2. Multitopes and multitopic sets

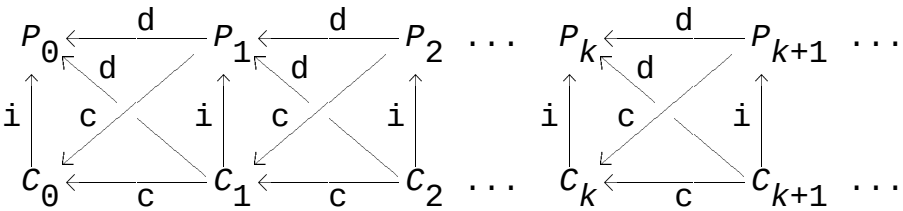
Here, I will recall the notions in the title; they are dealt with in [H/M/P] in complete detail. The present outline cannot quite eliminate the need for a detailed reference to the paper [H/M/P]. However, I hope that for a first orientation, it is of some use. In this section, references to sections are to [H/M/P].

A multitopic set $\mathcal{S}$ consists of *cell*s, each of a definite *dimension* that can be any non-negative integer; if the dimension of the cell a is n, a is said to be an *n-cell*. The set of k-cells is denoted by C_k, or, if $\mathcal{S}$ needs to be explicitly referred to, by $C_k(\mathcal{S})$.

For every positive n, each n-cell a has a *domain* da and a *codomain* ca; ca is an $(n-1)$-cell; da is a *pasting diagram of dimension* $(n-1)$, briefly, an $(n-1)$*-pd*. In general, a k-pd is a "meaningful formal composite" of k-cells, just what you would expect from the term "pasting diagram". Note that the idea of a pasting diagram (composable system of higher dimensional cells) involved here is essentially simpler than the general one, because of the fact that the cells making up the pd are all "many-to-one" (their codomain is a single cell), in contrast to the general "many-to-many" concept. The set of k-pd's is denoted P_k (or $P_k(\mathcal{S})$).

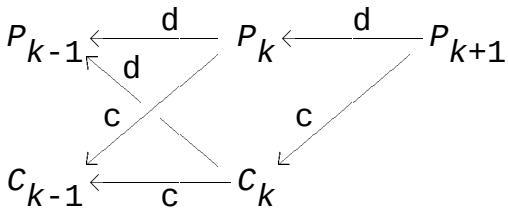
For given $\beta\in P_{n-1}$, $b\in C_{n-1}$, a necessary condition for the existence of $a:\beta\to b$ (which abbreviates $da=\beta$, $ca=b$) is that β and b be *parallel*, that is, either $n-1=0$, or $n-1>0$, and $d\beta=db$, $c\beta=cb$ (*globularity*). Note that we now used the *domain* $d\beta$ and the *codomain* $c\beta$ of a k-pd β, for positive $k\in\mathbb{N}$; $d\beta$ is a $(k-1)$-pd, $c\beta$ is a $(k-1)$-cell.

The items mentioned so far are organized into the following diagram of sets:



Here, we used some abbreviations. Besides omitting subscripts to distinguish the various maps

denoted by the same symbol (for instance, we have, in more detail, $d_{k+1}:P_{k+1}\to P_k$ for the arrow denoted d in the right-most position), we also have the *inclusions*, denoted i, of cells in pd's, and written d, c for the composites di, ci, respectively. The globularity condition is the commutativity of each instance of



Now, the basic fact about the concept of multitopic set is that globularity is the only limiting condition that regulates the existence of cells. Given any multitopic set $\mathcal{S}$, an $(n-1)$-pd β and an $(n-1)$-cell b in it, one can *adjoin* an n-cell $a:\beta\to b$ to $\mathcal{S}$ provided β and b are parallel; this adjunction will not effect anything else, that is, the cells in the new multitopic set will be the old ones plus the single new one a. Similarly, one can, for a fixed n, simultaneously adjoin any number of n-cells, each with prescribed domain and codomain, provided the latter are parallel.

An immediate consequence of this circumstance is the possibility of a recursive construction of multitopic sets. Assuming that we have all the cells of dimensions less than n, together with their domains and codomains, we now can introduce the n-cells by the process of simultaneous adjunction.

(Small) multitopic sets form a category $\mathtt{MltSet}$; the morphisms are the natural structure preserving maps.

It turns out (see Section 7) that $\mathtt{MltSet}$ is *presheaf-category*: there is a particular category, called the category of *multitopes*, and denoted by $\mathtt{Mlt}$, such that $\mathtt{MltSet}$ is equivalent to the category of Set-valued functors on $\mathtt{Mlt}$:

$$\mathtt{MltSet} \simeq \mathtt{Set}^{\mathtt{Mlt}}.$$

The situation is interesting from a logical point of view: it is *not* the case that the definition of "multitopic set" as "$\mathtt{Set}$-valued functor on $\mathtt{Mlt}$" is easier than the original, somewhat complex, one (see Section 6). The reason is that the definition of the category $\mathtt{Mlt}$ is not

direct. In fact, the objects of $\mathtt{Mlt}$, the *multitopes*, are defined to be the cells of the terminal multitopic set (the terminal object of $\mathtt{MltSet}$); the arrows of $\mathtt{Mlt}$ are then defined by an additional, geometrically straightforward, but algebraically rather tedious, manner (see Section 7). Moreover, the terminal multitopic set $\mathcal{T}$ (before we know that the category $\mathtt{MltSet}$ is a presheaf category!) is given by the recursive method mentioned above. The principle is simple: having introduced all $<n$-multitopes, that is, $<n$-cells in $\mathcal{T}$, we adjoin exactly one n-multitope a for each possible type (β, b), a parallel pair of an $(n\text{-}1)$-pd and an $(n\text{-}1)$-cell in $\mathcal{T}$, such that $\mathrm{d}a=\beta$, $\mathrm{c}a=b$. Moreover, one notes the fact that, by the uniqueness of the cells involved, here the cell b is determined by β by the requirement that it is to be parallel to β; thus, in fact, each multitope is uniquely determined by its domain, a pd of multitopes of one lower dimension.

Multitopes are the *shapes* of cells in multitopic sets. Indeed, for any multitopic set $\mathcal{S}$, the unique map $\mathcal{S}\to\mathcal{T}$ in $\mathtt{MltSet}$ assigns to each cell in $\mathcal{S}$ its *shape*. For ρ a multitope, and the multitopic set $\mathcal{S}$ construed as a functor $\mathcal{S}:\mathtt{Mlt}\to\mathtt{Set}$, $\mathcal{S}(\rho)$ is the set of cells whose shape is ρ.

To eliminate ambiguity, we adopt the concept of "multitopic set" meaning " $\mathtt{Set}$-valued functor on $\mathtt{Mlt}$ " as the official definition. The *total set* $|\mathcal{S}|$ of the multitopic set $\mathcal{S}$ is the sum

$$|\mathcal{S}| = \bigsqcup_{\rho\in\mathrm{Ob}(\mathtt{Mlt})} \mathcal{S}(\rho) .$$

However, often we write $a\in\mathcal{S}$ for $a\in|\mathcal{S}|$.

Throughout the paper, the Greek letters $\kappa, \pi, \rho, \sigma, \tau, \theta$ will denote multitopes.

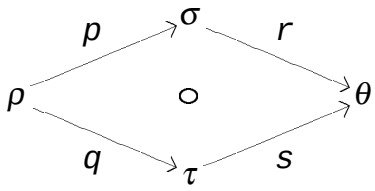
For a multitope ρ, we write $|\rho|$ for the set of all arrows in $\mathtt{Mlt}$ whose domain is ρ; the elements of $|\rho|$ are called the *faces* of ρ. For $p\in|\rho|$, we write K_p (as in the previous section) for the codomain of p.

Let $\mathcal{S}$ be a multitopic set, ρ a multitope, $a\in\mathcal{S}(\rho)$. For $p\in|\rho|$, we have that $a\hat{}p \underset{\mathrm{def}}{=} \mathcal{S}(p)(a) \in \mathcal{S}(\mathrm{K}_p)$. The pair $(p, a\hat{}p)$, and sometimes, more sloppily, the cell $a\hat{}p$ by itself also, is called the *p-face* of a. Intuitively speaking, the faces of a are the same as the *occurrences* of cells "involved" in a: these include a itself, the

one-lower-dimensional cell-occurrence that is the codomain of a, and the ones that are in the domain of a, and (inductively) all still lower dimensional cells involved in the ones previously mentioned.

Talking about faces of a multitope and talking about faces of a cell in a multitopic set are compatible; the former is a special case of the latter, done in the terminal multitopic set $\mathcal{T}$.

The composition structure of `Mlt` represents the geometry of the cells. For instance, if we have the commutative diagram



in `Mlt`, a cell a in $\mathcal{S}$ of shape ρ, and b=`a^p`, c=`a^q`, then d=`b^r`=`c^s`: the cells b and c are located in a at the places p and q, respectively, in such a way that they share an occurrence of d, namely the one at the place $rp=sq$; in other words, they overlap in such a way that d is in the overlap.

The category `Mlt` is generated by arrows each of which is of the form $\rho\to\tau$ where `dim(`ρ`)=dim(`τ`)+1`. Understanding by the *length* of an arrow $\rho\to\tau$ the difference `dim(`ρ`)-dim(`τ`)`, we have that, for any given multitope ρ of positive dimension, the length-1 elements of $|\rho|$ are in two disjoint classes $|\rho|_{\mathtt{c}}$ and $|\rho|_{\mathtt{d}}$. One of these is a singleton, $|\rho|_{\mathtt{c}}=\{\mathtt{c}_\rho:\rho\to\mathtt{c}\rho\}$; the other is $|\rho|_{\mathtt{d}}=\{\mathtt{d}_{\rho,\,p}:p\in|\langle\rho\rangle|\}$, where we used the notation of Section 7, items Mlt[2] and Mlt[1] [in that order; here we use a different notation for the codomain of $\mathtt{c}_\rho$. Here, I am not going to explain the notation $p\in|\langle\rho\rangle|$. Also, I use the letter p for denoting an element of $|\rho|$, and not for what it is used in [H/M/P] .) Elements of $|\rho|_{\mathtt{c}}$ are called *c-arrows*, those of $|\rho|_{\mathtt{d}}$ *d-arrows*. The cardinality of $|\rho|_{\mathtt{d}}$ can be any natural number, including zero. Let a be a cell of shape ρ, `dim(`ρ`)=`$n\geq$`1`. For $p\in|\rho|_{\mathtt{c}}$, the p-face of a is $\mathtt{c}a$, the codomain of a; and for $p\in|\rho|_{\mathtt{d}}$, the p-face of a is an `(`n`-1)`-cell in the domain $\mathtt{d}a$ of a.

The geometry of a single multitope ρ is encapsulated in the structure which is the slice-category `Mlt/`ρ (with objects the elements of $|\rho|$), endowed with a *sign-structure*

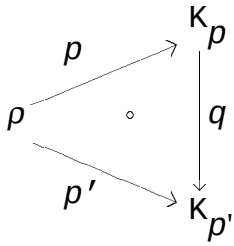
which is the label "c" or "d" on each of the length-1 arrows of $\mathtt{Mlt}/\rho$ (which are also arrows of Mlt). The fact that a multitope is "no more than its geometric representation" is expressed precisely in the statement that if $\mathtt{Mlt}/\rho$ and $\mathtt{Mlt}/\tau$ are isomorphic by an isomorphism (of categories) respecting the sign-structure, then $\rho=\tau$. We also have that the only sign-respecting automorphism of $\mathtt{Mlt}/\rho$ is the identity.

Mlt is a FOLDS signature; the multitopic sets are the Mlt-structures. Multitopic ω-categories (to be defined below) are multitopic sets with additional properties invariant under Mlt-equivalence. For a finite n , the concept of multitopic n-category (see below) is a first-order concept, which can be formulated in the language of FOLDS over a truncation of Mlt .

In what follows, we reformulate the ingredients of multitopic sets we started with in this section in terms of the definition of "multitopic set" as " Set-valued functor on Mlt ". The results of this exercise will help us to define, in section 6, the generalizations of the relevant concepts to "colored multitopic sets".

Let us write $\rho\|$ for the set of all arrows p with domain equal to ρ such that $p\neq \mathtt{id}_\rho$ and $p\neq \mathtt{c}_\rho$.

Let $k\geq 1$. A $(k\text{-}1)$-pd β in $\mathcal{S}$ is given by a k-dimensional multitope ρ , and a function β such that $\mathtt{dom}(\beta)=\rho\|$; $\beta(p)\in\mathcal{S}(\mathtt{K}_p)$ whenever $p\in\rho\|\mathtt{Mlt}$; and if



then $\mathcal{S}(q)(\beta(p))=\beta(p')$. The pd β is then of *shape* ρ , or, β is a *ρ-pd.*

In particular, it is clear that for any $a\in\mathcal{S}(\tau)$, the family $\langle b_p\rangle_{p\in\rho\|}$, where $b_p=\mathcal{S}(p)(a)$, defines a pd in the sense just described; this pd is what we denoted by $\mathtt{d}a$. We write $\mathcal{S}[\rho^-]$ for the set of pd's of shape ρ .

A useful way of looking at τ-pd's is that they are $[\rho^-]$-contexts in $\mathcal{S}$, where $[\rho^-]$ is the subfunctor of $\hat{\rho}=\mathtt{hom}(\rho,-)$ in which $[\rho^-](\rho)=\emptyset$, $[\rho^-](c\rho)=[\rho](c\rho)-\{c_\rho\}$, and $[\rho^-](\kappa)=[\rho](\kappa)$ for all $\kappa\in\mathtt{Ob}(\mathtt{Mlt})-\{\rho,c\rho\}$. The domain $\mathtt{d}a$ of a cell $a\in\mathcal{S}(\rho)$ is the composite $[\rho^-]\xrightarrow{\mathtt{incl}}\hat{\rho}\xrightarrow{\dot{a}}\mathcal{S}$, where $\dot{a}$ is the arrow corresponding to a by Yoneda: $\dot{a}_\rho(\mathtt{id}_\rho)=a$.

Note that the formal context $[\rho^-]$, and hence the set $\mathcal{S}[\rho^-]$ too, are (contravariant) functors of $\rho\in\mathtt{Mlt}$.

For a ρ-pd $\beta:[\rho^-]\to\mathcal{S}$, $\mathtt{d}\beta$, a pd of dimension one less than that of β, is obtained as the composite

$$\mathtt{d}\beta : [(c\rho)^-]\xrightarrow{\iota}[\rho^-]\xrightarrow{\beta}\mathcal{S},$$

where ι is an arrow determined by the commutative diagram

$$\begin{array}{ccc} [\rho^-] & \xrightarrow{\mathtt{incl}} & \hat{\rho} \\ {\scriptstyle\iota}\uparrow & \circ & \uparrow{\scriptstyle(c_\rho)^*} \\ [(c\rho)^-] & \xrightarrow[\mathtt{incl}]{} & \widehat{(c\rho)} \end{array}$$

Let θ be an arbitrary multitope; $\mathtt{dim}(\theta)=k-1$. There is a particular multitope θ^* of dimension k, uniquely determined by θ by the facts that $\theta=c\theta^*$, there are exactly two distinct non-identity arrows with domain θ^*, and both of them are of the form $\theta^*\to\theta$; one of them, of course, is c_{θ^*}; we denote the other arrow $\theta^*\to\theta$ by $\mathtt{d}_{\theta^*}:\theta^*\to\theta$. θ^* is the shape of a k-cell both of whose domain and codomain are of shape θ. A special feature of the pair

$$\theta^* \;\underset{c_{\theta^*}}{\overset{d_{\theta^*}}{\rightrightarrows}}\; \theta$$

of arrows is that they are equalized by any non-identity arrow out of their codomain.

Given any cell $a\in\mathcal{S}(\rho)$, with $\dim(\rho)=k$, we have the ρ^*-pd $\langle a\rangle\in\mathcal{S}[\rho^*]$ for which $\langle a\rangle_\rho(c_{\rho^*})=a$. $\langle a\rangle$ is the same as $i(a)$ for the "inclusion" $i:C_k(\mathcal{S})\to P_k(\mathcal{S})$ at the beginning of this section.

The codomain $c\beta$ of a ρ-pd β is $\beta_{cc\rho}(c_{c\rho}\circ c_\rho)$. When ρ is replaced with ρ^*, and $\beta\in\mathcal{S}[(\rho^*)^-]$ is taken to be $\beta=\langle a\rangle$ for $a\in\mathcal{S}(\rho)$, the formula for $c\beta$ reduces to $a\hat{}c_\rho$ as it should.

I will now discuss a particular class of multitopes, the shapes of the "empty" pd's.

There is a multitope θ^+ of dimension $k+1$ for which $c\theta^+=\theta^*$, and the only non-identity arrow out of θ^+ is $c_{\theta^+}:\theta^+\longrightarrow c\theta^+=\theta^*$; again, θ^+ is uniquely determined by said properties from θ. In any multitopic set, θ^+ is the shape of an *empty* pd of dimension k, one which is an identity arrow in the multicategory $\boldsymbol{c}_k$ on an object, a $(k\text{-}1)$-cell, of shape θ (see [H/M/P]. θ^+ is the shape of a $(k+1)$-cell whose domain is a pd of shape θ^+, and whose codomain is of shape θ^*. Writing ε for θ^+, we have that

$$\varepsilon \xrightarrow{c_\varepsilon} c\varepsilon \;\underset{c_{c\varepsilon}}{\overset{d_{c\varepsilon}}{\rightrightarrows}}\; cc\varepsilon \qquad (1)$$

is the same as

$$\theta^+ \xrightarrow{c\,\theta^+} \theta^* \underset{c\,\theta^*}{\overset{d\,\theta^*}{\rightrightarrows}} \theta \qquad (2)$$

Multitopes ε of the form θ^+ are called *empty-domain* multitopes.

Given any cell $a \in \mathcal{S}(\theta)$ in a multitopic set $\mathcal{S}$, there is a unique θ^+-shaped pd α for which $cc\alpha = a$. α is the *empty k-pd on* the $(k\text{-}1)$-cell a . It is called "empty" since there is no k-cell in it: $\theta^+\|$ has no element p with $\dim(K_p) = k$.

Among diagrams of the form

$$\rho \xrightarrow{f} \tau \underset{h}{\overset{g}{\rightrightarrows}} \kappa\,, \qquad (3)$$

the ones of the type (2) are anomalous because $d_{c\varepsilon} \circ c_{\varepsilon} = c_{c\varepsilon} \circ c_{\varepsilon}$, but $d_{c\varepsilon} \neq c_{c\varepsilon}$. In fact, we have that if (3) is such that $gf = hf$ and $g \neq h$, then there is an empty-domain multitope ε and $\rho \xrightarrow{e} \varepsilon$ such that (3) is obtained from (1) by precomposing with e : $f = c_{\varepsilon} \circ e$, $\{g, h\} = \{d_{c\varepsilon}, c_{c\varepsilon}\}$. In particular, arrows in $\mathtt{Mlt}$ are "usually" epimorphisms; in fact, an arrow is an epimorphism if and only if there is no empty-domain face in its codomain.

Multitopes that have no faces with empty-domain shapes are called *positive*. The codomain of an arrow out of a positive multitope is also positive. Several things in this paper allow simplified formulations when one restricts oneself to positive multitopes (that is, when one replaces $\mathtt{Mlt}$ be its full subcategory on the positive multitopes); unfortunately, this cannot be done without harm to the concepts we are aiming at.