3. Colored multitopes.

The question how to define the concept of weak n-category, for finite n, and in the second place, also, for $n=\omega$, has been around for a while; and several answers have already been proposed (see [B]. [B/D],...). However, one clearly wants more than that concept: one wants the definition of the total structure weak higher dimensional categories are elements of. For instance, one wants morphisms, or functors (some may want to say: "homomorphisms", since for the case of $n=2$ and $n=3$, the existing notions that we want to emulate and generalize are called "homomorphism" and "trihomomorphism", respectively), of n-categories ; for $n>0$, one wants an (appropriately higher dimensional) notion of (pseudo-)natural transformation of functors of weak n-categories; for still greater n, modifications of the latter, and so on.

This structure will be provided here in a uniform manner.

In the first place, the concept of multitopic ω-category will be defined and proposed as a concept of weak ω-category. Multitopic n-categories, for each finite n, will be the natural ("truncated") special cases of the ω-concept.

Secondly, we will define a (large) multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ whose 0-cells will be the (small) multitopic ω-categories. The 1-cells of $\mathtt{Mlt}\omega\mathtt{Cat}$ will be our construal of the functors (morphisms) of multitopic ω-categories. The 2-cells will be the "natural transformations"; but note that now we will have, as separate kinds of entities, binary, ternary, and, in general, k-ary (for all $k\in\mathbb{N}$, even for $k=0$!) "natural transformations". The 3-cells in $\mathtt{Mlt}\omega\mathtt{Cat}$ will stand for modifications; now, they come with somewhat complicated arities, which are exactly the 2-multitopes, shapes of "many-to-one" pasting diagrams of dimension 2. And so on, for all finite n . Borrowing a term from Sjoerd Crans, I will call an n-cell in $\mathtt{Mlt}\omega\mathtt{Cat}$ an *n-transfor*. (Crans uses this term in the context of *teisi*, "semi-strict" weak higher dimensional categories. Unfortunately however, there is mismatch in the assignment of the "order" n ; what I call n-transfor corresponds to Crans's $(n-1)$-transfor.) Thus, the, usually multiary, n-cells in the multitopic set $\mathtt{Mlt}\omega\mathtt{Cat}$ will be the n-transfors; the 0-transfors are the same as the (multitopic) ω-categories.

The main theorem announced in this paper is that $\mathtt{Mlt}\omega\mathtt{Cat}$ is a multitopic ω-category.

The uniformity of the theory lies in the fact that each n-transfor will itself be a multitopic set, endowed, however, with certain additional structure that we call *coloring*. Given a multitope π, we will define a particular category $\mathtt{Mlt}[\pi]$ of *π-colored multitopes*. The category $\mathtt{MltSet}[\pi]$ of *π-colored multitopic sets* will then be defined as the functor category $\mathtt{Set}^{\mathtt{Mlt}[\pi]}$. When $\pi=\rho^{(0)}$, the unique zero-dimensional multitope, then $\mathtt{Mlt}[\pi]$ and $\mathtt{MltSet}[\pi]$ are, essentially, the same as $\mathtt{Mlt}$ and $\mathtt{MltSet}$, respectively. An n-transfor of shape π, that is, an element of $\mathtt{Mlt}\omega\mathtt{Cat}(\pi)$, will be a π-colored multitopic set with certain properties (but with no additional data).

Recall that the objects of $\mathtt{Mlt}$ were the cells of $\mathcal{T}$, the terminal multitopic set. Generalizing, for each multitope π, we will define a particular multitopic set $\mathcal{T}[\pi]$; the objects of $\mathtt{Mlt}[\pi]$ will then be the cells of $\mathcal{T}[\pi]$.

Together with the multitopic sets $\mathcal{T}[\pi]$, we will also define, for each arrow $\pi\to\kappa$ in $\mathtt{Mlt}$, morphisms $\mathcal{T}[p]:\mathcal{T}[\kappa]\longrightarrow\mathcal{T}[\pi]$; in fact, we will define a functor

$$\mathcal{T}[\text{-}]:\mathtt{Mlt}^{\mathtt{op}}\longrightarrow\mathtt{MltSet}.$$

(As we will see, $\mathcal{T}[p]$ "recolors" every κ-colored multitope as a π-colored one, by replacing the color $r\in|\kappa|$ by $r\circ p\in|\pi|$.)

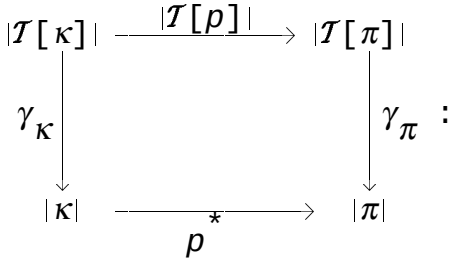
In addition, for each $\pi\in\mathtt{Mlt}$, we will define a mapping

$$\gamma_\pi:|\mathcal{T}[\pi]|\longrightarrow|\pi|,$$

on the total set

$$|\mathcal{T}[\pi]|\ \bar{\mathtt{def}}\ \coprod_{\tau\in\mathtt{Mlt}}\mathcal{T}[\pi](\tau)$$

of the multitopic set $\mathcal{T}[\pi]$ to the set of arrows out of π, called *coloring*. The coloring will be compatible with the functor $\mathcal{T}[\text{-}]$ in the natural sense that, for each $p:\pi\to\kappa$, the following diagram commutes:



here, the notation $|\mathcal{T}[p]|$ is self-explanatory; p^* is the mapping $r \mapsto r \circ p$.

The definition is by recursion. Fix $n \in \mathbb{N}$. Assume that the functor $\mathcal{T}[-]$ has been defined on arguments that are objects of dimensions less than n , and arrows between such objects; moreover, assume that said restriction of $\mathcal{T}[-]$ is a functor. Assume that the coloring γ_κ has been defined for all $\kappa \in \mathtt{Mlt}$ of dimension $<n$, with the relevant instances of the compatibility condition satisfied. Fix $\pi \in \mathtt{Mlt}$ such that $\dim(\pi)=n$; we will define $\mathcal{T}[\pi]$, γ_π , and $\mathcal{T}[p]$ for $p \in \pi|$, where $\pi|$ is the set of non-identity arrows in $\mathtt{Mlt}$ out of π .

Let $\pi|\mathtt{Mlt}$ denote the full subcategory of the comma-category $\pi\backslash\mathtt{Mlt}$ on objects the elements of $\pi| = |\pi| - \{\mathrm{id}_\pi\}$. Let $\mathtt{Mlt}\restriction_{<\pi}$ denote the full subcategory of $\mathtt{Mlt}$ on the objects $\mathtt{K}_p$ $(=\mathtt{cod}(p))$ for $p \in \pi|$. We have the forgetful functor $\varphi : \pi|\mathtt{Mlt} \longrightarrow \mathtt{Mlt}\restriction_{<\pi}$. Consider the composite functor

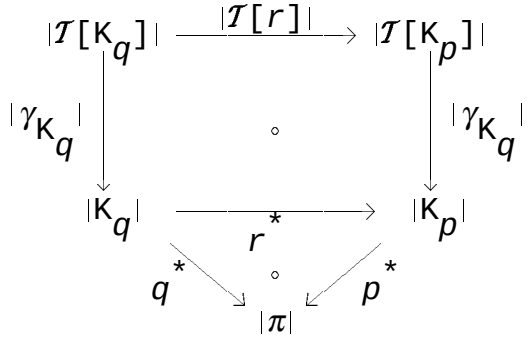
$$\pi|\mathtt{Mlt} \xrightarrow{\varphi} \mathtt{Mlt}\restriction_{<\pi} \xrightarrow{\mathcal{T}[-]\restriction_{<\pi}} \mathtt{MltSet} ,$$

and call it Φ ; note that by the recursion hypothesis, $\mathcal{T}[-]\restriction_{<\pi}$ is well-defined. Let $\mathcal{P}$ be the colimit $\mathtt{colim}(\Phi)$ in $\mathtt{MltSet}$ (when, later on, we want to refer $\mathcal{P}$ for various instances of π , we may write $\mathcal{P}[\pi]$ for $\mathcal{P}$). Since $\mathtt{MltSet}$ is (equivalent to) $\mathtt{Set}^{\mathtt{Mlt}}$, $\mathtt{colim}(\Phi)$ exists, and is computed "pointwise" as in $\mathtt{Set}$. In particular, for each multitope τ ,

$$\mathcal{P}(\tau) = \mathop{\mathtt{colim}}_{(p:\pi \to \mathtt{K}_p) \in \pi|\mathtt{Mlt}} \mathcal{T}[\mathtt{K}_p](\tau) .$$

We will define $\mathcal{T}[\pi]$ as a certain extension of the multitopic set $\mathcal{P}$; we will adjoin certain specific cells to $\mathcal{P}$, to obtain $\mathcal{T}[\pi]$. Recall that we also have to define the coloring $\gamma_\pi(a)$ for cells a in $\mathcal{T}[\pi]$. The coloring $\gamma_\pi(a)$ of cells a in $\mathcal{P}$ (a part of $\mathcal{T}[\pi]$) is given as

follows. For any $p, q \in |\pi|$, and $(r : p \to q) \in \mathrm{Arr}(\pi|\mathrm{Mlt})$, the commutativity of the diagram



ensures that we have a well-defined map γ from the colimit $|\mathcal{P}| = \mathrm{colim}(|\Phi|)$ to $|\pi|$ such that, for each colimit coprojection $\iota_p : \mathcal{T}[\mathrm{K}_p] \to \mathcal{P}$, we have $\gamma \circ |\iota_p| = p^* \circ \gamma_{\mathrm{K}_p}$. The coloring γ_π restricted to $\mathcal{P}$ is defined to be this γ.

All newly adjoined cells in $\mathcal{T}[\pi]$ are to have the γ_π-color id_π. Thus, the part $\mathcal{P}$ is distinguished inside $\mathcal{T}[\pi]$ as those cells whose color is different from id_π.

The extension of $\mathcal{P}$ to $\mathcal{T}[\pi]$ is done as follows. No cells of dimension less than n are adjoined to $\mathcal{P}$. The short way of saying how to adjoin cells of dimensions $\geq n$ is that we adjoin, recursively, for each $m \geq n$, for each $(m\text{-}1)$-pd α and for each $(m\text{-}1)$-cell a such that $\alpha \| a$, a new cell $b : \alpha \to a$ if and only if there is no $c \in \mathcal{P}$ such that $c : \alpha \to a$. The form of the definition that we give next, which we consider the official one, gives more complicated, but also more descriptive conditions for when an adjunction takes place; the Proposition stated later on will contain, among others, the fact of the equivalence of the two forms of the definition.

An $(n\text{-}1)$-pd $\beta = \langle \beta(r) \rangle_{r \in \tau\|}$ in $\mathcal{P}$ is called c_π-*free* if for all $r \in \tau\|$, we have $\gamma_\pi(\beta(r)) \neq \mathrm{c}_\pi$. The new n-cells adjoined to $\mathcal{P}$ are in a bijective correspondence with the pairs (β, b) where β is a c_π-free $(n\text{-}1)$-pd in $\mathcal{P}$, b is an $(n\text{-}1)$-cell in $\mathcal{P}$ such that $\gamma_\pi(b) = \mathrm{c}_\pi$, and β and b are parallel: $\mathrm{d}\beta = \mathrm{d}b$, $\mathrm{c}\beta = \mathrm{c}b$; for each such pair (β, b), we adjoin to $\mathcal{P}$ exactly one new cell a with $\mathrm{d}a = \beta$, $\mathrm{c}a = b$; we declare that $\gamma_\pi(a) = \mathrm{id}_\pi$.

For dimensions $m > n$, we proceed recursively. For any parallel pair (β, b) of an

$(m\text{-}1)$-pd β and an $(m\text{-}1)$-cell such that $\gamma_\pi(b)=\mathrm{id}_\pi$, we adjoin a single new m-cell a for which $da=\beta$, $ca=b$, and make $\gamma_\pi(a)$ equal to id_π.

This completes the definition of the multitopic set $\mathcal{T}[\pi]$. We have the inclusion $\iota:\mathcal{P}\to\mathcal{T}[\pi]$. For $(p:\pi\to\kappa)\in|\pi|$, we define $\mathcal{T}[p]:\mathcal{T}[\kappa]\to\mathcal{T}[\pi]$ as the composite $\mathcal{T}[p]=\iota\circ\iota_p$. The functorial nature of $\mathcal{T}[-]$ on arrows between objects in the set $\{K_p: p\in|\pi|\}$, and the compatibility of the various colorings indexed by elements of $|\pi|$ follow from the definition.

For the fixed n, performing the above construction simultaneously for all multitopes π of dimension n, gives the definition of all necessary items involving indices $\le n$, since all non-identity arrows out of a multitope are to multitopes of lower dimensions. The functor $\mathcal{T}:\mathtt{Mlt}^{\mathrm{op}}\to\mathtt{Set}^{\mathtt{Mlt}}$ has been defined.

Some remarks on the above.

Let $\pi,\tau\in\mathtt{Mlt}$, $a\in\mathcal{T}[\pi](\tau)$. The *total π-coloring* of a includes the colors of all the faces of a; it is defined to be

$$\bar{\gamma}_\pi(a) \;\overset{\mathrm{def}}{=}\; \langle \gamma_\pi(\mathcal{T}[\pi](s)(a))\rangle_{s\in|\tau|} \;\in\; |\pi|^{|\tau|}.$$

Using, for any $r:\kappa\to\lambda$, the naturality of $\mathcal{T}[r]:\mathcal{T}[\lambda]\longrightarrow\mathcal{T}[\kappa]$, we find that we have the commutativity

$$\begin{array}{ccc}
\mathcal{T}[\kappa](\tau) & \xrightarrow{\;\mathcal{T}[p]_\tau\;} & \mathcal{T}[\pi](\tau) \\
\Big\downarrow{\scriptstyle \bar{\gamma}_\kappa} & \circ & \Big\downarrow{\scriptstyle \bar{\gamma}_\pi} \\
|\kappa|^{|\tau|} & \xrightarrow[\;p^*\;]{} & |\pi|^{|\tau|}
\end{array}.$$

We can see that each map $\bar{\gamma}_\pi:\mathcal{T}[\pi](\tau)\longrightarrow|\pi|^{|\tau|}$ is 1-to-1. This fact expresses that a colored multitope is determined by its underlying (ordinary) multitope, that is, its shape, and its (total) coloring. Therefore, a π-colored multitope is given by an (ordinary) multitope and a

correct coloring of its faces by the faces of $|\pi|$. What constitutes a correct coloring is, of course, defined unambiguously by the above-given definition of `Mlt[-]`; however, this definition is not very direct (it uses a recursion). There is a simple description of what correctness means for the case when π is positive; for the general case, I do not have a simple alternative to the definition provided.

Let us note here that for each $\pi\in$`Mlt`, we have the canonical π-colored multitope $P=$`P[`π`]` for which $P\in$`Mlt[`π`](`π`)`, and for which $\bar{\gamma}_{\pi}(\pi)=\mathrm{id}_{\pi}$.

Let us make some further points of importance on colored multitopes. We have the

Proposition. With $n=$`dim(`π`)`, for any pd α and any cell a in the multitopic set $\mathcal{I}$`[`π`]` such that α and a are parallel, and `dim(`α`)=dim(`a`)`$\geq n-1$, we have exactly one cell b in $\mathcal{I}$`[`π`]` such that `d`$b=\alpha$, `c`$b=a$.

For the proof, which uses an induction on n, we prove the auxiliary statement saying that, with reference to the above definition, in the multitopic set $\mathcal{P}$, for each $m\geq n-1$, the canonical morphism

$$\underset{(p:\pi\to \mathtt{K}_p)\in\pi\,|\,\mathtt{Mlt}}{\mathtt{colim}}\ \mathtt{P}_m(\mathcal{I}[\mathtt{K}_p])\longrightarrow \mathtt{P}_m(\mathcal{P})$$

is an isomorphism. The other fact we need, proven by induction, is that, under the conditions given above for the case of adjoining a new n-cell $\alpha\to a$, there is no cell of the form $\alpha\to a$ in $\mathcal{P}$.

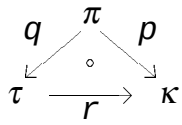
As it is obvious, and also will be pointed out in the examples, the proposition fails in dimensions (for b in the statement) less than n.

A fairly immediate consequence of the proposition is the

Corollary. With $n=$`dim(`π`)`, for any pd α in the multitopic set $\mathcal{I}$`[`π`]` such that `dim(`α`)`$\geq n$, we have exactly one cell b in $\mathcal{I}$`[`π`]` such that `d`$b=\alpha$, and in particular,

exactly one cell parallel to α.

Another general remark of importance is that for all $p:\pi\to\kappa$ in $\mathtt{Mlt}$, the induced map $\mathcal{T}[p]:\mathcal{T}[\kappa]\to\mathcal{T}[\pi]$ is a monomorphism. Moreover, if $\pi,\rho\in\mathtt{Mlt}$, $R\in\mathcal{T}[\pi](\rho)$, then for κ the codomain of the arrow $p=\gamma_\pi(R):\pi\to\kappa$, there is a (unique) cell $\dot{R}\in\mathcal{T}[\kappa](\rho)$, called the *root* of R, for which $\dot{R}\vdash\xrightarrow{\mathcal{T}[p]}R$. Furthermore, if $q:\pi\to\tau$, and $S\in\mathcal{T}[\tau](\rho)$ is such that $S\vdash\xrightarrow{\mathcal{T}[p]}R$, then, necessarily, the root of S is the same as that of R; in other words, there is a commmutative diagram



such that $\dot{R}\vdash\xrightarrow{\mathcal{T}[r]}S$. Let us call a cell T in $\mathcal{T}[\theta]$ *primitive* if $\gamma_\theta(T)=\mathtt{id}_\theta$. The roots of cells are always primitive; the primitive cells are the same as the ones that are their own roots. Note that if $T\in\mathcal{T}[\theta]$ is primitive, then $\mathtt{dim}(T)\geq\mathtt{dim}(\theta)$.

Let us give some examples.

It is clear that when $\pi=\rho^{(0)}$, $\mathcal{T}[\pi]$ is just $\mathcal{T}$: there is just one color.

The next example is when $\pi=\rho^{(1)}$, the unique $\mathtt{1}$-dimensional multitope. We have that $|\pi| = \{\mathtt{d},\mathtt{c}\}$, both with codomain $\rho^{(0)}$. The multitopic set $\mathcal{P}=\mathcal{P}[\pi]$ is the disjoint union of two copies of $\mathcal{T}$, one colored $\mathtt{d}$, the other colored $\mathtt{c}$. In $\mathcal{T}[\pi]$, there are exactly two $\mathtt{0}$-cells and three $\mathtt{1}$-cells; changing the notation somewhat, and representing π as

$$\mathtt{1}\xrightarrow{\mathtt{2}}\mathtt{3}$$

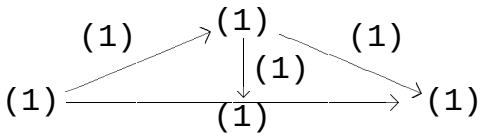
(which, of course, means that $\mathtt{1}=\mathtt{d}$, $\mathtt{3}=\mathtt{c}$ and $\mathtt{2}=\mathtt{id}_\pi$), the $\mathtt{1}$-cells in $\mathcal{T}[\rho^{(1)}]$ are

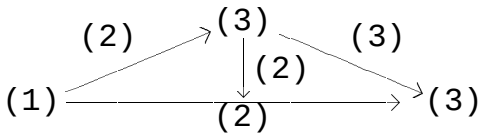
$$(1)\xrightarrow{(1)}(1),$$

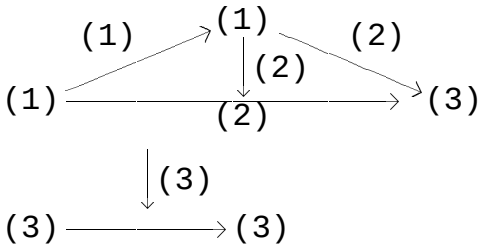
$(3) \xrightarrow{(3)} (3)$,

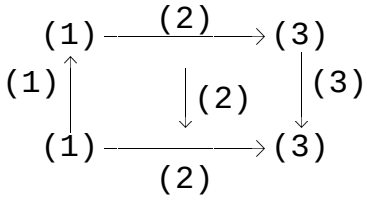
$(1) \xrightarrow{(2)} (3)$;

the parenthetical numbers are the colors of the cells in question. The first two come from $\mathcal{P}$, the last as the n-cell (n=1) adjoined to $\mathcal{P}$. Some examples of 2-cells in $\mathcal{I}[\rho^{(1)}]$:



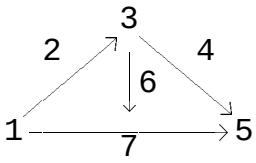






The last 2-cell has domain the empty 1-pd on the 0-cell (3) . Note that the 1-cell $(1) \xrightarrow{(2)} (3)$ cannot be the target of an empty-domain 2-cell, since $(1) \neq (3)$. The first and the last of these 2-cells are "from" $\mathcal{P}[\rho^{(1)}]$; the others are introduced by the third stage of the definition of $\mathcal{I}[\rho^{(1)}]$.

Next, consider the example of the following 2-cell for π :



In $\mathcal{I}[\pi]$, we have three 0-cells: (1) , (3) , (5) ; six 1-cells:

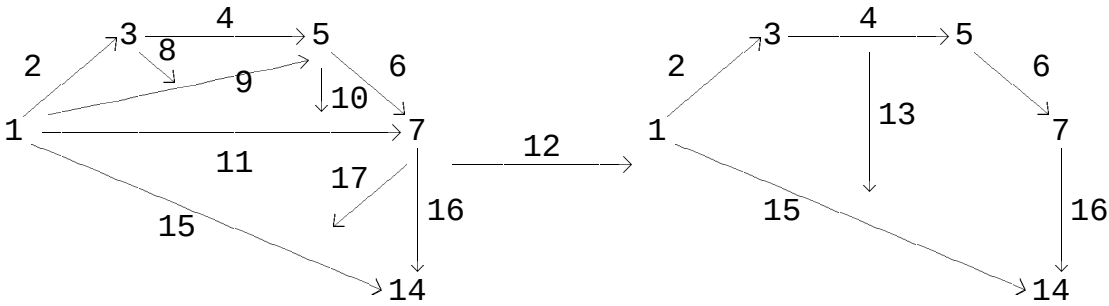
$(1) \xrightarrow{(1)} (1)$, $(1) \xrightarrow{(2)} (3)$,

$(3) \xrightarrow{(3)} (3)$, $(3) \xrightarrow{(4)} (5)$,

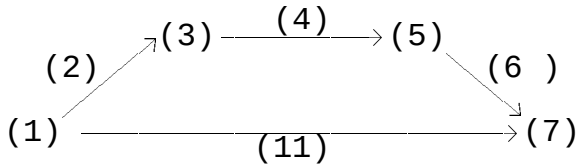
$(5) \xrightarrow{(5)} (5)$, $(1) \xrightarrow{(7)} (5)$.

As for 2-cells, they come in seven different colors. The ones of color (6) have as domain a 1-pd that is a sequence of 1-cells, starting in a succession (possibly of zero length) of (1)'s, followed by a single instance of $(1) \xrightarrow{(2)} (3)$, followed by a succession of (3)'s, followed by a single $(3) \xrightarrow{(4)} (5)$, ending in a succession of (5)'s. The 2-cells of color (6) all have as codomain the 1-cell $(1) \xrightarrow{(7)} (5)$.

For π the 3-dimensional multitope

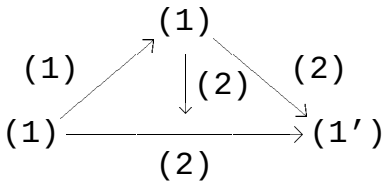


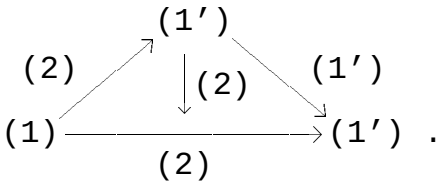
in $\mathtt{Mlt}[\pi]$, the parallel pair (α, a) of the 1-pd α and 1-cell a shown in



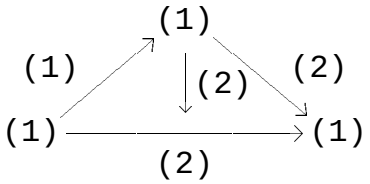
does not have a "filling" $b : \alpha \to a$.

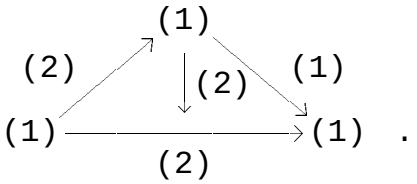
Let π be $0 \xrightarrow[1]{2} 0$, the simplest empty-domain multitope, and let κ be $0 \xrightarrow[1]{} 0'$, the unique 1-dimensional multitope $\rho^{(1)}$ (considered already above, with the notation $1 \xrightarrow{2} 3$). We have the map $p = c_{\pi} : \pi \to \kappa$ (unique of the form $\pi \to \kappa$) for which p^* maps 0 and 0′ to 0, 1 to 1. We have the κ-colored multitopes



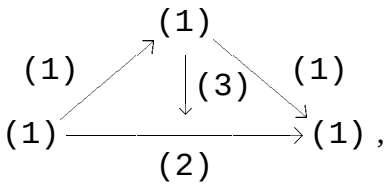


Their images under $\mathcal{I}[p]$ in $\mathcal{I}[\pi]$ are





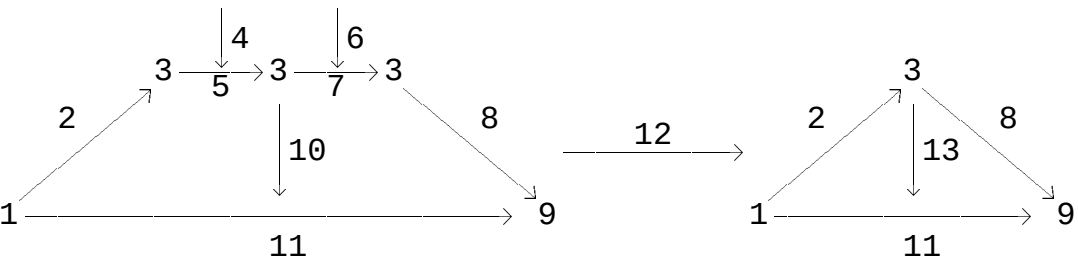
In $\mathcal{I}[\pi]$, we have the further cell



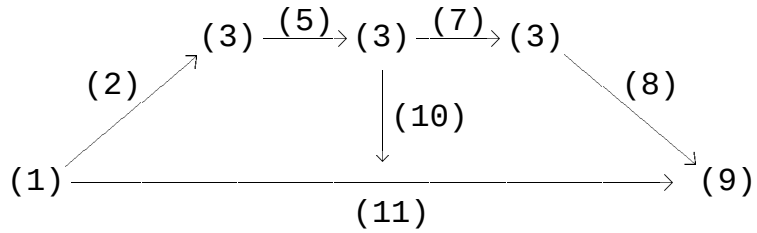
which is not obtained from $\mathcal{P}[\pi]$ under the canonical map into $\mathcal{I}[\pi]$.

Finally, let us take an example where the recursive character of the definition shows itself to be more essential than in the "positive" (empty-domain-free) examples considered so far.

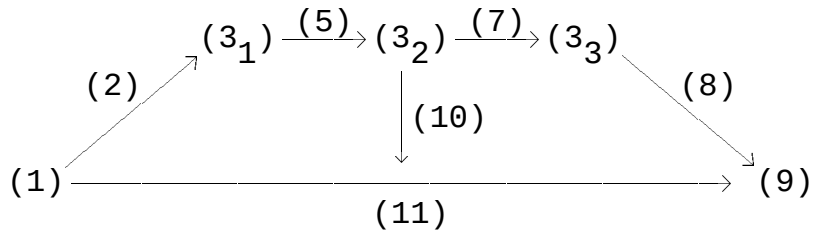
Let π be the 3-dimensional multitope



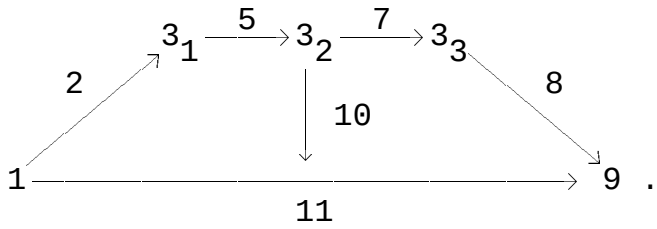
In $\mathtt{Mlt}[\pi]$, we have the 2-cell R:



The one in which (5) and (7) are interchanged, call it S, is not a correct coloring; it is not a cell in $\mathtt{Mlt}[\pi]$, despite the fact that locally it looks all right. The point is that R is the image of the cell



in $\mathtt{Mlt}[\tau]$ for τ the 2-cell



However, S cannot be represented as such an image.

Let $\pi\in\mathtt{Mlt}$. We put $\mathtt{Mlt}[\pi]$, the *category of π-colored multitopes*, to be $\mathtt{El}(\mathcal{I}[\pi])$, the category of elements of $\mathcal{I}[\pi]:\mathtt{Mlt}\to\mathtt{Set}$. That is, the objects of $\mathtt{Mlt}[\pi]$ are pairs (τ, a) where $a\in\mathcal{I}[\pi](\tau)$; and a morphism $f:(\tau, a)\longrightarrow(\theta, b)$ is an arrow $f:\tau\to\theta$ in $\mathtt{Mlt}$ such that $(\mathcal{I}[\pi](f))(a)=b$. In particular, we have the faithful forgetful functor $\|\ \|:\mathtt{Mlt}[\pi]\to\mathtt{Mlt}$. $\|\ \|$ is a discrete cofibration: the arrows out of any $R\in\mathtt{Mlt}[\pi]$ in $\mathtt{Mlt}[\pi]$ are in a bijective correspondence with the arrows out of $\|R\|$ in $\mathtt{Mlt}$.

For every $p:\pi\to\kappa$ in $\mathtt{Mlt}$, we have an induced functor

$$\mathtt{Mlt}[p]=\mathtt{El}(\mathcal{I}[p]):\mathtt{Mlt}[\kappa]\longrightarrow\mathtt{Mlt}[\pi]$$

("recoloring along p "). In fact, we have the functor

$$\mathtt{Mlt[-]} : \mathtt{Mlt}^{\mathtt{op}} \longrightarrow \mathtt{Cat} \ .$$

In other words, we have a strict fibration

$$\begin{array}{c} \mathtt{Mlt[Mlt]} \\ \big\downarrow \Pi \\ \mathtt{Mlt} \end{array}$$

whose fiber at each $\pi \in \mathtt{Mlt}$ is $\mathtt{Mlt}[\pi]$. The total category $\mathtt{Mlt[Mlt]}$ is a suitable category that has as objects all colored multitopes, colored by any (simple) multitope.

The *category* $\mathtt{Mlt}[\pi]\mathtt{Set}$ *of* π*-colored multitopic sets* is

$$\mathtt{Mlt}[\pi]\mathtt{Set} = \mathtt{Set}^{\mathtt{Mlt}[\pi]} \ .$$