta$ of a ρ-pd β is $\beta_{\mathrm{cc}\rho}(\mathrm{c}_{\mathrm{c}\rho}\circ\mathrm{c}_\rho)$. When ρ is replaced with ρ^*, and $\beta\in\mathcal{S}[(\rho^*)^-]$ is taken to be $\beta=\langle a\rangle$ for $a\in\mathcal{S}(\rho)$, the formula for $\mathrm{c}\beta$ reduces to $a\text{\textasciicircum}\mathrm{c}_\rho$ as it should.

I will now discuss a particular class of multitopes, the shapes of the "empty" pd's.

There is a multitope θ^+ of dimension $k+1$ for which $\mathrm{c}\theta^+=\theta^*$, and the only non-identity arrow out of θ^+ is $\mathrm{c}_{\theta^+}:\theta^+\longrightarrow\mathrm{c}\theta^+=\theta^*$; again, θ^+ is uniquely determined by said properties from θ. In any multitopic set, θ^+ is the shape of an *empty* pd of dimension k, one which is an identity arrow in the multicategory $\boldsymbol{C}_k$ on an object, a $(k-1)$-cell, of shape θ (see [H/M/P]. θ^+ is the shape of a $(k+1)$-cell whose domain is a pd of shape θ^+, and whose codomain is of shape θ^*. Writing ε for θ^+, we have that

$$\varepsilon \xrightarrow{\mathrm{c}_\varepsilon} \mathrm{c}\varepsilon \overset{\mathrm{d}_{\mathrm{c}\varepsilon}}{\underset{\mathrm{c}_{\mathrm{c}\varepsilon}}{\rightrightarrows}} \mathrm{cc}\varepsilon \qquad (1)$$

is the same as

$$\theta^+ \xrightarrow{\mathrm{c}_{\theta^+}} \theta^* \underset{\mathrm{c}_{\theta^*}}{\overset{\mathrm{d}_{\theta^*}}{\rightrightarrows}} \theta \qquad (2)$$

Multitopes ε of the form θ^+ are called *empty-domain* multitopes.

Given any cell $a \in \mathcal{S}(\theta)$ in a multitopic set $\mathcal{S}$, there is a unique θ^+-shaped pd α for which $\mathrm{c}\mathrm{c}\alpha = a$. α is the *empty* k*-pd on* the $(k-1)$-cell a . It is called "empty" since there is no k-cell in it: $\theta^+\|$ has no element p with $\dim(\mathrm{K}_p) = k$.

Among diagrams of the form

$$\rho \xrightarrow{f} \tau \underset{h}{\overset{g}{\rightrightarrows}} \kappa\,, \qquad (3)$$

the ones of the type (2) are anomalous because $\mathrm{d}_{\mathrm{c}\varepsilon} \circ \mathrm{c}_\varepsilon = \mathrm{c}_{\mathrm{c}\varepsilon} \circ \mathrm{c}_\varepsilon$, but $\mathrm{d}_{\mathrm{c}\varepsilon} \neq \mathrm{c}_{\mathrm{c}\varepsilon}$. In fact, we have that if (3) is such that $gf = hf$ and $g \neq h$, then there is an empty-domain multitope ε and $\rho \xrightarrow{e} \varepsilon$ such that (3) is obtained from (1) by precomposing with e : $f = \mathrm{c}_\varepsilon \circ e$, $\{g, h\} = \{\mathrm{d}_{\mathrm{c}\varepsilon}, \mathrm{c}_{\mathrm{c}\varepsilon}\}$. In particular, arrows in Mlt are "usually" epimorphisms; in fact, an arrow is an epimorphism if and only if there is no empty-domain face in its codomain.

Multitopes that have no faces with empty-domain shapes are called *positive*. The codomain of an arrow out of a positive multitope is also positive. Several things in this paper allow simplified formulations when one restricts oneself to positive multitopes (that is, when one replaces Mlt be its full subcategory on the positive multitopes); unfortunately, this cannot be done without harm to the concepts we are aiming at.

3. Colored multitopes.

The question how to define the concept of weak n-category, for finite n, and in the second place, also, for $n=\omega$, has been around for a while; and several answers have already been proposed (see [B]. [B/D],...). However, one clearly wants more than that concept: one wants the definition of the total structure weak higher dimensional categories are elements of. For instance, one wants morphisms, or functors (some may want to say: "homomorphisms", since for the case of $n=2$ and $n=3$, the existing notions that we want to emulate and generalize are called "homomorphism" and "trihomomorphism", respectively), of n-categories ; for $n>0$, one wants an (appropriately higher dimensional) notion of (pseudo-)natural transformation of functors of weak n-categories; for still greater n, modifications of the latter, and so on.

This structure will be provided here in a uniform manner.

In the first place, the concept of multitopic ω-category will be defined and proposed as a concept of weak ω-category. Multitopic n-categories, for each finite n, will be the natural ("truncated") special cases of the ω-concept.

Secondly, we will define a (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ whose 0-cells will be the (small) multitopic ω-categories. The 1-cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ will be our construal of the functors (morphisms) of multitopic ω-categories. The 2-cells will be the "natural transformations"; but note that now we will have, as separate kinds of entities, binary, ternary, and, in general, k-ary (for all $k\in\mathbb{N}$, even for $k=0$!) "natural transformations". The 3-cells in $\mathtt{Mlt}\omega\mathtt{Cat}$ will stand for modifications; now, they come with somewhat complicated arities, which are exactly the 2-multitopes, shapes of "many-to-one" pasting diagrams of dimension 2. And so on, for all finite n. Borrowing a term from Sjoerd Crans, I will call an n-cell in $\mathtt{Mlt}\omega\mathtt{Cat}$ an *n-transfor*. (Crans uses this term in the context of *teisi*, "semi-strict" weak higher dimensional categories. Unfortunately however, there is mismatch in the assignment of the "order" n; what I call n-transfor corresponds to Crans's $(n-1)$-transfor.) Thus, the, usually multiary, n-cells in the multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ will be the n-transfors; the 0-transfors are the same as the (multitopic) ω-categories.

The main theorem announced in this paper is that $\mathtt{Mlt}\omega\mathtt{Cat}$ is a multitopic ω-category.

The uniformity of the theory lies in the fact that each n-transfor will itself be a multitopic set, endowed, however, with certain additional structure that we call *coloring*. Given a multitope π, we will define a particular category $\mathtt{Mlt}[\pi]$ of *π-colored multitopes*. The category $\mathtt{MltSet}[\pi]$ of *π-colored multitopic sets* will then be defined as the functor category $\mathtt{Set}^{\mathtt{Mlt}[\pi]}$. When $\pi=\rho^{(0)}$, the unique zero-dimensional multitope, then $\mathtt{Mlt}[\pi]$ and $\mathtt{MltSet}[\pi]$ are, essentially, the same as $\mathtt{Mlt}$ and $\mathtt{MltSet}$, respectively. An n-transfor of shape π, that is, an element of $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$, will be a π-colored multitopic set with certain properties (but with no additional data).

Recall that the objects of $\mathtt{Mlt}$ were the cells of $\mathcal{T}$, the terminal multitopic set. Generalizing, for each multitope π, we will define a particular multitopic set $\mathcal{T}[\pi]$; the objects of $\mathtt{Mlt}[\pi]$ will then be the cells of $\mathcal{T}[\pi]$.

Together with the multitopic sets $\mathcal{T}[\pi]$, we will also define, for each arrow $\pi\to\kappa$ in $\mathtt{Mlt}$, morphisms $\mathcal{T}[p]:\mathcal{T}[\kappa]\longrightarrow\mathcal{T}[\pi]$; in fact, we will define a functor

$$\mathcal{T}[-] : \mathtt{Mlt}^{\mathrm{op}}\longrightarrow\mathtt{MltSet} .$$

(As we will see, $\mathcal{T}[p]$ "recolors" every κ-colored multitope as a π-colored one, by replacing the color $r\in|\kappa|$ by $r\circ p\in|\pi|$.)

In addition, for each $\pi\in\mathtt{Mlt}$, we will define a mapping

$$\gamma_{\pi}: |\mathcal{T}[\pi]| \longrightarrow |\pi| ,$$

on the total set

$$|\mathcal{T}[\pi]| \overset{\mathrm{def}}{=} \coprod_{\tau\in\mathtt{Mlt}} \mathcal{T}[\pi](\tau)$$

of the multitopic set $\mathcal{T}[\pi]$ to the set of arrows out of π, called *coloring*. The coloring will be compatible with the functor $\mathcal{T}[-]$ in the natural sense that, for each $p:\pi\to\kappa$, the following diagram commutes:

$$\begin{array}{ccc} |\mathcal{T}[\kappa]| & \xrightarrow{|\mathcal{T}[p]|} & |\mathcal{T}[\pi]| \\ \downarrow \gamma_\kappa & & \downarrow \gamma_\pi \\ |\kappa| & \xrightarrow[p^*]{} & |\pi| \end{array} :$$

here, the notation $|\mathcal{T}[p]|$ is self-explanatory; p^* is the mapping $r \mapsto r \circ p$.

The definition is by recursion. Fix $n \in \mathbb{N}$. Assume that the functor $\mathcal{T}[-]$ has been defined on arguments that are objects of dimensions less than n , and arrows between such objects; moreover, assume that said restriction of $\mathcal{T}[-]$ is a functor. Assume that the coloring γ_κ has been defined for all $\kappa \in \mathtt{Mlt}$ of dimension $<n$, with the relevant instances of the compatibility condition satisfied. Fix $\pi \in \mathtt{Mlt}$ such that $\mathtt{dim}(\pi)=n$; we will define $\mathcal{T}[\pi]$, γ_π , and $\mathcal{T}[p]$ for $p \in \pi|$, where $\pi|$ is the set of non-identity arrows in $\mathtt{Mlt}$ out of π .

Let $\pi|\mathtt{Mlt}$ denote the full subcategory of the comma-category $\pi \backslash \mathtt{Mlt}$ on objects the elements of $\pi| = |\pi| - \{\mathrm{id}_\pi\}$. Let $\mathtt{Mlt} \restriction <\pi$ denote the full subcategory of $\mathtt{Mlt}$ on the objects K_p $(=\mathtt{cod}(p))$ for $p \in \pi|$. We have the forgetful functor $\varphi : \pi|\mathtt{Mlt} \longrightarrow \mathtt{Mlt} \restriction <\pi$. Consider the composite functor

$$\pi|\mathtt{Mlt} \xrightarrow{\varphi} \mathtt{Mlt} \restriction <\pi \xrightarrow{\mathcal{T}[-] \restriction <\pi} \mathtt{MltSet} \ ,$$

and call it Φ ; note that by the recursion hypothesis, $\mathcal{T}[-] \restriction <\pi$ is well-defined. Let $\mathcal{P}$ be the colimit $\mathtt{colim}(\Phi)$ in $\mathtt{MltSet}$ (when, later on, we want to refer $\mathcal{P}$ for various instances of π , we may write $\mathcal{P}[\pi]$ for $\mathcal{P}$). Since $\mathtt{MltSet}$ is (equivalent to) $\mathtt{Set}^{\mathtt{Mlt}}$, $\mathtt{colim}(\Phi)$ exists, and is computed "pointwise" as in $\mathtt{Set}$. In particular, for each multitope τ ,

$$\mathcal{P}(\tau) = \underset{(p:\pi \to \mathrm{K}_p) \in \pi|\mathtt{Mlt}}{\mathtt{colim}} \mathcal{T}[\mathrm{K}_p](\tau) \ .$$

We will define $\mathcal{T}[\pi]$ as a certain extension of the multitopic set $\mathcal{P}$; we will adjoin certain specific cells to $\mathcal{P}$, to obtain $\mathcal{T}[\pi]$. Recall that we also have to define the coloring $\gamma_\pi(a)$ for cells a in $\mathcal{T}[\pi]$. The coloring $\gamma_\pi(a)$ of cells a in $\mathcal{P}$ (a part of $\mathcal{T}[\pi]$) is given as follows. For any $p, q \in \pi|$, and $(r{:}p \to q) \in \mathrm{Arr}(\pi|\mathrm{Mlt})$, the commutativity of the diagram

$$
\begin{array}{ccc}
|\mathcal{T}[\mathrm{K}_q]| & \xrightarrow{\ |\mathcal{T}[r]|\ } & |\mathcal{T}[\mathrm{K}_p]| \\
{\scriptstyle |\gamma_{\mathrm{K}_q}|}\downarrow & \circ & \downarrow{\scriptstyle |\gamma_{\mathrm{K}_q}|} \\
|\mathrm{K}_q| & \xrightarrow{\ \ r^{*}\ \ } & |\mathrm{K}_p| \\
{\scriptstyle q^{*}}\searrow & \circ & \swarrow{\scriptstyle p^{*}} \\
 & |\pi| &
\end{array}
$$

ensures that we have a well-defined map γ from the colimit $|\mathcal{P}| = \mathrm{colim}(|\Phi|)$ to $|\pi|$ such that, for each colimit coprojection $\iota_p{:}\mathcal{T}[\mathrm{K}_p] \to \mathcal{P}$, we have $\gamma \circ |\iota_p| = p^{*} \circ \gamma_{\mathrm{K}_p}$. The coloring γ_π restricted to $\mathcal{P}$ is defined to be this γ.

All newly adjoined cells in $\mathcal{T}[\pi]$ are to have the γ_π-color id_π. Thus, the part $\mathcal{P}$ is distinguished inside $\mathcal{T}[\pi]$ as those cells whose color is different from id_π.

The extension of $\mathcal{P}$ to $\mathcal{T}[\pi]$ is done as follows.

No cells of dimension less than n are adjoined to $\mathcal{P}$.

Next, we describe the n-cells added to $\mathcal{P}$.

An $(n-1)$-pd $\beta = \langle \beta(r) \rangle_{r \in \tau\|}$ in $\mathcal{P}$ is called c_π-*free* if for all $r \in \tau\|$, we have $\gamma_\pi(\beta(r)) \neq \mathrm{c}_\pi$. The new n-cells adjoined to $\mathcal{P}$ are in a bijective correspondence with the pairs (β, b) where β is a c_π-free $(n-1)$-pd in $\mathcal{P}$, b is an $(n-1)$-cell in $\mathcal{P}$ such that $\gamma_\pi(b) = \mathrm{c}_\pi$, and β and b are parallel: $\mathrm{d}\beta = \mathrm{d}b$, $\mathrm{c}\beta = \mathrm{c}b$; for each such pair (β, b), we adjoin to $\mathcal{P}$ exactly one new cell a with $\mathrm{d}a = \beta$, $\mathrm{c}a = b$; we declare that $\gamma_\pi(a) = \mathrm{id}_\pi$.

For dimensions $m > n$, we proceed recursively. For any parallel pair (β, b) of an $(m-1)$-pd β and an $(m-1)$-cell b such that $\gamma_\pi(b) = \mathrm{id}_\pi$, we adjoin a single new m-cell a for which $\mathrm{d}a = \beta$, $\mathrm{c}a = b$, and make $\gamma_\pi(a)$ equal to id_π.

This completes the definition of the multitopic set $\mathcal{T}[\pi]$. We have the inclusion $\iota:\mathcal{P}\to\mathcal{T}[\pi]$. For $(p:\pi\to\kappa)\in\pi|$, we define $\mathcal{T}[p]:\mathcal{T}[\kappa]\to\mathcal{T}[\pi]$ as the composite $\mathcal{T}[p]=\iota\circ\iota_p$. The functorial nature of $\mathcal{T}[-]$ on arrows between objects in the set $\{K_p:\ p\in|\pi|\}$, and the compatibility of the various colorings indexed by elements of $|\pi|$ are assured by the definition.

For the fixed n , performing the above construction simultaneously for all multitopes π of dimension n , gives the definition of all necessary items involving indices $\leq n$, since all non-identity arrows out of a multitope are to multitopes of lower dimensions. The functor $\mathcal{T}:\mathtt{Mlt}^{\mathtt{op}}\to\mathtt{Set}^{\mathtt{Mlt}}$ has been defined.

Let $\pi,\tau\in\mathtt{Mlt}$, $a\in\mathcal{T}[\pi](\tau)$. The *total* π*-coloring* of a includes the colors of all the faces of a ; it is defined to be

$$\bar{\gamma}_{\pi}(a)\ \overset{=}{\mathrm{def}}\ \langle\gamma_{\pi}(\mathcal{T}[\pi](s)(a))\rangle_{s\in|\tau|}\ \in\ |\pi|^{|\tau|}\ .$$

In other words, for $s\in|\tau|$, and for the s-face $b=\mathcal{T}[\pi](s)(a)$ of a , the value of the function $\bar{\gamma}_{\pi}(a)$ at s equals the π-color of b :

$$\bar{\gamma}_{\pi}(a)(s)\ =\ \gamma_{\pi}(b)\ .$$

Using, for any $p:\pi\to\kappa$, the naturality of $\mathcal{T}[p]:\mathcal{T}[\kappa]\longrightarrow\mathcal{T}[\pi]$, we find that we have the commutativity

$$\begin{array}{ccc}
\mathcal{T}[\kappa](\tau) & \xrightarrow{\ \mathcal{T}[p]_{\tau}\ } & \mathcal{T}[\pi](\tau) \\
\Big\downarrow{\scriptstyle\bar{\gamma}_{\kappa}} & \circ & \Big\downarrow{\scriptstyle\bar{\gamma}_{\pi}} \\
|\kappa|^{|\tau|} & \xrightarrow[\ p^{*}\ ]{} & |\pi|^{|\tau|}
\end{array}\ .$$

Next, we list some facts about colored multitopes.

(1) Each map $\bar{\gamma}_{\pi}:\mathcal{T}[\pi](\tau)\longrightarrow|\pi|^{|\tau|}$ is 1-to-1 .

This fact says that a colored multitope is determined by its underlying (ordinary) multitope, that is, its shape, and its (total) coloring. Therefore, a π-colored multitope is given by an (ordinary) multitope and a *correct* coloring of its faces by the faces of $|\pi|$. What constitutes a correct coloring is, of course, defined unambiguously by the above-given definition of $\mathtt{Mlt}[-]$; however, this definition is not very direct (it uses a recursion).

Let us note here that for each $\pi\in\mathtt{Mlt}$, we have the canonical π-colored multitope $P=P[\pi]$ for which $P\in\mathtt{Mlt}[\pi](\pi)$, and for which $\bar{\gamma}_{\pi}(\pi)=\mathrm{id}_{|\pi|}$.

(2) For all $p:\pi\to\kappa$ in $\mathtt{Mlt}$, the induced map $\mathcal{T}[p]:\mathcal{T}[\kappa]\to\mathcal{T}[\pi]$ is a monomorphism. Moreover, if $\pi,\rho\in\mathtt{Mlt}$, $R\in\mathcal{T}[\pi](\rho)$, then for κ the codomain of the arrow $p=\gamma_{\pi}(R):\pi\to\kappa$, there is a (unique) cell $\dot{R}\in\mathcal{T}[\kappa](\rho)$, called the *root* of R , for which $\dot{R}\overset{\mathcal{T}[p]}{\longmapsto}R$. Furthermore, if $q:\pi\to\tau$, and $S\in\mathcal{T}[\tau](\rho)$ is such that $S\overset{\mathcal{T}[p]}{\longmapsto}R$, then, necessarily, the root of S is the same as that of R ; in other words, there is a commmutative diagram

$$\begin{array}{ccc} & \pi & \\ {}^{q}\swarrow & \circ & \searrow^{p} \\ \tau & \xrightarrow[r]{} & \kappa \end{array}$$

such that $\dot{R}\overset{\mathcal{T}[r]}{\longmapsto}S$. Let us call a cell T in $\mathcal{T}[\theta]$ *primitive* if $\gamma_{\theta}(T)=\mathrm{id}_{\theta}$. The roots of cells are always primitive; the primitive cells are the same as the ones that are their own roots. Note that if $T\in\mathcal{T}[\theta]$ is primitive, then $\dim(T)\geq\dim(\theta)$.

(3) With reference to the multitopic set $\mathcal{P}$ in the above definition of $\mathcal{T}[\pi]$, for each $m\geq n$, the canonical morphism

$$\underset{(p:\pi\to K_p)\in\pi|\mathrm{Mlt}}{\mathrm{colim}} P_m(\mathcal{T}[K_p]) \longrightarrow P_m(\mathcal{P})$$

is an isomorphism.

This says that, for $m\geq n$, every m-pd in $\mathcal{P}$ is already a pd in a single component $\mathcal{T}[K_p]$ for a proper (non-top) color $p\in\pi|\mathrm{Mlt}$.

Essentially as a consequence of (3), we have (4) and (5):

(4) With $n=\dim(\pi)$, for any pd α and any cell a in the multitopic set $\mathcal{T}[\pi]$ such that α and a are parallel, and $\dim(\alpha)=\dim(a)\geq n$, we have exactly one cell b in $\mathcal{T}[\pi]$ such that $\mathrm{d}b=\alpha$, $\mathrm{c}b=a$.

(5) With $n=\dim(\pi)$, for any pd α in the multitopic set $\mathcal{T}[\pi]$ such that $\dim(\alpha)\geq n$, we have exactly one cell b in $\mathcal{T}[\pi]$ such that $\mathrm{d}b=\alpha$, and in particular, exactly one cell parallel to α.

Let us give some examples.

It is clear that when $\pi=\rho^{(0)}$, $\mathcal{T}[\pi]$ is just $\mathcal{T}$: there is just one color.

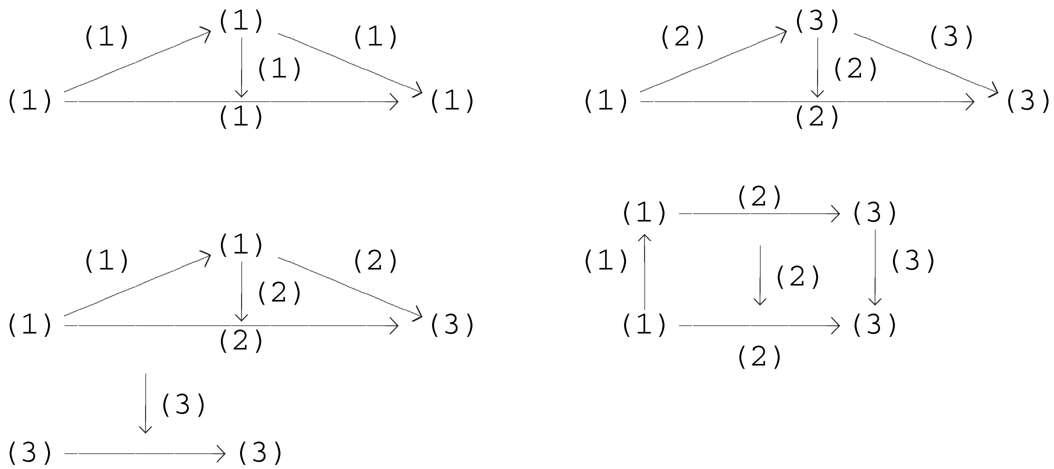
The next example is when $\pi=\rho^{(1)}$, the unique 1-dimensional multitope. We have that $\pi| = \{\mathrm{d},\mathrm{c}\}$, both with codomain $\rho^{(0)}$. The multitopic set $\mathcal{P}=\mathcal{P}[\pi]$ is the disjoint union of two copies of $\mathcal{T}$, one colored d, the other colored c. In $\mathcal{T}[\pi]$, there are exactly two 0-cells and three 1-cells; changing the notation somewhat, and representing π as

$$1 \xrightarrow{\;2\;} 3$$

(which, of course, means that `1=d` , `3=c` and $2=\mathrm{id}_\pi$), the `1`-cells in $\mathcal{T}[\rho^{(1)}]$ are

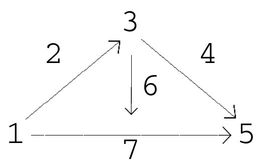
$$(1)\xrightarrow{(1)}(1)\ ,$$
$$(3)\xrightarrow{(3)}(3)\ ,$$
$$(1)\xrightarrow{(2)}(3)\ ;$$

the parenthetical numbers are the colors of the cells in question. The first two come from $\mathcal{P}$, the last as the n-cell ($n=1$) adjoined to $\mathcal{P}$. Some examples of `2`-cells in $\mathcal{T}[\rho^{(1)}]$:



The last `2`-cell has domain the empty `1`-pd on the `0`-cell `(3)` . Note that the `1`-cell $(1)\xrightarrow{(2)}(3)$ cannot be the target of an empty-domain `2`-cell, since $(1)\neq(3)$. The first and the last of these `2`-cells are "from" $\mathcal{P}[\rho^{(1)}]$; the others are introduced by the third stage of the definition of $\mathcal{T}[\rho^{(1)}]$.

Next, consider the example of the following `2`-cell for π :

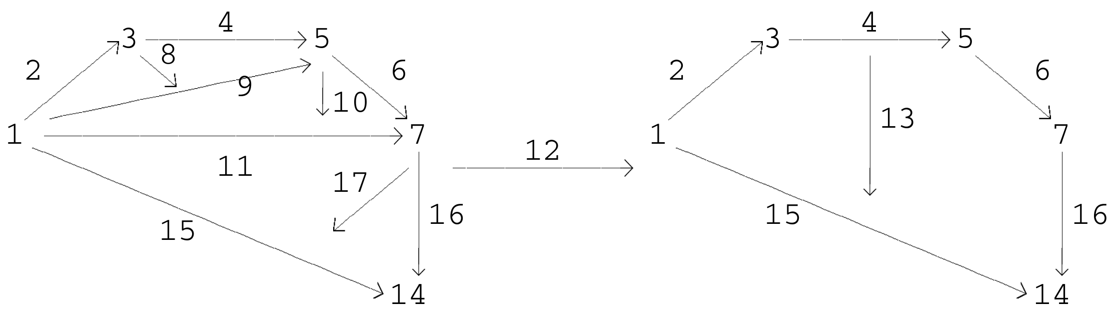


In $\mathcal{T}[\pi]$, we have three `0`-cells: `(1)` , `(3)` , `(5)` ; six `1`-cells:

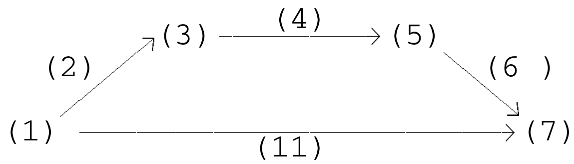
$$(1) \xrightarrow{(1)} (1)\,, \qquad (1) \xrightarrow{(2)} (3)\,,$$
$$(3) \xrightarrow{(3)} (3)\,, \qquad (3) \xrightarrow{(4)} (5)\,,$$
$$(5) \xrightarrow{(5)} (5)\,, \qquad (1) \xrightarrow{(7)} (5)\,.$$

As for 2-cells, they come in seven different colors. The ones of color (6) have as domain a 1-pd that is a sequence of 1-cells, starting in a succession (possibly of zero length) of (1)'s, followed by a single instance of $(1) \xrightarrow{(2)} (3)$, followed by a succession of (3)'s, followed by a single $(3) \xrightarrow{(4)} (5)$, ending in a succession of (5)'s. The 2-cells of color (6) all have as codomain the 1-cell $(1) \xrightarrow{(7)} (5)$.

For π the 3-dimensional multitope



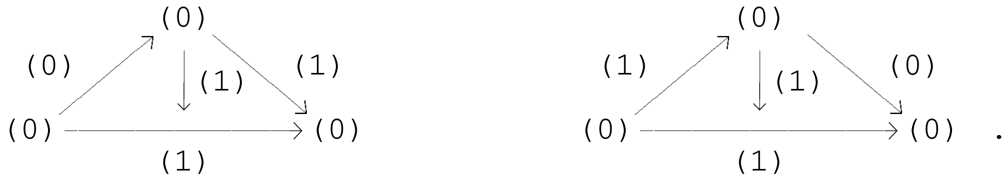
in $\mathrm{Mlt}[\pi]$, the parallel pair (α, a) of the 1-pd α and 1-cell a shown in



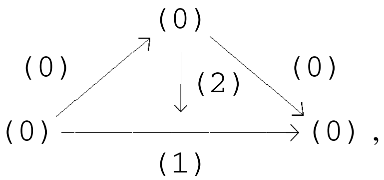
does not have a "filling" $b:\alpha \to a$.

Let π be $0 \xrightarrow[1]{\;\downarrow 2\;} 0$, the simplest empty-domain multitope, and let κ be $0 \xrightarrow[1]{} 0'$, the unique 1-dimensional multitope $\rho^{(1)}$ (considered already above, with the notation $1 \xrightarrow{2} 3$). We have the map $p=c_{\pi}:\pi \to \kappa$ (unique of the form $\pi \to \kappa$) for which p^* maps 0 and $0'$ to 0, 1 to 1. We have the κ-colored multitopes

Their images under $\mathcal{T}[p]$ in $\mathcal{T}[\pi]$ are



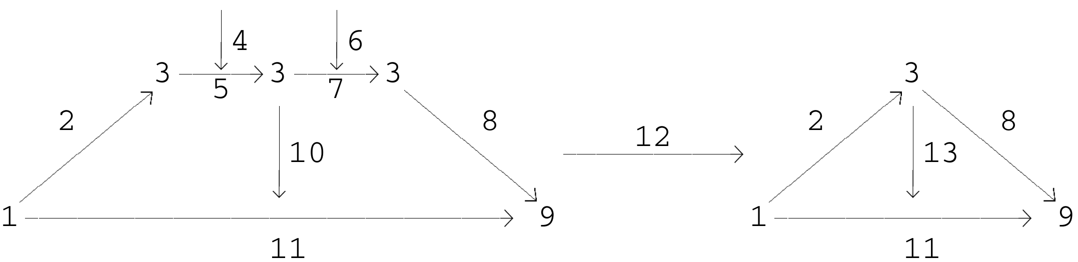
In $\mathcal{T}[\pi]$, we have the further cell



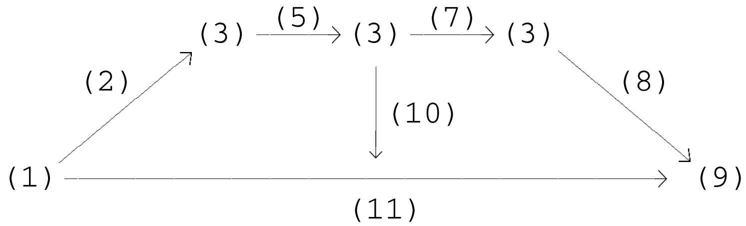
which is not obtained from $\mathcal{P}[\pi]$ under the canonical map into $\mathcal{T}[\pi]$.

Finally, let us take an example where the recursive character of the definition shows itself to be more essential than in the "positive" (empty-domain-free) examples considered so far.

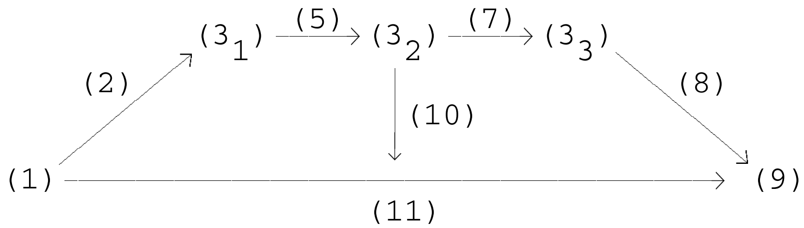
Let π be the 3-dimensional multitope



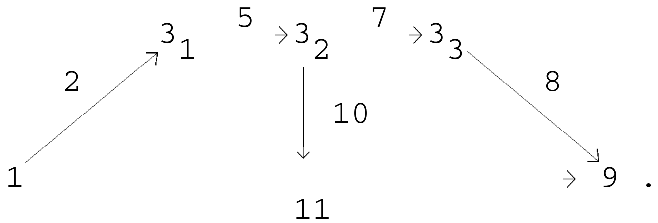
In $\mathtt{Mlt}[\pi]$, we have the 2-cell R :



The one in which (5) and (7) are interchanged, call it S, is not a correct coloring; it is not a cell in $\mathtt{Mlt}[\pi]$, despite the fact that locally it looks all right. The point is that R is the image of the cell



in $\mathtt{Mlt}[\tau]$ for τ the 2-cell



However, S cannot be represented as such an image.

Let $\pi \in \mathtt{Mlt}$. We put $\mathtt{Mlt}[\pi]$, the *category of π-colored multitopes*, to be $\mathtt{El}(\mathcal{T}[\pi])$, the category of elements of $\mathcal{T}[\pi]:\mathtt{Mlt}\to\mathtt{Set}$. That is, the objects of $\mathtt{Mlt}[\pi]$ are pairs (τ, a) where $a \in \mathcal{T}[\pi](\tau)$; and a morphism $f:(\tau, a)\longrightarrow(\theta, b)$ is an arrow $f:\tau\to\theta$ in $\mathtt{Mlt}$ such that $(\mathcal{T}[\pi](f))(a) = b$. In particular, we have the faithful forgetful functor $\| \; \|:\mathtt{Mlt}[\pi]\to\mathtt{Mlt}$. $\| \; \|$ is a discrete cofibration: the arrows out of any $R \in \mathtt{Mlt}[\pi]$ in $\mathtt{Mlt}[\pi]$ are in a bijective correspondence with the arrows out of $\|R\|$ in $\mathtt{Mlt}$.

For every $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we have an induced functor

$$\mathtt{Mlt}[p]=\mathtt{El}(\mathcal{T}[p]):\mathtt{Mlt}[\kappa]\longrightarrow\mathtt{Mlt}[\pi]$$

("recoloring along p"). In fact, we have the functor

$$\mathtt{Mlt}[-] : \mathtt{Mlt}^{\mathrm{op}}\longrightarrow\mathtt{Cat} \ .$$

In other words, we have a strict fibration

$$\begin{array}{c}\mathtt{Mlt}[\mathtt{Mlt}]\\ \downarrow{\scriptstyle\Pi}\\ \mathtt{Mlt}\end{array}$$

whose fiber at each $\pi\in\mathtt{Mlt}$ is $\mathtt{Mlt}[\pi]$. The total category $\mathtt{Mlt}[\mathtt{Mlt}]$ is a suitable category that has as objects all colored multitopes, colored by any (simple) multitope.

The *category* $\mathtt{Mlt}[\pi]\mathtt{Set}$ *of* π*-colored multitopic sets* is

$$\mathtt{Mlt}[\pi]\mathtt{Set} = \mathtt{Set}^{\mathtt{Mlt}[\pi]} \ .$$

4. Substitution

This section gives a treatment of the syntactic operation of *substitution* for multitopes and colored multitopes.

We are going to describe two types of substitution, the *d-type substitution*, and the *c-type substitution*. Whereas the two types of substitution are similar to each other, and also, they are "dual" to each other, in the sense of the "duality" of domain versus codomain, their symmetry is not perfect; d-type substitution is somewhat simpler, and also, more "basic". First, we describe d-type substitution.

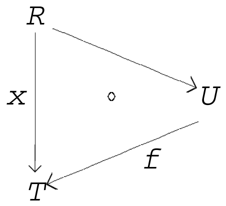
4.1. d–Type substitution

We fix a multitopic set set $\mathcal{R}$. We place ourselves in the category $\mathtt{El}(\mathcal{R})$ of elements of $\mathcal{R}$ ($\mathcal{R}$ being a functor $\mathcal{R}:\mathtt{Mlt}\longrightarrow\mathtt{Set}$). For simplicity in notation and terminology, we assume that for different objects ρ and σ of $\mathtt{Mlt}$, the sets $\mathcal{R}(\rho)$ and $\mathcal{R}(\sigma)$ are disjoint; this allows us to identify the objects of $\mathtt{El}(\mathcal{R})$ with the *cells* of $\mathcal{R}$, that is, with the elements $R\in\mathcal{R}(\rho)$ for varying $\rho\in\mathtt{Mlt}$.

The letters K, R, S, T, U are used for cells of $\mathcal{R}$; with the forgetful functor $\|\ \|:\mathtt{El}(\mathcal{R})\to\mathtt{Mlt}$, the values $\|K\|$, $\|R\|$, $\|S\|$, $\|T\|$, $\|U\|$ will be denoted by κ, ρ, σ, θ and τ, respectively. We write just f for $\|f\|$ for any $f:R\to T$. Of course, for a fixed cell R, the arrows $f:R\to T$ are in a bijective correspondence with the faces of R.

An arrow $f:R\to T$ is a *c-arrow* (*d-arrow*) iff $f:\rho\to\theta$ is one (see section 2).

An arrow $R\xrightarrow{x}T$ in $\mathtt{El}(\mathcal{R})$ is said to be (of) *d-type* (respectively, *c-type*) if $x\neq\mathrm{id}_\theta$, and there is no factorization of x of the form



where f is a c-arrow (respectively, a d-arrow). In other words, x is d-type if x is not the top face of ρ, and there is no face in ρ that would have the x-occurrence of θ as its codomain. "Dually", x is c-type if it is not the top face, and it is not in the domain of any face of one higher dimension. Note that any c-arrow is c-type, and any d-arrow is d-type.

We take a cell $S\in\mathcal{R}$ of dimension equal to $n\geq 1$, and keep it constant for most of what follows. Let us write $T \underset{\text{def}}{=} \mathrm{c}S$.

Let $\mathrm{El}(\mathcal{R})//T$ denote the full subcategory of the slice-category $\mathrm{El}(\mathcal{R})/T$ with objects the pairs (R, x) in which $\overset{R}{\underset{T}{x\downarrow}}$ is d-type T-shape face of R.

For the given $\mathcal{R}$-cell S, $T \underset{\text{def}}{=} \mathrm{c}S$, $n=\dim(S)$, assumed fixed, and for a variable pair (R, x), with $R\in\mathcal{R}$ and $\overset{R}{\underset{T}{x\downarrow}}$ of d-type, we wish to define an $\mathcal{R}$-cell $R[S/x]$, the "result of substituting the *domain* of S for the *codomain* T of S at the occurrence x in R." [since not the whole of S, but rather its "domain", is the one which is being substituted, I would like write $R[S^{-}/x]$; but I will stay with the simpler symbol].

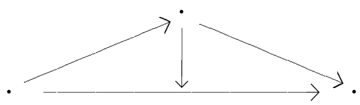
We proceed by describing requirements for the d-type substitution operation

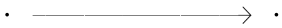
$$(R, x)\ \ (\overset{R}{\underset{T}{x\downarrow}}\ \text{of d-type}) \longmapsto R[S/x]\ . \qquad (1)$$

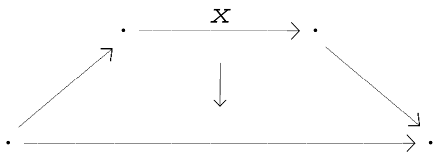
in the signature category $\mathrm{El}(\mathcal{R})$. In the case $\mathcal{R}=\mathcal{T}$, the terminal multitopic set, these requirements uniquely define said operation; d-type substitution in $\mathcal{R}=\mathcal{T}$ is what we use in section 5 for the definition of "multitopic category". In the general case of the basic multitopic set $\mathcal{R}$, the requirements amount to a definition of a possible "d-type substitution operation". For the definition of transfors of positive dimensions, we will need particular d-type substitution operations in cases

when $\mathcal{R}$ is the multitopic set $\mathcal{T}[\pi]$, for various examples of the multitope π. We will state additional data which determine these substitution operations.

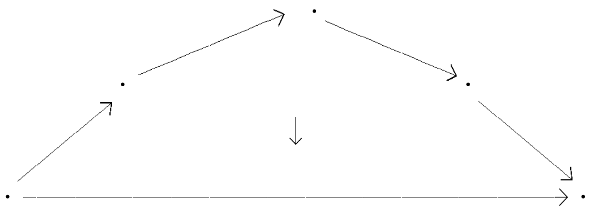
First, some examples for the case $\mathcal{R}=\mathcal{T}$. R, S, T are now multitopes, and we write the corresponding Greek letters for them ($\|\ \|:\mathcal{R}\to\mathcal{T}$ is the identity).

We let $n=2$. Let σ be the 2-dimensional multitope  .

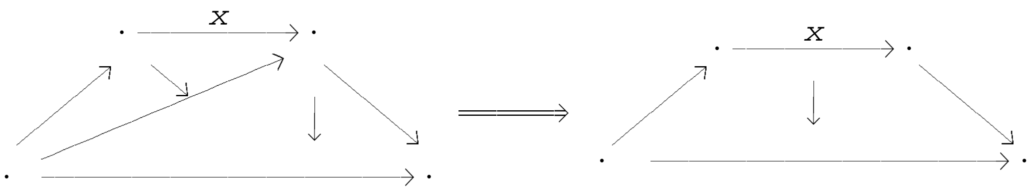
Then θ is 

If ρ is 

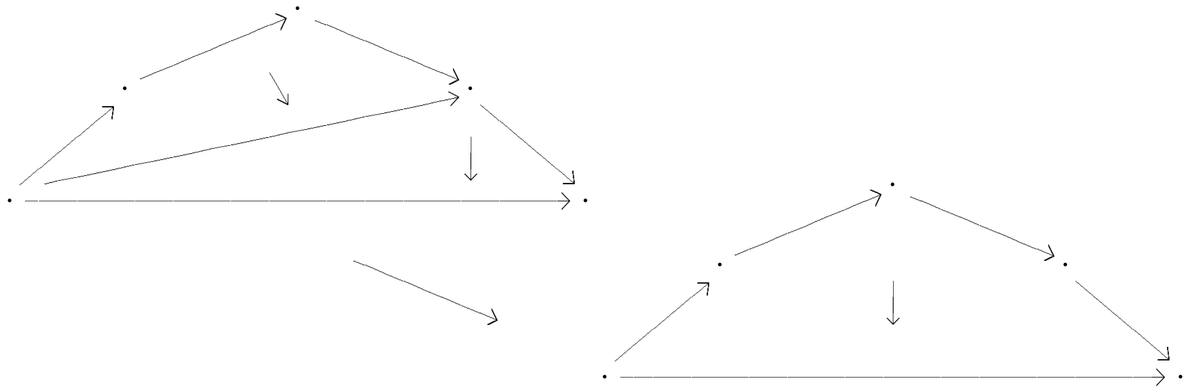
, with the face x as indicated, then $\rho[\sigma/x]$ is



If ρ is

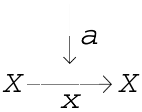


then $\rho[\sigma/x]$ is

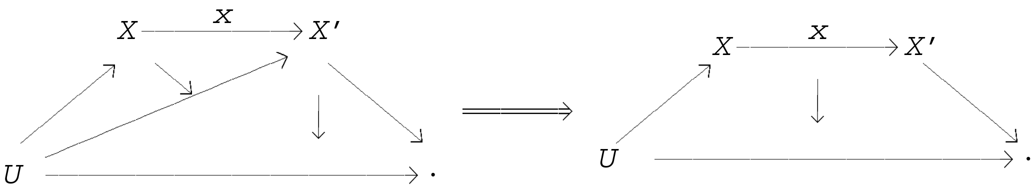


(in the last two diagrams, we gave two-dimensional renderings of three-dimensional objects; there are inevitable repetitions; all zero-dimensional faces, and all but one one-dimensional face, are drawn twice.)

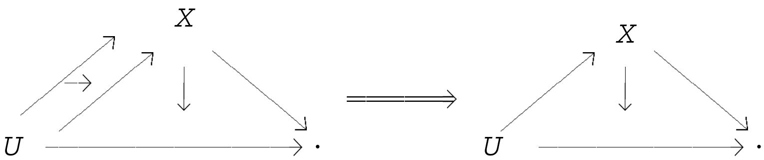
Let now σ be the empty-domain 2-multitope



Now, θ is $X \xrightarrow{x} X'$ (whereas the two X's in σ denote the *same* face, in θ, X and X' denote *different* faces). For ρ being



$\rho[\sigma/x]$ is



We return to the general case of $\mathcal{R}$ and S as given above, and proceed to give the requirements on d-type substitution (1).

Let $|S|$ denote the set of all arrows in $\mathrm{El}(\mathcal{R})$ whose domain is S. We write $S|$ for the set $|S|-\{\mathrm{id}_S\}$, and $S\|$ for $|S|-\{\mathrm{id}_S, \mathrm{c}_S\}$.

Let R be a cell of $\mathcal{R}$, $\begin{array}{c}R\\ x\downarrow\\ T\end{array}$ a d-type T-shaped face of R.

The faces of $R[S/x]$ are in three classes F_1, F_2 and F_3 given below; F_1 and F_2 are not disjoint, but their overlap will also be clarified; F_3 is disjoint from F_1 and F_2. We have

$$F_1 = \{\mathring{p}:R[S/x]\to \mathrm{K}_p : p\in S\|\} ;$$

the correspondence $p\mapsto \mathring{p}$ is a bijection between the p's as described and the elements of F_1.

Let $R\bot_x$ denote the set of all arrows $q:R\to K$ such that there is no y with $\begin{array}{ccc}R & \xrightarrow{q} & K\\ {\scriptstyle x}\downarrow & \circ & \swarrow{\scriptstyle y}\\ T & & \end{array}$;

that is, $q\in R\bot_x$ iff (intuitively speaking) the q-face of ρ does not contain x; the elements of $R\bot_x$ are called *x-free* faces. For the second class F_2, we have that

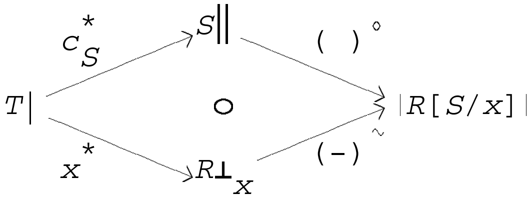
$$F_2 = \{\tilde{q}:R[S/x]\to \mathrm{K}_q : q\in R\bot_x\} ;$$

however, the correspondence $q\mapsto \tilde{q}$ is not always a bijection. Finally,

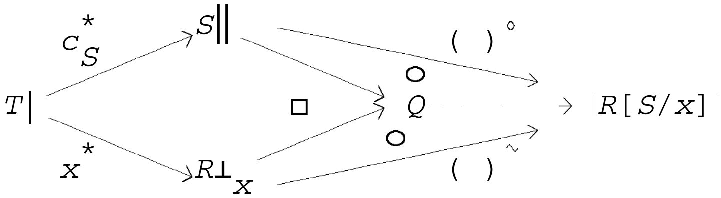
$$F_3 = \{f[S]:R[S/x]\longrightarrow U[S/y] : f:(R, x)\to(U, y)\} .$$

In other words, we have a mapping $f\mapsto f[S]$ of arrows in $\mathrm{El}(\mathcal{R})//T$ with domain (R, x) to certain faces of $R[S/x]$; when the codomain of f is (U, y), the shape of the face is $U[S/y]$. Note that an arrow $f:(R, x)\to(U, y)$ is the same thing as a face $f:R\to U$ of R which "contains" the face x.

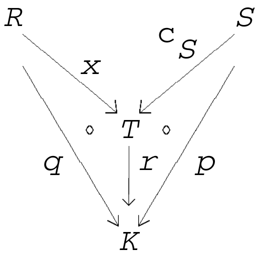
The identification of elements in F_1 and F_2 is described as follows. We have the diagram of sets



and in fact, when we take the pushout Q shown in the next diagram, the resulting canonical function $Q \longrightarrow |R[S/x]|$ in the diagram

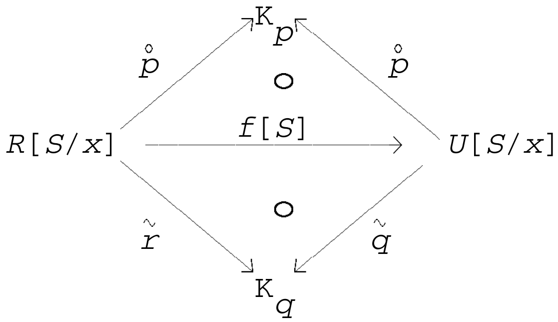


is 1-1 . In particular, if



,

then $\tilde{q} = \mathring{p}$. We should add that $r \circ \mathring{p} = (r \circ p)^\circ$, and $r \circ \tilde{q} = (r \circ q)^\sim$, every time $p \in S\|$, $q \in R\perp_x$ and the composites $r \circ p$, $r \circ q$ make sense. Finally, we have that, for all relevant parameters,



provided $q \circ f = r$.

Let us summarize substitution in a more functorial manner. For the present purpose, it is good to use *formal contexts*; see section 1.

We fix S and $T = cS$ as before. In the first place, substitution is a functor

$$\Sigma_S : \mathtt{El}(\mathcal{R})//T \longrightarrow \mathtt{El}(\mathcal{R})$$

$$\begin{array}{ccc} (R, x) & & R[S/x] \\ \downarrow f & \longmapsto & \downarrow f[S] \\ (U, y) & & U[S/y] \end{array}$$

Next, recall from section 1 the functor Π , associating to a concrete context in $\mathcal{R}: \mathtt{Mlt} \to \mathtt{Set}$ an abstract context of $\mathtt{El}(\mathcal{R})$. The cell $S \in \mathcal{R}(\sigma)$ gives rise (by Yoneda) to $\hat{S}: \hat{\sigma} \to \mathcal{R}$, and, with $[\sigma^{-}] \xrightarrow{\iota} \hat{\sigma}$ the inclusion of the formal context $[\sigma^{-}]$ for $\mathtt{Mlt}$, we have the composite

$$\hat{S} \circ \iota \; : \; [\sigma^{-}] \xrightarrow{\iota} \hat{\sigma} \xrightarrow{\hat{S}} \mathcal{R} \; .$$

Π applied to the $[\sigma^{-}]$-context $\hat{S} \circ \iota$ in $\mathcal{R}$ gives what we now denote by

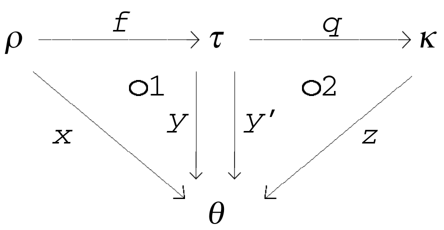
$$[S^{-}] \; : \; \mathtt{El}[\mathcal{R}] \longrightarrow \mathtt{Set} \; ,$$

an abstract context for $\mathtt{El}(\mathcal{R})$.

Let $(R, x) \in \mathrm{El}(\mathcal{R})//T$. We define the formal context $[R\bot_x]$ for $\mathrm{El}(\mathcal{R})$ as follows. It is a subfunctor of $\hat{R}$, and for any $K \in \mathrm{Ob}(\mathrm{El}(\mathcal{R}))$, $(q: R \to K) \in \hat{R}(K)$, we have $q \in [R\bot_x](K)$ iff $q \in R\bot_x$. The fact that this definition indeed gives a subfunctor $\hat{R}$ is expressed by the obvious circumstance that if $q \in R\bot_x$, then $r \circ q \in R\bot_x$. Moreover, $[R\bot_x]$ is a functor of $(R, x) \in \mathrm{El}(\mathcal{R})//T$:

$$[-\bot\_] : (\mathrm{El}(\mathcal{R})//T)^{\mathrm{op}} \to \mathrm{Set}^{\mathrm{El}(\mathcal{R})} , \tag{2}$$

Let us take a quick look at why the last assertion is true. Because of the map $\| \| : \mathcal{R} \to \mathcal{T}$, it suffices to show the assertion for the case $\mathcal{R} = \mathcal{T}$. We want that if $(f: (\rho, x) \longrightarrow (\tau, y)) \in \mathrm{Mlt}//\theta$, and $(q: \tau \to \kappa) \in \tau\bot_y$, then $(qf: \rho \to \kappa) \in \rho\bot_x$. Suppose not; then we have $z: \kappa \to \theta$ such that the outside large triangle in



commutes. Define y' by the commutativity ○2. We conclude that $yf = y'f$. $y' = y$ contradicts $(q: \tau \to \kappa) \in \tau\bot_y$; we have $y' \neq y$. As we said in the last section, this can happen only if there is an arrow $e: \rho \to \varepsilon$ to an empty-domain multitope ε such that, in

$$\rho \xrightarrow{e} \varepsilon \xrightarrow{\mathrm{c}_\varepsilon} \mathrm{c}\varepsilon \underset{\mathrm{d}_{\mathrm{c}\varepsilon}}{\overset{\mathrm{c}_{\mathrm{c}\varepsilon}}{\rightrightarrows}} \mathrm{cc}\varepsilon ,$$

we have $f = c_\varepsilon \circ e$, and either $y = \mathrm{d}_{\mathrm{c}\varepsilon}$ and $y' = \mathrm{c}_{\mathrm{c}\varepsilon}$, or $y' = \mathrm{d}_{\mathrm{c}\varepsilon}$ and $y = \mathrm{c}_{\mathrm{c}\varepsilon}$. In either case, since $y \circ f = y' \circ f = x$, we get a contradiction to x being d-type. This completes the proof.

In the next section, we will make good use of the two-variable functor

$$\begin{array}{rcl} \Lambda : (\mathrm{El}(\mathcal{R})//T)^{\mathrm{op}} \times \mathrm{El}(\mathcal{R}) & \longrightarrow & \mathrm{Set} \\ ((R, x), U) & \longmapsto & [R\bot_x](U) \end{array} \tag{3}$$

the exponential adjunct of (2).

For each $(R, x)\in \mathrm{El}(\mathcal{R})//T$, the operations

$$(\ )^{\circ} = (\ )^{\circ}_{(R,\ x)} : S\| \longrightarrow |R[S/x]|$$

$$(\ )^{\sim} = (\ )^{\sim}_{(R,\ x)} : R\bot_{x} \longrightarrow |R[S/x]|$$

are in fact maps of abstract contexts:

$$(\ )_{(R,\ x)}{}^{\circ} : [S^{-}] \longrightarrow (R[S/x])^{\wedge},$$

$$(\ )_{(R,\ x)}{}^{\sim} : [R\bot_{x}] \longrightarrow (R[S/x])^{\wedge};$$

and both of them are natural in $(R, x)\in \mathrm{El}(\mathcal{R})//T$:

$$(\mathrm{El}(\mathcal{R})//T)^{\mathrm{op}} \underset{(\ )^{\wedge}\circ\Sigma_{S}}{\overset{[\_^{-}]\circ\ulcorner S\urcorner}{\rightrightarrows}} \mathrm{Set}^{\mathrm{El}(\mathcal{R})//T}, \qquad \Downarrow(\ )^{\circ}$$

$$(\mathrm{El}(\mathcal{R})//T)^{\mathrm{op}} \underset{(\ )^{\wedge}\circ\Sigma_{S}}{\overset{[-\bot_{\_}]}{\rightrightarrows}} \mathrm{Set}^{\mathrm{El}(\mathcal{R})//T}, \qquad \Downarrow(\ )^{\sim}$$

We write the same symbol $(\ )^{\sim}$ for the exponential adjunct of $(\ )^{\sim}$:

$$(\mathrm{El}(\mathcal{R})//T)^{\mathrm{op}} \times \mathrm{El}(R) \underset{\hom_{\mathrm{El}(\mathcal{R})}(\Sigma_{S}-,-)}{\overset{\Lambda}{\rightrightarrows}} \mathrm{Set}. \qquad \Downarrow(\ )^{\sim}$$

This completes the general description of d-type substitution. Let's emphasize again that the above constitutes a definition of what a d-type substitution operation (1) is, for given $\mathcal{R}$ and S in $\mathcal{R}$, rather than defining one specific substitution operation.

Proposition

(i) When $\mathcal{R}=\mathcal{T}$, the terminal multitopic set, for any $S\in\mathcal{T}$, there is a uniquely determined d-type substitution operation (1) in $\mathrm{El}(\mathcal{R})\simeq\mathrm{Mlt}$.

(ii) Let π be a multitope, $\mathcal{R}=T[\pi]$. Let S be any primitive π-colored multitope: $\gamma_{\pi}(S)=\mathrm{id}_{\pi}$. There is a uniquely determined d-type substitution operation (1) in $\mathrm{El}(\mathcal{R})=\mathrm{Mlt}[\pi]$.

(iii) Let π be a multitope, $\mathcal{R}=T[\pi]$. Let $S\in\mathcal{R}$. Let k be the dimension of the π-color of S: for $p=\gamma_{\pi}(S)$, $p:\pi\to\kappa$, $k\underset{\mathrm{def}}{=}\dim(\kappa)$. Assume that $\dim(S)>k$. There is a uniquely determined d-type substitution operation (1) in $\mathrm{Mlt}[\pi]$ with the additional property that $\gamma_{\pi}(R[S/x])=\gamma_{\pi}(R)$ for all relevant pairs (R,x).

Note that (iii) specializes to (i) when π is the 0-dimensional multitope.

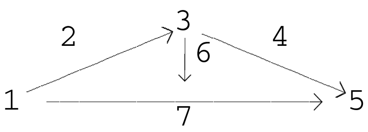
4.2. c–Type substitution

We next turn to c-type substitution.

In contrast to d-type substitution, the c-type substitution will be used (in this paper) only in very special circumstances. These are described as follows.

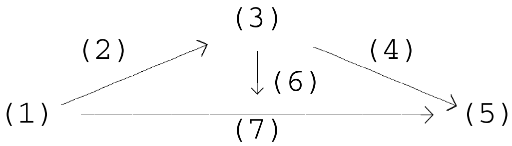
We fix $\pi\in\mathrm{Mlt}$ of dimension $n\geq 2$. In the applications, n will always be *even*; however, for the constructions we perform now, this will not be of importance. We put $\mathcal{R}=\mathrm{Mlt}[\pi]$. We let $P=P[\pi]\in\mathrm{Mlt}[\pi]$, the particular colored multitope whose underlying multitope $\|P\|$ is π, and for which the total coloring $\bar{\gamma}\colon|\pi|\to|\pi|$ is the identity.

We will consider the example when π is the multitope



(4)

P is then depicted by the picture



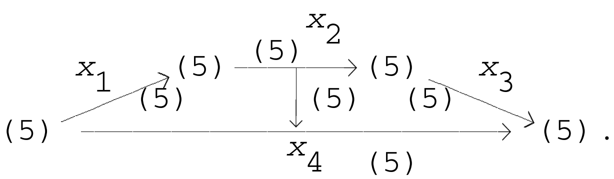
Here, and elsewhere, we employ the device of indicating the color of a face by putting the name of the color (in our case, one of the numbers $1,\ldots,7$) in parentheses onto the cell.

Returning to the general case, it is easy to see that for any $U\in\mathcal{R}$, there is a unique π-colored multitope $U^*\in\mathcal{R}$ such that $d(U^*)=\langle U\rangle$ and $c(U^*)=U$, and whose color is the same as that of U. (Recall, from section 2, the construction of the multitope θ^* out of any multitope θ.) We let $T \underset{\overline{\text{def}}}{=} (ccP)^*$. Here, $\langle ccP\rangle$ denotes the $(n-2)$-pd consisting of the single $(n-2)$-cell ccP. This is the same as $i_{n-2}(ccP)$ for the inclusion $i_{n-2}:C_{n-2}(\mathcal{R})\to P_{n-2}(\mathcal{R})$ mentioned at the beginning of section 2.

In our example started above, ccP is the zero-dimensional multitope colored 5 ; and T is

$$(5)\xrightarrow{(5)}(5)\quad .$$

Continuing the example, let R be



(5)

Then x_4 is c-type, but x_1, x_2 and x_3 are not. When R is the shape of a 3-cell with an empty 2-pd as domain, all colored (5), the one 1-dimensional face of R is c-type. .

Let us return to the general context. For later use, let me note the following map of abstract contexts:

$$
\begin{array}{ccc}
[T^{-}] & \xrightarrow{\quad\eta\quad} & [P^{-}] \\
\\
T & & P \\
{\scriptstyle c_T}\big\downarrow & & \big\downarrow{\scriptstyle c_{cP}\circ c_P} \\
\mathrm{cc}P & \longmapsto & \mathrm{cc}P \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
S & & S
\end{array}
$$

El($\mathcal{R}$)$\#T$ denotes the full subcategory of $\mathrm{El}(\mathcal{R})/T$ on the objects the c-type arrows. We have the functor

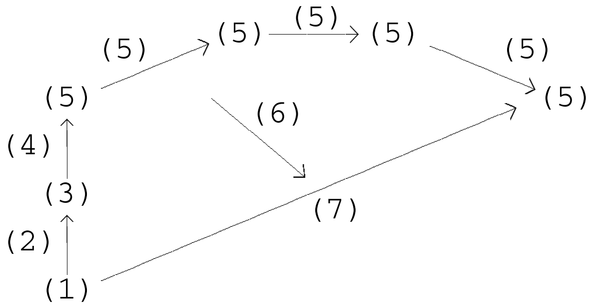
$$
\begin{array}{rcl}
\Lambda : (\mathrm{El}(\mathcal{R})\#T])^{\mathrm{op}}\times\mathrm{El}(\mathcal{R}) & \longrightarrow & \mathrm{Set} \\
\left(x\begin{array}{c} R \\ \big\downarrow \\ T\end{array},\quad U\right) & \longmapsto & [R\bot_x](U)
\end{array}
\tag{6}
$$

Here, $R\bot_x$ is the set of all " x-free arrows" out of R; $[R\bot_x]$ is the corresponding abstract context; these are defined as before. The action on arrows defined in the obvious way; the proof that this works is as in the analogous situation before.

We have a (*c-type*) *substitution* functor

$$
\begin{array}{rcl}
\Sigma^{\#}_{P} : \mathrm{El}(\mathcal{R})\#T & \longrightarrow & \mathrm{El}(\mathcal{R}) \\
(R,\ x) & \longmapsto & R[P\#x] \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f[P]} \\
(S,\ y) & \longmapsto & S[P\#y]
\end{array}
\quad .
$$

As an illustration: when, for instance, R is as in (5), and $x{=}x_4$, then (given that π is (4), and $\mathcal{R}{=}\mathrm{Mlt}[\pi]$) $R[P\#x]$ is



Intuitively speaking, the c-type substitution $R[P\#x]$ involves "splitting" P at the codomain of its codomain, causing the removal of cP, and the doubling of ccP, and then putting the resulting shape in place of the face x in R.

To explain the rest of the structure connected with c-type substitution, let us introduce a new formal context denoted $[P\#]$ (" P-split"). $[P\#]$, a modified ("split") version of $[P]$, is described by saying the following:

there is a monomorphism $\iota : [P^-] \longrightarrow [P\#]$ such that each component $\iota_S : [P^-](S) \longrightarrow [P\#](S)$ is a bijection except for $S{=}cP$ and $S{=}ccP$; for simplicity, we take ι to be the inclusion;

$[P\#](cP) = \{\#^{++}\}$ for a certain element $\#^{++}$ (note that $[P^-](cP) = \emptyset$);

$[P\#](ccP) = [P^-](ccP) \coprod \{\#^+\}$ for another element $\#^+$;

$[P\#](c_{cP})(\#^{++}) = \#^+$;

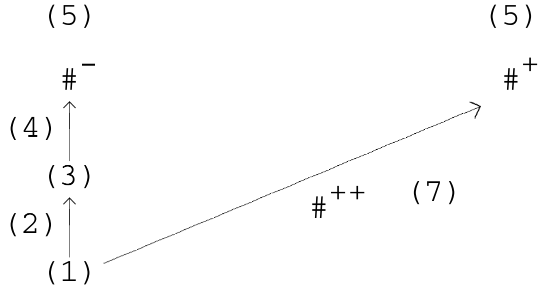
for any $r\in (cP)\|$, we have $[P\#](r)(\#^{++}) = r\circ c_P$;

for any $s\in (ccP)|$, $[P\#](s)(\#^+) = s\circ c_{cP}\circ c_P$.

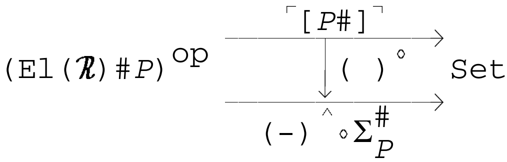
For the sake of symmetry, we write $\#^-$ for the element $c_{cP}\circ c_P$ in $[P^-](ccP) \subset [P\#](ccP)$.

These facts determine $[P\#]$ up to isomorphism.

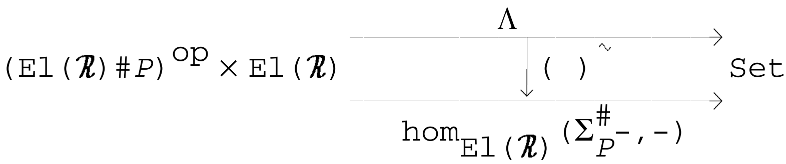
In the example, the formal context $[P\#]$ is depicted by the picture:



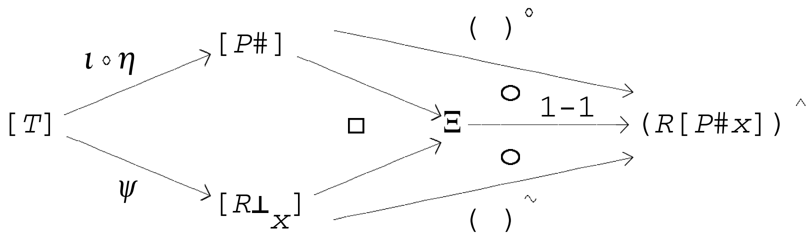
For each $(R, x) \in \mathrm{El}(\mathcal{R})\#P$, we have a map $(\ )^{\circ}_{(R,x)} : [P\#] \longrightarrow (R[P\#x])^{\wedge}$, which, as a function of (R, x), is natural:



On the other hand, we have



The identification of the images of $(\ )^{\circ}$ and $(\ )^{\sim}$ is explained by the diagram



where ψ is $(p:T\to S)\longmapsto p\circ x$, $\iota\circ\eta$ was introduced above in (4).

5. Omega-dimensional universal properties

The fundamental idea of multitopic categories, borrowed from the Baez/Dolan "opetopic" concept [B/D], is that *the composite of each composable system of cells is* to be *defined by a universal property*. Unlike in *loc.cit.* however, in this paper, the universal properties in question will be specified by uses of the concept of FOLDS equivalence (see Section 1 above). Another difference to *loc.cit.* is that we work in an untruncated multitopic set, aiming at weak ω-categories, and not just weak n-categories for finite n as in *loc.cit.* The finite-dimensional notion appears as a simple truncation of the ω-dimensional notion.

Let R be a commutative ring. Classically, the tensor product $A\otimes B$ of two R-modules A, B is defined by *using* a universal property: $A\otimes B$ is such that there exists a universal R-bilinear map $(A, B)\longrightarrow A\otimes B$. $A\otimes B$ is *defined* by making a specific choice of the particular module $A\otimes B$. (For the sake of the rest of the structure, a choice of a particular universal R-bilinear map $(A, B)\longrightarrow A\otimes B$ is also made.) The resulting monoidal category $R\text{-Mod}$ is, in fact, a bicategory with a single 0-cell in which the modules are the 1-cells, and the tensor product is the composition of 1-cells.

The fundamental difference of the present approach (already present in [B/D], and in special cases, even in the earlier [M1]), to the classical one is that we will have composition as a *virtual* (an adjective taken from [B/D]) operation. In the special case we chose for illustration, this means that we will be content to "define" $A\otimes B$ by the described universal property, without making specific choices of each and every $A\otimes B$. In particular, in the bicategory $R\text{-Mod}$, the result of the composition of 1-cells will be given only up to isomorphism only, in contrast to the classical concept of "bicategory" in which all compositions (of 1-cells and of 2-cells) are ordinary algebraic operations. It is important to realize that this procedure makes sense only if we adopt the structure of R-bilinear maps (and preferably, even that of k-ary R-linear maps of R-modules) as part of the structure of $R\text{-Mod}$.

The universal property of $A\otimes B$ is that there exists a system of bijections

$$\mathrm{hom}(A\otimes B, ?) \xrightarrow{\cong} \mathrm{hom}_{\mathrm{bilin}}((A, B), ?) \qquad (1)$$

for variable 1-cells $?$ (we continue talking about $R\text{-Mod}$ as a bicategory). When the dimension is raised, and we want to define the composite $A\otimes B$ of 1-cells in a *tri*category, say, then the isomorphisms in (1) will, presumably, be replaced by equivalences of hom-categories. And similarly for even higher dimensions, when the concept of "equivalence" will be taken to be the corresponding higher dimensional one.

More generally than the composition of two 1-cells discussed above, we will introduce a concept of composition of an arbitrary pasting diagram in a multitopic set. Let $\mathcal{S}$ be a multitopic set, let α be a k-dimensional pasting diagram in $\mathcal{S}$. We will define what it means for a k-cell a in $\mathcal{S}$ to be a *composite* of α. The first requirement is that a be parallel to α: $\mathrm{d}a=\mathrm{d}\alpha$, $\mathrm{c}a=\mathrm{c}\alpha$. The main part of the definition says, at present only allusively, that

the structure of arrows out of a (2)

is equivalent to (3)

the structure of arrows out of α . (4) .

This phrasing is intended to convey a general idea behind the special case (1) of the tensor product above: the "structure of arrows out of a ", when a is $A\otimes B$, is $\mathrm{hom}(A\otimes B, ?)$, the totality of the hom-sets, or hom-categories, or hom-bi-categories, etc, of arrows out of $A\otimes B$; the "structure of the arrows out of α ", now when $\alpha=(A, B)$, is $\mathrm{hom}_{\mathrm{bilin}}((A, B), ?)$. We will do two things: we will make precise the sense of the "structures" (2) and (4); and then we will deploy the general concept of FOLDS-equivalence to make sense of "equivalent" in (3).

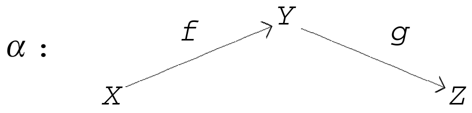
We will denote structure (2) by $\mathcal{S}\langle a\rangle$, structure (4) by $\mathcal{S}\langle\alpha\rangle$.

Before turning to the general definitions, we consider some examples.

In some, fixed, multitopic set $\mathcal{S}$, let a be the 1-cell

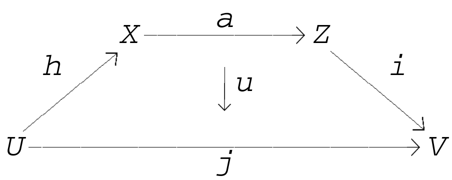
$$X \xrightarrow{a} Z$$

and α the 1-pd

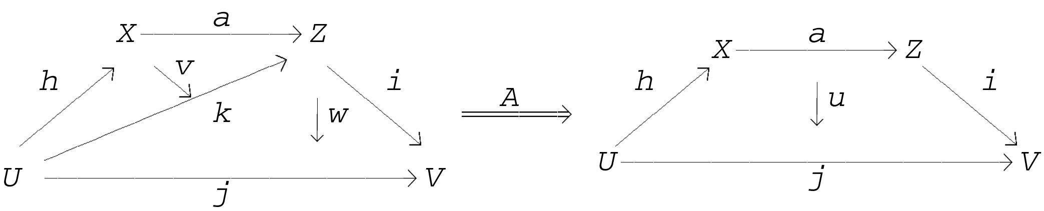


a and α have been taken to be parallel. Until further notice, let us fix the items listed so far. We will now explain parts of structures (2) and (4), specified by the items given so far, as well as what an equivalence (3) would be like.

Consider, for instance, the 2-cell u in $\mathcal{S}$:

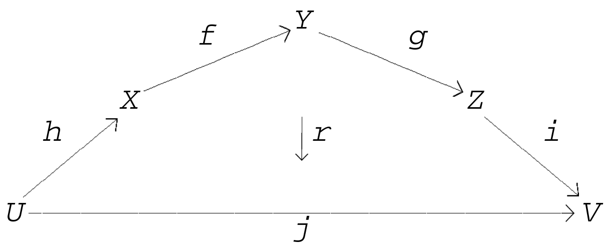


This is part of the "structure of arrows out of a", structure (2), also denoted $\mathcal{S}\langle a\rangle$. But also, for instance, the 3-cell A:

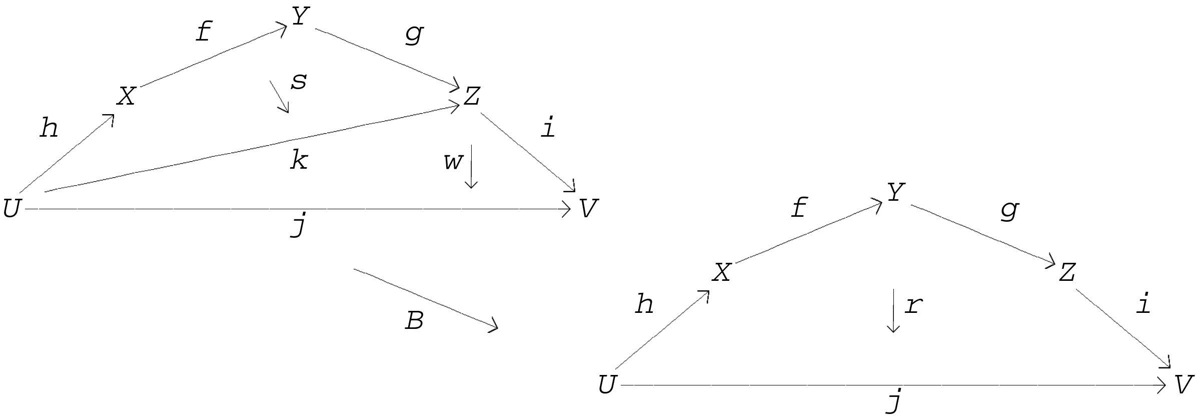


figures in structure (2).

In structure (4), there will be *corresponding* cells, r and B as follows:



and



The *kind* that the element u belongs to in $\mathcal{S}\langle a\rangle$ will be the *same* as the *kind* element r belongs to in $\mathcal{S}\langle \alpha\rangle$; let us call this kind K_1 . The same goes for A versus B ; let us call the common kind of A and B : K_2 . Here we use the word "kind" in the sense used in the context of FOLDS: an object of the signature category, the common signature of $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle \alpha\rangle$.

But more is true. In fact, the *sorts* u in $\mathcal{S}\langle a\rangle$ and r in $\mathcal{S}\langle a\rangle$ belong to are also the *same*. Remember that *sorts* are *dependent sorts*; a sort headed by a kind depends on an array of variables that is indexed by the arity of the kind, the non-identity arrows out of the kind. We note, first of all, that the new signature will extend $\mathtt{Mlt}$, the signature consisting of multitopes; the elements U, X, Z, V, h, i, j , the common faces of u and r , will appear in both $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle \alpha\rangle$ sorted as in $\mathcal{S}$. Now, the elements u and r both will be of the sort

$$K_1(U, X, Z, V, h, i, j) \; ,$$

albeit in different structures. In other words, the contexts for both u and r in the new structures will consist of the elements U, X, Z, V, h, i, j .

We see that the main act of making the element u of $\mathcal{S}$ an element of the $\mathcal{S}\langle a\rangle$ is to *remove* the dependence on the item a ; for r , we *remove* f, Y, g ; by these removals, the points of references of u and r became the same, and so, u and r will relatable in a (FOLDS-)equivalence. Of course, the removed items: a, f, Y, g are constant for $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle \alpha\rangle$, so there is no logical need for referring to them as values of variables.

Let us turn to the sortings of A and B . In $\mathcal{S}\langle a\rangle$, A will belong to the sort

$$K_2(U, X, Z, V; h, i, j, k; u, v, w) \ ,$$

while in $\mathcal{S}\langle\alpha\rangle$, B belongs to

$$K_2(U, X, Z, V; h, i, j, k; r, s, w) \ .$$

These are not the "same" sorts; they differ in two places: at u *vs* r and v *vs* s . Note that there will be two particular arrows $K_2 \xrightarrow{p} K_1$ and $K_2 \xrightarrow{q} K_1'$ (K_1 was mentioned above; K_1' is another new kind, corresponding to the shape of v and s) in the new signature; p "picks out" u ($(\mathcal{S}\langle a\rangle)(p)(A)=u$) , respectively r ; and q picks out v , respectively s .

A and B *may be related* by the equivalence (3) that we will have between $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$. In the first place, the equivalence will simply be the identity on the old part $\mathtt{Mlt}$ of the new signature; thus, denoting equivalence (3) by E ,

$$E : (\mathcal{S}\langle a\rangle; U, X, Z, V; h, i, j, k) \ \simeq \ (\mathcal{S}\langle\alpha\rangle; U, X, Z, V; h, i, j, k) \ .$$

Let us abbreviate the tuple $(U, X, Z, V; h, i, j)$ by $\vec{j}$, and $(U, X, Z; h, k)$ by $\vec{k}$. Since

$$E : (\mathcal{S}\langle a\rangle, \vec{j}) \ \simeq \ (\mathcal{S}\langle\alpha\rangle, \vec{j}) \ ,$$

if

$$u \in \mathcal{S}\langle a\rangle(K_1)(\vec{j}) \ , \qquad (5)$$

then there is at least one

$$r \in \mathcal{S}\langle\alpha\rangle(K_1)(\vec{j}) \qquad (6)$$

such that

$$E : (\mathcal{S}\langle a\rangle; \vec{j}, u) \ \simeq \ (\mathcal{S}\langle\alpha\rangle; \vec{j}, r) \ , \qquad (7)$$

and *vice versa*. Moreover, *if*

$$v \in \mathcal{S}\langle a \rangle (K_1)(\vec{k}) \ , \tag{8}$$

then there is at least one

$$s \in \mathcal{S}\langle \alpha \rangle (K_1)(\vec{k}) \tag{9}$$

such that

$$E : (\mathcal{S}\langle a \rangle; \vec{k}, v) \simeq (\mathcal{S}\langle \alpha \rangle; \vec{k}, s) \ . \tag{10}$$

Assume (5) to (10). Abbreviate $(U, X, Z, V; h, i, j, k; u, v, w)$, the context for A, by $\vec{v}$, and $(U, X, Z, V; h, i, j, k; r, s, w)$, the context for B, by $\vec{s}$. What we have assumed amounts to

$$E : (\mathcal{S}\langle a \rangle, \vec{v}) \simeq (\mathcal{S}\langle \alpha \rangle, \vec{s}) \ .$$

Therefore, *if* A is as we had it above, *then* there is B as above such that

$$E : (\mathcal{S}\langle a \rangle; \vec{v}, A) \simeq (\mathcal{S}\langle \alpha \rangle; \vec{s}, B) \ ,$$

and *vice versa*. This is what we meant by saying that A *may be* related to B in the equivalence (3).

The above examples were intended to give an idea of the workings of the items (2), (3) and (4) in a special case. Now we turn to the general case.

The first step is to specify the *common* (FOLDS-)signature of the structures (2) and (4). The said signature will be defined on the basis of the shape θ of the cell a.

Let θ be a multitope. Fix θ. Let us write k for $\mathrm{dim}(\theta)+1$; thus, $k \geq 1$. The first interesting example to bear in mind is when $k=2$, and θ is the shape $\cdot\to\cdot$, the unique 1-dimensional multitope). We will define the new (FOLDS-)signature $\mathrm{Mlt}\langle\theta\rangle$.

We will apply a general construction that may be called the "bipartite graph" construction, to the two-variable functor

$$\Lambda : (\mathrm{Mlt}//\theta)^{\mathrm{op}} \times \mathrm{Mlt} \longrightarrow \mathrm{Set} \qquad (10')$$

we introduced in the last section.

Here is the general, and well-known, construction. Let

$$\Lambda : \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{B} \longrightarrow \mathrm{Set}$$

be an arbitrary two-variable, Set-valued functor. We define the category $\langle\Lambda\rangle$ by the clauses that follow; A, A' denote objects of $\boldsymbol{A}$; B, B' those of $\boldsymbol{B}$.

$\mathrm{Ob}(\langle\Lambda\rangle) \underset{\mathrm{def}}{=} \mathrm{Ob}(\boldsymbol{A}) \coprod \mathrm{Ob}(\boldsymbol{B})$;
$\mathrm{hom}_{\langle\Lambda\rangle}(A, A') \underset{\mathrm{def}}{=} \mathrm{hom}_{\boldsymbol{A}}(A, A')$;
$\mathrm{hom}_{\langle\Lambda\rangle}(B, B') \underset{\mathrm{def}}{=} \mathrm{hom}_{\boldsymbol{B}}(B, B')$;
$\mathrm{hom}_{\langle\Lambda\rangle}(A, B) \underset{\mathrm{def}}{=} \Lambda(A, B)$;
$\mathrm{hom}_{\langle\Lambda\rangle}(B, A) \underset{\mathrm{def}}{=} \emptyset$;
the inclusions $\boldsymbol{A}\to\langle\Lambda\rangle$, $\boldsymbol{B}\to\langle\Lambda\rangle$ are (full and faithful) functors;
for $A' \xrightarrow{a} A \xrightarrow{f} B$ [$f\in\Lambda(A, B)$], $f\circ a \underset{\mathrm{def}}{=} \Lambda(a, B)(f) : A' \longrightarrow B$;
for $A \xrightarrow{f} B \xrightarrow{b} B'$, $b\circ f \underset{\mathrm{def}}{=} \Lambda(A, b)(f) : A \longrightarrow B'$.

This defines a category. The main case of the associative law is when $A' \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} B'$; $b(fa) = (bf)a$; it follows from Λ being a functor.

We have the cospan of full-and-faithful inclusions

$$\boldsymbol{A} \xrightarrow{\ \boldsymbol{a}\ } \langle\Lambda\rangle \xleftarrow{\ \boldsymbol{b}\ } \boldsymbol{B}. \qquad (11)$$

Every time we have a cospan

$$\boldsymbol{A} \xrightarrow{\boldsymbol{d}} \boldsymbol{C} \xleftarrow{\boldsymbol{e}} \boldsymbol{B} \qquad (12)$$

of functors, we have the induced functor

$$\Phi[\boldsymbol{d},\boldsymbol{e}] = \mathrm{hom}_{\boldsymbol{C}}(\boldsymbol{d}(-),\boldsymbol{e}(-)) : \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{B} \longrightarrow \mathrm{Set}\ .$$

In the case of the cospan (11), $\Phi[\boldsymbol{a},\boldsymbol{b}]$ is isomorphic to the original Λ ; we have

$$\lambda : \quad \Lambda \xrightarrow{\ \cong\ } \Phi[\boldsymbol{a},\boldsymbol{b}]\ . \qquad (13)$$

$$(A, B) \longmapsto \begin{array}{c} \Lambda(A, B) \\ \downarrow \mathrm{id} \\ \mathrm{hom}(\boldsymbol{a}A, \boldsymbol{b}B) \end{array}$$

$\langle\Lambda\rangle$ has a useful universal property, as follows. Suppose $\boldsymbol{c}:\langle\Lambda\rangle \to \boldsymbol{C}$ is a functor. Then, by composition, from (11), we get (12):

$$\boldsymbol{d} = \boldsymbol{ca}\ , \quad \boldsymbol{e} = \boldsymbol{cb}\ . \qquad (14)$$

Note that we have the induced natural transformation "apply-$\boldsymbol{c}$ ":

$$\dot{\boldsymbol{c}} : \mathrm{hom}_{\langle\Lambda\rangle}(\boldsymbol{a}-, \boldsymbol{b}-) \longrightarrow \mathrm{hom}_{\boldsymbol{C}}(\boldsymbol{d}-, \boldsymbol{e}-)$$

$$(A, B) \longmapsto \begin{array}{c} \mathrm{hom}(\boldsymbol{a}A, \boldsymbol{b}B) \\ \downarrow \\ \mathrm{hom}(\boldsymbol{d}A, \boldsymbol{e}B) \end{array} \qquad \begin{array}{c} \boldsymbol{a}A \xrightarrow{f} \boldsymbol{b}B \\ \downarrow \\ \boldsymbol{d}A \xrightarrow[\boldsymbol{c}(f)]{} \boldsymbol{e}B \end{array}$$

From λ in (13), we get, by composition with $\dot{\boldsymbol{c}}$, the natural transformation

$$\delta : \Lambda \longrightarrow \Phi[\boldsymbol{d},\boldsymbol{e}]\ :$$

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad\quad\delta\quad\quad} & \mathrm{hom}_{\boldsymbol{C}}(\boldsymbol{d}-,\boldsymbol{e}-) \\ & {}_{\lambda}\searrow \quad \circ \quad \nearrow_{\dot{\boldsymbol{c}}} & \\ & \mathrm{hom}_{\langle\Lambda\rangle}(\boldsymbol{a}-,\boldsymbol{b}-) & \end{array} \qquad (15)$$

Now, for the given $\Lambda:\boldsymbol{A}^{\mathrm{op}}\times\boldsymbol{B}\longrightarrow\mathtt{Set}$, the universal property of the data

$$(\boldsymbol{A}\xrightarrow{\boldsymbol{a}}\langle\Lambda\rangle\xleftarrow{\boldsymbol{b}}\boldsymbol{B}\ ,\ \lambda:\Lambda\to\Phi[\boldsymbol{a},\boldsymbol{b}])$$

is as follows. For any

$$(\boldsymbol{A}\xrightarrow{\boldsymbol{d}}\boldsymbol{C}\xleftarrow{\boldsymbol{e}}\boldsymbol{B}\ ,\ \delta:\Lambda\to\Phi[\boldsymbol{d},\boldsymbol{e}])\ ,$$

we have that, conversely to the above, there is a unique $\boldsymbol{c}:\langle\Lambda\rangle\longrightarrow\boldsymbol{C}$ such that (14) and (15) hold. In fact, there is a (straightforward) additional two-dimensional aspect of the full universal property; but we will not need it.

As we promised, we define $\mathtt{Mlt}\langle\theta\rangle$ to be $\langle\Lambda\rangle$, for the particular functor (10').

Next, let σ be a multitope, $\mathtt{dim}(\sigma)=k\geq 1$, and let $\theta=\mathtt{c}\sigma$, the shape of the codomain-face of σ. Let $\mathcal{S}$ be a multitopic set, let α be a $(k-1)$-pd of the shape σ in $\mathcal{S}$, a a $(k-1)$-cell of shape θ in $\mathcal{S}$, and assume that α and a are parallel. We will define "structure (4)" and "structure (2)", both $\mathtt{Mlt}\langle\theta\rangle$-structures. The first of these structures we denote by $\mathcal{S}\langle a\rangle$, the second by $\mathcal{S}\langle\alpha\rangle$. We will use the universal property of $\langle\Lambda\rangle$ given above. To connect to the notation above, we let (11) abbreviate

$$\mathtt{Mlt}//\theta\xrightarrow{\mathtt{incl}}\langle\Lambda\rangle\xleftarrow{\mathtt{incl}}\mathtt{Mlt}\ ;$$

and set $\boldsymbol{C}=\mathtt{Set}$.

The definition of $\mathcal{S}\langle a\rangle$ is the simpler one of the two. Letting $\Psi:\boldsymbol{A}=\mathtt{Mlt}//\theta\to\mathtt{Mlt}$ be the forgetful functor, we have the functors in

$$\boldsymbol{A}\ \underset{\ulcorner\mathcal{S}(\theta)\urcorner}{\overset{\mathcal{S}\circ\Psi}{\rightrightarrows}}\ \mathtt{Set}\qquad \varphi\downarrow\quad\downarrow\ulcorner a\urcorner$$

the lower one being the constant functor with value the set $\mathcal{S}(\theta)$. The natural transformation φ is defined thus. At $(\rho, x)\in\boldsymbol{A}$, its component $\varphi_{(\rho, x)}:\mathcal{S}(\rho)\longrightarrow\mathcal{S}(\theta)$ is $\varphi_{(\rho, x)}=\mathcal{S}(x):\mathcal{S}(\rho)\to\mathcal{S}(\theta)$; φ is clearly natural. $\ulcorner a\urcorner$ is the constant natural transformation, with value the element $a\in\mathcal{S}(\theta)$. In the category $\mathrm{Hom}(\boldsymbol{A},\mathrm{Set})$, we take the equalizer of the arrows

$$\boldsymbol{d}\xrightarrow{\ \varepsilon\ }\mathcal{S}\circ\Psi\ \underset{\ulcorner a\urcorner}{\overset{\varphi}{\rightrightarrows}}\ \ulcorner\mathcal{S}(\theta)\urcorner$$

On the other hand, for $\boldsymbol{e}:\boldsymbol{B}\longrightarrow\boldsymbol{C}$ we take $\boldsymbol{e}=\mathcal{S}:\mathrm{Mlt}\to\mathrm{Set}$. Finally, for $\delta:\Lambda\to\Phi[\boldsymbol{d},\boldsymbol{e}]$, we define the component

$$\delta_{((\rho, x);\ \tau)}:[\rho\bot_x](\tau)\longrightarrow\mathrm{hom}(\boldsymbol{d}((\rho, x)),\mathcal{S}(\tau))$$

to be the function $q\longmapsto\mathcal{S}(q)\circ\varepsilon_{(\rho, x)}$.

The functor $\boldsymbol{c}=\mathcal{S}\langle a\rangle:\mathrm{Mlt}\langle\theta\rangle\longrightarrow\mathrm{Set}$ is defined by the relations (14) and (15).

Perhaps it is not superfluous to describe $\mathcal{S}\langle a\rangle$ directly. On the full subcategory Mlt of $\mathrm{Mlt}\langle\theta\rangle$, $\mathcal{S}\langle a\rangle$ is the same as $\mathcal{S}$. For $(\rho, x)\in\mathrm{Mlt}//\theta$,

$$\mathcal{S}\langle a\rangle((\rho, x)) = \{u\in\mathcal{S}(\rho) : \mathcal{S}(x)(u) = a\};$$

in other words, the elements of kind (ρ, x) in $\mathcal{S}\langle a\rangle$ are those cells of shape ρ in $\mathcal{S}$ whose x-face is equal to a. For $(\rho, x)\in\mathrm{Mlt}//\theta$, $\tau\in\mathrm{Mlt}$, and $(\rho, x)\xrightarrow{q}\tau$, an arrow in $\mathrm{Mlt}\langle\theta\rangle$,

$$\mathcal{S}\langle a\rangle(q)(u) = \mathcal{S}(q)(u);$$

and for $f:(\rho, x)\longrightarrow(\tau, y)\in\mathrm{Arr}(\mathrm{Mlt}//\theta)$,

$$\mathcal{S}\langle a\rangle(f)(u) = \mathcal{S}(f)(u).$$

We make another application of the universal property of $\langle\Lambda\rangle$ to define $\boldsymbol{c} = \mathcal{S}\langle\alpha\rangle$. We define $\boldsymbol{e}$, as before, to be $\boldsymbol{e}=\mathcal{S}$. For $\boldsymbol{d}$, we invoke substitution. Let $(\rho, x)\in\boldsymbol{A}$. We have the maps of formal contexts

$$[\sigma^{-}] \xrightarrow{(\ )^{\circ}_{(\rho,\, x)}} (\rho[\sigma/x])^{\wedge} \ .$$

Taking $\mathrm{hom}_{\mathrm{Fun}}(-,\mathcal{S})$ on this, we get a map

$$\psi_{(\rho,\, x)} : \mathcal{S}(\rho[\sigma/x]) \longrightarrow \mathcal{S}[\sigma^{-}] \ .$$

This is the component at (ρ, x) of a natural transformation ψ in

$$\boldsymbol{A} \ \underset{\ulcorner\mathcal{S}[\sigma^{-}]\urcorner}{\overset{\mathcal{S}\circ\Sigma_{\sigma}}{\rightrightarrows}} \ \mathrm{Set} \ , \qquad \psi,\ \ulcorner\alpha\urcorner : \mathcal{S}\circ\Sigma_{\sigma} \Rightarrow \ulcorner\mathcal{S}[\sigma^{-}]\urcorner \ .$$

$\boldsymbol{d}:\boldsymbol{A}\to\boldsymbol{C}$ is defined as the equalizer

$$\boldsymbol{d} \xrightarrow{\ \varepsilon\ } \mathcal{S}\circ\Sigma_{\sigma} \underset{\ulcorner\alpha\urcorner}{\overset{\psi}{\rightrightarrows}} \ulcorner\mathcal{S}[\sigma^{-}]\urcorner \ .$$

Finally, we define $\delta:\Lambda\to\Phi[\boldsymbol{d}, \boldsymbol{e}]$ to be the composite

$$\Lambda \xrightarrow{(\ )^{\sim}} \mathrm{hom}_{\mathrm{Mlt}}(\Sigma_{\sigma}-,-) \xrightarrow{\dot{\mathcal{S}}} \mathrm{hom}_{\mathrm{Set}}(\mathcal{S}\circ\Sigma_{\sigma}-,\mathcal{S}-) \xrightarrow{(-)\circ\varepsilon} \mathrm{hom}(\boldsymbol{d}-,\boldsymbol{e}-) \ .$$

In concrete terms, we have this. $\mathcal{S}\langle\alpha\rangle:\mathrm{Mlt}\langle\theta\rangle\to\mathrm{Set}$ agrees with $\mathcal{S}$ on Mlt . For $(\rho, x)\in\mathrm{Mlt}//\theta$,

$$\mathcal{S}\langle\alpha\rangle((\rho, x)) = \{t\in\mathcal{S}(\rho[\sigma|x]) : t\hat{\ }\mathring{p}=\alpha(p) \ \text{ for all } \ p\in\sigma\|\} \ .$$

For $f:(\rho, x)\longrightarrow(\tau, y)$, and $t\in\mathcal{S}\langle\alpha\rangle((\rho, x))$,

$$\mathcal{S}\langle\alpha\rangle(f)(t) = \mathcal{S}(f[\sigma])(t) \ .$$

For $q\colon (\rho, x) \longrightarrow \tau \in \mathrm{Arr}(\mathrm{Mlt}\langle\theta\rangle)$, and t as before,

$$\mathcal{S}\langle\alpha\rangle(q)(t) = \mathcal{S}(\tilde{q})(t) .$$

Note that the restrictions of $M=\mathcal{S}\langle a\rangle$ and $N=\mathcal{S}\langle\alpha\rangle$ to the subcategory Mlt of $\mathrm{Mlt}\langle\theta\rangle$ are both equal to $\mathcal{S}$. An equivalence $(P, \mu . \nu) : M \simeq N$ is said to *extend the identity on* Mlt if its restriction to Mlt is equal to $(\mathcal{S}, \mathrm{id}_{\mathcal{S}}, \mathrm{id}_{\mathcal{S}}) : \mathcal{S} \simeq \mathcal{S}$.

We are ready for the first main definition.

Definition. A *multitopic* ω*-category* is a multitopic set $\mathcal{S}$ satisfying the following condition.

For every multitope σ, and every σ-shaped pasting diagram α in $\mathcal{S}$, there is at least one cell a parallel to α such that, for $\theta=\mathrm{c}\sigma$, the $\mathrm{Mlt}\langle\theta\rangle$-structures $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$ are $\mathrm{Mlt}\langle\theta\rangle$-equivalent by an equivalence span that extends the identity on Mlt .

The cell a as in the condition in the definition is said to be a *composite of* α ; composites are not uniquely determined. An equivalence span $E\colon\mathcal{S}\langle a\rangle\simeq\mathcal{S}\langle\alpha\rangle$ extending the identity on Mlt is said to be an *equipment* for a as a composite of α .

Assume that $\mathcal{S}$ is a multitopic ω-category. Let a be any cell in $\mathcal{S}$ of shape θ ; $\dim(a)=k$. A particular application of the definition of "multitopic ω-category" is that the empty k-pd β on a (see section 2) has at least one composite. Such a composite, a k-cell, is what we call an *identity arrow* on a . We denote it by $\mathrm{id}_a\colon a \to a$. However, we have to remember that, due to the virtual character of all operations in multitopic categories, id_a is not uniquely determined. The best way of looking at the idea of "identity arrow" is the same as we look at any concept determined by a universal property. Consider, for instance, "the" product $X\times Y$ of two objects X , Y in a category. The "full determination" of the product includes the projections. In our case, the full determination of "the" *identity arrow*

$\mathrm{id}_a: a \to a$ is an arrow as shown plus the *equipment* making the arrow an identity arrow: the equivalence-span $E:\mathcal{S}\langle \mathrm{id}_a \rangle \simeq \mathcal{S}\langle \beta \rangle$ extending the identity on Mlt.

Let us return to the case of a general pd α as in the definition, and let a be a composite of α; α is a pd of shape σ, a is a cell of shape $\theta = \mathrm{c}\sigma$. Suppose $E:\mathcal{S}\langle a \rangle \simeq \mathcal{S}\langle \alpha \rangle$ is an equipment for a being a composite of α. Now, consider (any instance of) the identity arrow $\mathrm{id}_a: a \to a$ on a. We may, and do, regard id_a to be an element in $\mathcal{S}\langle a \rangle$ of the kind

$$K \underset{\mathrm{def}}{=} (\theta^*, \mathrm{d}_{\theta^*}) \in \mathrm{Mlt}//\theta \subset \mathrm{Mlt}\langle \theta \rangle \quad .$$

The K-context $\vec{a}$ for id_a in $\mathcal{S}\langle a \rangle$ consists of (a and) the faces of a, all cells of kinds in the "old" part Mlt in the signature $\mathrm{Mlt}\langle \theta \rangle$. By definition, we have some $u \in \mathcal{S}\langle \alpha \rangle (K)(\vec{a})$ for which

$$E : (\mathcal{S}\langle a \rangle, \vec{a}, \mathrm{id}_a) \simeq (\mathcal{S}\langle \alpha \rangle, \vec{a}, u) \qquad (11)$$

Note that, as an element of $\mathcal{S}$, u is a $(k+1)$-cell $u: \alpha \to a$ of shape equal to σ.

We define an arrow $u: \alpha \to a$ to be *universal* if it is obtained in the way described: that is, if there exists an equipment E for (11).

A concept of "universal cell" is the basis for the definition of the opetopic weak n-category, for a finite n, in [B/D]. To establish the connection, I make some further definitions.

Let me say that a multitopic set $\mathcal{S}$ is *n-truncated* if for every $k \geq n+1$, every time we have in $\mathcal{S}$ a parallel pair (α, a) of a k-pd α and a k-cell a, there is at least one $(k+1)$-cell b such that $\mathrm{d}b = \alpha$, $\mathrm{c}b = a$. In brief, all $\geq n+2$-cells that may exist do exist. It is immediate to see that an n-truncated multitopic set $\mathcal{S}$ is determined, up to Mlt-equivalence, by its restriction to $\mathrm{Mlt}\restriction(n+1)$, the full subcategory of Mlt consisting of the multitopes of dimensions $\leq (n+1)$. I define a *multitopic n-category* to be a multitopic ω-category which is, at the same time, an n-truncated multitopic set, and, moreover, all $(n+1)$-cells are universal.

It is possible to "normalize" multitopic n-categories, that is, to find, for each multitopic n-category $\mathcal{S}$, another one, $\mathcal{S}^!$, which is $\mathtt{Mlt}$-equivalent to $\mathcal{S}$ (and hence, shares all the "essential" properties with $\mathcal{S}$), and which has some additional desirable properties; namely, $\mathcal{S}^!$ satisfies the following:

(i) k-hom-sets, for $k \geq n+1$, are 0-1 valued: for any $(k-1)$-pd α and n-cell a, there is at most one k-cell $b:\alpha \to a$;

(ii) the composition of n-pd's is single valued: for every n-pd α, the composite of α is uniquely determined (and in fact, the equipment for the composition of α is also uniquely determined).

It is possible to define "multitopic n-category", equivalently to the definition given above, in a way that closely parallels the definition of "opetopic n-category" in [B/D]. One defines a concept of "universal cell", by "backward" recursion, declaring all $(n+2)$-cells to be universal, and by proceeding recursively to determine when lower dimensional cells are recursive. Finally, one postulates the axiom that for $k \leq n$, and every k-pd α, there are (at least one) k-cell and a *universal* $(k+1)$-cell $u:\alpha \to a$. (With the appropriate definition of "universal", the property, postulated as an axiom [B/D], that the composite of universal cells is universal, becomes provable.)

The concept of "multitopic n-category" is first order definable; in fact, it is naturally formulated in FOLDS over $\mathtt{Mlt}\restriction(n+1)$; in particular, it is invariant under $\mathtt{Mlt}\restriction(n+1)$-equivalence. However, the concept of multitopic ω-category does not seem to be first order definable, although it is invariant under $\mathtt{Mlt}$-equivalence.

6. The transfors

Having defined what the 0-cells of the (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ are, we now complete the definition of $\mathtt{Mlt}\omega\mathtt{Cat}$, by describing what its cells, the *transfors*, in general are.

For any given multitope π, the cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ of shape π, the elements of $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$, are called the *$\pi$-transfors*.

A π-transfor is, by definition, a π-colored multitopic set, required to satisfy certain conditions.

Recall that for any $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we have the induced map $\hat{p}=\mathtt{Mlt}[p]:\mathtt{Mlt}[\kappa]\to\mathtt{Mlt}[\pi]$. Thus, if $\mathcal{S}$ is a π-colored multitopic set, $\mathcal{S}:\mathtt{Mlt}[\pi]\to\mathtt{Set}$, we can look at its *$p$-restriction* $\mathcal{S}\restriction p \underset{\text{def}}{=} \mathcal{S}\circ\hat{p}$, a κ-colored multitopic set. The definition of "$\mathcal{S}$ is a π-transfor" will be recursive in the sense that part of the condition will be that for each $p\in\pi|$ (thus, κ_p is of lower dimension than π), $\mathcal{S}\restriction p$ be a κ_p-transfor.

The conditions on a π-colored multitopic set to be a π-transfor will, again, all be of the form that certain entities, satisfying certain universal properties, should exist in $\mathcal{S}$. The conditions fall into two classes. One involves π-colored multitopes of dimensions k greater than n; the conditions in this class are straightforward generalizations of the ones we saw before. The other class pertains to dimensions k at most n. The conditions for $k<n$ are simply inherited by the recursion. For $k=n$, the new condition is of an interesting alternating character: it has two distinct forms depending on whether n is odd or even.

We now give some technical definitions, to be assembled later in the final definition.

A π-colored multitopic set is, as we know, a $\mathtt{Set}$-valued functor on $\mathtt{Mlt}[\pi]$. We now generalize concepts we had for multitopic sets to colored multitopic sets.

Let $\mathcal{S}$ be a π-colored multitopic set, $R\in\mathcal{T}[\pi](\rho)$. We call an element $a\in\mathcal{S}(R)$ an

R-*cell* in $\mathcal{S}$. For $p:R\to U$, the *p-face* of $a\in\mathcal{S}(R)$ is $a\hat{}p \underset{\mathrm{def}}{=} \mathcal{S}(p)(a)$.

Recall the formal context $[R^-]$ for the signature $\mathrm{Mlt}[\pi]=\mathrm{El}(\mathcal{T}[\pi])$ from section 4 (now, $\mathcal{R}=\mathcal{T}[\pi]$). We write $\mathcal{S}[R^-]$ for $\mathrm{hom}([R^-],\mathcal{S})$. An element $\beta\in\mathcal{S}[R^-]$ is what we may regard a *π-colored pasting diagram of type* R (an R-pd, for short). The *domain* $\mathrm{d}a$ of a cell $a\in\mathcal{S}(R\}$ is defined to be the composite $[R^-]\xrightarrow{\mathrm{incl}}\hat{R}\xrightarrow{\dot{a}}\mathcal{S}$, where $\dot{a}$ is defined as in the analogous case in section 2; $\mathrm{d}a$ is an R-pd.

Analogously to the uncolored case, we have the arrow $\iota:[(\mathrm{c}R)^-]\longrightarrow[R^-]$ determined by the commutativity of the diagram

$$\begin{array}{ccc} [R^-] & \xrightarrow{\mathrm{incl}} & \hat{R} \\ {\scriptstyle\iota}\uparrow & \circ & \uparrow{\scriptstyle(\mathrm{c}_R)\hat{}^{\,*}} \\ [(\mathrm{c}R)^-] & \xrightarrow[\mathrm{incl}]{} & \widehat{(\mathrm{c}R)} \end{array} .$$

Generalizing the uncolored case, we define, for an R-pd $\beta\in\mathcal{S}[R^-]$, the *domain* $\mathrm{d}\beta$ of β as the composite

$$\mathrm{d}\beta\ :\ [(\mathrm{c}R)^-]\xrightarrow{\ \iota\ }[R^-]\xrightarrow{\ \beta\ }\mathcal{S}\ .$$

Thus, the domain of an R-pd is defined to be a $(\mathrm{c}R)$-pd.

It was mentioned in section 4 that for any R in $\mathcal{T}[\pi]$, we have a specific $R^*\in\mathcal{T}[\pi]$ such that $\mathrm{d}(R^*)=\langle R\rangle$, $\mathrm{c}(R^*)=R$ and $\gamma_\pi(R^*)=\gamma_\pi(R)$. Any colored cell $a\in\mathcal{S}(R)$ gives rise to an R^*-pd $\langle a\rangle$ for which $\langle a\rangle_R(\mathrm{c}_{R^*})=a$. It is then the case that $\mathrm{d}\langle a\rangle=\mathrm{d}a$.

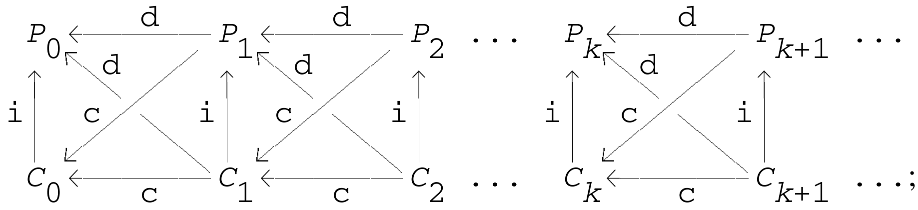
We define the *codomain* $\mathrm{c}\beta$ of $\beta\in\mathcal{S}[R^-]$ as $\mathrm{c}\beta=\beta_{\mathrm{cc}R}(\mathrm{c}_{\mathrm{c}R}\circ\mathrm{c}_R)$. When $\beta=\langle a\rangle$ with $a\in\mathcal{S}(R)$, we obtain that $\mathrm{c}\beta=a\hat{}\mathrm{c}_R \underset{\mathrm{def}}{=} \mathrm{c}a$.

Let us write C_k, P_k for the sets

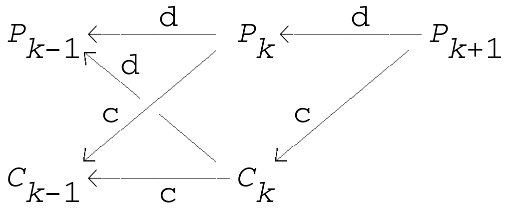
$$C_k = \coprod_{R \in \mathrm{C}_k(\mathrm{Mlt}[\pi])} \mathcal{S}(R) \ ,$$

$$P_k = \coprod_{R \in \mathrm{P}_k(\mathrm{Mlt}[\pi])} \mathcal{S}[R^{-}] \ .$$

With these definitions, we have, analogously to the uncolored case, the items in the following diagram of sets and functions:



and, as before, we have that the diagram



commutes for each $k \geq 1$; that is, $\mathrm{dd}=\mathrm{dc}$, $\mathrm{cd}=\mathrm{cc}$.

We generalize the concepts of "composite", "identity arrow", and "universal arrow" seen in section 5 for the uncolored context.

Let us fix a multitope π , write $n=\mathrm{dim}(\pi)$, and consider $\mathcal{R} = \mathcal{T}[\pi]$; then $\mathrm{Mlt}[\pi] = \mathrm{El}(\mathcal{R})$ is the signature for π-colored multitopic sets. Assume we have a cell S in $\mathcal{R}$ and a d-type substitution operation $(R, x) \longmapsto R[S/x]$ as in defined in section 4. (Examples are provided by the Proposition in 4.1.). As before, $T \stackrel{\mathrm{def}}{=} \mathrm{c}S$. We have the functor Λ given at (3) in section 4. We construct the category $\langle\Lambda\rangle$ according to the general prescription given in section 5 before (11); the signature $\mathtt{Mlt}[\pi]\langle T\rangle$ is defined to be $\langle\Lambda\rangle$.

Let S be a π-colored multitopic set. Fix the cell $a\in\mathcal{S}(T)$ and the pd

$\alpha\in\mathcal{S}[S^{-}]$ ($\overset{=}{\text{def}}$ $\mathrm{hom}([S^{-}],\mathcal{S})$) such that a and α are *parallel*: $\mathrm{d}a=\mathrm{d}\alpha$, $\mathrm{c}a=\mathrm{c}\alpha$. We define the $\mathtt{Mlt}[\pi]\langle T\rangle$-structures $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$ exactly as in the uncolored context in section 5. We say that a is an *S-composite of* α if there is an equivalence-span $E:\mathcal{S}\langle a\rangle\simeq\mathcal{S}\langle\alpha\rangle$ extending the identity on $\mathcal{S}$ (that is, the restriction to $\mathcal{R}$ of E is $(\mathcal{S},\mathrm{id}_{\mathcal{S}},\mathrm{id}_{\mathcal{S}})$).

Next, we define *identity arrows* on arbitrary colored cells.

Let $U\in\mathcal{T}[\pi]$. Put $\theta=\|U\|$, the underlying multitope of U. Let $k=\dim(U)=\dim(\theta)$. In section 2, we constructed the $(k+1)$-dimensional multitope θ^{*} such that $\mathrm{d}\theta^{*}=\langle\theta\rangle$ and $\mathrm{c}\theta^{*}=\theta$, and the $(k+2)$-dimensional multitope θ^{+} such that $\mathrm{c}\theta^{+}=\theta^{*}$ and $\mathrm{d}\theta^{+}$ is "empty" (has no $(k+1)$-dimensional cell in it). Returning to U, we have a unique $U^{+}\in\mathcal{T}[\pi]$ such that $\|U^{+}\|=\theta^{+}$ and $\gamma_{\pi}(U^{+})=\gamma_{\pi}(U)$; we have $\mathrm{c}(U^{+})=U^{*}$, $\dim(U^{+})=k+2$.

d-type U^{+}-substitution: $(R,x)\mapsto R[U^{+}/x]$ is well-defined in $\mathtt{Mlt}[\pi]$, according to 4.1 Proposition, part (iii), since

$$\dim(U^{+}) = k+2 > k=\dim(U)\geq\dim(\mathrm{cod}(\gamma_{\pi}(U))) = \dim(\mathrm{cod}(\gamma_{\pi}(U^{+}))) .$$

For a π-colored multitopic set $\mathcal{S}$ and a cell $a\in\mathcal{S}(R)$, we have the particular π-colored pd $\beta=\beta[a]\in\mathcal{S}[(R^{+})^{-}]$ ("with empty domain") determined by the condition that $\beta_{R}(\mathrm{c}_{R^{*}}\circ\mathrm{c}_{R^{+}})=a$. An *identity arrow on* a is any R^{+}-composite of $\beta[a]$. If i is an identity arrow on a, then i is of the form $i:\langle a\rangle\to a$.

By the notation id_{a}, we refer, ambiguously, to any identity arrow on a; of course, it may not exist.

We define when a cell in a colored multitopic set is *d-universal*.

Let $S\in\mathcal{T}[\pi]$, $\mathcal{S}\in\mathtt{Mlt}[\pi]\mathtt{Set}$, $u\in\mathcal{S}(S)$. Let us write $\alpha\underset{\mathrm{def}}{=}\mathrm{d}u$ and $a\underset{\mathrm{def}}{=}\mathrm{c}u$.

First, assume that S is primitive. By 4.1 Proposition, part (ii) , S-substitution $(R, x)\mapsto R[S/x]$ is now well-defined in $\mathtt{Mlt}[\pi]$.

u is *d-universal* if id_a exists, a is an S-composite of α , and in fact there exists an equivalence-span E witnessing the last fact in which u is related to id_a :

$$E : (\mathcal{S}\langle a\rangle,\mathrm{id}_a) \simeq (\mathcal{S}\langle\alpha\rangle, u) .$$

Next, consider a general S .

Let $\gamma_\pi(S)=(p:\pi\to\kappa)$. We have the functor $\mathtt{Mlt}[p]:\mathtt{Mlt}[\kappa]\longrightarrow\mathtt{Mlt}[\pi]$. We write $\mathcal{S}\restriction p\underset{\mathrm{def}}{=}\mathcal{S}\circ\mathtt{Mlt}[p]$. $\mathcal{S}\restriction p$ is a κ-colored multitopic set. $\dot{S}$, the root of S , is a primitive cell in $\mathcal{T}[\kappa]$ such that $\dot{S}\overset{\mathtt{Mlt}[p]}{\longmapsto}S$; see section 3. We have that $u\in(\mathcal{S}\restriction p)(\dot{S})$; u lives as a type-$\dot{S}$-cell in $\mathcal{S}$; moreover, $\dot{S}$ is primitive. We say that u is *d-universal in* $\mathcal{S}$ if it is d-universal in $\mathcal{S}\restriction p$.

Along with the "left-adjoint type" universal construction of the composite of a pd, we also need a "right-adjoint type" universal construction, a form of right lifting, or right Kan extension. We use the items of the c-type substitution introduced in section 4. As in section 4, we put $P=P[\pi]$ and $T=(\mathrm{cc}P)^*$. We have the functor

$$\Lambda : (\mathtt{Mlt}[\pi]\#\mathrm{T})^{\mathrm{op}}\times\mathtt{Mlt}[\pi]\longrightarrow\mathtt{Set}$$

$$\Big(x\Big\downarrow^{R}_{T} , U\Big) \longmapsto [R\perp_x](U)$$

as constructed in section 4.2 at (6). We define the category $\langle\Lambda\rangle$ by the general "bipartite-graph" construction explained in section 5. $\langle\Lambda\rangle$ is denoted as $\mathtt{Mlt}[\pi]\{P\}$; this is the signature needed to introduce the new ω-dimensional universal property.

Let $\mathcal{S}$ be a π-colored multitopic set. In the c-type substitution, the abstract context $[P\#]$ for $\mathtt{Mlt}[\pi]$ plays the main role. Before proceeding, we pause to explain what an element $\alpha\in\mathcal{S}[P\#] \underset{\text{def}}{=} \mathrm{hom}([P\#], \mathcal{S})$ amounts to.

For any $\alpha\in\mathcal{S}[P\#]$, if we let $\alpha^{+} \underset{\text{def}}{=} \alpha_{\mathrm{c}P}(\#^{++}) \in \mathcal{S}(\mathrm{c}P)$, and $\alpha^{-} \underset{\text{def}}{=} \alpha\restriction[P^{-}] \in \mathcal{S}[P^{-}]$, then $\mathrm{d}(\alpha^{+})=\mathrm{d}(\alpha^{-})$. Conversely, given any $\alpha^{+}\in\mathcal{S}(\mathrm{c}P)$ and $\alpha^{-}\in\mathcal{S}[P^{-}]$, then, first of all, $\mathrm{d}(\alpha^{+})$ and $\mathrm{d}(\alpha^{-})$ are both arrows of the form $[(\mathrm{c}P)^{-}]\longrightarrow\mathcal{S}$. The assertion, easily seen to be true, is that there is $\alpha\in\mathcal{S}[P\#]$ such that $\alpha^{+}=\alpha_{\mathrm{c}P}(\#^{++})$ and $\alpha^{-}=\alpha\restriction[P^{-}]$ iff $\mathrm{d}(\alpha^{+})=\mathrm{d}(\alpha^{-})$. Therefore, talking about an element $\alpha\in\mathcal{S}[P\#]$ is the same thing as talking about a pair (α^{+}, α^{-}) such that $\alpha^{+}\in\mathcal{S}(\mathrm{c}P)$, $\alpha^{-}\in\mathcal{S}[P^{-}]$ and $\mathrm{d}(\alpha^{+})=\mathrm{d}(\alpha^{-})$. In what follows, we use α^{+}, α^{-} for any $\alpha\in\mathcal{S}[P\#]$ in the sense just fixed.

Now, let $\mathcal{S}$ be π-colored multitopic set. Let $a\in\mathcal{S}(T)$ and $\alpha\in\mathcal{S}[P\#] \underset{\text{def}}{=} \mathrm{hom}([P\#], \mathcal{S})$ be such that

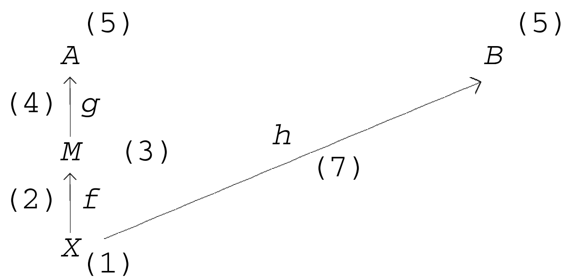
(*) (1) $\alpha^{-}_{\mathrm{cc}P}(\#^{-}) = a\hat{\ }\mathrm{d}_{T}$

and

(2) $\alpha^{+}\hat{\ }\mathrm{c}_{\mathrm{c}P} = a\hat{\ }\mathrm{c}_{T}$

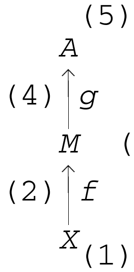
(both these elements belong to the set $\mathcal{S}(\mathrm{cc}P)$).

In the example of π being (2) in section 4, a is a 1-cell of the form $A\xrightarrow{a}B$, where each of the cells A, a, B is colored (5). α is

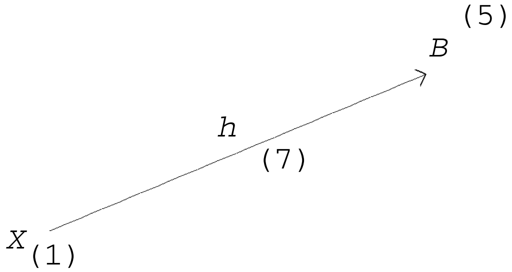


;

α^- is

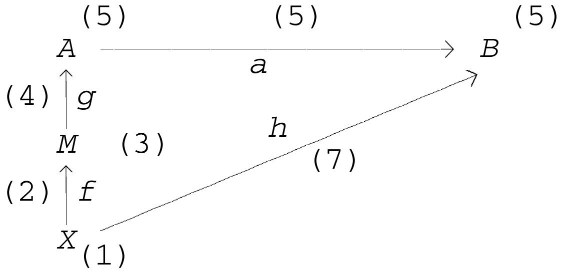


, and α^+ is



.

Note that a and α fit together into the diagram



Returning to the general case, we define the $\mathtt{Mlt}[\pi]\{P\}$-structures $\mathcal{S}\{a\}$, $\mathcal{S}\{\alpha\}$ by the universal property of $\mathtt{Mlt}[\pi]\{P\}=\langle\Lambda\rangle$, through some other data specified first, similarly to what we did before. Instead of going through the full definition, we confine ourselves to giving the clauses of the final elementary descriptions of these items (although in this way we are not giving the reasons why these items are well-defined). As a matter of fact, these elementary descriptions are exactly like the ones we had before for d-type substitution.

On the full subcategory $\mathtt{Mlt}[\pi]$ of $\mathtt{Mlt}\{P\}$, $\mathcal{S}\{a\}$ and $\mathcal{S}\{\alpha\}$ both agree with $\mathcal{S}$. For (R, x) , (S, y) in $\mathtt{Mlt}[\pi]\#T$, $U\in\mathtt{Mlt}[\pi]$, and for $(R, x)\xrightarrow{q}U$, $(R, x)\xrightarrow{f}(S, x)$, arrows in $\mathtt{Mlt}[\pi]\{P\}$,

$$\mathcal{S}\{a\}((R, x)) = \{u\in\mathcal{S}(R) : \mathcal{S}(x)(u)=a\} ,$$

$$\mathcal{S}\{a\}(q)(u) = \mathcal{S}(q)(u) ,$$

$$\mathcal{S}\{a\}(f)(u) = \mathcal{S}(f)(u) \qquad (u\in\mathcal{S}\{a\}((R, x)))$$

$$\mathcal{S}\{\alpha\}((R, x)) = \{t\in\mathcal{S}(R[P\#x]) : t\hat{\ }\mathring{s}=\alpha_S(s) \text{ for all } S\in\mathtt{Mlt}[\pi] \text{ and } s\in[P\#](S)\}$$

$$\mathcal{S}\{\alpha\}(q)(t) = \mathcal{S}(\tilde{p})(t) ,$$

$$\mathcal{S}\{\alpha\}(f)(t) = \mathcal{S}(f[P])(t) \qquad (t\in\mathcal{S}\{\alpha\}((R, x)))$$

For $a\in\mathcal{S}(T)$ and $\alpha\in\mathcal{S}[P\#]$ satisfying (*), we say that a *is a right lifting of* α^{+} *along* α^{-} if there is an equivalence-span $E:\mathcal{S}\{a\}\simeq\mathcal{S}\{\alpha\}$ extending the identity on $\mathtt{Mlt}[\pi]$.

One more piece of terminology. For $\alpha^{-}\in\mathcal{S}[P^{-}]$, we that α^{-} is *d-universal* if all codimension-1 faces of α^{-} (all cells of the form $\alpha^{-}_{K}(p)\in\mathcal{S}(K)$, with $(p:P\to K)\in[P^{-}](\kappa)$, and $\dim(K)=n-1$) are d-universal, in the sense specified above.

We are ready to give the definition of the (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$.

We define the (large) set $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$ by recursion on $\pi\in\mathtt{Mlt}$. Fix π ; let $n=\dim(\pi)$. $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$ is the class of all (small) π-colored multitopic sets $\mathcal{S}$ satisfying the following conditions (i) to (iii).

(i) For all $p:\pi\to\kappa\neq\mathrm{id}_{\pi}$, $\mathcal{S}\restriction p\underset{\mathrm{def}}{=}\mathcal{S}\circ\mathtt{Mlt}[p]$ belongs to $\mathtt{Mlt}\omega\mathtt{Cat}(\kappa)$;

(ii) For all $S\in\mathtt{Mlt}[\pi]$ such that, for $p=\gamma_{\pi}(S)$, $p:\pi\to\kappa$, $k=\dim(\kappa)$, we have $\dim(S)>k$, every pd $\alpha\in\mathcal{S}[S^{-}]$ has at least one S-composite in $\mathcal{S}$ [the notion of composite refers to the substitution operation in 4.1 Proposition (iii)];

(iii) **(a)** if n is odd, then for $P=P[\pi]$, every $\alpha^{-}\in\mathcal{S}[P^{-}]$ has at least one P-composite in $\mathcal{S}$;

(b) if n is even and positive, then for $P=P[\pi]$, and $T=(ccP)^{*}$, for every $\alpha\in\mathcal{S}(P\#)$ such that α^{-} is d-universal, α^{+} has a right lifting along α^{-} .

Having defined the action of

$$\mathtt{Mlt}\omega\mathtt{Cat}:\mathtt{Mlt}\longrightarrow\mathtt{SET}$$

on objects, for an arrow $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we define

$$\mathtt{Mlt}\omega\mathtt{Cat}(p)(\pi) = \langle\, \mathcal{S}\in\mathtt{Mlt}\omega\mathtt{Cat}[\pi] \longmapsto \mathcal{S}\upharpoonright p\in\mathtt{Mlt}\omega\mathtt{Cat}[\kappa] \,\rangle$$

Theorem $\mathtt{Mlt}\omega\mathtt{Cat}$ is a (large) multitopic ω-category.

References

[B] M. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics* **136** (1998), 39-103 .

[B/D] J. C. Baez and J. Dolan, Higher-dimensional algebra III. n-categories and the algebra of opetopes. *Advances in Mathematics* **135** (1998), 145-206.

[Ca] J. Cartmell, Generalized algebraic theories and contextual categories. *Annals of Pure and Applied Logic* **32** (1986), 209-243.

[Cr] S. E. Crans, Localizations of transfors, Preprint 98/237, Macquarie University.

[H/M/P] C. Hermida, M. Makkai and J. Power, On weak higher dimensional categories I. Accepted by: *Journal of Pure and Applied Algebra.*

[L] J. Lambek, Deductive systems and categories II. *Lecture Notes in Mathematics* **86** (1969), 76-122 (Springer-Verlag).

[M1] M. Makkai, Avoiding the axiom of choice in general category theory. *Journal of Pure and Applied Algebra* **108** (1996), 109-173.

[M2] M. Makkai, Towards a categorical foundation of mathematics. In: *Logic Colloquium '95* (J. A. Makowski and E. V. Ravve, editors). Lecture Notes in Logic, no. **11**, New York, Springer-Verlag; pp. 153-190.

[M3] M. Makkai, *First Order Logic with Dependent Sorts*. Research monograph, accepted by Lecture Notes in Logic (Springer-Verlag), 200 pages. Available electronically. Under revision.

[M-L] P. Martin-Lof, An Intuitionistic Theory of Types. In: *Proceedings of the Bristol Logic Colloquium*, North-Holland, 1973.