4. Substitution

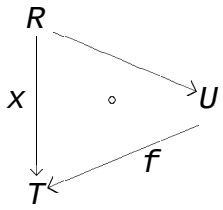
This section gives a treatment of the syntactic operation of *substitution* for multitopes and colored multitopes.

Let $m \in \mathbb{N}$. I call a multitopic set $\mathcal{R}$ *m-contractible* if for all $k \geq m$, and for all parallel pairs (α, a) of a k-pd α and a k-cell a in $\mathcal{R}$, there is exactly one $(k+1)$-cell $b : \alpha \to a$ in $\mathcal{R}$. Of course, m-contractibility implies m'-contractibility for all $m' > m$. 0-contractibility characterizes, up to isomorphism, the terminal multitopic set $\mathcal{T}$ (see section 2). For π an n-pd, $\mathcal{T}[\pi]$ is $(n-1)$-contractible; see the Proposition in section 3.

We fix a positive natural number n, and fix an $(n-1)$-contractible multitopic set $\mathcal{R}$. We place ourselves into the category $\mathrm{El}(\mathcal{R})$ of elements of $\mathcal{R}$ ($\mathcal{R}$ being a functor $\mathcal{R} : \mathrm{Mlt} \longrightarrow \mathrm{Set}$). The letters K, R, S, T, U are used for cells of $\mathcal{R}$; with the forgetful functor $\| \ \| : \mathrm{El}(\mathcal{R}) \to \mathrm{Mlt}$, the values $\|K\|$, $\|R\|$, $\|S\|$, $\|T\|$, $\|U\|$ will be denoted by κ, ρ, σ, θ and τ, respectively. We write just f for $\|f\|$ for any $f : R \to T$.

An arrow $f : R \to T$ is a *c-arrow* (*d-arrow*) iff $f : \rho \to \theta$ is one (see section 2).

An arrow $R \xrightarrow{x} T$ in $\mathrm{El}(\mathcal{R})$ is said to be (of) *d-type* (respectively, *c-type*) if $x \neq \mathrm{id}_\theta$, and there is no factorization of x of the form



where f is a c-arrow (respectively, a d-arrow). In other words, x is d-type if x is not the top face of ρ, and there is no face in ρ that would have the x-occurrence of θ as its codomain. "Dually", x is c-type if it is not the top face, and it is not in the domain of any face of one higher dimension. Note that any c-arrow is c-type, and any d-arrow is d-type.

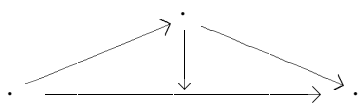
We are going to describe two types of substitution, the *d-type substitution*, and the *c-type substitution*. Whereas the two types of substitution are similar to each other, and also, they are

"dual" to each other, in the sense of the "duality" of domain versus codomain, their symmetry is not perfect; d-type substitution is somewhat simpler, and also, more "basic". First, we describe d-type substitution.

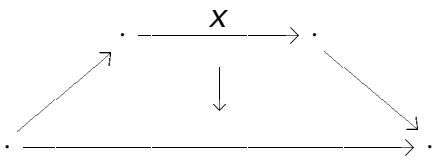
In addition to the data fixed so far, we take a cell $S \in \mathcal{R}$ of dimension equal to (the fixed) n, and keep it constant for most of what follows. Let us write $T \underset{\mathrm{def}}{=} \mathrm{c}S$.

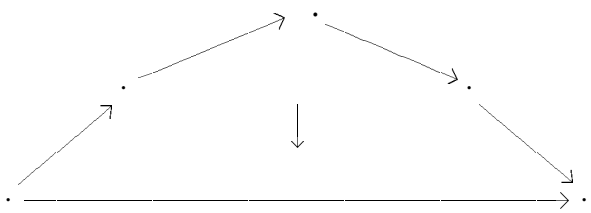
Let $\mathrm{El}(\mathcal{R})//T$ denote the full subcategory of the slice-category $\mathrm{El}(\mathcal{R})//T$ with objects the pairs (R, x) in which x is d-type.

For the given $\mathcal{R}$-cell S, $T \underset{\mathrm{def}}{=} \mathrm{c}S$, $n = \dim(S)$, assumed fixed, and for a variable (R, x), with $R \in \mathcal{R}$ and $\begin{array}{c} R \\ x \downarrow \\ T \end{array}$ of d-type, we define an $\mathcal{R}$-cell $R[S/x]$, the "result of substituting *the domain of* S in R" [since not the whole of S, but rather its "domain", is the one which is being substituted, I would like write $R[S^{-}/x]$; but I will stay with the simpler symbol]. First, some examples for the case $\mathcal{R} = \mathcal{T}$. R, S, T are now multitopes, and we write the corresponding Greek letters for them ($\| \ \| : \mathcal{R} \to \mathcal{T}$ is the identity).

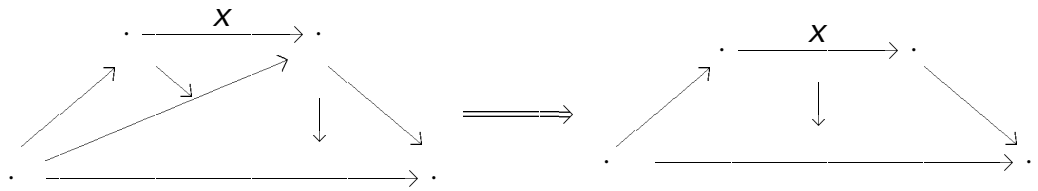
We let $n = 2$. Let σ be the 2-dimensional multitope  .

Then θ is

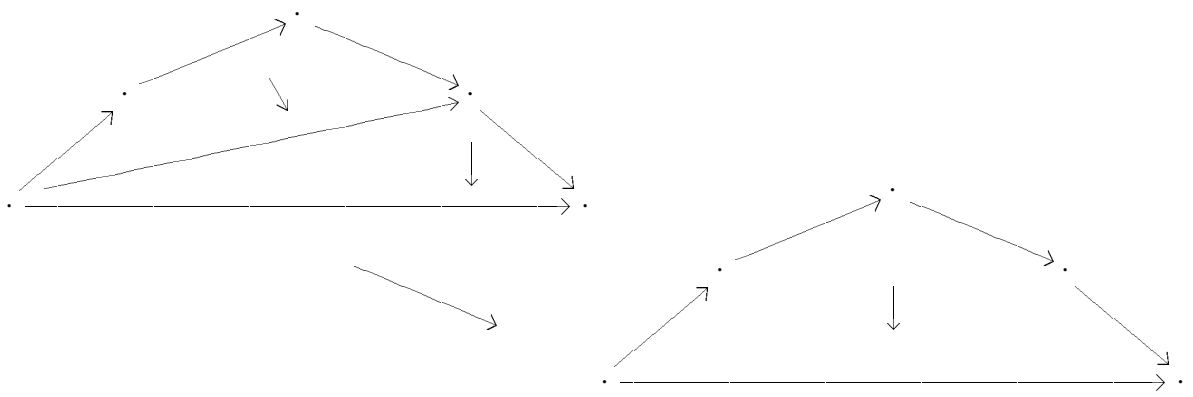
If ρ is  , with the face x as indicated, then $\rho[\sigma/x]$ is



If ρ is

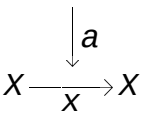


then $\rho[\sigma/x]$ is

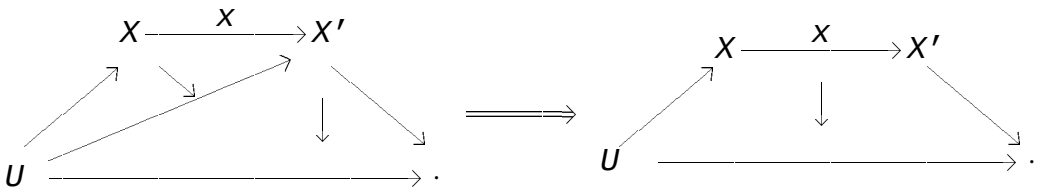


(in the last two diagrams, we gave two-dimensional renderings of three-dimensional objects; there are inevitable repetitions; all zero-dimensional faces, and all but one one-dimensional face, are drawn twice.)

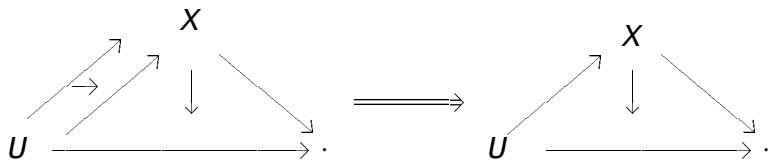
Let now σ be the empty-domain 2-multitope



Now, θ is $X \xrightarrow{x} X'$ (whereas the two X's in σ denote the *same* face, in θ, X and X' denote *different* faces). For ρ being



$\rho[\sigma/x]$ is



We return to the general case. I will not give the explicit definition of the operation $R \mapsto R[S/x]$, because doing so would require the detailed use of [H/M/P]. Instead, I will describe some properties, together with some accompanying structure, that collectively uniquely determine the operation in question; also, the facts given below will be needed later. On the other hand, the explicit definition is easy to figure out, since the intuitive meaning of substitution is clear. For the basic case $\mathcal{R}=\mathcal{T}$, that is, substitution of multitopes, what we say below amounts to describing the "geometry" of the multitope $\rho[\sigma/x]$, in the way of describing the comma-category $\mathtt{Mlt}/\rho[\sigma/x]$.

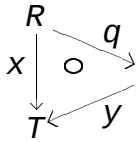
It should be emphasized that $(n\text{-}1)$-contractibility is essential for substitution to be well-defined. Also, it is clear that the condition that $x:\rho \to \theta$ be d-type is essential for $\rho[\sigma/x]$ to be well-defined as a multitope.

Let $|S|$ denote the set of all arrows in $\mathtt{El}(\mathcal{R})$ whose domain is S. We write $S|$ for the set $|S|\text{-}\{\mathrm{id}_S\}$, and $S\|$ for $|S|\text{-}\{\mathrm{id}_S, \mathrm{c}_S\}$.

The faces of $R[S/x]$ are in three classes F_1, F_2 and F_3 given below; F_1 and F_2 are not disjoint, but their overlap will also be clarified; F_3 is disjoint from F_1 and F_2. We have

$$F_1 = \{\mathring{p}:R[S/x] \to \mathrm{K}_p : p \in S\|\};$$

the correspondence $p \mapsto \mathring{p}$ is a bijection between the p's as described and the elements of F_1.

Let $R\perp_x$ denote the set of all arrows $q:R \to \kappa$ such that there is no y with 

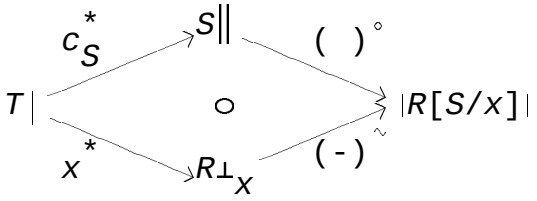
; that is, $q \in R\perp_x$ iff (intuitively speaking) the q-face of ρ does not contain x; the elements of $R\perp_x$ are called *x-free* faces. For the second class F_2, we have that

$$F_2 = \{\tilde{\hat{q}} : R[S/x] \to \mathsf{K}_q : q \in R\bot_x \} ;$$

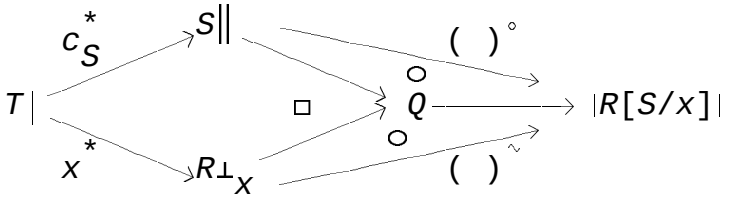
however, the correspondence $q \mapsto \tilde{\hat{q}}$ is not always a bijection. Finally,

$$F_3 = \{f[S] : R[S/x] \longrightarrow U[S/y] : f : (R, x) \to (S, y) \} .$$

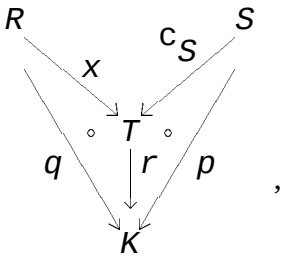
The identification of elements in F_1 and F_2 is described as follows. We have the diagram of sets



and in fact, when we take the evident pushout Q, the resulting canonical function $Q \longrightarrow |R[S/x]|$ in the diagram

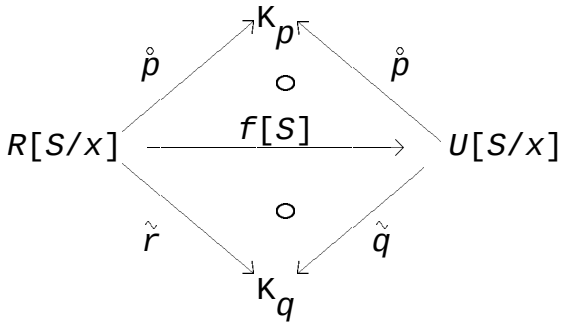


is 1-1 . In particular, if



,

then $\tilde{\hat{q}} = \mathring{p}$. We should add that $r \circ \mathring{p} = (r \circ p)^\circ$, and $r \circ \tilde{\hat{q}} = (r \circ q)^{\tilde{}}$, every time $p \in S\|$, $q \in R\bot_x$ and the composites $r \circ p$, $r \circ q$ make sense. Finally, we have that, for all relevant parameters,



provided $q \circ f = r$.

Let us summarize substitution in a more functorial manner. For the present purpose, it is good to use *formal contexts*; see section 1.

We fix S and $T = \mathtt{c}S$ as before. In the first place, substitution is a functor

$$\begin{array}{ccc} \Sigma_S : \mathtt{El}(\mathcal{R})//T & \longrightarrow & \mathtt{El}(\mathcal{R}) \\ (R, x) & & R[S/x] \\ \downarrow f & \longmapsto & \downarrow f[S] \\ (U, y) & & U[S/y] \end{array}$$

Next, recall from section 1 the functor Π , associating to a concrete context in $\mathcal{R} : \mathtt{Mlt} \to \mathtt{Set}$ an abstract context of $\mathtt{El}(\mathcal{R})$. The cell $S \in \mathcal{R}(\sigma)$ gives rise (by Yoneda) to $\hat{S} : \hat{\sigma} \to \mathcal{R}$, and, with $[\sigma^-] \xrightarrow{\iota} \hat{\sigma}$ the inclusion of the formal context $[\sigma^-]$ for $\mathtt{Mlt}$, we have the composite

$$\hat{S} \circ \iota : [\sigma^-] \xrightarrow{\iota} \hat{\sigma} \xrightarrow{\hat{S}} \mathcal{R} .$$

Π applied to the $[\sigma^-]$-context $\hat{S} \circ \iota$ in $\mathcal{R}$ gives what we now denote by

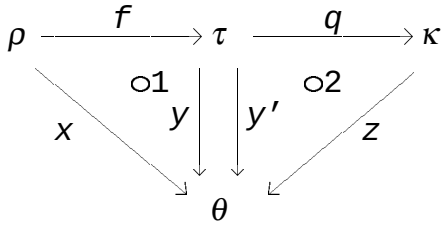
$$[S^-] : \mathtt{El}[\mathcal{R}] \longrightarrow \mathtt{Set} ,$$

an abstract context for $\mathtt{El}(\mathcal{R})$.

Let $(R, x)\in \mathtt{El}(\mathcal{R})//T$. We define the formal context $[R\perp_X]$ for $\mathtt{El}(\mathcal{R})$ as follows. It is a subfunctor of $\hat{R}$, and for any $K\in \mathtt{Ob}(\mathtt{El}(\mathcal{R}))$, $(q:R\to K)\in\hat{R}(K)$, we have $q\in[R\perp_X](K)$ iff $q\in R\perp_X$. The fact that this definition indeed gives a subfunctor $\hat{R}$ is expressed by the obvious circumstance that if $q\in R\perp_X$, then $r\circ q\in R\perp_X$. Moreover, $[R\perp_X]$ is a functor of $(R, x)\in\mathtt{El}(\mathcal{R})//T$:

$$[\text{-}\perp\_]:(\mathtt{El}(\mathcal{R})//T)^{\mathtt{op}}\to\mathtt{Set}^{\mathtt{El}(\mathcal{R})}, \tag{1}$$

Let us take a quick look at why the last assertion is true. Because of the map $\|\ \|:\mathcal{R}\to\mathcal{T}$, it suffices to show the assertion for the case $\mathcal{R}=\mathcal{T}$. We want that if $(f:(\rho, x)\longrightarrow(\tau, y))\in\mathtt{Mlt}//\theta$, and $(q:\tau\to\kappa)\in\tau\perp_Y$, then $(qf:\rho\to\kappa)\in\rho\perp_X$. Suppose not; then we have $z:\kappa\to\theta$ such that the outside large triangle in



commutes. Define y' by the commutativity $\circ 2$. We conclude that $yf=y'f$. $y'=y$ contradicts $(q:\tau\to\kappa)\in\tau\perp_Y$; we have $y'\neq y$. As we said in the last section, this can happen only if there is an arrow $e:\rho\to\varepsilon$ to an empty-domain multitope ε such that, in

$$\rho\xrightarrow{e}\varepsilon\xrightarrow{\mathtt{c}_\varepsilon}\mathtt{c}\varepsilon\underset{\mathtt{d}_{\mathtt{c}\varepsilon}}{\overset{\mathtt{c}_{\mathtt{c}\varepsilon}}{\rightrightarrows}}\mathtt{cc}\varepsilon,$$

we have $f=\mathtt{c}_\varepsilon\circ e$, and either $y=\mathtt{d}_{\mathtt{c}\varepsilon}$ and $y'=\mathtt{c}_{\mathtt{c}\varepsilon}$, or $y'=\mathtt{d}_{\mathtt{c}\varepsilon}$ and $y=\mathtt{c}_{\mathtt{c}\varepsilon}$. In either case, since $y\circ f=y'\circ f=x$, we get a contradiction to x being d-type. This completes the proof.

In the next section, we will make good use of the two-variable functor

$$\begin{array}{rcl} \Lambda : (\mathtt{El}(\mathcal{R})//T)^{\mathrm{op}} \times \mathtt{El}(\mathcal{R}) & \longrightarrow & \mathtt{Set} \\ ((R, x), U) & \longmapsto & [R\perp_x](U) \end{array} \qquad (1')$$

the exponential adjunct of (1).

For each $(R, x) \in \mathtt{El}(\mathcal{R})//T$, the operations

$$(\ )^{\circ} = (\ )^{\circ}_{(R, x)} : S\| \longrightarrow |R[S/x]|$$

$$(\ )^{\sim} = (\ )^{\sim}_{(R, x)} : R\perp_x \longrightarrow |R[S/x]|$$

are in fact maps of abstract contexts:

$$(\ )^{\circ}_{(R, x)} : [S^{-}] \longrightarrow (R[S/x])^{\wedge},$$

$$(\ )^{\sim}_{(R, x)} : [R\perp_x] \longrightarrow (R[S/x])^{\wedge};$$

and both of them are natural in $(R, x) \in \mathtt{El}(\mathcal{R})//T$:

$$(\mathtt{El}(\mathcal{R})//T)^{\mathrm{op}} \underset{(\ )^{\wedge} \circ \Sigma_S}{\overset{[\_^{-}] \circ \ulcorner S \urcorner}{\underset{\downarrow (\ )^{\circ}}{\longrightarrow}}} \mathtt{Set}^{\mathtt{El}(\mathcal{R})//T},$$

$$(\mathtt{El}(\mathcal{R})//T)^{\mathrm{op}} \underset{(\ )^{\wedge} \circ \Sigma_S}{\overset{[-\perp\_]}{\underset{\downarrow (\ )^{\sim}}{\longrightarrow}}} \mathtt{Set}^{\mathtt{El}(\mathcal{R})//T},$$

We write the same symbol $(\ )^{\sim}$ for the exponential adjunct of $(\ )^{\sim}$:

$$(\mathtt{El}(\mathcal{R})//T)^{\mathrm{op}} \times \mathtt{El}(R) \;\underset{\hom_{\mathtt{El}(\mathcal{R})}(\Sigma_S -,-)}{\overset{\Lambda}{\Downarrow (\ )^{\sim}}}\; \mathtt{Set}\ .$$

This completes the specification of d-type substitution.

We next turn to c-type substitution.

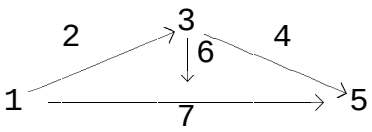
We still have the fixed n, and the $(n\text{-}1)$-contractible multitopic set $\mathcal{R}$ as before. Instead of S, now we use the symbol P for a fixed n-cell in $\mathcal{R}$.

Before going on to the description proper, I want to mention that, in contrast to d-type substitution, the c-type substitution will be used (in this paper) only in very special circumstances. These are described as follows.

We fix $\pi \in \mathtt{Mlt}$ of dimension $n \geq 2$. In the applications, n will always be *even*; however, for the constructions we perform now, this will not be of importance. We put $\mathcal{R} = \mathtt{Mlt}[\pi]$.

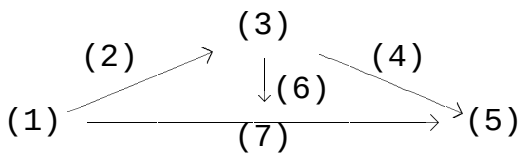
There is one particular colored multitope $P[\pi] = P \in \mathtt{Mlt}[\pi]$ whose underlying multitope $\|P\|$ is π, and for which the total coloring $\bar{\gamma}\colon |\pi| \to |\pi|$ is the identity. The P above will be $P = P[\pi]$.

We will consider the example when π is the multitope



(2)

P is then depicted by the picture



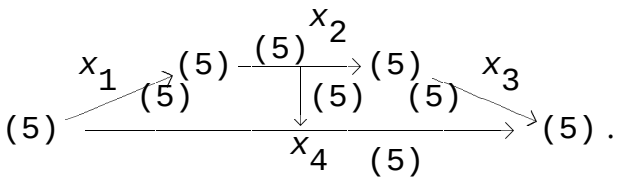
Here, and elsewhere, we employ the device of indicating the color of a face by putting the name of the color (in our case, one of the numbers 1,...,7) in parentheses onto the cell.

We now assume that $n \geq 2$, and postulate another piece of data. We assume the existence of, and fix, a cell T in $\mathcal{R}$ such that $\mathtt{d}T = \langle \mathtt{cc}P \rangle$, and $\mathtt{c}T = \mathtt{cc}P$. Here, $\langle \mathtt{cc}P \rangle$ denotes the $(n-2)$-pd consisting of the single $(n-2)$-cell $\mathtt{cc}P$. This is the same as $\mathtt{i}_{n-2}(\mathtt{cc}P)$ for the inclusion $\mathtt{i}_{n-2}:\mathtt{C}_{n-2}(\mathcal{R}) \to \mathtt{P}_{n-2}(\mathcal{R})$ mentioned at the beginning of section 2. When $\mathcal{R} = \mathcal{T}[\pi]$, then for any $U \in \mathcal{R}$, there is a unique π-colored multitope $U^* \in \mathcal{R}$ such that $\mathtt{d}(U^*) = \langle U \rangle$ and $\mathtt{c}(U^*) = U$, and whose color is the same as that of U. In this case, we will define $T \stackrel{\text{def}}{=} (\mathtt{cc}P)^*$. (Recall, from section 2, the construction of the multitope θ^* out of any multitope θ.)

In our example started above, $\mathtt{cc}P$ is the zero-dimensional multitope colored 5 ; and T is

$$(5) \xrightarrow{(5)} (5) \ .$$

Continuing the example, let R be



(3)

Then x_4 is c-type, but x_1, x_2 and x_3 are not. When R is the shape of a 3-cell with an empty 2-pd as domain, all colored (5) , the one 1-dimensional face of R is c-type. .

Let us return to the general context. For later use, let me note the following map of abstract contexts:

$$\begin{array}{ccc} [T^{-}] & \xrightarrow{\eta} & [P^{-}] \\ \\ \begin{array}{c} T \\ \downarrow{\scriptstyle c_T} \\ \mathtt{c}c P \\ {\scriptstyle p}\downarrow \\ S \end{array} & \longmapsto & \begin{array}{c} P \\ \downarrow{\scriptstyle c_{cP}\circ c_P} \\ \mathtt{c}c P \\ {\scriptstyle p}\downarrow \\ S \end{array} \end{array} \qquad (4)$$

$\mathtt{El}(\mathcal{R})\#T$ denotes the full subcategory of $\mathtt{El}(\mathcal{R})/T$ on the objects the c-type arrows. We have the functor

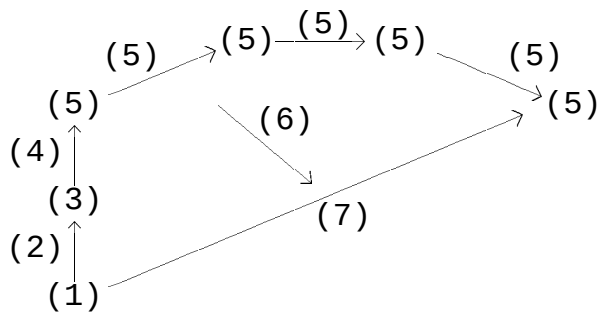
$$\begin{array}{ccc} \Lambda : (\mathtt{El}(\mathcal{R})\#T])^{\mathrm{op}} \times \mathtt{El}(\mathcal{R}) & \longrightarrow & \mathtt{Set} \\ \left(x \overset{R}{\downarrow} T \;,\; U\right) & \longmapsto & R\perp_X \end{array}$$

Here, $R\perp_X$ is the set of all " x-free arrows" out of R; $[R\perp_X]$ is the corresponding abstract context; these are defined as before. The action on arrows defined in the obvious way; the proof that this works is as in the analogous situation before.

We have a (*c-type*) *substitution* functor

$$\begin{array}{ccc} \Sigma_P^{\#} : \mathtt{El}(\mathcal{R})\#T & \longrightarrow & \mathtt{El}(\mathcal{R}) \\ (R, x) & \longmapsto & R[P\#x] \\ f\downarrow & & \downarrow f[P] \\ (S, y) & \longmapsto & S[P\#y] \end{array} \quad .$$

As an illustration: when, for instance, R is as in (3), and $x{=}x_4$, then (given that π is (2), and $\mathcal{R}{=}\mathtt{Mlt}[\pi]$) $R[P\#x]$ is



Intuitively speaking, the c-type substitution $R[P\#x]$ involves "splitting" P at the codomain of its codomain, causing the removal of $\mathrm{c}P$, and the doubling of $\mathrm{cc}P$, and then putting the resulting shape in place of the face x in R.

To explain the rest of the structure connected with c-type substitution, let us introduce a new formal context denoted $[P\#]$ ("P-split"). $[P\#]$, a modified ("split") version of $[P]$, is described by saying the following:

there is a monomorphism $\iota : [P^-] \longrightarrow [P\#]$ such that each component $\iota_S : [P^-](S) \longrightarrow [P\#](S)$ is a bijection except for $S=\mathrm{c}P$ and $S=\mathrm{cc}P$; for simplicity, we take ι to be the inclusion;

$[P\#](\mathrm{c}P) = \{\#^{++}\}$ for a certain element $\#^{++}$ (note that $[P^-](\mathrm{c}P) = \emptyset$);

$[P\#](\mathrm{cc}P) = [P^-](\mathrm{cc}P) \bigsqcup \{\#^{+}\}$ for another element $\#^{+}$;

$[P\#](\mathrm{c}_{\mathrm{c}P})(\#^{++}) = \#^{+}$;

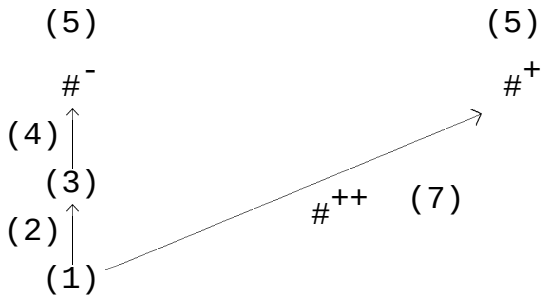
for any $r \in (\mathrm{c}P)\|$, we have $[P\#](r)(\#^{++}) = r \circ \mathrm{c}_P$;

for any $s \in (\mathrm{cc}P)|$, $[P\#](s)(\#^{+}) = s \circ \mathrm{c}_{\mathrm{c}P} \circ \mathrm{c}_P$.

For the sake of symmetry, we write $\#^-$ for the element $\mathrm{c}_{\mathrm{c}P} \circ \mathrm{c}_P$ in $[P^-](\mathrm{cc}P) \subset [P\#](\mathrm{cc}P)$.

These facts determine $[P\#]$ up to isomorphism.

In the example, the formal context $[P\#]$ is depicted by the picture:



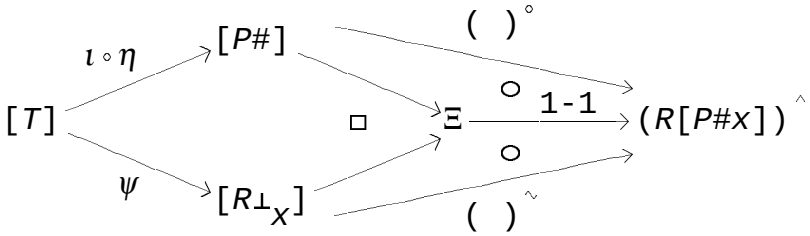
For each $(R, x) \in \mathrm{El}(\mathcal{R})\#P$, we have a map $(\ )^{\circ}_{(R,x)} : [P\#] \longrightarrow (R[P\#x])\hat{}$, which, as a function of (R, x), is natural:

$$(\mathrm{El}(\mathcal{R})\#P)^{\mathrm{op}} \xrightarrow[(-)\hat{}\circ\Sigma_P^{\#}]{\ulcorner [P\#] \urcorner \ (\ )^{\circ}\ } \mathrm{Set}$$

On the other hand, we have

$$(\mathrm{El}(\mathcal{R})\#P)^{\mathrm{op}} \times \mathrm{El}(\mathcal{R}) \xrightarrow[\mathrm{hom}_{\mathrm{El}(\mathcal{R})}(\Sigma_P^{\#}-,-)]{\Lambda\ (\ )\tilde{}\ } \mathrm{Set}$$

The identification of the images of $(\ )^{\circ}$ and $(\ )\tilde{}$ is explained by the diagram



where ψ is $(p : T \to S) \longmapsto p \circ x$, $\iota \circ \eta$ was introduced above in (4).