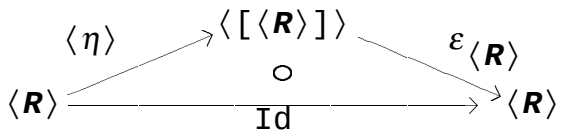
The first triangle identity

Let $\boldsymbol{R}$ be any ms. We want to show that the diagram



commutes. As before, we write $\boldsymbol{Y}$ for $\langle \boldsymbol{R} \rangle$, and use the notation of section 4 for $\boldsymbol{Y}$. We write $\boldsymbol{M}$ for $[\langle \boldsymbol{R} \rangle]=[\boldsymbol{Y}]$; this is the construction of section 2 for $\boldsymbol{Y}=\boldsymbol{X}$. We use the notation of section 2 with respect to this $\boldsymbol{M}$. Let us write $\boldsymbol{W}$ for the ω-category $\langle [\langle \boldsymbol{R} \rangle] \rangle = \langle [\boldsymbol{Y}] \rangle = \langle \boldsymbol{M} \rangle$. In the definition of $\varepsilon_{\langle \boldsymbol{R} \rangle}=\varepsilon_{\boldsymbol{Y}}$ as a special case of $\varepsilon_{\boldsymbol{X}}$, $\boldsymbol{X}$ is now $\boldsymbol{Y}$, and the role of $\boldsymbol{Y}$ is now played by $\boldsymbol{W}$.

Using the notation $[\![-]\!]$ for $\varepsilon(-)$ (as before), the assertion amounts to the identity

$$[\![\langle \eta \rangle \alpha]\!] = \alpha \qquad (\alpha \in Y_{\geq 0}) \qquad (33)$$

Remember that, for any $\delta \in W_n \subset P_n^+$, $[\![\delta]\!] = [\![\ulcorner \delta \urcorner]\!]$; and, for $\alpha \in Y_n$, $\langle \eta \rangle \alpha = \eta^+ \alpha$.

To prove (33), we need to do some work that parallels the work for the proof of the identity (18). In fact, it can be seen that (33) is a generalization of (18) (that is, (18) can be deduced from (33) by a short calculation). Thus, it may seem that one should give a proof of (33) in the first place, and deduce (18) from it. Instead, we give a proof of (33) that, although does go through some steps that are familiar from the proof of (18), also uses the result (18) itself.

A pair (α, p) is *of the second kind*, of *dimension* n and *length* $m \geq 1$, if $|\alpha| = m$, $|p| = m-1$, $\dim(\alpha_0)=n$, $\alpha_0 \in Y_n$, $\alpha_i \in S_{n+i}$ for all $0<i<m$, and $p_i \in |\langle \alpha_i \rangle|$, $c\alpha_{i+1}=\langle \alpha_i \rangle (p_i)$ for all $0 \leq i < m-1$. The length m being 1 means that we have a single n-pd $\alpha=\alpha_0 \in Y_n$, and no p.

The only difference to the pairs "of the first kind" from above is in the first element α_0. As a matter of fact, a pair (α, p) of the first kind may be regarded as a special case of that of the second kind: replace α_0 by α_0^* (thereby obtaining a special sort of element of Y_n), and modify p_0 appropriately.

If the length of the pair (α, p) of the 2nd kind is ≥ 2, then $(\alpha|, p|)$ is of the second kind, and $(|\alpha, |p)$ is of the first kind.

Very similarly to the previous use of the symbol $[-, -]$, for a pair (α, p) of the second kind, of length m and dimension n, we define $[\alpha, p] \in Y_{n+m-1}$, and $\rho(\alpha, p) : \langle \hat{\alpha} \rangle^{(-)} \xrightarrow{\cong} \langle [\alpha, p] \rangle^{-}$ (where the superscript $^{-}$ is present when $n=1$, and is not present otherwise), by recursion on m.

For $m=1$, $[\alpha, p] \underline{\underline{\text{def}}} \alpha_0$, and $\rho(\alpha, p) = \text{id} : \langle \alpha_0 \rangle^{-} \xrightarrow{\cong} \langle \alpha_0 \rangle^{-}$.

Given (α, p), $m=\text{lh}(\alpha, p)$, $n=\dim(\alpha, p) \geq 2$, consider the truncated pair $(\alpha|, p|)$, $\check{\alpha}=(\alpha|)\hat{}$, and the map $\rho' = \rho(\alpha|, p|) : \langle \check{\alpha} \rangle^{(-)} \to \langle [\alpha|, p|] \rangle^{-}$. Note that $\hat{p} \in |\langle \check{\alpha} \rangle^{(-)}|$. Let $\dot{p} = \rho'(\hat{p})$. Note also that

$$c\hat{\alpha} = \langle \check{\alpha} \rangle(\hat{p}) = \langle [\alpha|, p|] \rangle(q) = \langle d([\alpha|, p|]^{+}) \rangle(q) .$$

We define

$$[\alpha, p] \underline{\underline{\text{def}}} [\alpha|, p|]^{+} \circ_{\dot{p}} \hat{\alpha} .$$

and $\rho(\alpha, p)$ to be the canonical ρ-map

$$\rho = \rho(\alpha, p) = \rho[[\alpha|, p|]^{+}, \hat{\alpha}, \dot{p}] : \langle \hat{\alpha} \rangle \xrightarrow{\cong} \langle [\alpha, p] \rangle^{-} .$$

The following a slight generalization of (27); we omit the proof:

(34) Assume that $\alpha^1 \in Y_n$, and $[d\alpha^1 \hat{} \alpha^2 \hat{} \beta, j \hat{} p^2 \hat{} q]$ is a pair (of the second kind) of dimension $n-1$. Then,

$$\alpha^1 \circ_{n-1} [d\alpha^1 \hat{} \alpha^2 \hat{} \beta, j \hat{} p^2 \hat{} q] = [(\alpha^1 \circ_j \alpha^2) \hat{} \beta, (\rho p^2) \hat{} q] .$$

(35)$_n$ **Lemma** Let (α, p) be a pair of the second kind, of dimension n and length m. If $m=1$, then

$$\llbracket \ulcorner \eta^+ \alpha \urcorner \rrbracket = [\alpha, p] = \alpha ,$$

and if $m \geq 2$,

$$\llbracket \ulcorner \eta^+ \alpha_0 \urcorner \square_{\theta \eta^+ p_0} [\,|\alpha, \,|p]\rrbracket = [\alpha, p] .$$

Proof The proof is an induction on n. However, there is no induction on $\#\,|\langle \alpha_0 \rangle|$ involved.

Note first that we can always write α_0 in the form $\alpha_0 = \alpha^1 \circ_j \alpha^2$, with $\alpha^1 \in Y_{n-1}$, $\alpha^2 \in S_{n-1}$ and such that $p_0 \in \mathrm{Im}(\rho)$, where

$$\langle \alpha^1 \rangle^- \xrightarrow{\ \sigma\ } \langle \alpha \rangle^- \xleftarrow{\ \rho\ } \langle \alpha^2 \rangle$$

are the canonical maps. Let $p^2 \in |\langle \alpha^2 \rangle|$ such that $p_0 = \rho(p^2)$.

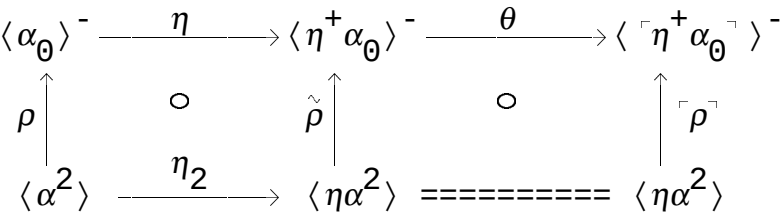
According to 3.(35'), we have

$$\ulcorner \eta^+ \alpha_0 \urcorner = \ulcorner \eta^+ \alpha^1 \urcorner \circ^-_{\theta_1 \eta_1^+ j} \eta \alpha^2 , \tag{36}$$

using the canonical maps

$$\langle d\alpha^1 \rangle^- \xrightarrow{\ \eta_1^+\ } \langle \eta^+ d\alpha^1 \rangle^- \xrightarrow{\ \theta_1\ } \langle \ulcorner \eta^+ d\alpha^1 \urcorner \rangle^- ,$$

and we have the following commutative diagram:



Therefore, $\theta\eta^{+}p_0 = \ulcorner\rho\urcorner\,\eta_2 p^2$. (37)

We have, in case $m=1$,

$$
\begin{aligned}
&[\![\ulcorner\eta^{+}\alpha\urcorner]\!] \overset{1}{=} [\![(\ulcorner\eta^{+}\alpha^1\urcorner \circ_{\theta_1\eta_1^{+}j} \eta\alpha^2)]\!] \\
&\overset{3}{=} [\![\ulcorner\eta^{+}\alpha^1\urcorner]\!] \circ_{n-1} [\![\mathrm{d}\ulcorner\eta^{+}\alpha^1\urcorner \square_{\theta_1\eta_1^{+}j} [\![\eta\alpha^2]\!]]\!] \\
&\overset{4}{=} [\![\ulcorner\eta^{+}\alpha^1\urcorner]\!] \circ_{n-1} [\![\ulcorner\eta^{+}\mathrm{d}\alpha^1\urcorner \square_{\theta_1\eta_1^{+}j} [\alpha^2, \emptyset]]\!] \\
&\overset{5}{=} \alpha^1 \circ_{n-1} [\mathrm{d}\alpha^1 \wedge \alpha^2, j] \\
&\overset{6}{=} [\alpha^1 \circ_j \alpha^2, \emptyset] = [\alpha, \emptyset]
\end{aligned}
$$

and in case $m\geq 2$,

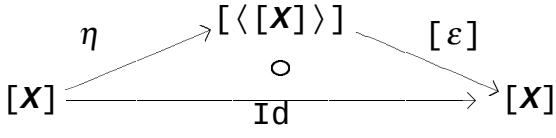
$$
\begin{aligned}
&[\![\ulcorner\eta^{+}\alpha_0\urcorner \square_{\theta\eta^{+}p_0} [\,|\alpha, \,|p]]\!] \overset{1}{=} [\![(\ulcorner\eta^{+}\alpha^1\urcorner \circ_{\theta_1\eta_1^{+}j} \eta\alpha^2) \square_{\theta\eta^{+}p_0} [\,|\alpha, \,|p]]\!] \\
&\overset{2}{=} [\![\ulcorner\eta^{+}\alpha^1\urcorner \circ_{\theta_1\eta_1^{+}j} (\eta\alpha^2 \square_{\eta_2 p^2} [\,|\alpha, \,|p])]\!] \\
&\overset{3}{=} [\![\ulcorner\eta^{+}\alpha^1\urcorner]\!] \circ_{n-1} [\![\mathrm{d}\ulcorner\eta^{+}\alpha^1\urcorner \square_{\theta_1\eta_1^{+}j} [\![\eta\alpha^2 \square_{\eta_2 p^2} [\,|\alpha, \,|p]]\!]]\!] \\
&\overset{4}{=} [\![\ulcorner\eta^{+}\alpha^1\urcorner]\!] \circ_{n-1} [\![\ulcorner\eta^{+}\mathrm{d}\alpha^1\urcorner \square_{\theta_1\eta_1^{+}j} [\alpha^2 \wedge |\alpha, p^2 \wedge |p]]\!] \\
&\overset{5}{=} \alpha^1 \circ_{n-1} [\mathrm{d}\alpha^1 \wedge \alpha^2 \wedge |\alpha, j \wedge p^2 \wedge |p]
\end{aligned}
$$

$$\overset{6}{=} [(\alpha^1 \circ_j \alpha^2)^\wedge \mid \alpha,\ \rho p^{2\wedge} \mid p] = [\alpha,\ p]$$

In both sequences, the justifications are as follows. 1 : (23); 2 : 1.(46) and (37); 3 : 2.(44); 4 : $(18)_n$, 5 : induction hypothesis: $(35)_{n-1}$; 6 : (34).

The second triangle identity

Let $\boldsymbol{X}$ an ω-category. We show:



As usual, we write $\boldsymbol{M}=[\boldsymbol{X}]$, with the notation of section 2; we also put $\boldsymbol{R}=\boldsymbol{M}$, and write $\boldsymbol{Y}=\langle \boldsymbol{R} \rangle$, with the notation of section 4.

For the next, preparatory, assertion, let us start with an arbitrary ω-category $\boldsymbol{X}$, and let $\boldsymbol{M}=[\boldsymbol{X}]$ according to §2; we apply the $*$-construction, described in an arbitrary ms $\boldsymbol{R}$ above, in $\boldsymbol{M}$. On the other hand, now we have the $\ulcorner - \urcorner$ -construction in $\boldsymbol{M}$ as explained at the beginning of this section. We can bring the two constructions together as follows.

(38) (i) For $b \in M_n$ and $u \in X_{\geq n}$ such that $\mathrm{d}^{(n-1)}u = [\![\mathrm{d}b]\!]$ and $\mathrm{c}^{(n-1)}u = [\![\mathrm{c}b]\!]$, we have $[\![\ulcorner b^* \urcorner \Box_* u]\!] = u$.

Here, $\{\&\}=\{\&_b\}= |\langle b \rangle|$, $\rho=\rho[b]:\langle b \rangle \to \langle b^* \rangle$, $\theta=\theta_{b^*}:\langle b^* \rangle \to \ulcorner \langle b^* \rangle \urcorner$, and $*=*_b \overset{\mathrm{def}}{=} \theta\rho(\&)$. The $\Box$-operation takes place in $\boldsymbol{G}_n=\mathbf{D}_n[\boldsymbol{N}_n^{(\boldsymbol{M})}]$.

(ii) For $\beta \in P_n$, $[\![\ulcorner \beta^* \urcorner]\!] = [\![\beta]\!]$.

Proof (i) Induction on n . When $n=0$ or $n=1$, $\ulcorner b^* \urcorner = b^* = b$, $\rho=\mathrm{id}$, and the assertion is clear.

Suppose $n \geq 2$. With $\# = {}^{*}_{cb}$, and $\% = \rho[cb]\&_{cb}$, we have $b^{*} = (cb)^{*+} \circ_{\%} b$, thus $\ulcorner b^{*} \urcorner = \ulcorner (cb)^{*+} \urcorner \circ_{\#} b$, and

$$\ulcorner b^{*} \urcorner \square_{*} u = (\ulcorner (cb)^{*+} \urcorner \circ_{\#} b) \square_{*} u = \ulcorner (cb)^{*+} \urcorner \circ_{\#} (b \square_{\&} u) ;$$

thus,

$$\begin{aligned} [\![\ulcorner b^{*} \urcorner \square_{*} u]\!] &= [\![\ulcorner (cb)^{*+} \urcorner]\!] \circ_{n-1} [\![d \ulcorner (cb)^{*+} \urcorner \square_{\#} [\![b \square_{\&} u]\!]]\!] \\ &\overset{1}{=} [\![d \ulcorner (cb)^{*+} \urcorner \square_{\#} [\![b \square_{\&} u]\!]]\!] \\ &\overset{2}{=} [\![\ulcorner (cb)^{*} \urcorner \square_{\#} u]\!] \\ &\overset{3}{=} u \end{aligned}$$

At 1 : we used that $[\![\ulcorner (cb)^{*+} \urcorner]\!]$ is an identity n-cell in $\boldsymbol{X}$. At 2 : see 3.(26). At 3 : induction hypothesis.

(ii) With $\%$ and $\#$ appropriately (and similarly to before) defined, we have $\beta^{*} = (c\beta)^{*+} \circ_{\%} \beta$, thus $\ulcorner \beta^{*} \urcorner = \ulcorner (c\beta)^{*+} \urcorner \circ_{\#} \beta$, thus

$$\begin{aligned} [\![\ulcorner \beta^{*} \urcorner]\!] &= [\![\ulcorner (c\beta)^{*+} \urcorner]\!] \circ_{n-1} [\![d \ulcorner (c\beta)^{*+} \urcorner \square_{\#} [\![\beta]\!]]\!] \\ &\overset{1}{=} [\![d \ulcorner (c\beta)^{*+} \urcorner \square_{\#} [\![\beta]\!]]\!] \\ &\overset{2}{=} [\![\ulcorner (c\beta)^{*} \urcorner \square_{\#} [\![\beta]\!]]\!] \\ &\overset{3}{=} [\![\beta]\!] \end{aligned}$$

At 1 and 2 : as above; at 3 : (i).

We can now show that $[\varepsilon] \circ \eta = \mathrm{Id}$ on the truncation $\boldsymbol{M}_n$ of $\boldsymbol{M}$, by induction on n. Suppose we know this for $n-1$. Let $a \in M_n$; $a = ([\![a]\!], da, ca)$. Then

$\eta a = (a^*, \eta \mathrm{d} a, \eta \mathrm{c} a)$, and $[\varepsilon](a^*, \eta \mathrm{d} a, \eta \mathrm{c} a) = (\varepsilon a^*, [\varepsilon]\eta \mathrm{d} a, [\varepsilon]\eta \mathrm{c} a) \,.\, \varepsilon a^* = [\![a^*]\!] = [\![a]\!]$ by (38), and $[\varepsilon]\eta \mathrm{d} a = \mathrm{d} a$, $[\varepsilon]\eta \mathrm{c} a = \mathrm{c} a$ by the induction hypothesis. We obtained that $[\varepsilon]\eta a = a$ as promised.