6. The transfors

Having defined what the 0-cells of the (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ are, we now complete the definition of $\mathtt{Mlt}\omega\mathtt{Cat}$, by describing what its cells, the *transfors*, in general are.

For any given multitope π, the cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ of shape π, the elements of $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$, are called the *$\pi$-transfors*.

A π-transfor is, by definition, a π-colored multitopic set, required to satisfy certain conditions.

Recall that for any $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we have the induced map $\hat{p}=\mathtt{Mlt}[p]:\mathtt{Mlt}[\kappa]\to\mathtt{Mlt}[\pi]$. Thus, if $\mathcal{S}$ is a π-colored multitopic set, $\mathcal{S}:\mathtt{Mlt}[\pi]\to\mathtt{Set}$, we can look at its *$p$-restriction* $\mathcal{S}\restriction p \underline{\underline{\mathrm{def}}} \mathcal{S}\circ\hat{p}$, a κ-colored multitopic set. The definition of "$\mathcal{S}$ is a π-transfor" will be recursive in the sense that part of the condition will be that for each $p\in\pi|$ (thus, K_p is of lower dimension than π), $\mathcal{S}\restriction p$ be a K_p-transfor.

The conditions on a π-colored multitopic set to be a π-transfor will, again, all be of the form that certain entities, satisfying certain universal properties, should exist in $\mathcal{S}$. The conditions fall into two classes. One involves π-colored multitopes of dimensions k greater than n; the conditions in this class are straightforward generalizations of the ones we saw before. The other class pertains to dimensions k at most n. The conditions for $k<n$ are simply inherited by the recursion. For $k=n$, the new condition is of an interesting alternating character: it has two distinct forms depending on whether n is odd or even.

We now give some technical definitions, to be assembled later in the final definition.

A π-colored multitopic set is, as we know, a $\mathtt{Set}$-valued functor on $\mathtt{Mlt}[\pi]$. We now generalize concepts we had for multitopic sets to colored multitopic sets.

Let $\mathcal{S}$ be a π-colored multitopic set, $R\in\mathcal{T}[\pi](\rho)$. We call an element $a\in\mathcal{S}(R)$ an

R-cell in $\mathcal{S}$. For $p:R\to U$, the *p-face* of $a\in\mathcal{S}(R)$ is $a\hat{}p_{\mathrm{def}}\mathcal{S}(p)(a)$.

Recall the formal context $[R^-]$ for the signature $\mathrm{Mlt}[\pi]=\mathrm{El}(\mathcal{T}[\pi])$ from section 4 (now, $\mathcal{R}=\mathcal{T}[\pi]$). We write $\mathcal{S}[R^-]$ for $\hom([R^-],\mathcal{S})$. An element $\beta\in\mathcal{S}[R^-]$ is what we may regard a *π-colored pasting diagram of type* R (an R-pd, for short). The *domain* $\mathrm{d}a$ of a cell $a\in\mathcal{S}(R\}$ is defined to be the composite $[R^-]\xrightarrow{\mathrm{incl}}\hat{R}\xrightarrow{\dot{a}}\mathcal{S}$, where $\dot{a}$ is defined as in the analogous case in section 2; $\mathrm{d}a$ is an R-pd.

Analogously to the uncolored case, we have the arrow $\iota:[(\mathrm{c}R)^-]\longrightarrow[R^-]$ determined by the commutativity of the diagram

$$\begin{array}{ccc}
[R^-] & \xrightarrow{\mathrm{incl}} & \hat{R} \\
{\scriptstyle\iota}\uparrow & \circ & \uparrow{\scriptstyle(\mathrm{c}_R)^*} \\
[(\mathrm{c}R)^-] & \xrightarrow[\mathrm{incl}]{} & (\mathrm{c}R)\hat{}
\end{array} .$$

Generalizing the uncolored case, we define, for an R-pd $\beta\in\mathcal{S}[R^-]$, the *domain* $\mathrm{d}\beta$ of β as the composite

$$\mathrm{d}\beta : [(\mathrm{c}R)^-]\xrightarrow{\iota}[R^-]\xrightarrow{\beta}\mathcal{S} .$$

Thus, the domain of an R-pd is defined to be a $(\mathrm{c}R)$-pd.

It was mentioned in section 4 that for any R in $\mathcal{T}[\pi]$, we have a specific $R^*\in\mathcal{T}[\pi]$ such that $\mathrm{d}(R^*)=\langle R\rangle$, $\mathrm{c}(R^*)=R$ and $\gamma_\pi(R^*)=\gamma_\pi(R)$. Any colored cell $a\in\mathcal{S}(R)$ gives rise to an R^*-pd $\langle a\rangle$ for which $\langle a\rangle_R(\mathrm{c}_{R^*})=a$. It is then the case that $\mathrm{d}\langle a\rangle=\mathrm{d}a$.

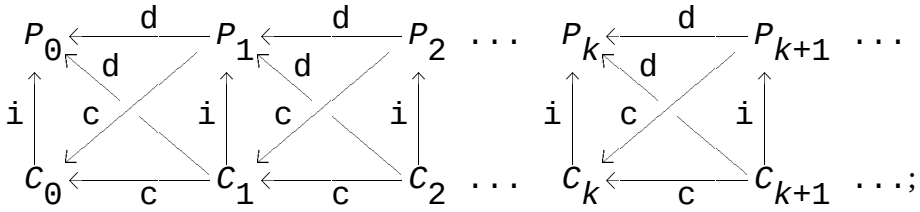
We define the *codomain* $\mathrm{c}\beta$ of $\beta\in\mathcal{S}[R^-]$ as $\mathrm{c}\beta=\beta_{\mathrm{c}cR}(\mathrm{c}_{\mathrm{c}R}\circ\mathrm{c}_R)$. When $\beta=\langle a\rangle$ with $a\in\mathcal{S}(R)$, we obtain that $\mathrm{c}\beta=a\hat{}\mathrm{c}_R\,\mathrm{d}_{\mathrm{def}}\,\mathrm{c}a$.

Let us write C_k, P_k for the sets

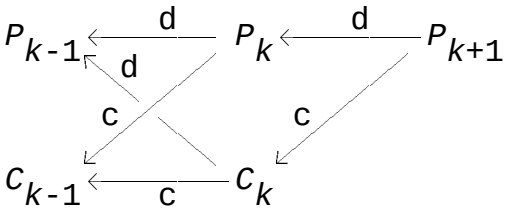
$$C_k = \coprod_{R \in C_k(\mathtt{Mlt}[\pi])} \mathcal{S}(R)\ ,$$

$$P_k = \coprod_{R \in P_k(\mathtt{Mlt}[\pi])} \mathcal{S}[R^-]\ .$$

With these definitions, we have, analogously to the uncolored case, the items in the following diagram of sets and functions:



and, as before, we have that the diagram



commutes for each $k \geq 1$; that is, $\mathtt{dd=dc}$, $\mathtt{cd=cc}$.

We generalize the concepts of "composite", "identity arrow", and "universal arrow" seen in section 5 for the uncolored context.

Let us fix a multitope π , write $n=\mathtt{dim}(\pi)$, and consider $\mathcal{R} = \mathcal{T}[\pi]$; then $\mathtt{Mlt}[\pi] = \mathtt{El}(\mathcal{R})$ is the signature for π-colored multitopic sets. Assume we have a cell S in $\mathcal{R}$ and a d-type substitution operation $(R, x) \longmapsto R[S/x]$ as in defined in section 4. (Examples are provided by the Proposition in 4.1.). As before, $T \stackrel{\mathtt{def}}{=} \mathtt{c}S$. We have the functor Λ given at (3) in section 4. We construct the category $\langle \Lambda \rangle$ according to the general

prescription given in section 5 before (11); the signature $\mathtt{Mlt}[\pi]\langle T\rangle$ is defined to be $\langle\Lambda\rangle$.

Let S be a π-colored multitopic set. Fix the cell $a\in\mathcal{S}(T)$ and the pd $\alpha\in\mathcal{S}[S^-]$ ($\underline{\underline{\mathtt{def}}}\ \mathtt{hom}([S^-],\mathcal{S})$) such that a and α are *parallel*: $\mathtt{d}a=\mathtt{d}\alpha$, $\mathtt{c}a=\mathtt{c}\alpha$. We define the $\mathtt{Mlt}[\pi]\langle T\rangle$-structures $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$ exactly as in the uncolored context in section 5. We say that a is an *$\mathcal{S}$-composite of α* if there is an equivalence-span $E:\mathcal{S}\langle a\rangle\simeq\mathcal{S}\langle\alpha\rangle$ extending the identity on $\mathcal{S}$ (that is, the restriction to $\mathcal{T}$ of E is $(\mathcal{S},\mathtt{id}_{\mathcal{S}},\mathtt{id}_{\mathcal{S}})$).

Next, we define *identity arrows* on arbitrary colored cells.

Let $U\in\mathcal{T}[\pi]$. Put $\theta=\|U\|$, the underlying multitope of U. Let $k=\mathtt{dim}(U)=\mathtt{dim}(\theta)$. In section 2, we constructed the $(k+1)$-dimensional multitope θ^* such that $\mathtt{d}\theta^*=\langle\theta\rangle$ and $\mathtt{c}\theta^*=\theta$, and the $(k+2)$-dimensional multitope θ^+ such that $\mathtt{c}\theta^+=\theta^*$ and $\mathtt{d}\theta^+$ is "empty" (has no $(k+1)$-dimensional cell in it). Returning to U, we have a unique $U^+\in\mathcal{T}[\pi]$ such that $\|U^+\|=\theta^+$ and $\gamma_\pi(U^+)=\gamma_\pi(U)$; we have $\mathtt{c}(U^+)=U^*$, $\mathtt{dim}(U^+)=k+2$.

d-type U^+-substitution: $(R,x)\mapsto R[U^+/x]$ is well-defined in $\mathtt{Mlt}[\pi]$, according to 4.1 Proposition, part (iii), since

$$\mathtt{dim}(U^+)=k+2>k=\mathtt{dim}(U)\geq\mathtt{dim}(\mathtt{cod}(\gamma_\pi(U))=\mathtt{dim}(\mathtt{cod}(\gamma_\pi(U^+)).$$

For a π-colored multitopic set $\mathcal{S}$ and a cell $a\in\mathcal{S}(R)$, we have the particular π-colored pd $\beta=\beta[a]\in\mathcal{S}[(R^+)^-]$ ("with empty domain") determined by the condition that $\beta_R(\mathtt{c}_{R^*}\circ\mathtt{c}_{R^+})=a$. An *identity arrow on a* is any R^+-composite of $\beta[a]$. If i is an identity arrow on a, then i is of the form $i:\langle a\rangle\to a$.

By the notation $\mathtt{id}_a$, we refer, ambiguously, to any identity arrow on a; of course, it may not exist.

We define when a cell in a colored multitopic set is *d-universal*.

Let $S \in \mathcal{T}[\pi]$, $\mathcal{S} \in \mathtt{Mlt}[\pi]\mathtt{Set}$, $u \in \mathcal{S}(S)$. Let us write $\alpha_{\mathtt{d\bar{e}f}} \mathtt{d}u$ and $a_{\mathtt{d\bar{e}f}} \mathtt{c}u$.

First, assume that S is primitive. By 4.1 Proposition, part (ii) , S-substitution $(R, x) \mapsto R[S/x]$ is now well-defined in $\mathtt{Mlt}[\pi]$.

u is *d-universal* if $\mathtt{id}_a$ exists, a is an S-composite of α , and in fact there exists an equivalence-span E witnessing the last fact in which u is related to $\mathtt{id}_a$:

$$E : (\mathcal{S}\langle a \rangle, \mathtt{id}_a) \simeq (\mathcal{S}\langle \alpha \rangle, u) \ .$$

Next, consider a general S .

Let $\gamma_\pi(S) = (p : \pi \to \kappa)$. We have the functor $\mathtt{Mlt}[p] : \mathtt{Mlt}[\kappa] \longrightarrow \mathtt{Mlt}[\pi]$. We write $\mathcal{S}\restriction p \ {}_{\mathtt{d\bar{e}f}} \mathcal{S} \circ \mathtt{Mlt}[p]$. $\mathcal{S}\restriction p$ is a κ-colored multitopic set. $\dot{S}$, the root of S , is a primitive cell in $\mathcal{T}[\kappa]$ such that $\dot{S} \overset{\mathtt{Mlt}[p]}{\longmapsto} S$; see section 3. We have that $u \in (\mathcal{S}\restriction p)(\dot{S})$; u lives as a type-$\dot{S}$-cell in $\mathcal{S}$; moreover, $\dot{S}$ is primitive. We say that u is *d-universal in* $\mathcal{S}$ if it is d-universal in $\mathcal{S}\restriction p$.

Along with the "left-adjoint type" universal construction of the composite of a pd, we also need a "right-adjoint type" universal construction, a form of right lifting, or right Kan extension. We use the items of the c-type substitution introduced in section 4. As in section 4, we put $P = P[\pi]$ and $T = (\mathtt{c}\mathtt{c}P)^*$. We have the functor

$$\Lambda : (\mathtt{Mlt}[\pi]\#\mathtt{T})^{\mathtt{op}} \times \mathtt{Mlt}[\pi] \longrightarrow \mathtt{Set}$$

$$\left(x \overset{R}{\downarrow} T \ , \ U\right) \longmapsto [R \perp_x](U)$$

as constructed in section 4.2 at (6). We define the category $\langle \Lambda \rangle$ by the general "bipartite-graph" construction explained in section 5. $\langle \Lambda \rangle$ is denoted as $\mathtt{Mlt}[\pi]\{P\}$; this is the signature needed to introduce the new ω-dimensional universal property.

Let $\mathcal{S}$ be a π-colored multitopic set. In the c-type substitution, the abstract context $[P\#]$ for $\mathtt{Mlt}[\pi]$ plays the main role. Before proceeding, we pause to explain what an element $\alpha\in\mathcal{S}[P\#]_{\mathtt{def}}\mathtt{hom}([P\#],\mathcal{S})$ amounts to.

For any $\alpha\in\mathcal{S}[P\#]$, if we let $\alpha^+{}_{\mathtt{def}}\alpha_{\mathtt{c}P}(\#^{++})\in\mathcal{S}(\mathtt{c}P)$, and $\alpha^-{}_{\mathtt{def}}\alpha\restriction[P^-]\in\mathcal{S}[P^-]$, then $\mathtt{d}(\alpha^+)=\mathtt{d}(\alpha^-)$. Conversely, given any $\alpha^+\in\mathcal{S}(\mathtt{c}P)$ and $\alpha^-\in\mathcal{S}[P^-]$, then, first of all, $\mathtt{d}(\alpha^+)$ and $\mathtt{d}(\alpha^-)$ are both arrows of the form $[(\mathtt{c}P)^-]\longrightarrow\mathcal{S}$. The assertion, easily seen to be true, is that there is $\alpha\in\mathcal{S}[P\#]$ such that $\alpha^+=\alpha_{\mathtt{c}P}(\#^{++})$ and $\alpha^-=\alpha\restriction[P^-]$ iff $\mathtt{d}(\alpha^+)=\mathtt{d}(\alpha^-)$. Therefore, talking about an element $\alpha\in\mathcal{S}[P\#]$ is the same thing as talking about a pair (α^+,α^-) such that $\alpha^+\in\mathcal{S}(\mathtt{c}P)$, $\alpha^-\in\mathcal{S}[P^-]$ and $\mathtt{d}(\alpha^+)=\mathtt{d}(\alpha^-)$. In what follows, we use α^+, α^- for any $\alpha\in\mathcal{S}[P\#]$ in the sense just fixed.

Now, let $\mathcal{S}$ be π-colored multitopic set. Let $a\in\mathcal{S}(T)$ and $\alpha\in\mathcal{S}[P\#]_{\mathtt{def}}\mathtt{hom}([P\#],\mathcal{S})$ be such that

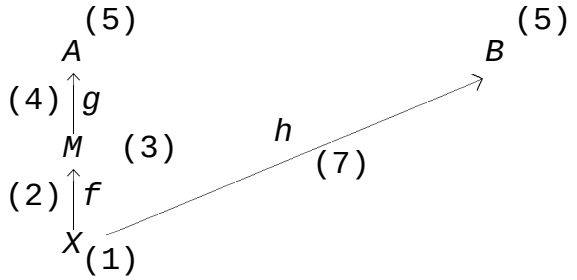
(*) (1) $\alpha^-_{\mathtt{cc}P}(\#^-)=a\hat{}\mathtt{d}_T$

and

(2) $\alpha^+\hat{}\mathtt{c}_{\mathtt{c}P}=a\hat{}\mathtt{c}_T$

(both these elements belong to the set $\mathcal{S}(\mathtt{cc}P)$).

In the example of π being (2) in section 4, a is a 1-cell of the form $A\xrightarrow{\ a\ }B$, where each of the cells A, a, B is colored (5). α is



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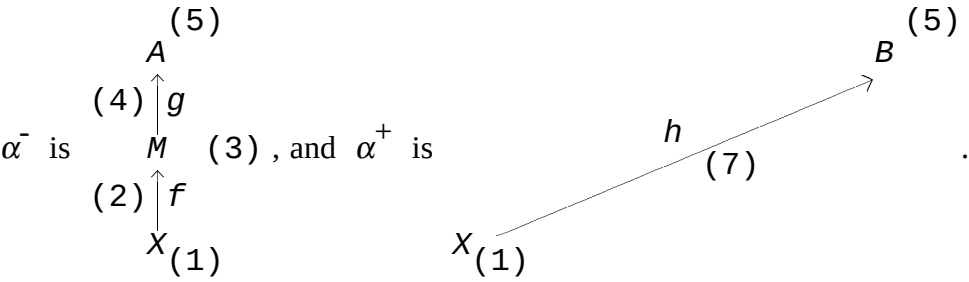
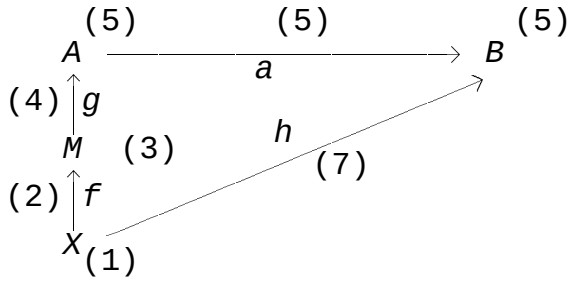


Note that a and α fit together into the diagram



Returning to the general case, we define the $\mathtt{Mlt}[\pi]\{P\}$-structures $\mathcal{S}\{a\}$, $\mathcal{S}\{\alpha\}$ by the universal property of $\mathtt{Mlt}[\pi]\{P\}=\langle\Lambda\rangle$, through some other data specified first, similarly to what we did before. Instead of going through the full definition, we confine ourselves to giving the clauses of the final elementary descriptions of these items (although in this way we are not giving the reasons why these items are well-defined). As a matter of fact, these elementary descriptions are exactly like the ones we had before for d-type substitution.

On the full subcategory $\mathtt{Mlt}[\pi]$ of $\mathtt{Mlt}\{P\}$, $\mathcal{S}\{a\}$ and $\mathcal{S}\{\alpha\}$ both agree with $\mathcal{S}$. For (R, x), (S, y) in $\mathtt{Mlt}[\pi]\#T$, $U\in\mathtt{Mlt}[\pi]$, and for $(R, x)\xrightarrow{q}U$, $(R, x)\xrightarrow{f}(S, x)$, arrows in $\mathtt{Mlt}[\pi]\{P\}$,

$$\mathcal{S}\{a\}((R, x)) = \{u\in\mathcal{S}(R) : \mathcal{S}(x)(u)=a\},$$
$$\mathcal{S}\{a\}(q)(u) = \mathcal{S}(q)(u),$$
$$\mathcal{S}\{a\}(f)(u) = \mathcal{S}(f)(u) \qquad (u\in\mathcal{S}\{a\}((R, x)))$$

$$\mathcal{S}\{\alpha\}((R, x)) = \{t\in\mathcal{S}(R[P\#x]) : t\hat{\ }\mathring{s}=\alpha_S(s) \text{ for all } S\in\mathtt{Mlt}[\pi] \text{ and } s\in[P\#](S)\}$$
$$\mathcal{S}\{\alpha\}(q)(t) = \mathcal{S}(\tilde{p})(t),$$
$$\mathcal{S}\{\alpha\}(f)(t) = \mathcal{S}(f[P])(t) \qquad (t\in\mathcal{S}\{\alpha\}((R, x)))$$

For $a \in \mathcal{S}(T)$ and $\alpha \in \mathcal{S}[P\#]$ satisfying (*), we say that a *is a right lifting of* α^+ *along* α^- if there is an equivalence-span $E : \mathcal{S}\{a\} \simeq \mathcal{S}\{\alpha\}$ extending the identity on $\mathtt{Mlt}[\pi]$.

One more piece of terminology. For $\alpha^- \in \mathcal{S}[P^-]$, we that α^- is *d-universal* if all codimension-1 faces of α^- (all cells of the form $\alpha^-_K(p) \in \mathcal{S}(K)$, with $(p : P \to K) \in [P^-](\kappa)$, and $\dim(K) = n-1$) are d-universal, in the sense specified above.

We are ready to give the definition of the (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$.

We define the (large) set $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$ by recursion on $\pi \in \mathtt{Mlt}$. Fix π; let $n = \dim(\pi)$. $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$ is the class of all (small) π-colored multitopic sets $\mathcal{S}$ satisfying the following conditions (i) to (iii).

(i) For all $p : \pi \to \kappa \neq \mathtt{id}_\pi$, $\mathcal{S}\restriction p \underline{\underline{\mathrm{def}}} \mathcal{S} \circ \mathtt{Mlt}[p]$ belongs to $\mathtt{Mlt}\omega\mathtt{Cat}(\kappa)$;

(ii) For all $S \in \mathtt{Mlt}[\pi]$ such that, for $p = \gamma_\pi(S)$, $p : \pi \to \kappa$, $k = \dim(\kappa)$, we have $\dim(S) > k$, every pd $\alpha \in \mathcal{S}[S^-]$ has at least one S-composite in $\mathcal{S}$ [the notion of composite refers to the substitution operation in 4.1 Proposition (iii)];

(iii) **(a)** if n is odd, then for $P = P[\pi]$, every $\alpha^- \in \mathcal{S}[P^-]$ has at least one P-composite in $\mathcal{S}$;

(b) if n is even and positive, then for $P = P[\pi]$, and $T = (\mathtt{cc}P)^*$, for every $\alpha \in \mathcal{S}(P\#)$ such that α^- is d-universal, α^+ has a right lifting along α^-.

Having defined the action of

$$\mathtt{Mlt}\omega\mathtt{Cat} : \mathtt{Mlt} \longrightarrow \mathtt{SET}$$

on objects, for an arrow $p : \pi \to \kappa$ in $\mathtt{Mlt}$, we define

$$\mathtt{Mlt}\omega\mathtt{Cat}(p)(\pi) = \langle \mathcal{S} \in \mathtt{Mlt}\omega\mathtt{Cat}[\pi] \longmapsto \mathcal{S} \upharpoonright p \in \mathtt{Mlt}\omega\mathtt{Cat}[\kappa] \rangle$$

Theorem $\mathtt{Mlt}\omega\mathtt{Cat}$ is a (large) multitopic ω-category.