3. Colored multitopes.

The question how to define the concept of weak n-category, for finite n, and in the second place, also, for $n=\omega$, has been around for a while; and several answers have already been proposed (see [B]. [B/D],...). However, one clearly wants more than that concept: one wants the definition of the total structure weak higher dimensional categories are elements of. For instance, one wants morphisms, or functors (some may want to say: "homomorphisms", since for the case of $n=2$ and $n=3$, the existing notions that we want to emulate and generalize are called "homomorphism" and "trihomomorphism", respectively), of n-categories ; for $n>0$, one wants an (appropriately higher dimensional) notion of (pseudo-)natural transformation of functors of weak n-categories; for still greater n, modifications of the latter, and so on.

This structure will be provided here in a uniform manner.

In the first place, the concept of multitopic ω-category will be defined and proposed as a concept of weak ω-category. Multitopic n-categories, for each finite n, will be the natural ("truncated") special cases of the ω-concept.

Secondly, we will define a (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ whose 0-cells will be the (small) multitopic ω-categories. The 1-cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ will be our construal of the functors (morphisms) of multitopic ω-categories. The 2-cells will be the "natural transformations"; but note that now we will have, as separate kinds of entities, binary, ternary, and, in general, k-ary (for all $k\in\mathbb{N}$, even for $k=0$!) "natural transformations". The 3-cells in $\mathtt{Mlt}\omega\mathtt{Cat}$ will stand for modifications; now, they come with somewhat complicated arities, which are exactly the 2-multitopes, shapes of "many-to-one" pasting diagrams of dimension 2. And so on, for all finite n . Borrowing a term from Sjoerd Crans, I will call an n-cell in $\mathtt{Mlt}\omega\mathtt{Cat}$ an *n-transfor*. (Crans uses this term in the context of *teisi*, "semi-strict" weak higher dimensional categories. Unfortunately however, there is mismatch in the assignment of the "order" n ; what I call n-transfor corresponds to Crans's $(n-1)$-transfor.) Thus, the, usually multiary, n-cells in the multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ will be the n-transfors; the 0-transfors are the same as the (multitopic) ω-categories.

The main theorem announced in this paper is that $\mathtt{Mlt}\omega\mathtt{Cat}$ is a multitopic ω-category.

The uniformity of the theory lies in the fact that each n-transfor will itself be a multitopic set, endowed, however, with certain additional structure that we call *coloring*. Given a multitope π, we will define a particular category $\mathtt{Mlt}[\pi]$ of *π-colored multitopes*. The category $\mathtt{MltSet}[\pi]$ of *π-colored multitopic sets* will then be defined as the functor category $\mathtt{Set}^{\mathtt{Mlt}[\pi]}$. When $\pi=\rho^{(0)}$, the unique zero-dimensional multitope, then $\mathtt{Mlt}[\pi]$ and $\mathtt{MltSet}[\pi]$ are, essentially, the same as $\mathtt{Mlt}$ and $\mathtt{MltSet}$, respectively. An n-transfor of shape π, that is, an element of $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$, will be a π-colored multitopic set with certain properties (but with no additional data).

Recall that the objects of $\mathtt{Mlt}$ were the cells of $\mathcal{T}$, the terminal multitopic set. Generalizing, for each multitope π, we will define a particular multitopic set $\mathcal{T}[\pi]$; the objects of $\mathtt{Mlt}[\pi]$ will then be the cells of $\mathcal{T}[\pi]$.

Together with the multitopic sets $\mathcal{T}[\pi]$, we will also define, for each arrow $\pi\to\kappa$ in $\mathtt{Mlt}$, morphisms $\mathcal{T}[p]:\mathcal{T}[\kappa]\longrightarrow\mathcal{T}[\pi]$; in fact, we will define a functor

$$\mathcal{T}[\text{-}]:\mathtt{Mlt}^{\mathtt{op}}\longrightarrow\mathtt{MltSet}.$$

(As we will see, $\mathcal{T}[p]$ "recolors" every κ-colored multitope as a π-colored one, by replacing the color $r\in|\kappa|$ by $r\circ p\in|\pi|$.)

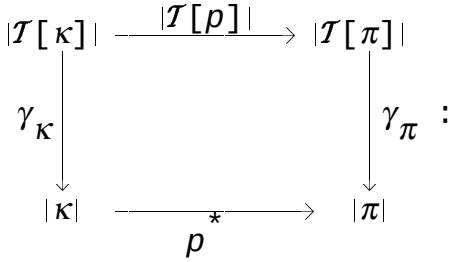
In addition, for each $\pi\in\mathtt{Mlt}$, we will define a mapping

$$\gamma_\pi:|\mathcal{T}[\pi]|\longrightarrow|\pi|,$$

on the total set

$$|\mathcal{T}[\pi]|\ \bar{\mathtt{def}}\ \coprod_{\tau\in\mathtt{Mlt}}\mathcal{T}[\pi](\tau)$$

of the multitopic set $\mathcal{T}[\pi]$ to the set of arrows out of π, called *coloring*. The coloring will be compatible with the functor $\mathcal{T}[\text{-}]$ in the natural sense that, for each $p:\pi\to\kappa$, the following diagram commutes:



here, the notation $|\mathcal{I}[p]|$ is self-explanatory; p^* is the mapping $r \mapsto r \circ p$.

The definition is by recursion. Fix $n \in \mathbb{N}$. Assume that the functor $\mathcal{I}[-]$ has been defined on arguments that are objects of dimensions less than n , and arrows between such objects; moreover, assume that said restriction of $\mathcal{I}[-]$ is a functor. Assume that the coloring γ_κ has been defined for all $\kappa \in \mathtt{Mlt}$ of dimension $<n$, with the relevant instances of the compatibility condition satisfied. Fix $\pi \in \mathtt{Mlt}$ such that $\mathtt{dim}(\pi)=n$; we will define $\mathcal{I}[\pi]$, γ_π , and $\mathcal{I}[p]$ for $p \in \pi|$, where $\pi|$ is the set of non-identity arrows in $\mathtt{Mlt}$ out of π .

Let $\pi|\mathtt{Mlt}$ denote the full subcategory of the comma-category $\pi\backslash\mathtt{Mlt}$ on objects the elements of $\pi| = |\pi| - \{\mathrm{id}_\pi\}$. Let $\mathtt{Mlt}\restriction_{<\pi}$ denote the full subcategory of $\mathtt{Mlt}$ on the objects $\mathtt{K}_p$ $(=\mathtt{cod}(p))$ for $p \in \pi|$. We have the forgetful functor $\varphi : \pi|\mathtt{Mlt} \longrightarrow \mathtt{Mlt}\restriction_{<\pi}$. Consider the composite functor

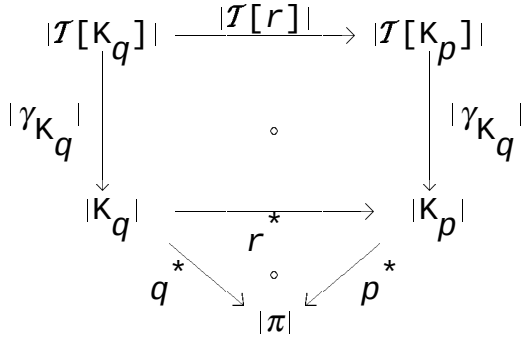
$$\pi|\mathtt{Mlt} \xrightarrow{\varphi} \mathtt{Mlt}\restriction_{<\pi} \xrightarrow{\mathcal{I}[-]\restriction_{<\pi}} \mathtt{MltSet} ,$$

and call it Φ ; note that by the recursion hypothesis, $\mathcal{I}[-]\restriction_{<\pi}$ is well-defined. Let $\mathcal{P}$ be the colimit $\mathtt{colim}(\Phi)$ in $\mathtt{MltSet}$ (when, later on, we want to refer $\mathcal{P}$ for various instances of π , we may write $\mathcal{P}[\pi]$ for $\mathcal{P}$). Since $\mathtt{MltSet}$ is (equivalent to) $\mathtt{Set}^{\mathtt{Mlt}}$, $\mathtt{colim}(\Phi)$ exists, and is computed "pointwise" as in $\mathtt{Set}$. In particular, for each multitope τ ,

$$\mathcal{P}(\tau) = \mathop{\mathtt{colim}}_{(p:\pi \to \mathtt{K}_p) \in \pi|\mathtt{Mlt}} \mathcal{I}[\mathtt{K}_p](\tau) .$$

We will define $\mathcal{I}[\pi]$ as a certain extension of the multitopic set $\mathcal{P}$; we will adjoin certain specific cells to $\mathcal{P}$, to obtain $\mathcal{I}[\pi]$. Recall that we also have to define the coloring $\gamma_\pi(a)$ for cells a in $\mathcal{I}[\pi]$. The coloring $\gamma_\pi(a)$ of cells a in $\mathcal{P}$ (a part of $\mathcal{I}[\pi]$) is given as

follows. For any $p, q \in \pi|$, and $(r : p \to q) \in \mathtt{Arr}(\pi|\mathtt{Mlt})$, the commutativity of the diagram



ensures that we have a well-defined map γ from the colimit $|\mathcal{P}| = \mathtt{colim}(|\Phi|)$ to $|\pi|$ such that, for each colimit coprojection $\iota_p : \mathcal{T}[\mathtt{K}_p] \to \mathcal{P}$, we have $\gamma \circ |\iota_p| = p^* \circ \gamma_{\mathtt{K}_p}$. The coloring γ_π restricted to $\mathcal{P}$ is defined to be this γ.

All newly adjoined cells in $\mathcal{T}[\pi]$ are to have the γ_π-color $\mathtt{id}_\pi$. Thus, the part $\mathcal{P}$ is distinguished inside $\mathcal{T}[\pi]$ as those cells whose color is different from $\mathtt{id}_\pi$.

The extension of $\mathcal{P}$ to $\mathcal{T}[\pi]$ is done as follows.

No cells of dimension less than n are adjoined to $\mathcal{P}$.

Next, we describe the n-cells added to $\mathcal{P}$.

An $(n\text{-}1)$-pd $\beta = \langle \beta(r) \rangle_{r \in \tau||}$ in $\mathcal{P}$ is called $\mathtt{c}_\pi$-*free* if for all $r \in \tau||$, we have $\gamma_\pi(\beta(r)) \neq \mathtt{c}_\pi$. The new n-cells adjoined to $\mathcal{P}$ are in a bijective correspondence with the pairs (β, b) where β is a $\mathtt{c}_\pi$-free $(n\text{-}1)$-pd in $\mathcal{P}$, b is an $(n\text{-}1)$-cell in $\mathcal{P}$ such that $\gamma_\pi(b) = \mathtt{c}_\pi$, and β and b are parallel: $\mathtt{d}\beta = \mathtt{d}b$, $\mathtt{c}\beta = \mathtt{c}b$; for each such pair (β, b), we adjoin to $\mathcal{P}$ exactly one new cell a with $\mathtt{d}a = \beta$, $\mathtt{c}a = b$; we declare that $\gamma_\pi(a) = \mathtt{id}_\pi$.

For dimensions $m > n$, we proceed recursively. For any parallel pair (β, b) of an $(m\text{-}1)$-pd β and an $(m\text{-}1)$-cell b such that $\gamma_\pi(b) = \mathtt{id}_\pi$, we adjoin a single new m-cell a for which $\mathtt{d}a = \beta$, $\mathtt{c}a = b$, and make $\gamma_\pi(a)$ equal to $\mathtt{id}_\pi$.

This completes the definition of the multitopic set $\mathcal{T}[\pi]$. We have the inclusion $\iota : \mathcal{P} \to \mathcal{T}[\pi]$. For $(p : \pi \to \kappa) \in \pi|$, we define $\mathcal{T}[p] : \mathcal{T}[\kappa] \to \mathcal{T}[\pi]$ as the composite $\mathcal{T}[p] = \iota \circ \iota_p$. The functorial nature of $\mathcal{T}[-]$ on arrows between objects in the set $\{K_p : p \in |\pi|\}$, and the compatibility of the various colorings indexed by elements of $|\pi|$ are assured by the definition.

For the fixed n, performing the above construction simultaneously for all multitopes π of dimension n, gives the definition of all necessary items involving indices $\leq n$, since all non-identity arrows out of a multitope are to multitopes of lower dimensions. The functor $\mathcal{T} : \mathtt{Mlt}^{\mathrm{op}} \to \mathtt{Set}^{\mathtt{Mlt}}$ has been defined.

Let $\pi, \tau \in \mathtt{Mlt}$, $a \in \mathcal{T}[\pi](\tau)$. The *total π-coloring* of a includes the colors of all the faces of a; it is defined to be

$$\bar{\gamma}_\pi(a) \;\overline{\mathtt{def}}\; \langle \gamma_\pi(\mathcal{T}[\pi](s)(a)) \rangle_{s \in |\tau|} \;\in\; |\pi|^{|\tau|} .$$

In other words, for $s \in |\tau|$, and for the s-face $b = \mathcal{T}[\pi](s)(a)$ of a, the value of the function $\bar{\gamma}_\pi(a)$ at s equals the π-color of b:

$$\bar{\gamma}_\pi(a)(s) = \gamma_\pi(b) .$$

Using, for any $p : \pi \to \kappa$, the naturality of $\mathcal{T}[p] : \mathcal{T}[\kappa] \longrightarrow \mathcal{T}[\pi]$, we find that we have the commutativity

$$\begin{array}{ccc} \mathcal{T}[\kappa](\tau) & \xrightarrow{\mathcal{T}[p]_\tau} & \mathcal{T}[\pi](\tau) \\ \Big\downarrow{\scriptstyle \bar{\gamma}_\kappa} & \circ & \Big\downarrow{\scriptstyle \bar{\gamma}_\pi} \\ |\kappa|^{|\tau|} & \xrightarrow[p^*]{} & |\pi|^{|\tau|} \end{array} .$$

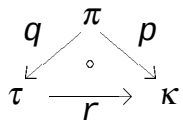
Next, we list some facts about colored multitopes.

(1) Each map $\bar{\gamma}_\pi : \mathcal{I}[\pi](\tau) \longrightarrow |\pi|^{|\tau|}$ is 1-to-1 .

This fact says that a colored multitope is determined by its underlying (ordinary) multitope, that is, its shape, and its (total) coloring. Therefore, a π-colored multitope is given by an (ordinary) multitope and a *correct* coloring of its faces by the faces of $|\pi|$. What constitutes a correct coloring is, of course, defined unambiguously by the above-given definition of `Mlt[-]` ; however, this definition is not very direct (it uses a recursion).

Let us note here that for each $\pi \in \mathtt{Mlt}$, we have the canonical π-colored multitope $P=P[\pi]$ for which $P \in \mathtt{Mlt}[\pi](\pi)$, and for which $\bar{\gamma}_\pi(\pi) = \mathtt{id}_{|\pi|}$.

(2) For all $p : \pi \to \kappa$ in `Mlt` , the induced map $\mathcal{I}[p] : \mathcal{I}[\kappa] \to \mathcal{I}[\pi]$ is a monomorphism. Moreover, if $\pi, \rho \in \mathtt{Mlt}$, $R \in \mathcal{I}[\pi](\rho)$, then for κ the codomain of the arrow $p = \gamma_\pi(R) : \pi \to \kappa$, there is a (unique) cell $\dot{R} \in \mathcal{I}[\kappa](\rho)$, called the *root* of R , for which $\dot{R} \overset{\mathcal{I}[p]}{\longmapsto} R$. Furthermore, if $q : \pi \to \tau$, and $S \in \mathcal{I}[\tau](\rho)$ is such that $S \overset{\mathcal{I}[p]}{\longmapsto} R$, then, necessarily, the root of S is the same as that of R ; in other words, there is a commmutative diagram



such that $\dot{R} \overset{\mathcal{I}[r]}{\longmapsto} S$. Let us call a cell T in $\mathcal{I}[\theta]$ *primitive* if $\gamma_\theta(T) = \mathtt{id}_\theta$. The roots of cells are always primitive; the primitive cells are the same as the ones that are their own roots. Note that if $T \in \mathcal{I}[\theta]$ is primitive, then $\dim(T) \geq \dim(\theta)$.

(3) With reference to the multitopic set $\mathcal{P}$ in the above definition of $\mathcal{I}[\pi]$, for each $m \geq n$, the canonical morphism

$$\operatorname{colim}_{(p:\pi\to \mathtt{K}_p)\in \pi\mid \mathtt{Mlt}} \mathtt{P}_m(\mathcal{I}[\mathtt{K}_p]) \longrightarrow \mathtt{P}_m(\mathcal{P})$$

is an isomorphism.

This says that, for $m\geq n$, every m-pd in $\mathcal{P}$ is already a pd in a single component $\mathcal{I}[\mathtt{K}_p]$ for a proper (non-top) color $p\in \pi\mid \mathtt{Mlt}$.

Essentially as a consequence of (3), we have (4) and (5):

(4) With $n=\mathtt{dim}(\pi)$, for any pd α and any cell a in the multitopic set $\mathcal{I}[\pi]$ such that α and a are parallel, and $\mathtt{dim}(\alpha)=\mathtt{dim}(a)\geq n$, we have exactly one cell b in $\mathcal{I}[\pi]$ such that $\mathtt{d}b=\alpha$, $\mathtt{c}b=a$.

(5) With $n=\mathtt{dim}(\pi)$, for any pd α in the multitopic set $\mathcal{I}[\pi]$ such that $\mathtt{dim}(\alpha)\geq n$, we have exactly one cell b in $\mathcal{I}[\pi]$ such that $\mathtt{d}b=\alpha$, and in particular, exactly one cell parallel to α.

Let us give some examples.

It is clear that when $\pi=\rho^{(0)}$, $\mathcal{I}[\pi]$ is just $\mathcal{I}$: there is just one color.

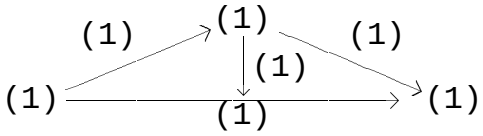
The next example is when $\pi=\rho^{(1)}$, the unique $\mathtt{1}$-dimensional multitope. We have that $\pi\mid = \{\mathtt{d},\mathtt{c}\}$, both with codomain $\rho^{(0)}$. The multitopic set $\mathcal{P}=\mathcal{P}[\pi]$ is the disjoint union of two copies of $\mathcal{I}$, one colored $\mathtt{d}$, the other colored $\mathtt{c}$. In $\mathcal{I}[\pi]$, there are exactly two $\mathtt{0}$-cells and three $\mathtt{1}$-cells; changing the notation somewhat, and representing π as

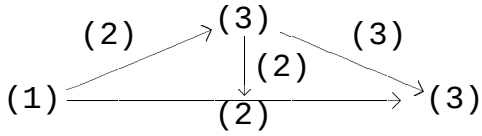
$$1 \xrightarrow{\ 2\ } 3$$

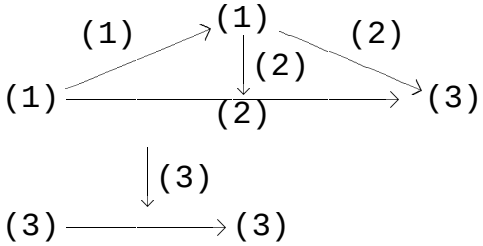
(which, of course, means that 1=d , 3=c and 2=id_π), the 1-cells in $\mathcal{I}[\rho^{(1)}]$ are

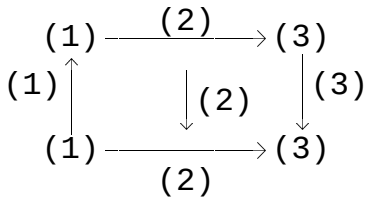
$$(1) \xrightarrow{(1)} (1)\,,$$
$$(3) \xrightarrow{(3)} (3)\,,$$
$$(1) \xrightarrow{(2)} (3)\,;$$

the parenthetical numbers are the colors of the cells in question. The first two come from $\mathcal{P}$, the last as the n-cell (n=1) adjoined to $\mathcal{P}$. Some examples of 2-cells in $\mathcal{I}[\rho^{(1)}]$:



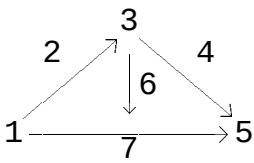






The last 2-cell has domain the empty 1-pd on the 0-cell (3) . Note that the 1-cell $(1) \xrightarrow{(2)} (3)$ cannot be the target of an empty-domain 2-cell, since (1)≠(3) . The first and the last of these 2-cells are "from" $\mathcal{P}[\rho^{(1)}]$; the others are introduced by the third stage of the definition of $\mathcal{I}[\rho^{(1)}]$.

Next, consider the example of the following 2-cell for π :



In $\mathcal{I}[\pi]$, we have three 0-cells: (1) , (3) , (5) ; six 1-cells:

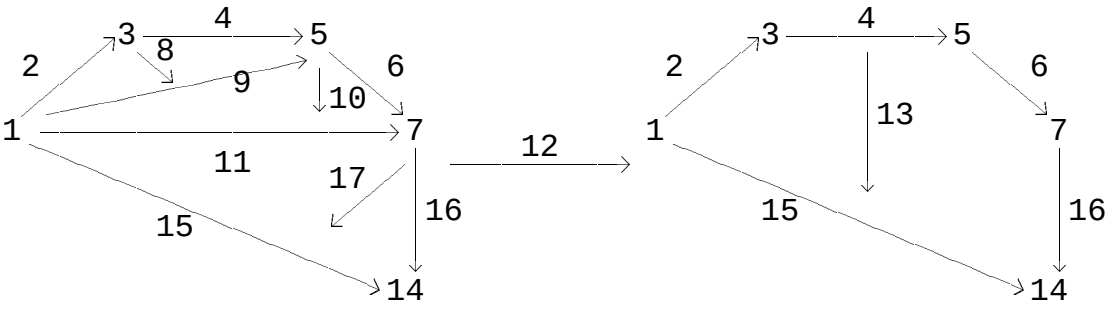
$(1) \xrightarrow{(1)} (1)$, $(1) \xrightarrow{(2)} (3)$,

$(3) \xrightarrow{(3)} (3)$, $(3) \xrightarrow{(4)} (5)$,

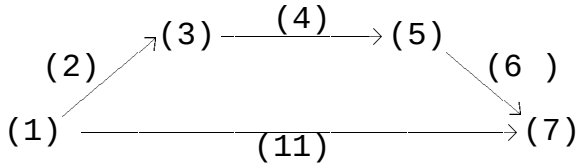
$(5) \xrightarrow{(5)} (5)$, $(1) \xrightarrow{(7)} (5)$.

As for 2-cells, they come in seven different colors. The ones of color (6) have as domain a 1-pd that is a sequence of 1-cells, starting in a succession (possibly of zero length) of (1)'s, followed by a single instance of $(1) \xrightarrow{(2)} (3)$, followed by a succession of (3)'s, followed by a single $(3) \xrightarrow{(4)} (5)$, ending in a succession of (5)'s. The 2-cells of color (6) all have as codomain the 1-cell $(1) \xrightarrow{(7)} (5)$.

For π the 3-dimensional multitope

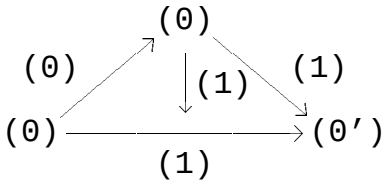


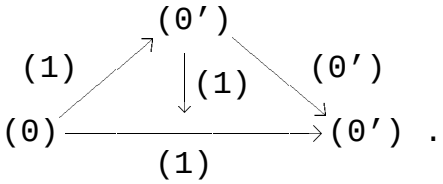
in Mlt[π], the parallel pair (α, a) of the 1-pd α and 1-cell a shown in



does not have a "filling" $b : \alpha \to a$.

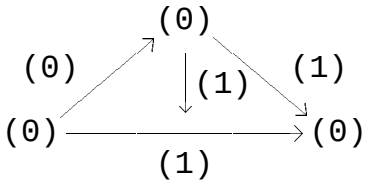
Let π be $0 \xrightarrow[1]{2} 0$, the simplest empty-domain multitope, and let κ be $0 \xrightarrow{1} 0'$, the unique 1-dimensional multitope $\rho^{(1)}$ (considered already above, with the notation $1 \xrightarrow{2} 3$). We have the map $p = c_\pi : \pi \to \kappa$ (unique of the form $\pi \to \kappa$) for which p^* maps 0 and 0′ to 0, 1 to 1. We have the κ-colored multitopes

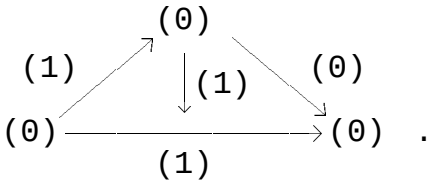




.

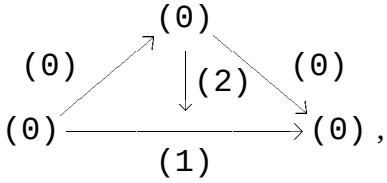
Their images under $\mathcal{I}[p]$ in $\mathcal{I}[\pi]$ are





.

In $\mathcal{I}[\pi]$, we have the further cell

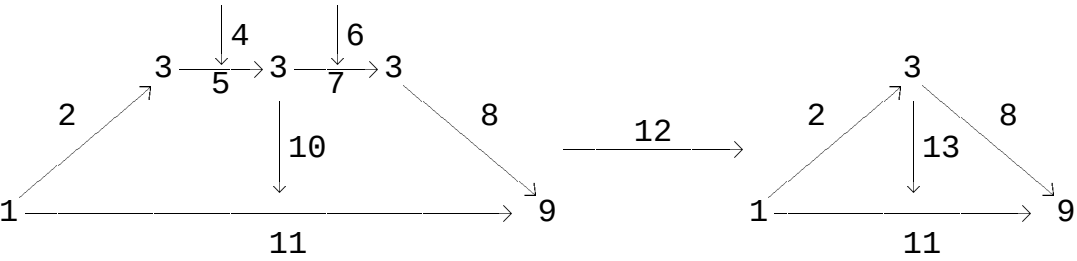


,

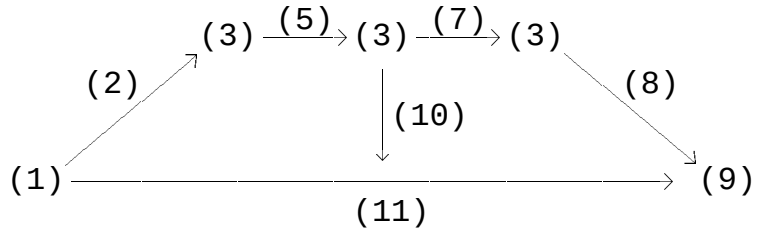
which is not obtained from $\mathcal{P}[\pi]$ under the canonical map into $\mathcal{I}[\pi]$.

Finally, let us take an example where the recursive character of the definition shows itself to be more essential than in the "positive" (empty-domain-free) examples considered so far.

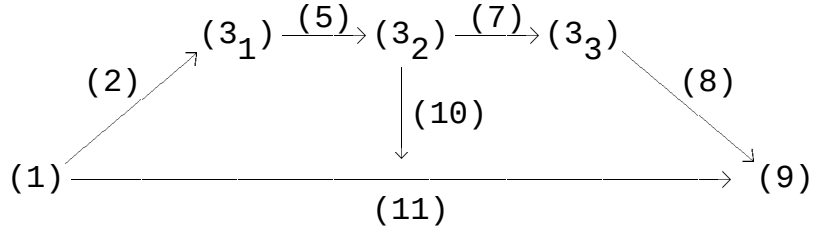
Let π be the 3-dimensional multitope



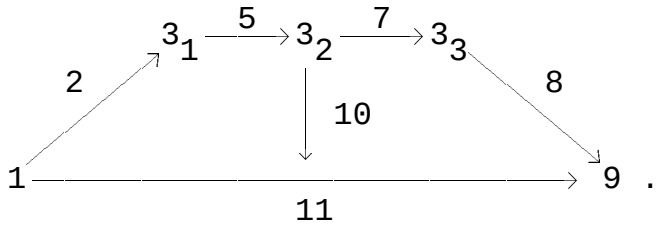
In $\mathtt{Mlt}[\pi]$, we have the 2-cell R:



The one in which (5) and (7) are interchanged, call it S, is not a correct coloring; it is not a cell in $\mathtt{Mlt}[\pi]$, despite the fact that locally it looks all right. The point is that R is the image of the cell



in $\mathtt{Mlt}[\tau]$ for τ the 2-cell



However, S cannot be represented as such an image.

Let $\pi\in\mathtt{Mlt}$. We put $\mathtt{Mlt}[\pi]$, the *category of π-colored multitopes*, to be $\mathtt{El}(\mathcal{I}[\pi])$, the category of elements of $\mathcal{I}[\pi]:\mathtt{Mlt}\to\mathtt{Set}$. That is, the objects of $\mathtt{Mlt}[\pi]$ are pairs (τ, a) where $a\in\mathcal{I}[\pi](\tau)$; and a morphism $f:(\tau, a)\longrightarrow(\theta, b)$ is an arrow $f:\tau\to\theta$ in $\mathtt{Mlt}$ such that $(\mathcal{I}[\pi](f))(a) = b$. In particular, we have the faithful forgetful functor $\|\ \|:\mathtt{Mlt}[\pi]\to\mathtt{Mlt}$. $\|\ \|$ is a discrete cofibration: the arrows out of any $R\in\mathtt{Mlt}[\pi]$ in $\mathtt{Mlt}[\pi]$ are in a bijective correspondence with the arrows out of $\|R\|$ in $\mathtt{Mlt}$.

For every $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we have an induced functor

$$\mathtt{Mlt}[p]=\mathtt{El}(\mathcal{T}[p]):\mathtt{Mlt}[\kappa]\longrightarrow\mathtt{Mlt}[\pi]$$

("recoloring along p"). In fact, we have the functor

$$\mathtt{Mlt}[\text{-}]:\mathtt{Mlt}^{\mathtt{op}}\longrightarrow\mathtt{Cat}\ .$$

In other words, we have a strict fibration

$$\begin{array}{c}\mathtt{Mlt}[\mathtt{Mlt}]\\ \big\downarrow{\scriptstyle\Pi}\\ \mathtt{Mlt}\end{array}$$

whose fiber at each $\pi\in\mathtt{Mlt}$ is $\mathtt{Mlt}[\pi]$. The total category $\mathtt{Mlt}[\mathtt{Mlt}]$ is a suitable category that has as objects all colored multitopes, colored by any (simple) multitope.

The *category* $\mathtt{Mlt}[\pi]\mathtt{Set}$ *of* π*-colored multitopic sets* is

$$\mathtt{Mlt}[\pi]\mathtt{Set}=\mathtt{Set}^{\mathtt{Mlt}[\pi]}\ .$$