The unit map

We fix an arbitrary ms $\boldsymbol{R}$; let $\boldsymbol{Y}=\langle \boldsymbol{R}\rangle$ as constructed in §4; and let $\boldsymbol{M}=[\boldsymbol{Y}]$ as constructed in §2. We construct the map $\eta=\eta_{\boldsymbol{R}}:\boldsymbol{R}\longrightarrow \boldsymbol{M}$ of ms's.

Let $r\in R_n$. We should have $\eta r\in M_n$; that is, ηr is to be an $(n\text{-}1)$-framed n-cell of $\boldsymbol{Y}$; in particular, $[\![\eta r]\!]\in Y_n$. We first construct $[\![\eta r]\!]$, and denote it by r^*. Thus, we need a particular $r^*\in Y_n$. Recall that $Y_n\subset S_n^+=\mathtt{P}_n(\boldsymbol{R}^+)$; thus, r^* is to be a particular n-pd in $\boldsymbol{R}$ *with indeterminates*.

It will be useful to construct, more generally, $\alpha^*\in Y_n$ for any $\alpha\in S_n$.

When $n=0$ or 1, $\alpha^*=\alpha$.

For $n\geq 1$, together with α^*, we also define a map $\rho=\rho[\alpha]:\langle\alpha\rangle\longrightarrow\langle\alpha^*\rangle$. Note that for $\alpha=r\in R_n\subset S_n$, $|\langle r\rangle|$ is a singleton, $|\langle r\rangle|=\{*\}$; thus, in this case, the map ρ amounts to a particular place $\rho(*)\in|\langle r^*\rangle|$ such that $\langle r^*\rangle(\rho(*))=r$.

We proceed recursively. Let $n\geq 2$, and let $\alpha\in S_n$. We already have $\mathtt{c}\alpha\in R_{n-1}$, $(\mathtt{c}\alpha)^*\in Y_{n-1}$, and $\langle\mathtt{c}\alpha\rangle\xrightarrow{\rho}\langle(\mathtt{c}\alpha)^*\rangle$. Recall the $\gamma\mapsto\gamma^+:S_{n-1}^+\longrightarrow S_n^+$ construction (beginning of this section). We define

$$\alpha^*\ \mathtt{d\bar{e}f}\ (\mathtt{c}\alpha)^{*+}\circ_{\rho(*)}\alpha \qquad (13)$$

The composition takes place in the multicategory $\mathbf{C}_n(\boldsymbol{R}^+)$; since $\mathtt{d}(\mathtt{c}\alpha)^{*+}=(\mathtt{c}\alpha)^*$, and $\langle(\mathtt{c}\alpha)^*\rangle(\rho(*))=\mathtt{c}\alpha$, the composite is well-defined.

We define $\rho=\rho[\alpha]$ to be the ρ-component of the (ρ,σ)-pair

$$\langle(\mathtt{c}\alpha)^{*+}\rangle\xrightarrow{\sigma}\langle\alpha^*\rangle=\langle(\mathtt{c}\alpha)^{*+}\circ_{\rho(*)}\alpha\rangle\xleftarrow{\rho}\langle\alpha\rangle$$

in the HAA-structure of $\mathbf{C}_n(\boldsymbol{R}^+)$.

By induction on the dimension, it is clear that $\alpha^* \in Y_n$: from $(c\alpha)^* \in Y_{n-1}$, by (13) and 4.(3), it follows that $\alpha^* \in Y_n$. Moreover, the mapping $(\alpha \in S_n) \mapsto \alpha^*$ is 1-1 : α can be recovered from α^* as $\overrightarrow{\alpha^*}(\#)$ where $\#$ is the unique element of $|\langle \hat{c}(\alpha^*)\rangle^-|$ ($\hat{c}(\alpha^*) = (c\alpha)^*$ and $\#=\rho(*)$ in terms of (13)).

The following characterizes the elements of Y_n of the form α^* or r^* .

(14) For $\gamma \in Y_n$, $\gamma=\alpha^*$ for a (necessarily unique) $\alpha \in S_n$ iff the set $|\langle \hat{c}^{(k)}\gamma\rangle^-|$ is a singleton for every $0 \le k < n$, and $\gamma=r^*$ for some $r \in R_n$ iff the same holds for all $0 \le k \le n$.

Proof This is pretty clear -- but, here are the proofs of the "if" directions.

The second assertion follows from the first.

Assume $\gamma \in Y_n$ and $|\langle \hat{c}^{(k)}\gamma\rangle^-|$ is a singleton for every $0 \le k < n$. Let $\beta = \hat{c}\gamma$. We have $\gamma=\beta^+ \circ \vec{\gamma}$ and $\beta \in Y_{n-1}$. By induction, $\beta=b^*$ for $b=\langle\beta\rangle(\#) \in R_{n-1}$, $\{\#\}=|\langle\beta\rangle^-|$. For $\alpha = \vec{\gamma}(\#)$, we have $c\alpha=\beta$ because $\beta^+ \circ \vec{\gamma}$ is well-defined. Thus, $\gamma=b^{*+} \circ \vec{\gamma}=b^{*+} \circ_{\#} \alpha=(c\alpha)^{*+} \circ_{\#} \alpha=\alpha^*$.

(15) We have that $(\gamma_1 \circ_j \gamma_2)^* = \gamma_1^* \circ_{\rho j} \gamma_2$, for $\rho=\rho[\gamma_1]$, and the following commutes:

$$\begin{array}{ccccc}
\langle \gamma_1 \rangle & \xrightarrow{\sigma} & \langle \gamma_1 \circ_j \gamma_2 \rangle & \xleftarrow{\rho} & \langle \gamma_2 \rangle \\
\rho \downarrow & \circ & \rho \downarrow & \circ & \downarrow \rho \\
\langle \gamma_1^* \rangle & \xrightarrow[\sigma]{} & \langle \gamma_1^* \circ_{\rho j} \gamma_2 \rangle & \xleftarrow[\rho]{} & \langle \gamma_2^* \rangle
\end{array}$$

(16) For $n\geq 1$, $\alpha\in S_n$, we have
$\hat{d}(\alpha^*) = d(\alpha^*) = (d\alpha)^*$, and $\hat{c}(\alpha^*) = dP(\alpha^*) = (c\alpha)^*$.

Proof We deal with the first equality; the second one is obvious by (13).

For $n=1$, this is obvious. Let $n\geq 2$. We have

$$d(\alpha^*) = d((c\alpha)^{*+} \circ_{\rho(*)} \alpha) = d((c\alpha)^{*+}) \square_{\rho(*)} d\alpha = (c\alpha)^* \square_{\rho(*)} d\alpha. \qquad (17)$$

When $n=2$, and thus $c\alpha\in R_1$, we have $(c\alpha)^* = c\alpha$, $|\langle c\alpha\rangle|$ is a singleton; by 1.(46), $d(\alpha^*)=d\alpha=(d\alpha)^*$.

Let $n\geq 3$. With $\tilde{\rho}=\rho[cc\alpha]$, continuing from (17),

$$(c\alpha)^* \square_{\rho(*)} d\alpha = ((cc\alpha)^{*+} \circ_{\tilde{\rho}(*)} c\alpha) \square_{\rho(*)} d\alpha \overset{1}{=} (cc\alpha)^{*+} \circ_{\tilde{\rho}(*)} (c\alpha \square_? d\alpha)$$
$$\overset{2}{=} (cc\alpha)^{*+} \circ_{\tilde{\rho}(*)} d\alpha = (cd\alpha)^{*+} \circ_{\tilde{\rho}(*)} d\alpha = (d\alpha)^*;$$

here, 1: by 1.(46); 2: $|\langle c\alpha\rangle|$ being a singleton, by 1.(46) $c\alpha \square_? d\alpha = d\alpha$.

As usual, η is defined recursively on the sets R_n; we assume that n is fixed and η is defined on the truncation $\boldsymbol{R}_{n-1}$.

Recall that an element a of M_n, for $n\geq 1$, is a triple $a=(\llbracket a\rrbracket, da, ca)$, where da and ca are the domain and codomain of a in the sense of the ms $\boldsymbol{M}$; the triple has to satisfy two conditions which we will recall soon. Now, as a morphism of ms's, $\eta:\boldsymbol{R}\to\boldsymbol{M}$, initially defined on the set $R_{\geq 0}$ of all cells of $\boldsymbol{R}$, extends to pd's of $\boldsymbol{R}$, and as such it preserves the operations d and c. It follows that, for $r\in R_n$, in defining $\eta(r)=(\llbracket\eta(r)\rrbracket, d\eta(r), c\eta(r))$, we must have $d\eta(r)=\eta(dr)$, $c\eta(r)=\eta(cr)$. Thus, $\eta(r)$ is now defined, since we already declared that $\llbracket\eta(r)\rrbracket = r^*$:

$$\eta(r) \underset{\mathrm{def}}{=} (r^*, \eta(dr), \eta(cr)) \ . \qquad (14')$$

However. we have to check that the two conditions 2.[4](a) and (b) are satisfied. The first one: $\eta(dr)$ and $\eta(cr)$ being parallel, is automatic since dr and cr are parallel. What we need is the second condition, which amounts to the equalities marked ? in the following two lines:

$$d_{\boldsymbol{Y}}(r^*) = d(r^*) = (dr)^* \overset{?}{=} [\![\eta(dr)]\!] \ ,$$
$$c_{\boldsymbol{Y}}(r^*) = \hat{c}(r^*) = dP(r^*) = (cr)^* \overset{?}{=} [\![\eta(cr)]\!] \ .$$

Therefore, we need $(18)_{n-1}$, where $(18)_n$ is

$(18)_n$ For $n \geq 0$ and $\alpha \in S_n$, we have $[\![\eta\alpha]\!] = \alpha^*$.

Unfortunately, this requires a somewhat elaborate induction.

The following part including Lemma (19) takes part entirely in $\boldsymbol{R}$ and $\boldsymbol{Y}$; with no reference to $\boldsymbol{M}$ and η .

We introduce some notation.

By a *sequence* s , we mean a function whose domain $|s|$ is $|s| = \{i \in \mathbb{N} : i < m\}$ for some $m \in \mathbb{N}$.

In what follows, α denotes a sequence of pd's, $\alpha = \langle \alpha_i \rangle_{i<m}$; we write $|\alpha|$ for m ; also, p is a sequence of "places"; $p = \langle p_i \rangle_{i<|p|}$. We write $\alpha|$ for $\langle \alpha_i \rangle_{i<|\alpha|-1}$; $\hat{\alpha} = \alpha_{|\alpha|-1}$ the last term of α , and similarly for $p|$ and $\hat{p}$. $|\alpha = \beta = \langle \beta_i \rangle_{i<|\gamma|-1}$ where $\beta_i = \alpha_{i+1}$.

A pair (α, p) is called *proper*, of *dimension* n and *length* $m \geq 1$, if $|\alpha| = m$, $|p| = m-1$, $\dim(\alpha_0) = n$, $\alpha_i \in S_{n+i}$ for all $i < m$, and $p_i \in |\langle \alpha_i \rangle|$, $c\alpha_{i+1} = \langle \alpha_i \rangle(p_i)$ for all

$0 \le i < m-1$. The length m being 1 means that we have a single n-pd α_0 , and no p .

If the length of the proper pair is ≥ 2 , then $(\alpha|, p|)$ and $(|\alpha, |p)$ are also proper pairs.

For a proper pair (α, p) , of length m and dimension n , we define $[\alpha, p] \in Y_{n+m-1}$, and $\rho(\alpha, p) : \langle \hat{\alpha} \rangle \xrightarrow{\cong} \langle [\alpha, p] \rangle^-$ by recursion on m .

For $m=1$, $[\alpha, p] \operatorname{\bar{\bar{d}}ef} \alpha_0^*$, and $\rho(\alpha, p) = \rho[\alpha] : \langle \alpha_0 \rangle \xrightarrow{\cong} \langle \alpha_0^* \rangle^-$.

Given (α, p) , $m = \mathrm{lh}(\alpha, p)$, $n = \dim(\alpha, p) \ge 2$, consider the truncated pair $(\alpha|, p|)$, $\check{\alpha} = (\alpha|)\hat{}$, and the map $\rho' = \rho(\alpha|, p|) : \langle \check{\alpha} \rangle \to \langle [\alpha|, p|] \rangle^-$. Note that $\hat{p} \in |\langle \check{\alpha} \rangle|$. Let $\dot{p} = \rho'(\hat{p})$. Note also that

$$\mathrm{c}\hat{\alpha} = \langle \check{\alpha} \rangle(\hat{p}) = \langle [\alpha|, p|] \rangle(q) = \langle \mathrm{d}([\alpha|, p|]^+) \rangle(q) \ .$$

We define

$$[\alpha, p] \operatorname{\bar{\bar{d}}ef} [\alpha|, p|]^+ \circ_{\dot{p}} \hat{\alpha} \ .$$

and $\rho(\alpha, p)$ to be the canonical ρ-map $\rho = \rho(\alpha, p) = \rho([\alpha|, p|]^+, \hat{\alpha}, \dot{p}]$,

$$\langle \hat{\alpha} \rangle \xrightarrow[\cong]{\rho} \langle [\alpha, p] \rangle^- \ .$$

(19) Lemma Let $(\alpha^1 \hat{} \beta, p^1 \hat{} q)$ be a proper pair of dimension n and length $m \ge 2$. Assume $\alpha^1 \circ_j \alpha^2 \in S_n$ is well-defined. Then

$$[\alpha^1 \hat{} \beta, p^1 \hat{} q] \circ_{n-1} ((\mathrm{d}\alpha^1)^{*+} \circ_{\rho j} \alpha^2) = [(\alpha^1 \circ_j \alpha^2) \hat{} \beta, (\sigma p^1) \hat{} q] \ ; \ (20)$$

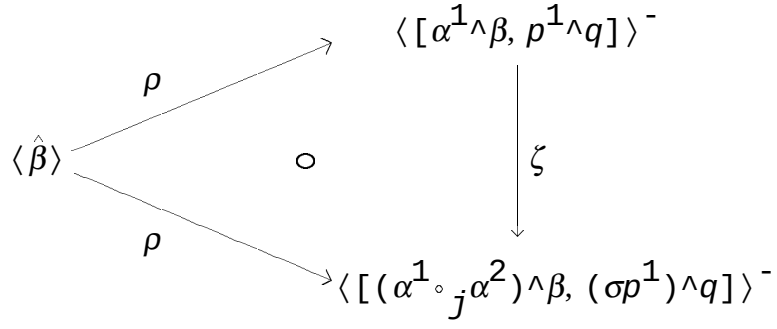
here,

$$\rho=\rho[\mathrm{d}\alpha^1]:\langle \mathrm{d}\alpha^1\rangle \longrightarrow \langle (\mathrm{d}\alpha^1)^*\rangle$$

and

$$\sigma:\langle \alpha^1\rangle \longrightarrow \langle \alpha^1 \circ_j \alpha^2\rangle \ .$$

Moreover, $(\alpha^1 \wedge \beta)\hat{} = ((\alpha^1 \circ_j \alpha^2)\wedge\beta)\hat{} = \hat{\beta}$, and we have



(21)

here, ζ is the canonical ζ-map associated with the equality (20).

Proof The proof is by induction on the length m of $(\alpha^1\wedge\beta, p^1\wedge q)$.

Let us write $\delta = (\mathrm{d}\alpha^1)^{*+} \circ_{\rho j} \alpha^2$. We note the following identity:

$$(\alpha^1)^* \circ_{n-1} \delta = (\alpha^1)^* \circ \overrightarrow{((\mathrm{d}\alpha^1)^{*+} \circ_{\rho j} \alpha^2)} = (\alpha^1 \circ_j \alpha^2)^* \qquad (22)$$

Let first $m=2$. β is a singleton, $\hat{\beta}=\beta$, and q is empty.

The LHS in (20) is

$$[\alpha^1\wedge\beta, p^1]\circ_{n-1}\delta = ([\alpha^1, \emptyset]^+ \circ_{p^1\cdot} \beta)\circ_{n-1}\delta = (\alpha^{1*+} \circ_{p^1\cdot} \beta)\circ_{n-1}\delta$$
$$= (\alpha^{1*}\circ_{n-1}\delta)^+ \circ_{\zeta(p^1\cdot)} \beta \ . \qquad (23)$$

The RHS is

$$[(\alpha^1 \circ_j \alpha^2)\wedge\beta,\ \sigma p^1] = [\alpha^1 \circ_j \alpha^2, \emptyset] \circ_{\sigma(p^1)\cdot} \beta$$
$$= (\alpha^1 \circ_j \alpha^2)^{*+} \circ_{(\sigma p^1)\cdot} \beta \qquad (24)$$

By (22), the first factors in (23) and (24) are equal. Thus, to see that the expressions in (23) and (24) are equal, we need to see that the two subscripts are equal. Let us spell out the ingredients in those subscripts:

$$\begin{array}{ccccc}
\langle \alpha^1 \rangle^- & \xrightarrow{\qquad\qquad\rho\qquad\qquad} & & \langle \alpha^{1*} \rangle^- = \langle (c\alpha^1)^{*+} \circ_* \alpha^1 \rangle^- & \\
\sigma\downarrow & p^1 \longmapsto & p^{1\cdot} & \zeta\downarrow \quad = \quad \sigma\downarrow & \\
 & \sigma p^1 \longmapsto (\sigma p^1)^{\cdot} \overset{!}{=} \xi(p^{1\cdot}) & & \langle ((c\alpha^1)^{*+} \circ_* \alpha^1) \circ_{\rho j} \alpha^2 \rangle^- & \\
 & & & \| & \\
\langle \alpha^1 \circ_j \alpha^2 \rangle^- & \xrightarrow[\rho]{\qquad\qquad} & \langle (\alpha^1 \circ_j \alpha^2)^* \rangle^- = \langle \alpha^{1*} \circ_{n-1} \delta \rangle = & \langle (c\alpha^1)^{*+} \circ_* (\alpha^1 \circ_j \alpha^2) \rangle^- &
\end{array}$$

The identification of ζ with the canonical σ-map on the right is the definition of ζ. The desired equality marked ! is a consequence of the commutativity of the displayed rectangle, which is an instance of "associativity coherence", 1.(1)(vi).

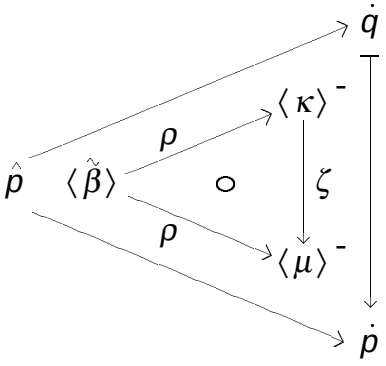
Let $m>2$. Assume the result for $m-1$, and write:

$$\delta = (d\alpha^1)^{*+} \circ_{\rho j} \alpha^2,$$
$$\kappa = [\alpha^1\wedge\beta\,|,\ p^1\wedge q\,|\,],$$
$$\lambda = [\alpha^1\wedge\beta,\ p^1\wedge q],$$
$$\mu = [(\alpha^1 \circ_j \alpha^2)\wedge\beta\,|,\ (\sigma p^1)\wedge q\,|\,],$$
$$\nu = [(\alpha^1 \circ_j \alpha^2)\wedge\beta,\ (\sigma p^1)\wedge q],$$

We have

$$\overset{1}{\lambda} = \kappa^+ \circ_{\dot{q}} \hat{\beta}, \qquad \overset{2}{\mu} = \kappa \circ_{n-1} \delta, \qquad \overset{3}{\nu} = \mu^+ \circ_{\dot{p}} \hat{\beta} \qquad (25)$$

with $\dot{q}$, $\dot{p}$ defined by the following diagram:



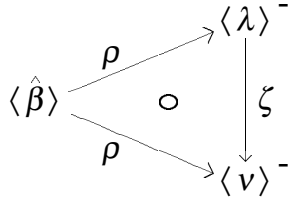
(26)

In (25), `1` and `3` are instances of the definition of the symbol `[-, -]`; `2` holds by induction hypothesis (for the length m`-1`).

We have

$$\kappa \circ_{n-1} \delta = (\lambda^+ \circ_{\dot{q}} \hat{\beta}) \circ_{n-1} \delta \overset{*}{=} (\lambda \circ_{n-1} \delta)^+ \circ_{\zeta\dot{q}} \hat{\beta} = \mu^+ \circ_{\zeta\dot{q}} \hat{\beta} = \mu^+ \circ_{\dot{p}} \hat{\beta} = \nu ;$$

at $*$ we employed a special case of 4.(28); this is the required equality (20). The commutativity (21) to be proved,



is an instance of the commutativity in 4.(28).

The proof of the following is analogous to that of (19).

(27) Lemma Assume that $\alpha^1 \in S_n$, and $[d\alpha^1 \wedge \alpha^2 \wedge \beta,\ j \wedge p^2 \wedge q]$ is a proper pair of dimension n`-1` . Then, the expressions in the following identity are well-defined, and the identity is true:

$$\alpha^{1*} \circ_{n-1} [d\alpha^1 \wedge \alpha^2 \wedge \beta,\ j \wedge p^2 \wedge q] = [(\alpha^1 \circ_j \alpha^2) \wedge \beta,\ (\rho p^2) \wedge q] \ .$$

The following lemma is a generalization of $(18)_n$.

$(28)_n$ Lemma Let (α, p) be a proper pair of dimension n and length m. We assume that $(18)_{n-1}$ is true, and thus, η is well-defined on the n-truncation of $\boldsymbol{R}$.

If $m=1$, then

$$[\![\eta\alpha_0]\!] \ = \ [\alpha, p] = (\alpha_0)^* \ , \qquad (29)$$

(this is just $(18)_n$) ; and if $m \geq 2$,

$$[\![\eta\alpha_0 \square_{\eta p_0} [\,|\alpha, \,|p]]\!] \ = \ [\alpha, p] \ . \qquad (30)$$

including the fact that the $\square$-composite on the left-hand side of (30) is well-defined.

Note that $[\,|\alpha, \,|p]$ is a cell in the ω-category $\boldsymbol{Y}$. The operation $\square_{\eta p_0}$ takes place in the multicategory $\boldsymbol{G}_n = \mathbf{D}_n[\boldsymbol{N}_n]$; the expression $\eta\alpha_0 \square_{\eta p_0} [\,|\alpha, \,|p]$ is formed according to the notation $\alpha \square_p b$ in 2.(19); now, $\boldsymbol{X}$ is $\boldsymbol{Y}$.

Proof. The proof is an induction on n ; we fix n , and use $(28)_{n-1}$ to prove $(28)_n$.

Before we get to the induction proper, we observe that when $|\langle \alpha_0 \rangle| = \{p_0\}$ is a singleton, then

$$[\![\eta\alpha_0 \square_{\eta p_0} [\,|\alpha, \,|p]]\!] = [\,|\alpha, \,|p] = [\alpha, p] \ ;$$

the second equality follows from $\alpha_0^{*+} \circ_{\rho p_0} \alpha_1 = \alpha_1^*$, a consequence of $|\langle \alpha_0 \rangle| = \{p_0\}$. Thus, in this case, the required equality is true.

Let $n \geq 0$; we assume that the equalities have been proved for all dimensions ≥ 0 less than n , and for all lengths m . We mostly treat (29) as a degenerate case of (30), inasmuch for $m=1$ (when $|\alpha$ is empty) we take $[\![\eta\alpha_0 \square_{\eta p_0} [\,|\alpha, |p]]\!]$ to be $[\![\eta\alpha_0]\!]$.

Let first $n=0$.

In case $m=1$, both sides of (29) are equal to α_0 .

Let $m \geq 2$. Now ($n=0$), our first observation applies in general, and gives the result.

Assume $n \geq 1$. We perform an induction on $\#\,|\langle \alpha_0 \rangle|$. The minimal value is 1 ; in this case the assertion is true by the initial observation.

Next, let us assume that $\#\,|\langle \alpha_0 \rangle| > 1$. We can write $\alpha_0 = \alpha^1 \circ_j \alpha^2$ with $\#\,|\langle \alpha^\varepsilon \rangle| < \#\,|\langle \alpha_0 \rangle|$ for both $\varepsilon = 1$ and 2 . There are two cases: either $p_0 = \sigma(p^1)$ (**case 1**), or $p_0 = \rho(p^2)$ (**case 2**) for appropriate $p^1 \in |\langle \alpha^1 \rangle|$, $p^2 \in |\langle \alpha^2 \rangle|$, with reference to

$$\begin{array}{ccccc}
|\langle \alpha^1 \rangle| & \xrightarrow{\ \sigma\ } & |\langle \alpha_0 \rangle| & \xleftarrow{\ \rho\ } & |\langle \alpha^2 \rangle| \\
\eta \downarrow \cong & \circ & \eta \downarrow \cong & \circ & \eta \downarrow \cong \\
|\langle \eta\alpha^1 \rangle| & \xrightarrow[\tilde{\sigma}]{} & |\langle \eta\alpha_0 \rangle| & \xleftarrow[\tilde{\rho}]{} & |\langle \eta\alpha^2 \rangle|
\end{array}$$

Case 1: $[\![\eta\alpha_0 \square_{\eta p_0} [\,|\alpha, |p]]\!]$

$$\begin{aligned}
&\overset{1}{=} [\![(\eta\alpha^1 \square_{\eta p^1} [\,|\alpha, |p]) \circ_{\eta j} \eta\alpha^2]\!] \\
&\overset{2}{=} [\![\eta\alpha^1 \square_{\eta p^1} [\,|\alpha, |p]]\!] \circ_{n-1} [\![\mathrm{d}(\eta\alpha^1 \square_{\eta p^1} [\,|\alpha, |p]) \square_{\eta j} (\alpha^2)^*]\!] \\
&\overset{3}{=} [\alpha^1 \wedge |\alpha, p^1 \wedge |p] \circ_{n-1} [\![\eta \mathrm{d}\alpha^1 \square_{\eta j} (\alpha^2)^*]\!] \\
&\overset{4}{=} [\alpha^1 \wedge |\alpha, p^1 \wedge |p] \circ_{n-1} ((\mathrm{d}\alpha^1)^{*+} \circ_{\rho j} \alpha^2) \\
\overset{5}{=}\ & [(\alpha^1 \circ_j \alpha^2) \wedge |\alpha, p_0 \wedge |p] = [\alpha_0 \wedge |\alpha, p_0 \wedge |p] = [\alpha, p] \ .
\end{aligned}$$

Here, 1 : 1.(46); 2 : 2.(44);
3 : we are using two things:

the induction hypothesis for the shorter α^1 in place of α ; and

the equality $\mathrm{d}(\eta\alpha^1 \square_{\eta p^1} [\,|\,\alpha,\ |\,p]) = \mathrm{d}(\eta\alpha^1) = \eta(\mathrm{d}\alpha^1)$; for the second equality, note that $(28)_{n-1}$, in particular $(18)_{n-1}$ being assumed, η is well-defined on the n-truncation of $\boldsymbol{R}$; said equality is part of $\mathrm{d}:\boldsymbol{R}_n \to \boldsymbol{M}_n$ being a map of ms's.
4 : $(28)_{n-1}$ for length $= 2$.
5 : (19).

The equality $[\![\eta\alpha_0]\!] = \alpha_0^*$ is proved, in Case 1, by a similar sequence of equalities, which is obtained essentially by deleting $[\,|\,\alpha,\ |\,p]$ and superfluous $\square$-symbols from the sequence before.

Case 2: $[\![\eta\alpha_0 \square_{\eta p_0} [\,|\,\alpha,\ |\,p]]\!]$

$$\begin{aligned}
&\overset{1}{=} [\![\eta\alpha^1 \circ_{\eta j} (\eta\alpha^2 \square_{\eta p^2} [\,|\,\alpha,\ |\,p])]\!] \\
&\overset{2}{=} [\![\eta\alpha^1]\!] \circ_{n-1} [\![\eta\mathrm{d}\alpha^1 \square_{\eta j} [\![\eta\alpha^2 \square_{\eta p^2} [\,|\,\alpha,\ |\,p]]\!] \\
&\overset{3}{=} [\![\eta\alpha^1]\!] \circ_{n-1} [\![\eta\mathrm{d}\alpha^1 \square_{\eta j} [\alpha^2\hat{}\,|\,\alpha,\ p^2\hat{}\,|\,p]]\!] \\
&\overset{4}{=} \alpha^{1*} \circ_{n-1} [\mathrm{d}\alpha^1\hat{}\alpha^2\hat{}\,|\,\alpha,\ j\hat{}p^2\hat{}\,|\,p] \\
&\overset{5}{=} [(\alpha^1 \circ_j \alpha^2)\hat{}\,|\,\alpha,\ p_0\hat{}\,|\,p] = [\alpha_0\hat{}\,|\,\alpha,\ p_0\hat{}\,|\,p] = [\alpha,\ p] \ .
\end{aligned}$$

3 : induction hypothesis for α^2 ;
4 : $(28)_{n-1}$ (with length $m+1$ (!))
5 : (27)

Once again, the proof in Case 2 of $[\![\eta\alpha_0]\!] = \alpha_0^*$ is a reduced form of the proof just gone through.

This completes the definition of $\eta : \boldsymbol{R} \to \boldsymbol{M}$.

Let's turn to the naturality of $\eta : [\langle - \rangle] \longrightarrow \mathrm{Id}$. With an arbitrary $f : \boldsymbol{R}^1 \longrightarrow \boldsymbol{R}^2$ in $\mathtt{mltSet}$, we want

$$\begin{array}{ccc} \boldsymbol{R}^1 & \xrightarrow{\ \eta\ } & [\langle \boldsymbol{R}^1 \rangle] \\ {\scriptstyle f}\downarrow & \circ & \downarrow{\scriptstyle [\langle f \rangle]} \\ \boldsymbol{R}^2 & \xrightarrow{\ \eta\ } & [\langle \boldsymbol{R}^2 \rangle] \end{array} \tag{31}$$

Recall $f^+ : \boldsymbol{R}^{1+} \longrightarrow \boldsymbol{R}^{2+}$ from the end of section 4. We leave it to the reader to check the equalities $f^+(\gamma^+) = (f^+(\gamma))^+$ for any $\gamma \in S_n^{1+}$, and

$$f^+(\alpha^*) = (f\alpha)^* \quad \text{for any } \alpha \in S_n^1 . \tag{32}$$

It is enough to show (31) for $n \in \mathbb{N}$ and n-dimensional $\boldsymbol{R}^1$, $\boldsymbol{R}^2$, by induction on n. The commutativity (31) on R_n^1 is verified by using the definition of η in (14'), the definition of $\Phi = [\langle f \rangle]$ from $\Gamma = \langle f \rangle$ in 2.(41'), the commutativity (31) on the $(n\text{-}1)$-truncations, and (32).