1. FOLDS-signatures and FOLDS-equivalence

FOLDS is an acronym for "first order logic with dependent sorts"; see [M2]. FOLDS has a syntax and a semantics. Its syntax will not concern us here; but it is in fact quite completely described by the name. "Dependent sorts" are meant in a sense similar to the dependent types of [M-L] and [Ca].

FOLDS is based on (FOLDS-)*signatures* that are categories of a special kind. A *signature* is a category that satisfies the following two conditions:

(1) it is reverse well-founded: there are no infinite chains

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{k-1}} X_k \xrightarrow{f_k} X_{k+1} \xrightarrow{f_{k+1}} \ldots$$

of *non-identity* arrows f_k ($k \in \mathbb{N}$);

(2) it has finite fan-out: for every object X, the set of arrows with domain equal to X is finite.

Let L be a signature.

L is a skeletal category; also, every endomorphism $X \to X$ is the identity. Moreover, $\mathrm{Ob}(L)$ is the disjoint union $\dot{\bigcup}_{n \in \mathbb{N}} L_n$ of *levels* L_n, so that for every non-identity arrow $X \to Y$, we have $X \in L_n$, $Y \in L_m$ with $n > m$ (and if $n > 0$, and $X \in L_n$, then there is at least one arrow $X \to Y$ with $Y \in L_{n-1}$).

An *L-structure* M is a functor $M : L \to \mathrm{Set}$.

The semantics of FOLDS includes a theory of *identity* for structures: there is a separate concept of identity for each signature L, *L-equivalence*, $\simeq_L$, a relation between L-structures. The main part of the general theory of FOLDS deals with the exact matching of the syntax of FOLDS with the concept of FOLDS-equivalence. Roughly speaking, for any given FOLDS-signature L, the properties of L-structures formulated in a first order language possibly extending L that are invariant under L-equivalence are exactly the ones

that can be expressed in FOLDS over the given L.

There is a *higher-order logic with dependent sorts over* L. It involves using signatures extending L; properties formulated in it are also invariant under L-equivalence. In fact, our definition of "multitopic ω-category" is given in terms of an existential second-order property of multitopic sets, expressible in higher-order logic with dependent sorts over L, and thus invariant under L-equivalence; here, L is $\mathtt{Mlt}$, the category of multitopes, the signature for multitopic sets (see below).

In [M3], it is shown (it is also sketched in [M2]) how FOLDS equivalence specializes to the various notions of "equivalence" used in category theory: equivalence of categories, biequivalence of bicategories, and so on. The said concepts turn out to be cases of L-equivalence with suitable choices of the signature L. In this paper, L-equivalence, for various specially constructed signatures L, is used for the definition of the concepts of multitopic ω-category, and other related structures.

To define L-equivalence, I introduce some auxiliary concepts. Let K be a *kind*, that is an object of L. By $|K|$, we mean the set of all arrows in L with domain equal to K; and we let $K|L = |K| - \{\mathrm{id}_K\}$. The set $K|L$ is the *arity* of K; when K is used to form a dependent sort, one uses one variable x_p of the appropriate sort for each $p \in K|L$ to fill in the "places" of K. We write K_p for the codomain of the arrow p.

Consider the elements of $K|L$ to be the objects of a category, also denoted by $K|L$, in which an arrow $r : p \to q$ is any $r : K_p \to K_q$ such that $rp = q$; $K|L$ is a full subcategory of the slice-category $K \backslash L$. Let $\varphi_K : K|L \to L$ be the forgetful functor. Given any L-structure M, the limit of the composite $M \circ \varphi_K$, a set, is denoted by $M[K]$, and its elements are called the *K-contexts* in M. A K-context α is a "compatible" family $\alpha = \langle \alpha(p) \rangle_{p \in K|L}$ of elements $\alpha(p) \in M(K_p)$ indexed by the arity $K|L$ of K. "Compatibility" means that if $p, q \in K|L$ and $r \circ p = q$, then $M(r)(\alpha(p)) = \alpha(q)$.

$M(K)$ is fibered over $M[K]$ by the canonical mapping $\lambda_K : M(K) \longrightarrow M[K]$; for $a \in M(K)$, $\lambda_K(a) \in M[K]$ is the *K-context of* a. The fiber $\lambda_K^{-1}(\alpha)$ of $\alpha \in M[K]$ is denoted as $M(K)(\alpha)$.

When $a \in M(K)(\alpha)$, we also say is that a is *of sort* $K(\alpha)$; to be sure, the sort $K(\alpha)$ is an *interpreted* dependent sort; the places of the kind K heading the sort are not filled by variables, but by elements $\alpha(p)$ interpreting such variables.

Let $\mu : P \to M$ be a morphism of L-structures: a natural transformation of functors $L \to \mathtt{Set}$. For any kind K, we have the induced map $\mu_{[K]} : P[K] \longrightarrow M[K]$, and also, for any $\xi \in P[K]$ and $\alpha \stackrel{\mathrm{def}}{=} \mu_{[K]}(\xi) \in M[K]$, the induced map $\mu_{K(\xi)} : P(K)(\xi) \longrightarrow M(K)(\alpha)$, which is a restriction of $\mu_K : P(K) \longrightarrow M(K)$. We say that μ is *fiberwise surjective* if $\mu_{K(\xi)}$ is a surjective function for all choices of $K \in L$ and $\xi \in P[K]$.

In order to formulate a property of fiberwise surjective maps, we introduce some auxiliary concepts.

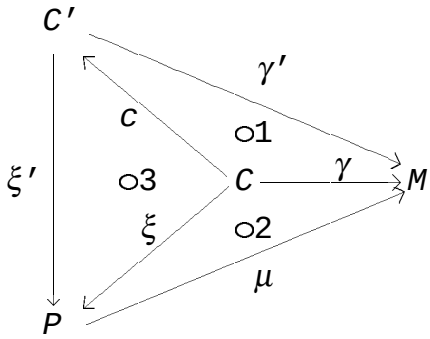
Let us define the cardinality $\#C$ of a $\mathtt{Set}$-valued functor $C : L \to \mathtt{Set}$ to be $\#C = \#(\coprod_{K \in \mathrm{Ob}(L)} C(K)) = \sum_{K \in \mathrm{Ob}(L)} \#C(K)$. A functor $C : L \to \mathtt{Set}$ is said to be *finite* if $\#C$ is finite; this is to say that $C(K)$ is empty for all but finitely many kinds K, and $C(K)$ is finite for all K. Finite functors $L \to \mathtt{Set}$ will now be called *formal contexts*. For a formal context C, and an L-structure M, a *C-context* in M is a natural transformation $\gamma : C \to M$. We write $M(C)$ or just MC for $\hom(C, M)$, the set of C-contexts in M. Sometimes we call C-contexts in M collectively for all C *concrete* contexts in M.

The representable functor $\hat{K} = \hom(K, \text{-})$ is a formal context, and so is the subfunctor $[K]$ of $\hat{K}$, defined by the condition that for every $U \in L$, the component $\iota_U : [K](U) \to \hat{K}(U)$ of the inclusion $\iota : [K] \to \hat{K}$ is an identity function, except for $U = K$ itself, when it is the inclusion of the empty set into the singleton $\{\mathrm{id}_K\}$. These facts are consequences of the L being a FOLDS signature. What we called a K-context above is the same as a $[K]$-context: $M[K] \cong \hom([K], M)$.

$[K]$ is a functor of $K : [\text{-}] : L^{\mathrm{op}} \to \mathtt{Set}^L$, since if $p : K_1 \to K_2$, then $\hom(K, p) : \hat{K}_1 \to \hat{K}_2$ restricts to a map $[p] : [K_1] \to [K_2]$; this, again, is something that depends on the special nature of the category L.

It is immediate now that $M[K]$ is a functor of $K : M[\ ] : L \to \mathtt{Set}$.

Suppose $\mu : P \to M$ is fiberwise surjective. Let $c : C \to C'$ be a map (natural transformation) of formal contexts *which is a monomorphism* (in $\mathbf{Set}^L$); let γ be a C-context, γ' a C'-context in M, γ' extending γ via c as shown by commutativity $\circ 1$ in the next diagram:



Suppose further that ξ is a C-context in P that lifts γ via μ (commutativity $\square 2$). Then, the assertion is, that there exists (at least one) C'-context ξ' in P lifting ξ (see $\circ 3$).

This last property of fiberwise surjective maps is a generalization of the definition; the defining property is obtained when $c : C \to C'$ is taken to be $\iota : [K] \to \hat{K}$. L being a FOLDS-signature is needed to have the generalized property follow from the definition.

Given L-structures M and N, an *L-equivalence-span E between M and N* (in notation: $E : M \simeq_L N$) is a span

$$E = (P, \mu, \nu) = (M \xleftarrow{\mu} P \xrightarrow{\nu} N)$$

of natural transformations in which μ and ν are fiberwise surjective. We write $M \simeq_L N$, and say that M and N are *L-equivalent* if there exists E such that $E : M \simeq_L N$.

We may omit the subscript L, and write $E : M \simeq N$, and similarly in other similar notations.

Suppose $E=(P, \mu, \nu) : M \simeq_L N$. For a formal context C, and C-contexts α, β and ξ in M, N and P, respectively, we write

$$(E,\ \xi)\ :\ (M,\ \alpha)\simeq_L(N,\ \beta) \qquad (3)$$

if ξ is a lifting of α via μ, and also a lifting of β via ν. We may omit ξ, and write

$$E:(M,\ \alpha)\simeq_L(N,\ \beta)\ ; \qquad (4)$$

this, of course, means that there exists ξ such that (3) holds. When (4) holds, we say that α and β are *related* by E.

For $K\in L$, and elements $a\in M(K)$, $b\in N(K)$, $x\in P(K)$, we write $(E,\ x):(M,\ a)\simeq_L(N,\ b)$ to mean that $\mu_K(x)=a$, $\nu_K(x)=b$. This last notation is in fact a special case of the previous one involving contexts: take $C=\hat{K}$, and $\alpha:\hat{K}\to M$ the unique natural transformation for which $\alpha_K(\mathrm{id}_K)=a$, and similarly for β and ξ; then $(E,\ x):(M,\ a)\simeq_L(N,\ b)$ iff (3).

A notation of the form

$$E:(M,\ \vec{a})\simeq_L(N,\ \vec{b})$$

where $\vec{a}$ and $\vec{b}$ are tuples of elements in M and N, respectively, will be used as an abbreviation for a relation (4); $\vec{a}$ lists the values of a suitable context α, and similarly for $\vec{b}$ and β.

The defining property of L-equivalence-span can be expressed, in a slightly incomplete way, thus. Let $K\in L$, $\alpha\in M[K]$, $\beta\in N[K]$, and assume that

$$E:(M,\ \alpha)\simeq_L(N,\ \beta)$$

Then for any $a\in M(K)(\alpha)$, there is $b\in N(K)(\beta)$ such that

$$E:(M,\ \alpha,\ a)\simeq_L(N,\ \beta,\ b)\ .$$

and *vice versa*.

We note that the concept of L-equivalence naturally extends to " L-structures" with value-categories other than the category of sets. The extended concept is important in formulating when two different notions of higher dimensional category are essentially equivalent.

We discuss a general construction of new signatures out of an old one.

Suppose L is a signature, $M:L\to\mathrm{Set}$ an L-structure. Then the category $\mathrm{El}(M)$ of elements is a signature too. $\mathrm{El}(M)$ has as objects pairs (K, x) where $K\in\mathrm{Ob}(L)$ and $x\in M(K)$; an arrow $(K, x)\longrightarrow(H, y)$ is an arrow $f:K\to H$ such that $F(f)(x)=y$. There is an obvious forgetful functor $\Sigma:\mathrm{El}(M)\to L$. Verifying that $\mathrm{El}(M)$ satisfies the conditions for "signature" if L does is immediate; in fact, the "fan-out" of the object (K, x) has the same cardinality as that of K .

Let us remind the reader of a simple general fact. This is the equivalence of categories

$$\Pi\ :\ \mathrm{Hom}(L,\mathrm{Set})/M \xrightarrow{\ \simeq\ } \mathrm{Hom}(\mathrm{El}(M),\mathrm{Set})$$

$$(L\xrightarrow{C}\mathrm{Set},\gamma:C\to M)\ \longmapsto\ \left[\ f\Big\downarrow\ \longmapsto\ \begin{array}{c}(K, x)\longmapsto\{a\in C(K):\gamma_K(a)=x\}\\ \Big\downarrow\ \ \overset{a}{\underset{C(f)(a)}{\rotatebox{-90}{$\longmapsto$}}}\\ (H, y)\longmapsto\{b\in C(H):\gamma_H(b)=y\}\end{array}\ \right]$$

holding for an arbitrary category L and functor $M:L\longrightarrow\mathrm{Set}$. On arrows $(C, \gamma)\xrightarrow{\varphi}(D, \delta)$, Π acts as follows:

$$\Pi(\varphi)_{(K, x)}(a)=\varphi_K(a)\ .$$

The quasi-inverse Π^{-1} of Π has, for $\Gamma:\mathrm{El}(M)\to\mathrm{Set}$, $\Pi^{-1}(\Gamma)=(C, \gamma)$ where, for $f:K\to H$ in L ,

$$C(K) = \coprod_{x \in M(K)} \Gamma((K, x)) = \{(x, s) : x \in M(K),\ s \in \Gamma((K, x))$$

$$C(f) \downarrow \qquad = \qquad \downarrow \qquad : \qquad \frac{(x, s)}{\downarrow}$$

$$(M(f)(x), \Gamma(f)(s))$$

$$C(H) = \coprod_{y \in M(H)} \Gamma((H, y)) = \{(y, t) : y \in M(H),\ t \in \Gamma((H, y))$$

and

$$\gamma_K((x, s)) = x\ .$$

It is immediate that if $\Pi((C, \gamma)) = \Gamma$, then $\#\Gamma = \#C$. We conclude that, in case L is a signature, and we write $L[M]$ for $\mathbf{El}(M)$, then

the category of formal contexts of $L[M]$ *is canonically equivalent to the category of concrete contexts in* M,

where (of course) we mean by the latter category the corresponding subcategory of $\mathbf{Hom}(L,\mathbf{Set})/M$.

The same equivalence Π expresses also that

an $L[M]$*-structure is essentially the same thing as an* L*-structure* N *together with a map* $N \to M$.