The counit map

Let us fix the ω-category $\boldsymbol{X}$. By §2, we construct $\boldsymbol{M} = [\boldsymbol{X}]$. Then, by §4, for $\boldsymbol{R} \overset{\text{def}}{=} \boldsymbol{M}$, we construct $\boldsymbol{Y} = \langle \boldsymbol{M} \rangle = \langle [\boldsymbol{X}] \rangle$. In this subsection, we will construct the *counit* map $\varepsilon_{\boldsymbol{X}} : \boldsymbol{Y} \to \boldsymbol{X}$.

This means defining $\varepsilon_{\boldsymbol{X}}(\alpha) \in X_n$ for any $n \in \mathbb{N}$, $\alpha \in Y_n$. We will write $[\![\alpha]\!]$ for $\varepsilon_{\boldsymbol{X}}(\alpha)$. Remember that we already have $[\![\alpha]\!] \in X_n$ for $\alpha \in P_n$; where the two notations may conflict, on $Y_0 = P_0$ and $Y_1 = P_1$, they will have the same meaning.

It turns out that, for the induction used in proving the requisite properties of the map $\alpha \mapsto [\![\alpha]\!]$, it is convenient to extend the definition of this map. For each n, we will define a set $\hat{Y}_n$ extending Y_n: $Y_n \subset \hat{Y}_n$; and we will define a map $\alpha \mapsto [\![\alpha]\!] : \hat{Y}_n \longrightarrow X_{\geq n}$.

We need to "update" some definitions of section 4, in our context when $\boldsymbol{R} = \boldsymbol{M} = [\boldsymbol{X}]$; we mention $\boldsymbol{M}$, and suppress the letter $\boldsymbol{R}$.

We have that $R_n = M_n$ and $S_n = P_n$; we will not need the notations R_n, S_n. $\boldsymbol{R}^+$ is defined as it was before: $\boldsymbol{R}^+ = \boldsymbol{M}^+$.

We have, for each n, the $\leq n$-truncated ms $\boldsymbol{N}_n$; we can construct $\boldsymbol{N}_n^+$ by the prescription in §4. However, we can and do make sure that all the *indeterminates* in $\boldsymbol{N}_n^+$ (the cells in $\boldsymbol{N}_n^+$ but not in $\boldsymbol{N}_n$) of dimension $\leq n$ are already in $\boldsymbol{M}^+$. (As a matter of fact, we do not use indeterminates of dimension higher than n in $\boldsymbol{N}_n^+$ at all). We can say equivalently that $\boldsymbol{M}_n^+$ defined as the n-truncation of $\boldsymbol{M}^+$ is a sub-ms of $\boldsymbol{N}_n^+$.

We have the set Q_n^+, the set of n-pd's of $\boldsymbol{N}_n^+$. P_n^+ is a subset of Q_n^+; the $(n\text{-}1)$-truncation of both P_n^+ and Q_n^+ is P_{n-1}^+. The difference of P_n^+ and Q_n^+ is that, for $\alpha \in Q_n^+$, the $\boldsymbol{X}$-dimension of any $\langle \alpha \rangle (p)$ may be any $m \geq n$, rather than just n itself.

We are going to use section 3, part 3, in particular, the Z's, and also the $\ulcorner - \urcorner$-construction, in the multitopic set $\boldsymbol{N}_n^+$; recall that in §4, we used the Z's in $\boldsymbol{R}^+$. We will write Z_n^+ for $Z_n(\boldsymbol{N}_n^+)$ (*not* for $Z_n(\boldsymbol{R}^+)$).

We define the set $\tilde{Y}_n$. $\tilde{Y}_n$ will be a subset of $Z_n^+ \subset Q_n^+$. The definition of $\tilde{Y}_n$ is like that of Y_n in 4.(2), except that in 4.(2)(iv), now we require $\vec{\alpha} \in \overrightarrow{Q_n}$ instead of $\vec{\alpha} \in \overrightarrow{P_n}$ (remember that "$S=P$"). In other words, for $\alpha \in Q_n^+$, we have $\alpha \in \tilde{Y}_n$ iff

(2̃) **(i)** α is positive and $\alpha = \mathrm{P}\alpha \circ \vec{\alpha}$;

(ii) $\mathrm{P}\alpha$ is an indeterminate, $\mathrm{cP}\alpha$ is an indeterminate;

(iii) $\mathrm{dP}\alpha \in Y_{n-1}$;

(iv) $\vec{\alpha} \in \overrightarrow{Q_n}$.

The terms used in the definition are to be understood in the context of the ms $\boldsymbol{N}_n^+$, rather than in the ms $\boldsymbol{M}^+ = \boldsymbol{R}^+$ as for Y_n.
Note that Y_n is a subset of $\tilde{Y}_n$, which in turn is a subset of Z_n^+. Note also that the definition is not "inductive", since in (2)(iii), we do not say "$\mathrm{dP}\alpha \in \tilde{Y}_{n-1}$", we just say "$\mathrm{dP}\alpha \in Y_{n-1}$". In fact, since for any $\beta \in Q_n^+$, we have that $\mathrm{d}\beta \in P_{n-1}^+$, and $\tilde{Y}_{n-1} \cap P_{n-1}^+ = Y_{n-1}$, $\mathrm{dP}\alpha \in \tilde{Y}_{n-1}$ iff $\mathrm{dP}\alpha \in Y_{n-1}$ anyway.

We have, as in §4, the following facts.

(3̃) Suppose $\alpha \in \tilde{Y}_n$, Φ is a polyarrow in $\boldsymbol{N}_n$, and $\alpha \circ_\kappa \Phi$ is well-defined. Then $\alpha \circ_\kappa \Phi \in \tilde{Y}_n$.

(4̃) If $\alpha \in \tilde{Y}_n$, Φ is a polyarrow in $\boldsymbol{N}_n$, and $\alpha \square_\lambda \Phi$ is well-defined, then $\alpha \square_\lambda \Phi$

is in $\tilde{Y}_n$ as well.

(5) For $\alpha\in\tilde{Y}_n$, $n>0$, we have $\mathrm{d}\alpha\in Y_{n-1}$.

We extend most, although not all, of the structure of the ω-category $\boldsymbol{Y}$ to the totality of the "cells" in $\bigcup_{n\in\mathbb{N}}\tilde{Y}_n$. In other words, we define a structure $\tilde{\boldsymbol{Y}}$, extending $\boldsymbol{Y}$, which is almost, but not quite, an ω-category (it is a *quasi-ω-category*), and whose n-cells are the elements of $\tilde{Y}_n$. The one thing that will be missing for $\tilde{\boldsymbol{Y}}$ to be an honest ω-category is the existence of identity cells. Since we do not actually need all the properties of $\tilde{\boldsymbol{Y}}$ as a quasi-ω-category, we refrain from formulating the exact concept of quasi-ω-category in general.

First of all, for $n>0$, we repeat the definitions of $\hat{\mathrm{d}}:Y_n\to Y_{n-1}$, $\hat{\mathrm{c}}:Y_n\to Y_{n-1}$ verbatim to get $\hat{\mathrm{d}}:\tilde{Y}_n\to Y_{n-1}$, $\hat{\mathrm{c}}:\tilde{Y}_n\to Y_{n-1}$ (note that there are no tildes on the codomain Y's). In particular, $\hat{\mathrm{d}}\alpha\underline{\underline{\mathrm{def}}}\mathrm{d}\alpha$, with d on the right-hand side understood as the map $\mathrm{d}:Q_n^+\to P_{n-1}$ (coming from the multitopic set $\boldsymbol{F}_n$).

By (5), $\alpha\in\tilde{Y}_n$ implies $\hat{\mathrm{d}}\alpha\in Y_{n-1}$.

The definition of $\hat{\mathrm{c}}\alpha$ as $\hat{\mathrm{c}}\alpha=\mathrm{dP}\alpha$, for $n\geq 2$, is "more innocent", since $\mathrm{P}\alpha\in P_n^+$. The fact that $\alpha\in\tilde{Y}_n$ implies $\hat{\mathrm{c}}\alpha\in Y_{n-1}$ is clear,

Next, we extend the composition operations $(\alpha,\beta)\mapsto\alpha\circ_k\beta$, for $0\leq k\leq n-1$, to arguments $\alpha,\beta\in\tilde{Y}_n$ subject to $\hat{\mathrm{c}}\beta=\hat{\mathrm{d}}\alpha$. The formalism of the definition does not change from section 4. We have $\alpha\circ_k\beta\in\tilde{Y}_n$. Also, we define the $(\xi.\zeta)$-maps as before. Lemma 4.(16) remains true, with Y_n replaced with $\tilde{Y}_n$, and the condition $\Phi\in\overrightarrow{P_n}$ [remember that "$S=P$" now] replaced with $\Phi\in\overrightarrow{Q_n}$.

The "domain/codomain" laws remain true.

As a matter of fact, the associativity and middle-exchange laws remain true for $\tilde{\tilde{Y}}$ too, but we don't need this fact.

We use an auxiliary construction in $\boldsymbol{M}$: $(\text{-})^{@}: P_n \longrightarrow M_{n+1}$ (to which we'll apply section 3, part 3, where the operation $(\text{-})^{@}$ was an essentially arbitrary parameter). For $\gamma \in P_n$,

$$\gamma^{@} \;\bar{\bar{\text{def}}}\; (\text{id}_{[\![\gamma]\!]}, \gamma, \|\gamma\|) \in M_{n+1} \qquad (6)$$

See §2.[3](1) and [4] for M_{n+1}, [10] and [3](1) for $\|\text{-}\|: P_n \longrightarrow M_n$. [4](a) for $\gamma^{@}$ is satisfied by [10]; [4](b) is clear; thus, $\gamma^{@} \in M_{n+1}$. Let us record:

$$[\![\gamma^{@}]\!] = \text{id}_{[\![\gamma]\!]}\,,\quad \text{d}\gamma^{@} = \gamma\,,\quad \text{c}\gamma^{@} = \|\gamma\|\,.$$

As in part 3 of section 3 (taking the $\boldsymbol{R}$ of part 3 of section 3 to be $\boldsymbol{N}_n^{+}$), we define, for all $\alpha \in Z_n^{+}$ (in particular, for all $\alpha \in \tilde{\tilde{Y}}_n$, and even more particularly, for all $\alpha \in Y_n$), the pd $\ulcorner \alpha \urcorner \in Q_n^{+}$, together with an isomorphism $\theta_\alpha : \langle \alpha \rangle^{-} \xrightarrow{\cong} \langle \ulcorner \alpha \urcorner \rangle^{-}$ in $(N_n)^{\#}$. We note that, for $\alpha \in \tilde{\tilde{Y}}_n$, we have $\ulcorner \alpha \urcorner \in Q_n$ (no indeterminates). Finally, we put, for $\alpha \in \tilde{\tilde{Y}}_n$,

$$[\![\alpha]\!] \;\bar{\bar{\text{def}}}\; [\![\ulcorner \alpha \urcorner]\!]\,.$$

The following fact (7) shows that we could have made the definition of $[\![\alpha]\!]$ for $\alpha \in \tilde{\tilde{Y}}_n$ more direct, in particular, without the $(\text{-})^{@}$-, $\ulcorner\text{-}\urcorner$-constructions. However, we do need the original definition directly; the alternative (recursive) definition in (7) is not sufficient (at least, not in the present treatment).

(7) For $\alpha \in \tilde{\tilde{Y}}_n$, $\quad [\![\alpha]\!]_{(n)} = [\![\hat{\text{c}}\alpha \,\square\, \text{B}\|\vec{\alpha}\|]\!]_{(n-1)}$.

[Some reminders on the notation. On the left , we have $[\![-]\!]_{(n)} : \tilde{Y}_n \longrightarrow X_{\geq n}$; on the right, we have $[\![-]\!]_{(n-1)} : Y_{n-1} \longrightarrow X_{\geq n-1}$. $\hat{c}\alpha$ belongs to $Y_{n-1} \subset P^+_{n-1} = \mathsf{P}_{n-1}(\boldsymbol{M}^+)$. On the right, $\| - \| : Q_n \longrightarrow N_n$, and $\mathsf{B} : N_n \longrightarrow N_{n-1}$; both $\| - \|$ and B are being applied pointwise to tuples of arguments in their original domains, resulting in tuples of values in their original codomains. The $\square$-composition on the right takes place in the multicategory $\boldsymbol{G}_{n-1} = \mathbf{D}_{n-1}(\boldsymbol{N}_{n-1})$.]

Proof of (7) With $\theta = \theta_{\mathsf{dP}\alpha}$, by 3.(38),

$$[\![\alpha]\!] = [\![\ulcorner \alpha \urcorner]\!] = [\![\ulcorner \mathsf{P}\alpha \urcorner \circ_\theta \vec{\alpha}]\!] = [\![\ulcorner \mathsf{P}\alpha \urcorner]\!] \circ_{n-1} [\![\mathsf{d} \ulcorner \mathsf{P}\alpha \urcorner \square_\theta \mathsf{B} \| \vec{\alpha} \|]\!] \quad .$$

By the definition of $\ulcorner \mathsf{P}\alpha \urcorner$, $[\![\ulcorner \mathsf{P}\alpha \urcorner]\!]$ is an identity n-cell in $\boldsymbol{X}$; thus,

$$[\![\alpha]\!] \;=\; [\![\mathsf{d} \ulcorner \mathsf{P}\alpha \urcorner \square_\theta \mathsf{B} \| \vec{\alpha} \|]\!] \quad . \tag{8}$$

On the other hand, by 3.(25'),

$$\ulcorner \hat{c}\alpha \square \mathsf{B} \| \vec{\alpha} \| \urcorner \;=\; \ulcorner \mathsf{dP}\alpha \square \mathsf{B} \| \vec{\alpha} \| \urcorner \;=\; \ulcorner \mathsf{dP}\alpha \urcorner \square_\theta \mathsf{B} \| \vec{\alpha} \|$$

and so

$$[\![\hat{c}\alpha \square \mathsf{B} \| \vec{\alpha} \|]\!] = [\![\ulcorner \hat{c}\alpha \square \mathsf{B} \| \vec{\alpha} \| \urcorner]\!] = [\![\ulcorner \mathsf{dP}\alpha \urcorner \square_\theta \mathsf{B} \| \vec{\alpha} \|]\!] \tag{9}$$

(8), (9) and 3.(26) show (7).

Note that in proving (7), we have also shown, as an intermediate result, the following fact.

(7') For $\alpha \in \tilde{Y}_n$, $\quad [\![\alpha]\!]_{(n)} \;=\; [\![\ulcorner \hat{c}\alpha \urcorner \square_{\theta_{\hat{c}\alpha}} \mathsf{B} \| \vec{\alpha} \|]\!]_{(n-1)}$.

We note that, from the given arbitrary ω-category $\boldsymbol{X}$, we can construct a simple-minded "quasi-ω-category" $\dot{\boldsymbol{X}}$, an extension of $\boldsymbol{X}$, whose n-cells are all the $\geq n$-cells of $\boldsymbol{X}$. More precisely, the n-cells of $\dot{\boldsymbol{X}}$ are pairs (n, u) where u is an m-cell for some $m \geq n$. We put $\mathsf{d}_{(\dot{\boldsymbol{X}})}(n, u) = \mathsf{d}^{(n-1)}u$, and similarly for "codomain". Composition is as in $\boldsymbol{X}$: for $k<n$, (n, u) and (n, v) are composable, in the sense of $(n, u) \circ_k^{(\dot{\boldsymbol{X}})} (n, v)$ being defined, if and only if $u \circ_k^{(\boldsymbol{X})} v$ is defined, and the former is equal to the latter with the marker n added as the first component of a pair.

Now we can show that $[\![-]\!] : \tilde{\tilde{\boldsymbol{Y}}} \longrightarrow \dot{\boldsymbol{X}}$ (hence, also $[\![-]\!] : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$) preserves "domain" and "codomain": for $\alpha \in \tilde{\tilde{Y}}_n$,

$$[\![\hat{\mathsf{d}}\alpha]\!] \stackrel{?}{=} \mathsf{d}^{(n-1)}[\![\alpha]\!] , \qquad [\![\hat{\mathsf{c}}\alpha]\!] \stackrel{?}{=} \mathsf{c}^{(n-1)}[\![\alpha]\!] .$$

Let $\alpha \in \tilde{\tilde{Y}}_n$. We may assume $n \geq 2$. We have, since $\ulcorner \alpha \urcorner \in Q_n$, that $\mathsf{d}^{(n-1)}[\![\ulcorner \alpha \urcorner]\!] = [\![\mathsf{d}\ulcorner \alpha \urcorner]\!]$ (2.[10](1'), 2.[4](b')). Now,

$$[\![\hat{\mathsf{d}}\alpha]\!] = [\![\mathsf{d}\alpha]\!] = [\![\ulcorner \mathsf{d}\alpha \urcorner]\!] \underset{\uparrow}{=} [\![\mathsf{d}\ulcorner \alpha \urcorner]\!] = \mathsf{d}[\![\ulcorner \alpha \urcorner]\!] = \mathsf{d}^{(n-1)}[\![\alpha]\!] ,$$

$\uparrow$ by 3.(26).

On the other hand, writing $\gamma = \mathsf{d}\mathsf{P}\alpha$, and using $\ulcorner \alpha \urcorner = (\ulcorner \gamma \urcorner)^{@} \circ \theta_\alpha \vec{\alpha}$, we have

$$[\![\hat{\mathsf{c}}\alpha]\!] = [\![\gamma]\!] = [\![\ulcorner \gamma \urcorner]\!] = [\![\mathsf{c}(\ulcorner \gamma \urcorner)^{@}]\!] = [\![\mathsf{c}\ulcorner \alpha \urcorner]\!] = \mathsf{c}^{(n-1)}[\![\ulcorner \alpha \urcorner]\!] = \mathsf{c}^{(n-1)}[\![\alpha]\!] .$$

Next we prove that $[\![-]\!] : \tilde{\tilde{\boldsymbol{Y}}} \to \dot{\boldsymbol{X}}$ preserves composition, in particular that $[\![-]\!] : \boldsymbol{Y} \to \boldsymbol{X}$ preserves composition. (Once again, the inductive proof requires treating $[\![-]\!] : \tilde{\tilde{\boldsymbol{Y}}} \to \dot{\boldsymbol{X}}$, even if we just want $[\![-]\!] : \boldsymbol{Y} \to \boldsymbol{X}$.) That is, for $\alpha, \beta \in \tilde{\tilde{Y}}_n$, $k<n$ such that $\alpha \circ_k \beta$ is well-defined, we want to show that

$$[\![\alpha \circ_k \beta]\!] \stackrel{?}{=} [\![\alpha]\!] \circ_k [\![\beta]\!] \ . \tag{10}$$

The proof is an induction on n .

Case 1: $k=n-1$. Now,

$$[\![\alpha \circ_{n-1} \beta]\!] = [\![\alpha \circ \vec{\beta}]\!] = [\![\ulcorner \alpha \circ \vec{\beta} \urcorner]\!]$$

By 3.(25), we have $\ulcorner \alpha \circ \vec{\beta} \urcorner = \ulcorner \alpha \urcorner \circ_\lambda \vec{\beta}$, where $\lambda=\varphi' \varphi^{-1}$, with the canonical isomorphisms

$$\varphi : \langle d\vec{\alpha} \rangle \xrightarrow{\cong} \langle d\alpha \rangle^- \ , \ \varphi' : \langle d\vec{\alpha} \rangle \xrightarrow{\cong} \langle d \ulcorner \alpha \urcorner \rangle^- \ .$$

Thus, by 3.(38) applied to $\ulcorner \alpha \urcorner \circ_\lambda \vec{\beta}$, and by $[\![\alpha]\!] = [\![\ulcorner \alpha \urcorner]\!]$,

$$[\![\alpha \circ_{n-1} \beta]\!] = [\![\alpha]\!] \circ_{n-1} [\![d \ulcorner \alpha \urcorner \square_\lambda B \| \vec{\beta} \|]\!] \ . \tag{11}$$

On the other hand, with $\theta=\theta_{\hat{c}\beta}$, by (7'),

$$[\![\beta]\!] = [\![\ulcorner \hat{c}\beta \urcorner \square_\theta B \| \vec{\beta} \|]\!] \tag{12}$$

By 3.(26), $\ulcorner \hat{c}\beta \urcorner = \ulcorner \hat{d}\alpha \urcorner = \ulcorner d\alpha \urcorner = d \ulcorner \alpha \urcorner$. By 3.(27), we have $\lambda=\theta$. Thus, (11) and (12) give the desired (10) in Case 1.

Case 2: $k \le n-2$; in particular, $n \ge 2$.

Let $\gamma \stackrel{\text{def}}{=} \hat{c}\alpha \circ_k \hat{c}\beta$, $\theta = \theta_\gamma$. Then $\alpha \circ_k \beta = \gamma^+ \circ (\vec{\alpha} \wedge \vec{\beta})$, and

$$\ulcorner \alpha \circ_k \beta \urcorner = \ulcorner \gamma^+ \urcorner \circ_\theta (\vec{\alpha} \wedge \vec{\beta}) \ .$$

In what follows, we abbreviate $B \| \Lambda \|$ by $B\Lambda$. Thus, by 3.(38),

$$[\![\alpha \circ_k \beta]\!] = [\![\ulcorner \alpha \circ_k \beta \urcorner]\!] = [\![\ulcorner \gamma^+ \urcorner]\!] \circ_{n-1} [\![\mathtt{d} \ulcorner \gamma^+ \urcorner \,\square_\theta \mathtt{B}(\vec{\alpha} \wedge \vec{\beta})]\!]_{(n-1)} .$$

$[\![\ulcorner \gamma^+ \urcorner]\!]$ is an identity n-cell, so it can be ignored. $\mathtt{d} \ulcorner \gamma^+ \urcorner = \ulcorner \mathtt{d} \gamma^+ \urcorner = \ulcorner \gamma \urcorner$. Of course, $\mathtt{B}(\vec{\alpha} \wedge \vec{\beta}) = \mathtt{B}\vec{\alpha} \wedge \mathtt{B}\vec{\beta}$. By 3.(25'), $\ulcorner \gamma \urcorner \,\square_\theta \mathtt{B}(\vec{\alpha} \wedge \vec{\beta}) = \ulcorner \gamma \square \mathtt{B}(\vec{\alpha} \wedge \vec{\beta}) \urcorner$. Thus, 1 holds in what follows:

$$\begin{aligned} [\![\alpha \circ_k \beta]\!]_{(n)} &\overset{1}{=} [\![(\hat{\mathtt{c}}\alpha \circ_k \hat{\mathtt{c}}\beta) \square (\mathtt{B}\vec{\alpha} \wedge \mathtt{B}\vec{\beta})]\!]_{(n-1)} \overset{2}{=} [\![(\hat{\mathtt{c}}\alpha \square \mathtt{B}\vec{\alpha}) \circ_k (\hat{\mathtt{c}}\beta \square \mathtt{B}\vec{\beta})]\!]_{(n-1)} \\ &\overset{3}{=} [\![(\hat{\mathtt{c}}\alpha \square \mathtt{B}\vec{\alpha})]\!]_{(n-1)} \circ_k [\![(\hat{\mathtt{c}}\beta \square \mathtt{B}\vec{\beta})]\!]_{(n-1)} \\ &\overset{4}{=} [\![\alpha]\!]_{(n)} \circ_k [\![\beta]\!]_{(n)} ; \end{aligned}$$

Here, 2 is by (the generalized version of) 4.(16), 3 by the induction hypothesis, and 4 by (7).

This completes the proof of (10).

Let's see that $\varepsilon_{\boldsymbol{X}} : \boldsymbol{Y} \to \boldsymbol{X}$ preserves identities. Let $\gamma \in Y_{n-1}$;

$$[\![\mathtt{id}_\gamma]\!] = [\![\gamma^+]\!] \overset{1}{=} [\![\hat{\mathtt{c}}(\gamma^+) \square \mathtt{B}\overrightarrow{\gamma^+}]\!] = [\![\gamma \square \mathtt{B}\overrightarrow{\gamma^+}]\!] \overset{2}{=} \mathtt{id}_{[\![\gamma]\!]} ;$$

1 by (7); for 2 , note that all components of the polyarrow $\overrightarrow{\gamma^+}$ are identity n-pd's in $\boldsymbol{M}$; from this, 2 follows, e.g. by an induction based on 3.(36).

We have completed the definition of the counit map $\varepsilon_{\boldsymbol{X}} : \boldsymbol{Y} \to \boldsymbol{X}$.

The naturality of $\varepsilon : \langle [\text{-}] \rangle \longrightarrow \mathtt{Id}$: given $\Gamma : \boldsymbol{X}^1 \to \boldsymbol{X}^2$, with $\boldsymbol{Y}^i = \langle [\boldsymbol{X}^i] \rangle$, $\Phi = [\Gamma]$, the diagram

$$\begin{array}{ccc} \boldsymbol{Y}^1 & \xrightarrow{\quad\varepsilon\quad} & \boldsymbol{X}^1 \\ {\scriptstyle \langle \Phi \rangle}\downarrow & \circ & \downarrow{\scriptstyle \Gamma} \\ \boldsymbol{Y}^2 & \xrightarrow[\quad\varepsilon\quad]{} & \boldsymbol{X}^2 \end{array}$$

commutes. For $\alpha \in Y_n^1$, we have $\ulcorner \Phi^+ \alpha \urcorner = \Phi \ulcorner \alpha \urcorner$, as easily seen. Then

$$\varepsilon \langle \Phi \rangle \alpha = [\![\Phi^+ \alpha]\!] = [\![\ulcorner \Phi^+ \alpha \urcorner]\!] = [\![\Phi \ulcorner \alpha \urcorner]\!] = \Gamma [\![\ulcorner \alpha \urcorner]\!] = \Gamma [\![\alpha]\!] = \Gamma \varepsilon \alpha$$

as desired.