Let $n\in\mathbb{N}$, $\boldsymbol{X}$ be an $(n\text{-}1)$-category. Let $U$ be a set, and $u\mapsto \mathtt{d}u$, $u\mapsto \mathtt{c}u$ two functions $U\longrightarrow X_{n\text{-}1}$ such that, for each $u\in U$, $\mathtt{d}u\Vert\mathtt{c}u$.

The elements of $U$ are each an $n$-(dimensional) *indeterminate* with a prescribed domain and codomain in $X_{n\text{-}1}$, the latter two required to be parallel; we say that $U$ is *attached to* $\boldsymbol{X}$ by the pair $(\mathtt{d},\mathtt{c})$ of maps. Note that the notations $\mathtt{d}u$, $\mathtt{c}u$ for the values of these functions is only suggestive at the moment; said functions are arbitrary, except for the stated conditions. Also recall that, by convention, we have the "cell" $*$ of dimension $\text{-}1$ in every $\omega$-category $\boldsymbol{X}$ so that $\mathtt{d}x=\mathtt{c}x=*$ for all $x\in X_{\geq 0}$; for $u\in U_0$, $\mathtt{d}u=\mathtt{c}u=*$; by convention, $*\Vert *$.

We have the $n$-category $\boldsymbol{X}[U]$ obtained by freely adjoining the indeterminates in $U$ to $\boldsymbol{X}$. The definition, via a universal property, is as follows.

With the given $\boldsymbol{X}$, $U$, $\mathtt{d}$, $\mathtt{c}$, consider the category $\boldsymbol{A}=\boldsymbol{A}\langle \boldsymbol{X}, U, d, c\rangle$ whose objects are the triples $(\boldsymbol{Y}, \boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{Y}, U\xrightarrow{\Lambda} Y_{\geq 0})$ [with $\Gamma$ a morphism in $n\mathtt{Cat}$, the category of $n$-categories, which is, essentially, a full subcategory of $\omega\mathtt{Cat}$] such that the following induced diagram in $\mathtt{Set}$ commutes:

$$\begin{array}{ccc} U & \xrightarrow{\quad\Lambda\quad} & Y_{\geq 0} \\ {\scriptstyle\langle \mathtt{d},\,\mathtt{c}\rangle}\downarrow & \circ & \downarrow{\scriptstyle\langle \mathtt{d},\,\mathtt{c}\rangle} \\ X_{\geq -1}\times X_{\geq -1} & \xrightarrow{\quad\Gamma\quad} & Y_{\geq -1}\times Y_{\geq -1} \end{array}$$

An arrow $(\boldsymbol{Y},\Gamma,\Lambda)\longrightarrow(\boldsymbol{Y}',\Gamma',\Lambda')$ of $\boldsymbol{A}$ is, as expected, a morphism $\Xi:\boldsymbol{Y}\to\boldsymbol{Y}'$ such that $\Xi\circ\Gamma=\Gamma'$ and $\Xi\circ\Lambda=\Lambda'$.

An initial object of $\boldsymbol{A}$ is denoted by $(\boldsymbol{X}[U], \boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{X}[U], U\xrightarrow{\Lambda}\boldsymbol{X}[U]_{\geq 0})$. The existence of said initial object is a consequence of general principles; also, below we give a concrete construction for it.

We want to state a result to the effect that, from $\boldsymbol{X}[U]$ as a mere $\omega$-category, we can recover $\boldsymbol{X}$ and $U$ (at least in a special case, which case is, however, sufficient for us).

Let $\boldsymbol{Y}$ be any $\leq\omega$-category.

Let $y\in Y_k$. We write $\mathtt{id}^0_y=y$, and $\mathtt{id}^m_y=\mathtt{id}_{\mathtt{id}^{m-1}_y}$ if $m\geq 1$. We have that $\mathtt{id}^m_y\in Y_{k+m}$; we say that $\mathtt{id}^m_y$ is a $k$-to-$k+m$ *identity* cell.

Let $n\in\mathbb{N}$. We consider the following condition $(\mathtt{C}_n)$ on $\boldsymbol{Y}$:

$(\mathtt{C}_n)$ Whenever $\ell<n$, $k<n$, $x, y\in Y_n$, and $x\circ_\ell y$ is well-defined, if $x\circ_\ell y$ is a $k$-to-$n$ identity, then both $x$ and $y$ are $k$-to-$n$ identities.

Let $n\in\mathbb{N}$ and $x\in Y_n$. Let us say that $x$ is *indecomposable* if the following hold:

**(i)** $x\neq\mathtt{id}_y$ for all $y\in Y_{n-1}$; and

**(ii)** whenever $y, z\in Y_n$, $k<n$ and $x=y\circ_k z$, we have that either $y$ or $z$ is a $k$-to-$n$ identity (and $x=z$, respectively, $x=y$).

**(1) Lemma** For any $(n-1)$-category $\boldsymbol{X}$ satisfying $(\mathtt{C}_{n-1})$, and any set $U$ of $n$-indeterminates attached to $\boldsymbol{X}$, we have the following:

**(1.1)** $\boldsymbol{X}[U]$ satisfies $(\mathtt{C}_n)$.

**(1.2)** The canonical map $\Lambda: U\longrightarrow(\boldsymbol{X}[U])_n$ is one-to-one.

**(1.3)** The image of $\Lambda$ consists exactly of the indecomposable $n$-cells of $\boldsymbol{X}[U]$.

**(1.4)** The canonical inclusion $\Gamma:\boldsymbol{X}\to\boldsymbol{X}[U]$ is an isomorphism onto the $(n-1)$-truncation of $\boldsymbol{X}[U]$.

**Proof.** Here is a complete construction of $\boldsymbol{X}[U]$ geared towards proving the lemma.

We define what we mean by a *word* $w$, and what are $\mathrm{d}w$, $\mathrm{c}w$; the latter two entities are elements of $X_{n-1}$, and they are parallel.

The words come in three disjoint sets, $W_0$, $W_1$ and $W_2$; the set $W$ of all words is $W=W_1\dot\cup W_2\dot\cup W_3$.

$W_0$ consists of all the expressions of the form $\hat{\mathrm{id}}_x$ where $x\in X_{n-1}$ (identity words); $\mathrm{d}(\hat{\mathrm{id}}_x)\ \bar{\bar{\mathrm{def}}}\ \mathrm{c}(\hat{\mathrm{id}}_x)\ \bar{\bar{\mathrm{def}}}\ x$.

For $k<n$, a word $v\in W_0$ is said to be a $k$-to-$n$ *identity* if $v=\hat{\mathrm{id}}_x$ for a $k$-to-$n$-1 identity $x\in X_{n-1}$ in $\boldsymbol{X}$.

$W_1$ equals $U$, the given set of indeterminates; $\mathrm{d}u$ and $\mathrm{c}u$ are as given in the attachment of $U$ to $\boldsymbol{X}$.

$W_2$ is defined inductively: for words $v$, $w$ and $k<n$ such that $\mathrm{d}^{(k)}v=\mathrm{c}^{(k)}w$ [here, $\mathrm{d}^{(k)}v\ \bar{\bar{\mathrm{def}}}\ \mathrm{d}^{(k-1)}\mathrm{d}v$, and similarly for $\mathrm{c}$ ],

the formal expression $v\hat\circ_k w$ belongs to $W_2$ iff

either $v\in W_1\cup W_2$, or $w\in W_1\cup W_2$;

and

neither $v$ nor $w$ is a $k$-to-$n$ identity.

Moreover, $\mathrm{d}(v\hat\circ_k w)\ \bar{\bar{\mathrm{def}}}\ \mathrm{d}w$ if $k=n-1$, and $\mathrm{d}(v\hat\circ_k w)\ \bar{\bar{\mathrm{def}}}\ \mathrm{d}v\circ_k\mathrm{d}w$ when $k<n-1$, and similarly for $c$. We do have that $\mathrm{d}(v\hat\circ_k w)\|\mathrm{c}(v\hat\circ_k w)$.

In other words, for words $v$ and $w$, the formal expression $v\hat\circ_k w$ is well-defined as a word, and belongs to $W_2$ provided $v$ and $w$ are "$k$-composable", and it is not the case that both $v$ and $w$ are identity words (in $W_0$), and, moreover, even if one of them is an identity word, it is not a $k$-to-$n$ identity word.

We write $v \| w$ for $\mathtt{d}v=\mathtt{d}w \,\&\, \mathtt{c}v=\mathtt{c}w$.

It is understood, of course, that two words are equal iff they are formally identical; e.g., $v \hat{\circ}_k w = v' \hat{\circ}_\ell w'$ only if $k=\ell$, $v=v'$ and $w=w'$.

The words in $W_2$ are characterized among all words by the fact that they are of the form $v \hat{\circ}_k w$ for words $v$ and $w$.

We give a "fully defined" composition of words: a word $v \circ_k w$ whenever $v, w \in W$, $k<n$ and $\mathtt{d}^{(k)}v=\mathtt{c}^{(k)}w$.

$v \circ_k w \;\bar{\bar{\mathtt{def}}}\; v \hat{\circ}_k w$ whenever $v \hat{\circ}_k w$ is well-defined.

If $v=\hat{\mathtt{i}}\mathtt{d}_x$, $w=\hat{\mathtt{i}}\mathtt{d}_y$, both from $W_0$, then $v \circ_k w \;\bar{\bar{\mathtt{def}}}\; \hat{\mathtt{i}}\mathtt{d}_x=\hat{\mathtt{i}}\mathtt{d}_y=v=w$ when $k=n\text{-}1$ (and $x=y$), and $v \circ_k w \;\bar{\bar{\mathtt{def}}}\; \hat{\mathtt{i}}\mathtt{d}_{x \circ_k y}$ when $k<n\text{-}1$.

If $v$ is a $k$-to-$n$ identity, $v \circ_k w \;\bar{\bar{\mathtt{def}}}\; w$;
If $w$ is a $k$-to-$n$ identity, $v \circ_k w \;\bar{\bar{\mathtt{def}}}\; v$.

In the instances when two of the last three clauses apply, the definitions give the same result.

Notice that we always have that $\mathtt{d}(v \circ_k w)=\mathtt{d}w$ when $k=n\text{-}1$, and $\mathtt{d}(v \circ_k w)=\mathtt{d}v \circ_k \mathtt{d}w$ when $k<n\text{-}1$, and similarly for $\mathtt{c}$.

Note that if either $v$ or $w$ is in $W_2$, and $v \circ_k w$ is well-defined, then $v \circ_k w \in W_2$; and if either $v$ or $w$ is in $W_1$, and $v \circ_k w$ is well-defined, then $v \circ_k w \in W_1 \cup W_2$.

Let us show the following version of (1.1):

**(1.1)**$^*$ Whenever $\ell<n$, $k<n$, $v, w \in W$, and $v \circ_\ell w$ is well-defined, if $v \circ_\ell w$ is a $k$-to-$n$ identity, then both $v$ and $w$ are $k$-to-$n$ identities.

Assume the hypotheses, including that $v \circ_\ell w$ is a $k$-to-$n$ identity. First of all, we must have that $v$ and $w$ are in $W_0$, $v=\hat{\mathtt{i}}\mathtt{d}_x$ and $w=\hat{\mathtt{i}}\mathtt{d}_y$, $x, y \in X_{n-1}$. When $\ell=n\text{-}1$, we have

$v=w=v\circ_\ell w$, and the assertion is clear. When $\ell<n-1$, we have $v\circ_\ell w = \hat{\mathrm{id}}_{x\circ_\ell y}$; this being a $k$-to-$n$ identity, $x\circ_\ell y$ is a $k$-to-$n-1$ identity in $\boldsymbol{X}$; therefore, by $(C_{n-1})$ for $\boldsymbol{X}$, $x$ and $y$ are both $k$-to-$n-1$ identities in $\boldsymbol{X}$, and the assertion follows.

By $W_2{}^{\times}{}_{\langle d,c\rangle}W_2$, we mean the pullback $\{(v, w)\in W_2\times W_2 : v \| w\}$.

We define the relation $\approx$, an equivalence relation on $W_2$, as the least relation $\approx \subset W_2{}^{\times}{}_{\langle d,c\rangle}W_2$ satisfying (i) to (iv) below; the variables $v$, $v'$, $w$, $w'$ range over $W_2$, $v_1$, $v_2$, $v_3$, $v_4$ over $W$.

**(i)** $\approx$ is an equivalence.

**(ii)** $v\approx v'$ and $w\approx w'$ imply $v\circ_k w \approx v'\circ_k w'$ provided one of the composites, hence both, are well-defined.

**(iii)** $v\approx v'$ and $w\in W_0\cup W_1$ imply $v\circ_k w \approx v'\circ_k w$ provided one of the composite words is (hence both of them are) well-defined.

**(iv)** If $v$ and $w$ are in $W_2$, and

either $v=(v_1\circ_k v_2)\circ_k v_3$ and $w=v_1\circ_k(v_2\circ_k v_3)$ for some $k<n$ and $v_1$, $v_2$, $v_3\in W$,

or $v=(v_1\circ_k v_2)\circ_\ell(v_3\circ_k v_4)$ and $w=(v_1\circ_\ell v_3)\circ_k(v_2\circ_\ell v_4)$ for some $\ell<k<n$ and $v_1$, $v_2$, $v_3$, $v_4\in W$,

then $v\approx w$.

One should note that each closure condition generates pairs in $\approx$ that are in the set $W_2{}^{\times}{}_{\langle d,c\rangle}W_2$.

We extend the equivalence $\approx$ to $\approx^*$ on the whole of $W$ by declaring that for $v, w\in W$, $v\approx^* w$ iff either $v=w$, or $v, w\in W_2$ and $v\approx w$.

We **claim** that the conditions (i) to (iv) remain true for $\approx^*$. More precisely,

**(i)*** $\approx^*$ is an equivalence on $W$.

**(ii)*** $v, v', w, w'\in W$, $v\approx^* v'$ and $w\approx^* w'$ imply $v\circ_k w \approx^* v'\circ_k w'$

provided one of the composites, hence both, are well-defined.

**(iv)*** If $v, w \in W$, and

either **(a)** $v=(v_1 \circ_k v_2)\circ_k v_3$ and $w=v_1\circ_k(v_2\circ_k v_3)$ for some $k<n$ and $v_1, v_2, v_3 \in W$,

or **(b)** $v=(v_1\circ_k v_2)\circ_\ell(v_3\circ_k v_4)$ and $w=(v_1\circ_\ell v_3)\circ_k(v_2\circ_\ell v_4)$ for some $\ell<k<n$ and $v_1, v_2, v_3, v_4 \in W$,

then $v \approx^* w$.

(What would be (iii)* is subsumed under (ii)*.)

(i)* is true.

For (ii)*, assume the hypotheses. We need to show that if $v\circ_k w$ is in $W_0 \cup W_1$, then $v'\circ_k w' = v\circ_k w$. By inspecting the definition of $v\circ_k w$, we see that $v\circ_k w \in W_0$ only if both $v$ and $w$ are in $W_0$, in which case $v'=v$ and $w'=w$, and the desired conclusion is reached; and similarly if $v\circ_k w \in W_1$.

For (iv)*: the case (a) is similar and simpler than (b); we discuss (b) only; assume the hypotheses in (b).

If $v$ or $w$ is in $W_0$, then all of $v_1, v_2, v_3, v_4$ must be in $W_0$, and we have $v=w$.

If $v$ or $w$ is in $W_1$, that is, $v=u$ or $w=u$ for some $u \in U$, then, first of all, clearly, one of $v_1, v_2, v_3, v_4$ must be equal to $u$, and the others must be elements of $W_0$.

Assume $v=u \in U$. Suppose, e.g., $v_1=u$. $(u\circ_k v_2)\circ_\ell(v_3\circ_k v_4) = u$ implies, on the one hand, that $u\circ_k v_2=u$ and thus that $v_2$ is a $k$-to-$n$ identity; on the other hand, that $v_3\circ_k v_4$ is an $\ell$-to-$n$ identity. By $(C_{n-1})$ being true for $\boldsymbol{X}$, it follows that both $v_3$ and $v_4$ are $\ell$-identities. It follows that

$$w = (u\circ_\ell v_3)\circ_k(v_2\circ_\ell v_4) = u\circ_k v_2 = u .$$

The other cases: $v_i=u$ ($i \in \{2.3.4\}$) are similar. Also, the argument is similar when we start with $w=u \in U$.

The (equivalence) class of the word $w \in W$ under $\approx^*$ is denoted by $[w]$. Of course, the class $[v]$ for $v \in W_0 \dot\cup W_1$ is $[v] = \{v\}$.

We define the $n$-category $\boldsymbol{Y}$ as follows. The $(n\text{-}1)$-truncation of $\boldsymbol{Y}$ is $\boldsymbol{X}$.

The set of $n$-cells of $\boldsymbol{Y}$ is $Y_n \stackrel{\text{def}}{=} W/\approx = \{[w] : w \in W\}$.

We put, for $w \in W$, $\mathrm{d}[w] = \mathrm{d}w$, $\mathrm{c}[w] = \mathrm{c}w$; these are clearly well-defined.

The identity $n$-cells are given by the elements of $W_2$: for $x \in X_{n-1}$, $\mathrm{id}_x = [\hat{\mathrm{id}}_x]$.

For the composition of $n$-cells of $\boldsymbol{Y}$, we put

$$[v] \circ_k [w] \stackrel{\text{def}}{=} [v \circ_k w] \qquad (k<n,\ v, w \in W,\ \mathrm{d}^{(k)}v = \mathrm{c}^{(k)}w)$$

The well-definedness of composition is assured by (ii)$^*$.

The domain/codomain laws were effectively pointed out above.

The identity laws, one for each $k<n$, concerning composition of $n$-cells, holds on the level of words already.

The remaining laws: associativity and middle exchange in dimension $n$ are true as a consequence of (iv)$^*$.

We have the obvious inclusion maps $\Gamma : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, $\Lambda : U \longrightarrow Y_n$; $(\boldsymbol{Y}, \Gamma, \Lambda)$ is an object of $\boldsymbol{A} = \boldsymbol{A}[\boldsymbol{X}, U, d, c]$. We claim that $(\boldsymbol{Y}, \Gamma, \Lambda)$ is in fact an initial object of $\boldsymbol{A}$. This is verified by inspection; intuitively, we did not generate elements of $\boldsymbol{Y}$, and we did not make identifications between them, unless it was so dictated by the $\omega$-category laws.

(1.1) holds by (1.1)$^*$. The assertions concerning $\Gamma$ and $\Lambda$ are true directly by the construction. This completes the proof of the Lemma.

Let us note that without assuming $(C_{n-1})$ for $X$, even when we drop (1.1) from the assertion, the lemma becomes false.

*n-computads*, for $n \in \mathbb{N}$, are defined recursively as follows; each $n$-computad is, in particular, an $n$-category.

A $0$-*computad* is a $0$-category: a set.

An *n-computad* is any $\omega$-category isomorphic to one of the form $X[U]$, where $X$ is an $(n-1)$-computad, and $U$ is a set of $n$-indeterminates attached to $X$.

A *computad* is an $\omega$-category whose $n$-truncation is an $n$-computad, for each $n \in \mathbb{N}$.

It is important to realize that the indeterminates are not "lost" in the wording of the definition; this is a consequence of (1).

We may rephrase the definition as follows.

Let $X$ be an $\omega$-category. Let us call a cell of $X$ an *indeterminate* if it is indecomposable. Let $X_n$ the $n$-truncation of $X$, and let $U_n$ denote the set of all $n$-indeterminates in $X$, attached to $X_n$ is the obvious way. Then $X$ is a computad iff, for all $n \geq 1$, $(X_n, \Gamma : X_{n-1} \to X_n, \Lambda : U_n \to X_n)$ with $\Gamma$, $\Lambda$ denoting inclusions is an initial object in $A[X_{n-1}, U_n, d, c]$.

A corollary is

**(2)** If $X$ is a computad, and $X'$ is an $\omega$-category isomorphic to $X$, then $X'$ is a computad too. Moreover, any isomorphism $f : X \xrightarrow{\cong} X'$ of $\omega$-categories between computads $X$, $X'$ takes any indeterminate in $X$ to an indeterminate in $X'$.

Next, we show that for any multitopic set $\boldsymbol{R}$, $\boldsymbol{Y}_{\mathsf{def}} \langle \boldsymbol{R} \rangle$ is a computad.

The proof is by induction: we show by induction on $n$ that for an $n$-dimensional ms $\boldsymbol{R}$, the $n$-category $\boldsymbol{Y}_{\mathsf{def}} \langle \boldsymbol{R} \rangle$ is an $n$-computad; this clearly suffices.

Let $\boldsymbol{R}$ be an $n$-dimensional ms, $\boldsymbol{R}_{n-1}$ its $(n-1)$-truncation. By induction hypothesis, $\boldsymbol{Y}_{n-1} = \langle \boldsymbol{R}_{n-1} \rangle$ is an $(n-1)$-computad.

Let $\boldsymbol{L} = [\boldsymbol{Y}] = [\langle \boldsymbol{R} \rangle]$; we have the unit map $\eta : \boldsymbol{R} \to \boldsymbol{L}$, and its component $\eta : R_n \to L_n$. On the other hand, we have $[\![ - ]\!] : L_n \to Y_n$ (the map $[\![ - ]\!] : M_n \to X_n$ of §2 in the present context). As in §5, we denote the composite $[\![ - ]\!] \circ \eta$ by $(-)^* : R_n \to Y_n$. We'll show that the $n$-indeterminates in $\boldsymbol{Y}_n$ are the elements $a^*$ for $a \in R_n$. That is, we claim that $\boldsymbol{Y}$ is a computad of the form $\boldsymbol{Y}_{n-1}[R_n]$, with the canonical $\Gamma : \boldsymbol{Y}_{n-1} \to \boldsymbol{Y}$ the inclusion, $\Lambda : R_n \to Y_n$ the map $(-)^*$.

Necessarily, the attachment of the set $U = R_n$ of indeterminates is defined by

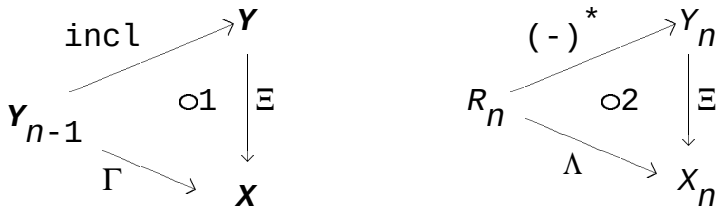
$$\mathsf{d}_{(a)} a = \hat{\mathsf{d}}(a^*) = (\mathsf{d}a)^*,$$
$$\mathsf{c}_{(a)} a = \hat{\mathsf{c}}(a^*) = (\mathsf{c}a)^*$$

(see 5.(13')).

To verify the claim, let $\boldsymbol{X}$ be an $\omega$-category, $\Gamma : \boldsymbol{Y}_{n-1} \to \boldsymbol{X}$, and $\Lambda : R_n \to X_n$ such that

$$\begin{array}{ccc} R_n & \xrightarrow{\quad\Lambda\quad} & X_n \\ {\scriptstyle \langle (-)^* \mathsf{d},\, (-)^* \mathsf{c} \rangle} \downarrow & \circ & \downarrow {\scriptstyle \langle \mathsf{d},\, \mathsf{c} \rangle} \\ Y_{n-1} \times Y_{n-1} & \xrightarrow{\quad\Gamma\quad} & X_{n-1} \times X_{n-1} \end{array}, \qquad (3)$$

to obtain $\Xi : \boldsymbol{Y} \to \boldsymbol{X}$ with the following two commutativities:




(4?)

Let $\boldsymbol{M}=[\boldsymbol{X}]$ , and consider the following items:

$$\begin{array}{l} \dfrac{\langle \boldsymbol{R}_{n-1}\rangle=\boldsymbol{Y}_{n-1}\xrightarrow{\ \Gamma\ }\boldsymbol{X}}{\boldsymbol{R}_{n-1}\xrightarrow{\ \Phi\ }[\boldsymbol{X}]=\boldsymbol{M}}\ 1 \\[2ex] \dfrac{\boldsymbol{R}_{n-1}\xrightarrow{\ \Phi\ }[\boldsymbol{X}]=\boldsymbol{M}}{\boldsymbol{R}\xrightarrow{\ \Theta\ }[\boldsymbol{X}]=\boldsymbol{M}}\ 2 \\[2ex] \dfrac{\boldsymbol{R}\xrightarrow{\ \Theta\ }[\boldsymbol{X}]=\boldsymbol{M}}{\langle \boldsymbol{R}\rangle=\boldsymbol{Y}\xrightarrow{\ \Xi\ }\boldsymbol{X}}\ 3 \end{array} \qquad (5)$$

Here, $\Phi$ , respectively $\Xi$ , is obtained by the adjunction $\langle - \rangle \dashv [-]$ from $\Gamma$ , respectively from $\Theta$ . $\Theta$ is the extension of $\Phi$ , obtained as follows.

Recall (1.(50)) that, to define the morphism $\Theta$ of ms's, we need to specify $\Theta(a)=\bar{a}\in M_n$ for each $a\in R_n$ such that

$$d\bar{a}=\Phi(da)\ ,\ c\bar{a}=\Phi(ca) \qquad (6)$$

(we write $\Phi$ for the $\Psi$ in *loc.cit.*). But, by the definition of $\boldsymbol{M}=[\boldsymbol{R}]$ , $\bar{a}=(\ [\![\bar{a}]\!]\ ,d\bar{a},c\bar{a})=(\ [\![\bar{a}]\!]\ ,\Phi(da),\Phi(ca))$ ; it remains to specify $[\![\bar{a}]\!]\in X_n$ , which we define as $[\![\bar{a}]\!]=\Lambda a$ . However, there are two conditions to be verified: (a) and (b) of 2.[4]: $d\bar{a}\overset{?}{\|}c\bar{a}$ and $d\,[\![\bar{a}]\!]\overset{?}{=}[\![d\bar{a}]\!]$ , $c\,[\![\bar{a}]\!]\overset{?}{=}[\![c\bar{a}]\!]$ . The first relation is clear; for the rest, we argue as follows.

By its definition, $\Phi$ is the composite

$$\begin{array}{ccccc} \boldsymbol{R}_{n-1} & \xrightarrow{\ \eta\ } & [\boldsymbol{Y}_{n-1}] & \xrightarrow{\ [\Gamma]\ } & [\boldsymbol{X}]\ . \\ & \searrow & \circ & \nearrow & \\ & & \Phi & & \end{array} \qquad (7)$$

Thus,

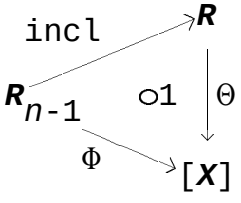
$$\mathrm{d}\,[\![\bar{a}]\!] \overset{1}{=} \mathrm{d}\Lambda a \overset{2}{=} \Gamma(\mathrm{d}a)^{*} = \Gamma\,[\![\eta \mathrm{d}a]\!] \overset{3}{=} [\![[\Gamma]\eta \mathrm{d}a]\!] \overset{4}{=} [\![\Phi \mathrm{d}a]\!] \overset{5}{=} [\![\mathrm{d}\bar{a}]\!] \;;$$

here, 1 by the definition of $[\![\bar{a}]\!]$, 2 by (3), 3 by the definition of $[\Gamma]$ from $\Gamma$, 4 by (7), 5 by (6). The equality $\mathrm{c}\,[\![\bar{a}]\!] = [\![\mathrm{c}\bar{a}]\!]$ is seen similarly.

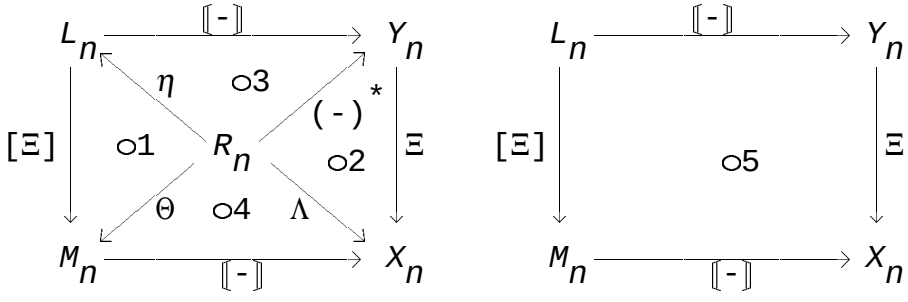
Let us record that the definition of $\Theta$ amounts to the equality

$$[\![\Theta(a)]\!] = \Lambda(a) \qquad (\, a \in R_n \,) \qquad (8)$$

The relation $\circ 1$ in (4) follows by adjunction from




Let us write $L = [Y]$. $\circ 2$ in (4) follows from commutativities




Here, $\circ 1$ is a consequence of line 3 in (5), $\circ 3$ is the definitional relation between $\eta$ and $(-)^{*}$, $\circ 4$ is (8), and $\circ 5$ is the definition of $[\Xi]$ from $\Xi$. $\circ 2$ follows; this is $\circ 2$ in (4).

It is not hard to see, going through the above in a different order, that $\Xi : Y \to X$ is uniquely determined by (4).

Let us summarize:

**(9)** For any $\boldsymbol{R}\in\mathtt{mltSet}$, $\langle\boldsymbol{R}\rangle$ is a computad. The indeterminates of $\langle\boldsymbol{R}\rangle$ are exactly the cells of the form $a^*=[\![\eta a]\!]$ for $a\in R_{\geq 0}$. We have $\langle\boldsymbol{R}\rangle=\langle\boldsymbol{R}_{n-1}\rangle[R_n]$ with attachment-assignments

$$\mathtt{d}_{(\mathtt{a})}a=\hat{\mathtt{d}}(a^*)=(\mathtt{d}a)^*,$$
$$\mathtt{c}_{(\mathtt{a})}a=\hat{\mathtt{c}}(a^*)=(\mathtt{c}a)^* \qquad (a\in R_n)$$

But, is $\langle\boldsymbol{R}\rangle$ any kind of computad? No; it is a *many-to-one* computad.

An indeterminate is *many-to-one* if either it is of dimension $\mathtt{0}$, or its codomain is an indeterminate. A *many-to-one* computad is a computad in which every indeterminate is many-to-one.

It is clear that in $\boldsymbol{Y}=\langle\boldsymbol{R}\rangle$, every indeterminate is many-to-one: $\hat{\mathtt{c}}(a^*)=(\mathtt{c}a)^*$, an indeterminate (see (9)). Thus, $\boldsymbol{Y}$ is a many-to-one computad.

But the converse also holds. Let $\boldsymbol{Y}$ be any many-to-one computad, to show that $\boldsymbol{Y}\cong\langle\boldsymbol{R}\rangle$ for some $\boldsymbol{R}\in\mathtt{mltSet}$. Proceeding inductively, we may assume that $\boldsymbol{Y}_{n-1}=\langle\boldsymbol{R}_{n-1}\rangle$ for an $(n\text{-}1)$-dimensional ms $\boldsymbol{R}_{n-1}$, and $\boldsymbol{Y}$ is a many-to-one $n$-computad. We have $\boldsymbol{Y}=\boldsymbol{Y}_{n-1}[U]$ for a set $U$ of many-to-one indeterminates attached to $\boldsymbol{Y}_{n-1}$: for $u\in U$, $\mathtt{d}_{(\mathtt{a})}u\in Y_{n-1}$, $\mathtt{c}_{(\mathtt{a})}u\in Y_{n-1}$ (the subscripts for "attachment") and $\mathtt{c}_{(\mathtt{a})}u$ is an indeterminate in $\boldsymbol{Y}_{n-1}$. We have $\mathtt{d}_{(\mathtt{a})}u\,\|\,\mathtt{c}_{(\mathtt{a})}u$ in the sense of $\boldsymbol{Y}_{n-1}$, that is, $\hat{\mathtt{d}}\mathtt{d}_{(\mathtt{a})}u=\hat{\mathtt{d}}\mathtt{c}_{(\mathtt{a})}u$, and $\hat{\mathtt{c}}\mathtt{d}_{(\mathtt{a})}u=\hat{\mathtt{c}}\mathtt{c}_{(\mathtt{a})}u$.

We apply 5.(14). One obvious consequence of the characterization 5.(14) of the elements of $Y_{n-1}$ of the form $\alpha^*$ is that any cell parallel (in $\boldsymbol{Y}_{n-1}$) to one of the form $\alpha^*$ is again one of that form. In fact, $\gamma$ being of the form $\alpha^*$ depends only on $\hat{\mathtt{c}}\gamma$. Furthermore, for $\alpha,\beta\in S_{n-1}$, $\alpha\|\beta$ in $\boldsymbol{R}$ iff $\alpha^*\|\beta^*$ in $\boldsymbol{Y}$; this is seen by 5.(16) and the injectivity of the

map $(\text{-})^*$.

Let $u \in U$. By assumption and (9), $\mathrm{c}_{(a)}u$ is of the form $(a_u)^*$, $a_u \in R_{n-1}$. Therefore, there is a (unique) $\alpha_u \in S_{n-1}$ such that $\mathrm{d}_{(a)}u = (\alpha_u)^*$; since $\mathrm{d}_{(a)}u \| \mathrm{c}_{(a)}u$, we have that $\alpha_u \| a_u$ in $\boldsymbol{R}_{n-1}$.

Now, consider the following $n$-dimensional ms $\boldsymbol{R}$: its $(n\text{-}1)$-truncation is $\boldsymbol{R}_{n-1}$, and its $n$-cells are exactly the elements of $U$, with $\mathrm{d}u = \mathrm{d}_{\boldsymbol{R}}u = \alpha_u$ and $\mathrm{c}u = \mathrm{c}_{\boldsymbol{R}}u = a_u$.

By (9), $\langle \boldsymbol{R} \rangle$ is a computad; in fact, it is the computad $\langle \boldsymbol{R}_{n-1} \rangle[U]$ with attachments

$$\begin{aligned} \mathrm{d}^{\langle \boldsymbol{R} \rangle}_{(a)}\, u &= \hat{\mathrm{d}}(u^*) = (\mathrm{d}u)^* = (\alpha_u)^* = \mathrm{d}_{(a)}u\,, \\ \mathrm{c}^{\langle \boldsymbol{R} \rangle}_{(a)}\, u &= \hat{\mathrm{c}}(a^*) = (\mathrm{c}a)^* = (a_u)^* = \mathrm{d}_{(a)}u \qquad (u \in U) \end{aligned}$$

That is, $\langle \boldsymbol{R} \rangle$ is, up to isomorphism, the same as $\boldsymbol{Y}$, as desired.

We have proved:

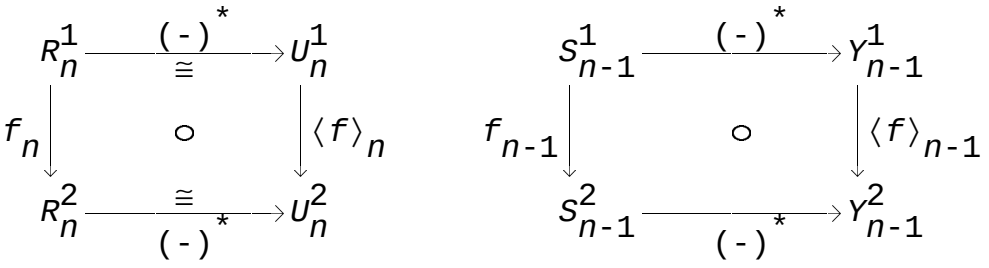
**(10)** An $\omega$-category is a many-to-one computad if and only if it is isomorphic to $\langle \boldsymbol{R} \rangle$ for a multitopic set $\boldsymbol{R}$.

A *morphism* of computads is an $\omega$-category morphism which, in addition, maps every indeterminate in its domain to an indeterminate in its codomain.

The category $\mathtt{Comp}$ of computads is a non-full subcategory of $\omega\text{-}\mathtt{Cat}$. Although the inclusion $\mathtt{Comp} \to \omega\text{-}\mathtt{Cat}$ is not full, by (2), it is full with respect to isomorphisms.

$\mathtt{Comp}_{m/1}$ denotes the full subcategory of $\mathtt{Comp}$ with objects the many-to-one computads.

Let $f : \boldsymbol{R}^1 \to \boldsymbol{R}^2$, a morphism in $\mathtt{mltSet}$. Let $n \in \mathbb{N}$. Writing $\boldsymbol{Y}^i = \langle \boldsymbol{R}^i \rangle$, and $U^i_n$ for the set of $n$-indeterminates in $\boldsymbol{Y}^i$, we have



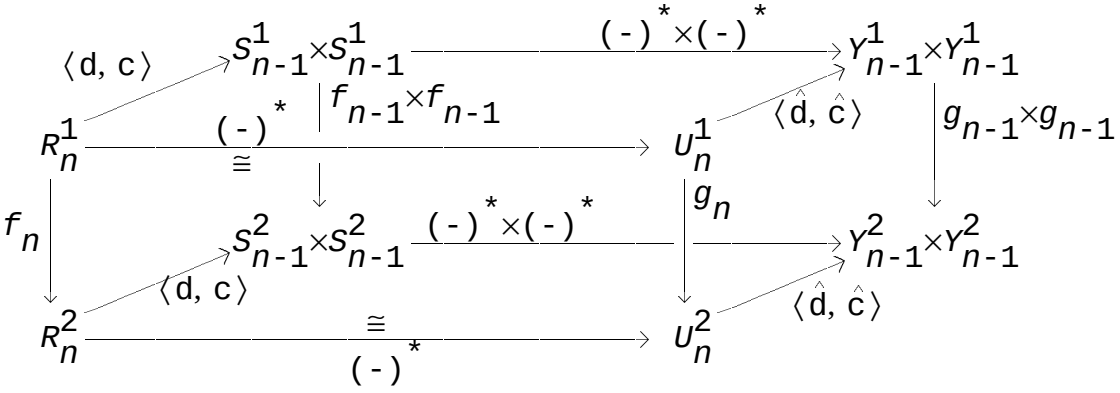
(11)

as a consequence of the naturality of $\eta$. This shows that

**(12)** The map $f \mapsto \langle f \rangle$ is a bijection

$$\hom_{\mathtt{mltSet}}(\boldsymbol{R}^1, \boldsymbol{R}^2) \longrightarrow \hom_{\mathtt{Comp}}(\boldsymbol{Y}^1, \boldsymbol{Y}^2) .$$

To see this in detail, first, we may and do assume that $\boldsymbol{R}^1$, $\boldsymbol{R}^2$ are of dimension $n$. Fix the map $f_{n-1} : \boldsymbol{R}^1_{n-1} \to \boldsymbol{R}^2_{n-1}$, let $g_{n-1} = \langle f_{n-1} \rangle : \boldsymbol{Y}^1_{n-1} \to \boldsymbol{Y}^2_{n-1}$, and with $f_n : R^1_n \to R^2_n$, $g_n : U^1_n \to U^2_n$ variable maps of sets, consider the following diagram:



The Top, Bottom and Back faces are constant, and they commute; for the Back this is the second diagram in (11). Since the horizontal edges of the Front are bijections, the commutativity of the Front defines a bijection $f_n \mapsto g_n$. If $f_n \mapsto g_n$ in this bijection, then the Left commutes iff the Right commutes.

On the other hand, note that a map $f : \boldsymbol{R}^1 \longrightarrow \boldsymbol{R}^2$ of multitopic sets is determined by its

$(n\text{-}1)$-truncation $f_{n-1}$ and its effect $f_n$ on $n$-cells subject to the commutativity of the Left. Similarly, a map

$$g : \boldsymbol{Y}^1 = \boldsymbol{Y}^1_{n-1}[U^1_n] \longrightarrow \boldsymbol{Y}^2 = \boldsymbol{Y}^2_{n-1}[U^2_n]$$

of computads is determined by its truncation $g_{n-1}$ and its effect $g_n$ on $n$-indeterminates subject to the commutativity of the Right. Therefore, we have a bijection

$$(f \mapsto g) : \mathtt{hom}(\boldsymbol{R}^1, \boldsymbol{R}^2) \longrightarrow \mathtt{hom}(\boldsymbol{Y}^1, \boldsymbol{Y}^2)$$

defined by saying that $f \mapsto g$ iff the Front is commutative. The first diagram in (11) shows that this bijection is the same as $f \mapsto \langle f \rangle$ .

(10) and (12) tells us that

**(13) Theorem** $\langle\text{-}\rangle : \mathtt{mltSet} \longrightarrow \omega\mathtt{Cat}$ induces an equivalence

$$\langle\text{-}\rangle : \mathtt{mltSet} \xrightarrow{\simeq} \mathtt{Comp}_{\mathtt{m/1}} \ .$$

We have proved the Theorem of the Introduction.