5. Omega-dimensional universal properties

The fundamental idea of multitopic categories, borrowed from the Baez/Dolan "opetopic" concept [B/D], is that *the composite of each composable system of cells is* to be *defined by a universal property*. Unlike in *loc.cit.* however, in this paper, the universal properties in question will be specified by uses of the concept of FOLDS equivalence (see Section 1 above). Another difference to *loc.cit.* is that we work in an untruncated multitopic set, aiming at weak ω-categories, and not just weak n-categories for finite n as in *loc.cit.* The finite-dimensional notion appears as a simple truncation of the ω-dimensional notion.

Let R be a commutative ring. Classically, the tensor product $A \otimes B$ of two R-modules A, B is defined by *using* a universal property: $A \otimes B$ is such that there exists a universal R-bilinear map $(A, B) \longrightarrow A \otimes B$. $A \otimes B$ is *defined* by making a specific choice of the particular module $A \otimes B$. (For the sake of the rest of the structure, a choice of a particular universal R-bilinear map $(A, B) \longrightarrow A \otimes B$ is also made.) The resulting monoidal category $R\text{-}\mathtt{Mod}$ is, in fact, a bicategory with a single $\mathtt{0}$-cell in which the modules are the $\mathtt{1}$-cells, and the tensor product is the composition of $\mathtt{1}$-cells.

The fundamental difference of the present approach (already present in [B/D], and in special cases, even in the earlier [M1]), to the classical one is that we will have composition as a *virtual* (an adjective taken from [B/D]) operation. In the special case we chose for illustration, this means that we will be content to "define" $A \otimes B$ by the described universal property, without making specific choices of each and every $A \otimes B$. In particular, in the bicategory $R\text{-}\mathtt{Mod}$, the result of the composition of $\mathtt{1}$-cells will be given only up to isomorphism only, in contrast to the classical concept of "bicategory" in which all compositions (of $\mathtt{1}$-cells and of $\mathtt{2}$-cells) are ordinary algebraic operations. It is important to realize that this procedure makes sense only if we adopt the structure of R-bilinear maps (and preferably, even that of k-ary R-linear maps of R-modules) as part of the structure of $R\text{-}\mathtt{Mod}$.

The universal property of $A \otimes B$ is that there exists a system of bijections

$$\mathtt{hom}(A \otimes B, ?) \xrightarrow{\cong} \mathtt{hom}_{\mathtt{bilin}}((A, B), ?) \qquad (1)$$

for variable $\mathtt{1}$-cells $?$ (we continue talking about $R\text{-}\mathtt{Mod}$ as a bicategory). When the

dimension is raised, and we want to define the composite $A \otimes B$ of 1-cells in a *tri*category, say, then the isomorphisms in (1) will, presumably, be replaced by equivalences of hom-categories. And similarly for even higher dimensions, when the concept of "equivalence" will be taken to be the corresponding higher dimensional one.

More generally than the composition of two 1-cells discussed above, we will introduce a concept of composition of an arbitrary pasting diagram in a multitopic set. Let $\mathcal{S}$ be a multitopic set, let α be a k-dimensional pasting diagram in $\mathcal{S}$. We will define what it means for a k-cell a in $\mathcal{S}$ to be a *composite* of α. The first requirement is that a be parallel to α: $da = d\alpha$, $ca = c\alpha$. The main part of the definition says, at present only allusively, that

the structure of arrows out of a (2)
is equivalent to (3)
the structure of arrows out of α. (4).

This phrasing is intended to convey a general idea behind the special case (1) of the tensor product above: the "structure of arrows out of a", when a is $A \otimes B$, is $\hom(A \otimes B, ?)$, the totality of the hom-sets, or hom-categories, or hom-bi-categories, etc, of arrows out of $A \otimes B$; the "structure of the arrows out of α", now when $\alpha = (A, B)$, is $\hom_{\mathtt{bilin}}((A, B), ?)$. We will do two things: we will make precise the sense of the "structures" (2) and (4); and then we will deploy the general concept of FOLDS-equivalence to make sense of "equivalent" in (3).

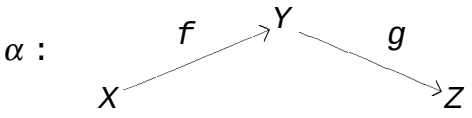
We will denote structure (2) by $\mathcal{S}\langle a \rangle$, structure (4) by $\mathcal{S}\langle \alpha \rangle$.

Before turning to the general definitions, we consider some examples.

In some, fixed, multitopic set $\mathcal{S}$, let a be the 1-cell

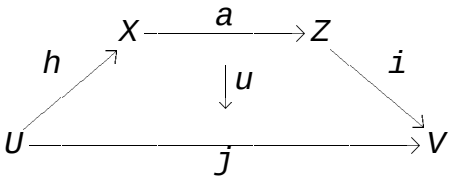
$$X \xrightarrow{\ a\ } Z$$

and α the 1-pd

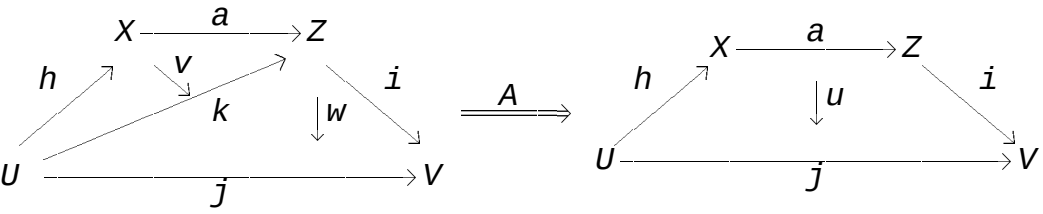


a and α have been taken to be parallel. Until further notice, let us fix the items listed so far. We will now explain parts of structures (2) and (4), specified by the items given so far, as well as what an equivalence (3) would be like.

Consider, for instance, the 2-cell u in $\mathcal{S}$:

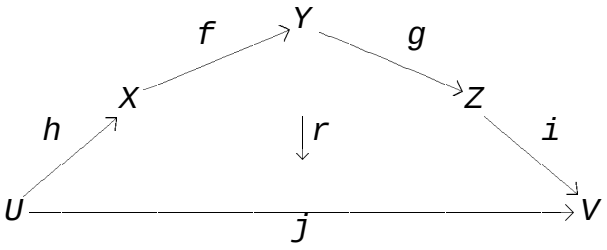


This is part of the "structure of arrows out of a ", structure (2), also denoted $\mathcal{S}\langle a\rangle$. But also, for instance, the 3-cell A :

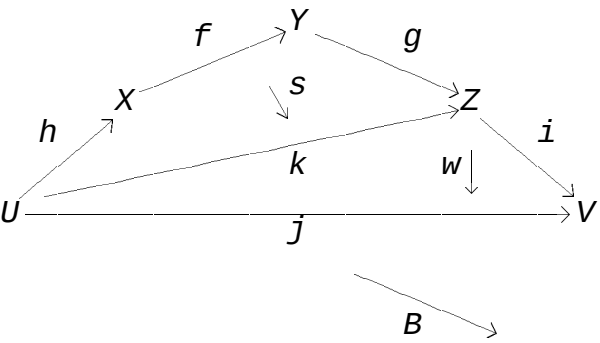


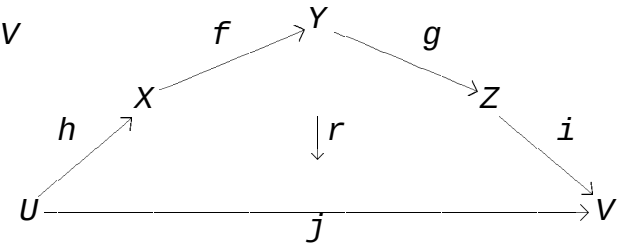
figures in structure (2).

In structure (4), there will be *corresponding* cells, r and B as follows:



and





The *kind* that the element u belongs to in $\mathcal{S}\langle a\rangle$ will be the *same* as the *kind* element r belongs to in $\mathcal{S}\langle\alpha\rangle$; let us call this kind K_1 . The same goes for A versus B ; let us call the common kind of A and B : K_2 . Here we use the word "kind" in the sense used in the context of FOLDS: an object of the signature category, the common signature of $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$.

But more is true. In fact, the *sorts* u in $\mathcal{S}\langle a\rangle$ and r in $\mathcal{S}\langle a\rangle$ belong to are also the *same*. Remember that *sorts* are *dependent sorts*; a sort headed by a kind depends on an array of variables that is indexed by the arity of the kind, the non-identity arrows out of the kind. We note, first of all, that the new signature will extend $\mathtt{Mlt}$, the signature consisting of multitopes; the elements U, X, Z, V, h, i, j , the common faces of u and r , will appear in both $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$ sorted as in $\mathcal{S}$. Now, the elements u and r both will be of the sort

$$K_1(U, X, Z, V, h, i, j) ,$$

albeit in different structures. In other words, the contexts for both u and r in the new structures will consist of the elements U, X, Z, V, h, i, j .

We see that the main act of making the element u of $\mathcal{S}$ an element of the $\mathcal{S}\langle a\rangle$ is to *remove* the dependence on the item a ; for r , we *remove* f, Y, g ; by these removals, the points of references of u and r became the same, and so, u and r will relatable in a (FOLDS-)equivalence. Of course, the removed items: a, f, Y, g are constant for $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$, so there is no logical need for referring to them as values of variables.

Let us turn to the sortings of A and B . In $\mathcal{S}\langle a\rangle$, A will belong to the sort

$$K_2(U, X, Z, V; h, i, j, k; u, v, w) ,$$

while in $\mathcal{S}\langle\alpha\rangle$, B belongs to

$$K_2(U, X, Z, V; h, i, j, k; r, s, w) \ .$$

These are not the "same" sorts; they differ in two places: at u *vs* r and v *vs* s . Note that there will be two particular arrows $K_2 \xrightarrow{p} K_1$ and $K_2 \xrightarrow{q} K_1'$ (K_1 was mentioned above; K_1' is another new kind, corresponding to the shape of v and s) in the new signature; p "picks out" u ($(\mathcal{S}\langle a\rangle)(p)(A)=u$) , respectively r ; and q picks out v , respectively s .

A and B *may be related* by the equivalence (3) that we will have between $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$. In the first place, the equivalence will simply be the identity on the old part `Mlt` of the new signature; thus, denoting equivalence (3) by E ,

$$E : (\mathcal{S}\langle a\rangle; U, X, Z, V; h, i, j, k) \simeq (\mathcal{S}\langle\alpha\rangle; U, X, Z, V; h, i, j, k) \ .$$

Let us abbreviate the tuple $(U, X, Z, V; h, i, j)$ by $\vec{j}$, and $(U, X, Z; h, k)$ by $\vec{k}$. Since

$$E : (\mathcal{S}\langle a\rangle, \vec{j}) \simeq (\mathcal{S}\langle\alpha\rangle, \vec{j}) ,$$

if

$$u \in \mathcal{S}\langle a\rangle(K_1)(\vec{j}) , \qquad (5)$$

then there is at least one

$$r \in \mathcal{S}\langle\alpha\rangle(K_1)(\vec{j}) \qquad (6)$$

such that

$$E : (\mathcal{S}\langle a\rangle; \vec{j}, u) \simeq (\mathcal{S}\langle\alpha\rangle; \vec{j}, r) , \qquad (7)$$

and *vice versa*. Moreover, *if*

$$v \in \mathcal{S}\langle a \rangle(K_1)(\vec{k}) \,, \tag{8}$$

then there is at least one

$$s \in \mathcal{S}\langle \alpha \rangle(K_1)(\vec{k}) \tag{9}$$

such that

$$E : (\mathcal{S}\langle a \rangle; \vec{k}, v) \simeq (\mathcal{S}\langle \alpha \rangle; \vec{k}, s) \,. \tag{10}$$

Assume (5) to (10). Abbreviate $(U, X, Z, V; h, i, j, k; u, v, w)$, the context for A, by $\vec{v}$, and $(U, X, Z, V; h, i, j, k; r, s, w)$, the context for B, by $\vec{s}$. What we have assumed amounts to

$$E : (\mathcal{S}\langle a \rangle, \vec{v}) \simeq (\mathcal{S}\langle \alpha \rangle, \vec{s}) \,.$$

Therefore, *if* A is as we had it above, *then* there is B as above such that

$$E : (\mathcal{S}\langle a \rangle; \vec{v}, A) \simeq (\mathcal{S}\langle \alpha \rangle; \vec{s}, B) \,,$$

and *vice versa*. This is what we meant by saying that A *may be* related to B in the equivalence (3).

The above examples were intended to give an idea of the workings of the items (2), (3) and (4) in a special case. Now we turn to the general case.

The first step is to specify the *common* (FOLDS-)signature of the structures (2) and (4). The said signature will be defined on the basis of the shape θ of the cell a.

Let θ be a multitope. Fix θ. Let us write k for $\dim(\theta)+1$; thus, $k \geq 1$. The first

interesting example to bear in mind is when $k=2$, and θ is the shape $\cdot \to \cdot$, the unique 1-dimensional multitope). We will define the new (FOLDS-)signature $\mathtt{Mlt}\langle\theta\rangle$.

We will apply a general construction that may be called the "bipartite graph" construction, to the two-variable functor

$$\Lambda : (\mathtt{Mlt}//\theta)^{\mathrm{op}} \times \mathtt{Mlt} \longrightarrow \mathtt{Set} \qquad (10')$$

we introduced in the last section.

Here is the general, and well-known, construction. Let

$$\Lambda : \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{B} \longrightarrow \mathtt{Set}$$

be an arbitrary two-variable, $\mathtt{Set}$-valued functor. We define the category $\langle\Lambda\rangle$ by the clauses that follow; A, A' denote objects of $\boldsymbol{A}$; B, B' those of $\boldsymbol{B}$.

$\mathtt{Ob}(\langle\Lambda\rangle) \stackrel{\mathrm{def}}{=} \mathtt{Ob}(\boldsymbol{A}) \sqcup \mathtt{Ob}(\boldsymbol{B})$;
$\mathtt{hom}_{\langle\Lambda\rangle}(A, A') \stackrel{\mathrm{def}}{=} \mathtt{hom}_{\boldsymbol{A}}(A, A')$;
$\mathtt{hom}_{\langle\Lambda\rangle}(B, B') \stackrel{\mathrm{def}}{=} \mathtt{hom}_{\boldsymbol{B}}(B, B')$;
$\mathtt{hom}_{\langle\Lambda\rangle}(A, B) \stackrel{\mathrm{def}}{=} \Lambda(A, B)$;
$\mathtt{hom}_{\langle\Lambda\rangle}(B, A) \stackrel{\mathrm{def}}{=} \emptyset$;
the inclusions $\boldsymbol{A} \to \langle\Lambda\rangle$, $\boldsymbol{B} \to \langle\Lambda\rangle$ are (full and faithful) functors;
for $A' \xrightarrow{a} A \xrightarrow{f} B$ [$f \in \Lambda(A, B)$], $f \circ a \stackrel{\mathrm{def}}{=} \Lambda(a, B)(f) : A' \longrightarrow B$;
for $A \xrightarrow{f} B \xrightarrow{b} B'$, $b \circ f \stackrel{\mathrm{def}}{=} \Lambda(A, b)(f) : A \longrightarrow B'$.

This defines a category. The main case of the associative law is when
$A' \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} B'$; $b(fa) = (bf)a$; it follows from Λ being a functor.

We have the cospan of full-and-faithful inclusions

$$\boldsymbol{A} \xrightarrow{\ \boldsymbol{a}\ } \langle\Lambda\rangle \xleftarrow{\ \boldsymbol{b}\ } \boldsymbol{B}. \qquad (11)$$

Every time we have a cospan

$$\boldsymbol{A} \xrightarrow{\boldsymbol{d}} \boldsymbol{C} \xleftarrow{\boldsymbol{e}} \boldsymbol{B} \qquad (12)$$

of functors, we have the induced functor

$$\Phi[\boldsymbol{d},\boldsymbol{e}] = \hom_{\boldsymbol{C}}(\boldsymbol{d}(-),\boldsymbol{e}(-)) : \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{B} \longrightarrow \mathrm{Set}\ .$$

In the case of the cospan (11), $\Phi[\boldsymbol{a},\boldsymbol{b}]$ is isomorphic to the original Λ ; we have

$$\begin{array}{rccc} \lambda : & \Lambda & \xrightarrow{\cong} & \Phi[\boldsymbol{a}, \boldsymbol{b}] \ . \\ & & & \Lambda(A, B) \\ & (A, B) & \longmapsto & \Big\downarrow \mathrm{id} \\ & & & \hom(\boldsymbol{a}A, \boldsymbol{b}B) \end{array} \qquad (13)$$

$\langle \Lambda \rangle$ has a useful universal property, as follows. Suppose $\boldsymbol{c} : \langle \Lambda \rangle \to \boldsymbol{C}$ is a functor. Then, by composition, from (11), we get (12):

$$\boldsymbol{d} = \boldsymbol{c}\boldsymbol{a}\ ,\ \ \boldsymbol{e} = \boldsymbol{c}\boldsymbol{b}\ . \qquad (14)$$

Note that we have the induced natural transformation "apply-$\boldsymbol{c}$ ":

$$\begin{array}{rccc} \dot{\boldsymbol{c}} : & \hom_{\langle \Lambda \rangle}(\boldsymbol{a}\text{-}, \boldsymbol{b}\text{-}) & \longrightarrow & \hom_{\boldsymbol{C}}(\boldsymbol{d}\text{-}, \boldsymbol{e}\text{-}) \\ & & & \hom(\boldsymbol{a}A, \boldsymbol{b}B) \\ & (A, B) & \longmapsto & \Big\downarrow \\ & & & \hom(\mathbf{d}A, \mathbf{e}B) \end{array} \qquad \begin{array}{ccc} \boldsymbol{a}A & \xrightarrow{f} & \boldsymbol{b}B \\ \Big\downarrow & & \Big\downarrow \\ \boldsymbol{d}A & \xrightarrow[\boldsymbol{c}(f)]{} & \boldsymbol{e}B \end{array}$$

From λ in (13), we get, by composition with $\dot{\boldsymbol{c}}$, the natural transformation

$$\delta : \Lambda \longrightarrow \Phi[\boldsymbol{d},\boldsymbol{e}] :$$

$$\begin{array}{ccc} \Lambda & \xrightarrow{\qquad\qquad\delta\qquad\qquad} & \hom_{\boldsymbol{C}}(\boldsymbol{d}\text{-},\boldsymbol{e}\text{-}) \\ & \searrow^{\lambda} \quad \circ \quad \nearrow_{\dot{\boldsymbol{c}}} & \\ & \hom_{\langle \Lambda \rangle}(\boldsymbol{a}\text{-},\boldsymbol{b}\text{-}) & \end{array} \qquad (15)$$

Now, for the given $\Lambda : \boldsymbol{A}^{\mathrm{op}} \times \boldsymbol{B} \longrightarrow \mathtt{Set}$, the universal property of the data

$$(\boldsymbol{A} \xrightarrow{\boldsymbol{a}} \langle \Lambda \rangle \xleftarrow{\boldsymbol{b}} \boldsymbol{B}\ ,\ \lambda : \Lambda \to \Phi[\boldsymbol{a}, \boldsymbol{b}])$$

is as follows. For any

$$(\boldsymbol{A} \xrightarrow{\boldsymbol{d}} \boldsymbol{C} \xleftarrow{\boldsymbol{e}} \boldsymbol{B}\ ,\ \delta : \Lambda \to \Phi[\boldsymbol{d}, \boldsymbol{e}])\ ,$$

we have that, conversely to the above, there is a unique $\boldsymbol{c} : \langle \Lambda \rangle \longrightarrow \boldsymbol{C}$ such that (14) and (15) hold. In fact, there is a (straightforward) additional two-dimensional aspect of the full universal property; but we will not need it.

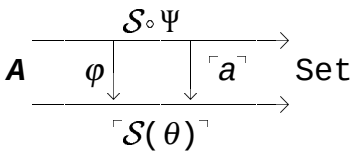
As we promised, we define $\mathtt{Mlt}\langle \theta \rangle$ to be $\langle \Lambda \rangle$, for the particular functor (10').

Next, let σ be a multitope, $\mathtt{dim}(\sigma) = k \geq 1$, and let $\theta = \mathtt{c}\sigma$, the shape of the codomain-face of σ. Let $\mathcal{S}$ be a multitopic set, let α be a $(k\text{-}1)$-pd of the shape σ in $\mathcal{S}$, a a $(k\text{-}1)$-cell of shape θ in $\mathcal{S}$, and assume that α and a are parallel. We will define "structure (4)" and "structure (2)", both $\mathtt{Mlt}\langle \theta \rangle$-structures. The first of these structures we denote by $\mathcal{S}\langle a \rangle$, the second by $\mathcal{S}\langle \alpha \rangle$. We will use the universal property of $\langle \Lambda \rangle$ given above. To connect to the notation above, we let (11) abbreviate

$$\mathtt{Mlt}/\!/\theta \xrightarrow{\mathtt{incl}} \langle \Lambda \rangle \xleftarrow{\mathtt{incl}} \mathtt{Mlt}\ ;$$

and set $\boldsymbol{C} = \mathtt{Set}$.

The definition of $\mathcal{S}\langle a \rangle$ is the simpler one of the two. Letting $\Psi : \boldsymbol{A} = \mathtt{Mlt}/\!/\theta \to \mathtt{Mlt}$ be the forgetful functor, we have the functors in



the lower one being the constant functor with value the set $\mathcal{S}(\theta)$. The natural transformation φ is defined thus. At $(\rho, x) \in \boldsymbol{A}$, its component $\varphi_{(\rho, x)} : \mathcal{S}(\rho) \longrightarrow \mathcal{S}(\theta)$ is $\varphi_{(\rho, x)} = \mathcal{S}(x) : \mathcal{S}(\rho) \to \mathcal{S}(\theta)$; φ is clearly natural. $\ulcorner a \urcorner$ is the constant natural transformation, with value the element $a \in \mathcal{S}(\theta)$. In the category $\mathtt{Hom}(\boldsymbol{A}, \mathtt{Set})$, we take the equalizer of the arrows

$$\boldsymbol{d} \xrightarrow{\ \varepsilon\ } \mathcal{S} \circ \Psi \underset{\ulcorner a \urcorner}{\overset{\varphi}{\rightrightarrows}} \ulcorner \mathcal{S}(\theta) \urcorner$$

On the other hand, for $\boldsymbol{e} : \boldsymbol{B} \longrightarrow \boldsymbol{C}$ we take $\boldsymbol{e} = \mathcal{S} : \mathtt{Mlt} \to \mathtt{Set}$. Finally, for $\delta : \Lambda \to \Phi[\boldsymbol{d}, \boldsymbol{e}]$, we define the component

$$\delta_{((\rho, x);\, \tau)} : [\rho \perp_x](\tau) \longrightarrow \mathtt{hom}(\boldsymbol{d}((\rho, x)), \mathcal{S}(\tau))$$

to be the function $q \longmapsto \mathcal{S}(q) \circ \varepsilon_{(\rho, x)}$.

The functor $\boldsymbol{c} = \mathcal{S}\langle a \rangle : \mathtt{Mlt}\langle \theta \rangle \longrightarrow \mathtt{Set}$ is defined by the relations (14) and (15).

Perhaps it is not superfluous to describe $\mathcal{S}\langle a \rangle$ directly. On the full subcategory $\mathtt{Mlt}$ of $\mathtt{Mlt}\langle \theta \rangle$, $\mathcal{S}\langle a \rangle$ is the same as $\mathcal{S}$. For $(\rho, x) \in \mathtt{Mlt}//\theta$,

$$\mathcal{S}\langle a \rangle((\rho, x)) = \{u \in \mathcal{S}(\rho) : \mathcal{S}(x)(u) = a\};$$

in other words, the elements of kind (ρ, x) in $\mathcal{S}\langle a \rangle$ are those cells of shape ρ in $\mathcal{S}$ whose x-face is equal to a. For $(\rho, x) \in \mathtt{Mlt}//\theta$, $\tau \in \mathtt{Mlt}$, and $(\rho, x) \xrightarrow{q} \tau$, an arrow in $\mathtt{Mlt}\langle \theta \rangle$,

$$\mathcal{S}\langle a \rangle(q)(u) = \mathcal{S}(q)(u);$$

and for $f : (\rho, x) \longrightarrow (\tau, y) \in \mathtt{Arr}(\mathtt{Mlt}//\theta)$,

$$\mathcal{S}\langle a \rangle(f)(u) = \mathcal{S}(f)(u).$$

We make another application of the universal property of $\langle\Lambda\rangle$ to define $\boldsymbol{C} = \mathcal{S}\langle\alpha\rangle$. We define $\boldsymbol{e}$, as before, to be $\boldsymbol{e}=\mathcal{S}$. For $\boldsymbol{d}$, we invoke substitution. Let $(\rho, x)\in\boldsymbol{A}$. We have the maps of formal contexts

$$[\sigma^-] \xrightarrow{(\ )^\circ(\rho, x)} (\rho[\sigma/x])^\wedge \ .$$

Taking $\hom_{\mathsf{Fun}}(-,\mathcal{S})$ on this, we get a map

$$\psi_{(\rho, x)} : \mathcal{S}(\rho[\sigma/x]) \longrightarrow \mathcal{S}[\sigma^-] \ .$$

This is the component at (ρ, x) of a natural transformation ψ in

$$\boldsymbol{A} \ \underset{\ulcorner\mathcal{S}[\sigma^-]\urcorner}{\overset{\mathcal{S}\circ\Sigma_\sigma}{\Downarrow \psi \quad \Downarrow \ulcorner\alpha\urcorner}} \ \mathsf{Set} \ .$$

$\boldsymbol{d}:\boldsymbol{A}\to\boldsymbol{C}$ is defined as the equalizer

$$\boldsymbol{d} \xrightarrow{\ \varepsilon\ } \mathcal{S}\circ\Sigma_\sigma \underset{\ulcorner\alpha\urcorner}{\overset{\psi}{\rightrightarrows}} \ulcorner\mathcal{S}[\sigma^-]\urcorner \ .$$

Finally, we define $\delta:\Lambda\to\Phi[\boldsymbol{d}, \boldsymbol{e}]$ to be the composite

$$\Lambda \xrightarrow{(\ )^\sim} \hom_{\mathsf{Mlt}}(\Sigma_\sigma{}^-, -) \xrightarrow{\dot{\mathcal{S}}} \hom_{\mathsf{Set}}(\mathcal{S}\circ\Sigma_\sigma{}^-, \mathcal{S}-) \xrightarrow{(-)\circ\varepsilon} \hom(\boldsymbol{d}-, \boldsymbol{e}-) \ .$$

In concrete terms, we have this. $\mathcal{S}\langle\alpha\rangle:\mathsf{Mlt}\langle\theta\rangle\to\mathsf{Set}$ agrees with $\mathcal{S}$ on Mlt. For $(\rho, x)\in\mathsf{Mlt}//\theta$,

$$\mathcal{S}\langle\alpha\rangle((\rho, x)) = \{t\in\mathcal{S}(\rho[\sigma|x]) : t\hat{\ }\mathring{p}=\alpha(p) \text{ for all } p\in\sigma\|\} \ .$$

For $f:(\rho, x)\longrightarrow(\tau, y)$, and $t\in\mathcal{S}\langle\alpha\rangle((\rho, x))$,

$$\mathcal{S}\langle\alpha\rangle(f)(t) = \mathcal{S}(f[\sigma])(t) \ .$$

For $q:(\rho, x) \longrightarrow \tau \in \mathtt{Arr}(\mathtt{Mlt}\langle\theta\rangle)$, and t as before,

$$\mathcal{S}\langle\alpha\rangle(q)(t) = \mathcal{S}(\hat{\tilde{q}})(t) .$$

Note that the restrictions of $M=\mathcal{S}\langle a\rangle$ and $N=\mathcal{S}\langle\alpha\rangle$ to the subcategory $\mathtt{Mlt}$ of $\mathtt{Mlt}\langle\theta\rangle$ are both equal to $\mathcal{S}$. An equivalence $(P, \mu.\nu) : M \simeq N$ is said to *extend the identity on* $\mathtt{Mlt}$ if its restriction to $\mathtt{Mlt}$ is equal to $(\mathcal{S}, \mathtt{id}_{\mathcal{S}}, \mathtt{id}_{\mathcal{S}}) : \mathcal{S} \simeq \mathcal{S}$.

We are ready for the first main definition.

Definition. A *multitopic* ω*-category* is a multitopic set $\mathcal{S}$ satisfying the following condition.

For every multitope σ, and every σ-shaped pasting diagram α in $\mathcal{S}$, there is at least one cell a parallel to α such that, for $\theta=\mathtt{c}\sigma$, the $\mathtt{Mlt}\langle\theta\rangle$-structures $\mathcal{S}\langle a\rangle$ and $\mathcal{S}\langle\alpha\rangle$ are $\mathtt{Mlt}\langle\theta\rangle$-equivalent by an equivalence span that extends the identity on $\mathtt{Mlt}$.

The cell a as in the condition in the definition is said to be a *composite of* α; composites are not uniquely determined. An equivalence span $E:\mathcal{S}\langle a\rangle \simeq \mathcal{S}\langle\alpha\rangle$ extending the identity on $\mathtt{Mlt}$ is said to be an *equipment* for a as a composite of α.

Assume that $\mathcal{S}$ is a multitopic ω-category. Let a be any cell in $\mathcal{S}$ of shape θ; $\mathtt{dim}(a)=k$. A particular application of the definition of "multitopic ω-category" is that the empty k-pd β on a (see section 2) has at least one composite. Such a composite, a k-cell, is what we call an *identity arrow* on a. We denote it by $\mathtt{id}_a : a \to a$. However, we have to remember that, due to the virtual character of all operations in multitopic categories, $\mathtt{id}_a$ is not uniquely determined. The best way of looking at the idea of "identity arrow" is the same as we look at any concept determined by a universal property. Consider, for instance, "the" product $X \times Y$ of two objects X, Y in a category. The "full determination" of the product includes the projections. In our case, the full determination of "the" *identity arrow*

$\mathtt{id}_a : a \to a$ is an arrow as shown plus the *equipment* making the arrow an identity arrow: the equivalence-span $E : \mathcal{S}\langle \mathtt{id}_a \rangle \simeq \mathcal{S}\langle \beta \rangle$ extending the identity on $\mathtt{Mlt}$.

Let us return to the case of a general pd α as in the definition, and let a be a composite of α; α is a pd of shape σ, a is a cell of shape $\theta = \mathtt{c}\sigma$. Suppose $E : \mathcal{S}\langle a \rangle \simeq \mathcal{S}\langle \alpha \rangle$ is an equipment for a being a composite of α. Now, consider (any instance of) the identity arrow $\mathtt{id}_a : a \to a$ on a. We may, and do, regard $\mathtt{id}_a$ to be an element in $\mathcal{S}\langle a \rangle$ of the kind

$$K_{\bar{\mathtt{def}}}(\theta^*, \mathtt{d}_{\theta^*}) \in \mathtt{Mlt}//\theta \subset \mathtt{Mlt}\langle \theta \rangle \ .$$

The K-context $\vec{a}$ for $\mathtt{id}_a$ in $\mathcal{S}\langle a \rangle$ consists of (a and) the faces of a, all cells of kinds in the "old" part $\mathtt{Mlt}$ in the signature $\mathtt{Mlt}\langle \theta \rangle$. By definition, we have some $u \in \mathcal{S}\langle \alpha \rangle(K)(\vec{a})$ for which

$$E : (\mathcal{S}\langle a \rangle, \vec{a}, \mathtt{id}_a) \simeq (\mathcal{S}\langle \alpha \rangle, \vec{a}, u) \qquad (11)$$

Note that, as an element of $\mathcal{S}$, u is a $(k+1)$-cell $u : \alpha \to a$ of shape equal to σ.

We define an arrow $u : \alpha \to a$ to be *universal* if it is obtained in the way described: that is, if there exists an equipment E for (11).

A concept of "universal cell" is the basis for the definition of the opetopic weak n-category, for a finite n, in [B/D]. To establish the connection, I make some further definitions.

Let me say that a multitopic set $\mathcal{S}$ is *n-truncated* if for every $k \geq n+1$, every time we have in $\mathcal{S}$ a parallel pair (α, a) of a k-pd α and a k-cell a, there is at least one $(k+1)$-cell b such that $\mathtt{d}b = \alpha$, $\mathtt{c}b = a$. In brief, all $\geq n+2$-cells that may exist do exist. It is immediate to see that an n-truncated multitopic set $\mathcal{S}$ is determined, up to $\mathtt{Mlt}$-equivalence, by its restriction to $\mathtt{Mlt}\restriction(n+1)$, the full subcategory of $\mathtt{Mlt}$ consisting of the multitopes of dimensions $\leq (n+1)$. I define a *multitopic n-category* to be a multitopic ω-category which is, at the same time, an n-truncated multitopic set, and, moreover, all $(n+1)$-cells are universal.

It is possible to "normalize" multitopic n-categories, that is, to find, for each multitopic n-category $\mathcal{S}$, another one, $\mathcal{S}^!$, which is $\mathtt{Mlt}$-equivalent to $\mathcal{S}$ (and hence, shares all the "essential" properties with $\mathcal{S}$), and which has some additional desirable properties; namely, $\mathcal{S}^!$ satisfies the following:

(i) k-hom-sets, for $k \geq n+1$, are 0-1 valued: for any $(k-1)$-pd α and n-cell a, there is at most one k-cell $b : \alpha \to a$;

(ii) the composition of n-pd's is single valued: for every n-pd α, the composite of α is uniquely determined (and in fact, the equipment for the composition of α is also uniquely determined).

It is possible to define "multitopic n-category", equivalently to the definition given above, in a way that closely parallels the definition of "opetopic n-category" in [B/D]. One defines a concept of "universal cell", by "backward" recursion, declaring all $(n+2)$-cells to be universal, and by proceeding recursively to determine when lower dimensional cells are recursive. Finally, one postulates the axiom that for $k \leq n$, and every k-pd α, there are (at least one) k-cell and a *universal* $(k+1)$-cell $u : \alpha \to a$. (With the appropriate definition of "universal", the property, postulated as an axiom [B/D], that the composite of universal cells is universal, becomes provable.)

The concept of "multitopic n-category" is first order definable; in fact, it is naturally formulated in FOLDS over $\mathtt{Mlt}\restriction(n+1)$; in particular, it is invariant under $\mathtt{Mlt}\restriction(n+1)$-equivalence. However, the concept of multitopic ω-category does not seem to be first order definable, although it is invariant under $\mathtt{Mlt}$-equivalence.