The multitopic ω-category of all multitopic ω-categories

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Introduction. Despite its considerable length, this paper is only an announcement. It gives two definitions: that of "multitopic ω-category", and that of "the (large) multitopic set of all (small) multitopic ω-categories". It also announces the theorem that the latter is a multitopic ω-category.

The work has two direct sources. One is the paper [H/M/P] in which, among others, the concept of "multitopic set" was introduced. The other is the author's work on FOLDS, first order logic with dependent sorts. The latter was reported on in [M2]. A detailed account of the work on FOLDS is in [M3]. For the understanding of the present paper, what is contained in [M2] suffices. In fact, section 1 of this paper gives the definitions of all that's needed in this paper; so, probably, there won't be even a need to consult [M2].

The concept of multitopic set, the main contribution of [H/M/P], was, in turn, inspired by the work of J. Baez and J. Dolan [B/D]. Multitopic sets are a variant of opetopic sets of *loc. cit.* The name "multitopic set" refers to multicategories, a concept originally due to J. Lambek [L], and given an only moderately generalized formulation in [H/M/P]. The earlier "opetopic set" is based on a concept of operad; see [B/D]. I should say that the exact relationship of the two concepts ("multitopic set" and "opetopic set") is still not clarified. The main aspect in which the theory of multitopic sets is in a more advanced state than that of opetopic sets is that, in [H/M/P], there is an explicitly defined *category* `Mlt` *of multitopes*, with the property that the category of multitopic sets is equivalent to the category of Set-valued functors on `Mlt`, a result given a detailed proof in [H/M/P]. The corresponding statement on opetopic sets and opetopes is asserted in [B/D], but the category of opetopes is not described. In this paper, the category of multitopes plays a basic role.

Multitopic sets and multitopes are described in section 2 of this paper; for a complete treatment, the paper [H/M/P] should be consulted.

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The indebtedness of the present work to the work of Baez and Dolan goes further than that of [H/M/P]. The second ingredient of the Baez/Dolan definition, after that of "opetopic set", is the concept of "universal cell". The Baez/Dolan definition of weak n-category achieves the remarkable feat of specifying the composition structure by universal properties taking place in an opetopic set. In particular, a (weak) opetopic (higher dimensional) category is an opetopic set with additional *properties* (but with no additional data): the *existence* of "sufficiently many" of the various types of universal cells. In [H/M/P], no universal cells were defined, although it was mentioned that their definition could be supplied without much difficulty by imitating [B/D]. In this paper, the "universal structure" is supplied, albeit in a somewhat unexpected manner, by using the concept of FOLDS-equivalence introduced in [M2].

In [M2], the concepts of "(FOLDS-)signature" and "(FOLDS-)equivalence" are introduced. A signature is a category with certain special properties; for a signature L, an L-structure is a $\mathtt{Set}$-valued functor on L. To each signature L, a particular relation $\simeq_L$, called L-equivalence, between two variable L-structures is defined. The slogan of the work on FOLDS is that *all meaningful properties of L-structures should be invariant under L-equivalence*. The main effort in [M3] goes into specifying a language, First Order Logic with Dependent Sorts, and showing that the *first order* properties invariant under L-equivalence are precisely the ones that can be defined in FOLDS. It is a basic realization of [M3] that the usual concepts of "equivalence" in category theory, including the higher dimensional variants such as "biequivalence", are all special cases of L-equivalence, upon suitable, and natural, choices of the signature L.

In this paper, the language of FOLDS plays no role. The concepts of "signature" and "equivalence" are fully described in section 1 of this paper.

The definition of multitopic ω-category goes, in outline, as follows. For an arbitrary multitope σ of dimension ≥ 2, for a multitopic set $\mathcal{S}$, for a pasting diagram α in $\mathcal{S}$ of shape the domain of σ and a cell a in $\mathcal{S}$ of the shape the codomain of σ such that a and α are parallel, we define what it means to say that a is a *composite* of α. First, we define a FOLDS signature $\mathrm{L}\langle\sigma\rangle$ extending $\mathtt{Mlt}$, the signature of multitopic sets. Next, we define structures $\mathcal{S}\langle a\rangle$ and $\mathrm{S}\langle\alpha\rangle$, both of signature $\mathrm{L}\langle\sigma\rangle$, the first constructed from the data $\mathcal{S}$ and a, the second from $\mathcal{S}$ and α, both structures extending $\mathcal{S}$. a *is a composite* of α if there is a FOLDS-equivalence between $\mathcal{S}\langle a\rangle$ and $\mathrm{S}\langle\alpha\rangle$ that restricts to the identity equivalence from $\mathcal{S}$ to $\mathcal{S}$; below, I'll refer to this FOLDS equivalence as an equipment for a being a composite of α.

. A multitopic set $\mathcal{S}$ is *a multitopic ω-category* if every pasting diagram in it has a composite.

The analog of the universal arrows in the Baez/Dolan style definition is as follows. A *universal arrow* is defined to be an arrow of the form $b:\alpha\to a$ where a is a composite of α via an equipment that relates b with the identity arrow on a; in turn, the identity arrow on a is a composite of the empty pasting diagram of dimension $\mathrm{dim}(a)+1$ on a.

A new feature in the present treatment is that what are being defined directly are ω-dimensional weak categories; the finite dimensional ones are obtained as truncated versions of the ω-dimensional concept. It is important to emphasize that a multitopic ω-category is still just a multitopic set with additional properties, with no additional data.

The definition of "multitopic ω-category" is given in section 5. It uses sections 1, 2 and 4, but not 3.

The second main thing done in the paper is the definition of $\mathrm{Mlt}\omega\mathrm{Cat}$. This is a particular large multitopic set. Its definition is completed only by the end of the paper. The 0-cells of $\mathrm{Mlt}\omega\mathrm{Cat}$ are the small multitopic ω-categories, defined in section 5. Its 1-cells, which we call *1-transfors* (thereby borrowing, and altering the meaning of, a term used by Sjoerd Crans [Cr]) are what stand for "morphisms", or "functors", of multitopic ω-categories. There are n-dimensional transfors for each $n\in\mathbb{N}$. For each multitope (that is, "shape" of a higher dimensional cell) π, we have the *π-transfors*, the cells of shape π in $\mathrm{Mlt}\omega\mathrm{Cat}$. For instance, when π is the 2-dimensional multitope depicted as

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\downarrow"] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd} ,$$

the π-transfors are the "binary" "natural transformations". In the following diagram:

$$\begin{tikzcd} & \boldsymbol{A} \arrow[dr, "G"] \arrow[d, phantom, "\downarrow k"] & \\ \boldsymbol{X} \arrow[ur, "F"] \arrow[rr, "H"'] & & \boldsymbol{M} \end{tikzcd} ,$$

representing a π-transfor for the given π, $\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{M}$ denote multitopic ω-categories, F, G, H are "functors" between them, and $k:(F, G)\to H$ is a "bi-" (as in "bilinear") "natural transformation".

For each fixed multitope π, a π-transfor is what we call a *π-colored multitopic set*, with additional properties. There is a particular multitopic set $\mathcal{T}[\pi]$ the cells of which are the *π-colored multitopes*. As $\mathcal{T}[\pi]$ is a Set-valued functor on Mlt, we can take $\mathrm{El}(\mathcal{T}[\pi])$, the category of elements of $\mathcal{T}[\pi]$; this is the (FOLDS-)signature for the π-colored multitopic sets.

Section 3 describes $\mathcal{T}[\pi]$, and related "syntactical" concepts.

Section 6 gives the definition of "π-transfor", by supplying the properties a π-colored multitopic set is to satisfy to qualify being a π-transfor. Just as in the case of "multitopic ω-category" (= " 0-transfor"), these properties consist in the *existence* of items satisfying certain universal properties with respect to specific data, the universal properties being defined via FOLDS equivalence. However, there is one new element. For $\dim(\pi)\geq 2$, the concept of π-transfor involves a universal property which is an "omega-dimensional, FOLDS-style" generalization of the concept of Kan-extension (right lifting, in the terminology used by Ross Street). This is a "right-adjoint" type universal property, contrasted with the "left-adjoint" type involved in the concept of composite.

The main theorem announced, but not proved, in this paper is that the large multitopic set $\mathrm{Mlt}\omega\mathrm{Cat}$ is a multitopic ω-category. The proof of the theorem will be given in a sequel to this paper.

The material in this paper has been applied to give formulations of ω-dimensional versions of various concepts of homotopy theory; details will appear elsewhere.

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