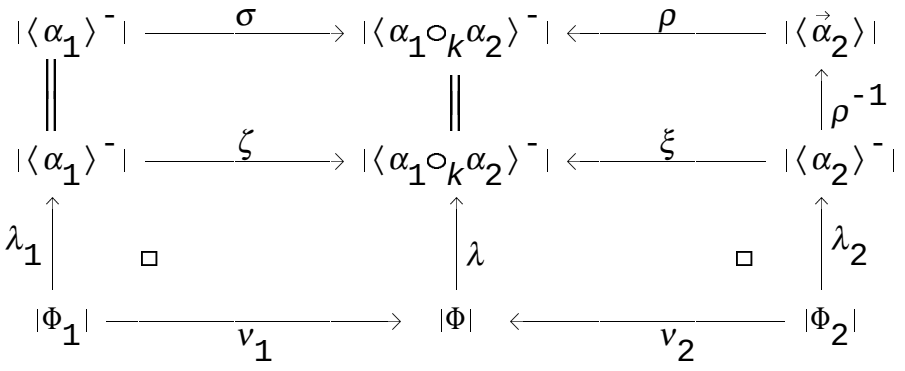


using the way the (σ, ρ)-pair is given in this case. On the other hand, by (6.2),

$$\overrightarrow{\alpha_2 \square_{\lambda_2} \Phi_2} = \vec{\alpha}_2 \square_{\rho^{-1}\lambda_2} \Phi_2 \qquad (16)$$

(here, ρ is the same as in (15)!). (15) and (16) give

$$(\alpha_1 \circ_{n-1} \alpha_2) \square_\lambda \Phi = (\alpha_1 \square_\lambda \Phi_1) \circ \overrightarrow{\alpha_2 \square_{\lambda_2} \Phi_2} = (\alpha_1 \square_\lambda \Phi_1) \circ_{n-1} (\alpha_2 \square_{\lambda_2} \Phi_2)$$

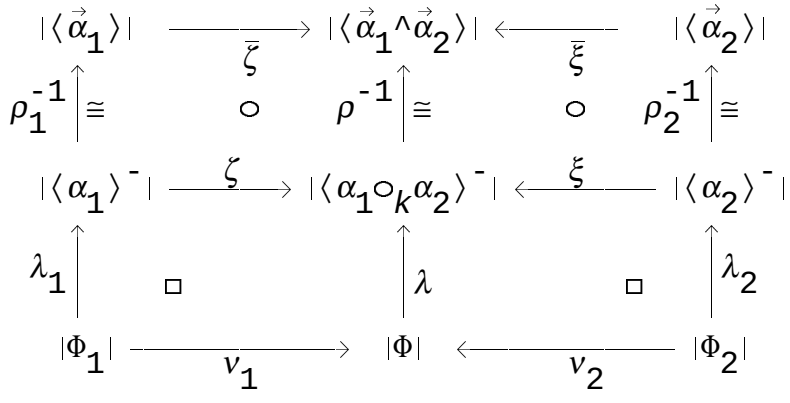
as required.

Next, assume $k \leq n-2$. We have:

$$\begin{aligned}
(\alpha_1 \circ_k \alpha_2) \square_\lambda \Phi \quad & \overset{1}{=} ((\hat{c}\alpha_1 \circ_k \hat{c}\alpha_2)^+ \circ (\vec{\alpha}_1 \wedge \vec{\alpha}_2)) \square_\lambda \Phi \\
& \overset{2}{=} (\hat{c}\alpha_1 \circ_k \hat{c}\alpha_2)^+ \circ ((\vec{\alpha}_1 \wedge \vec{\alpha}_2) \square_{\rho^{-1}\lambda} \Phi) \\
& \overset{3}{=} (\hat{c}\alpha_1 \circ_k \hat{c}\alpha_2)^+ \circ ((\vec{\alpha}_1 \square_{\rho_1^{-1}\lambda_1} \Phi) \wedge (\vec{\alpha}_2 \square_{\rho_2^{-1}\lambda_2} \Phi)) \\
& \overset{4}{=} (\hat{c}\alpha_1 \circ_k \hat{c}\alpha_2)^+ \circ ((\overrightarrow{\alpha_1 \square_{\lambda_1} \Phi}) \wedge (\overrightarrow{\alpha_2 \square_{\lambda_2} \Phi})) \\
& \overset{5}{=} (\alpha_1 \square_{\lambda_1} \Phi_1) \circ_k (\alpha_2 \square_{\lambda_2} \Phi_2)
\end{aligned}$$

Explanations: 1 : definition of $\alpha_1 \circ_k \alpha_2$; 2 : 3.(5). 3 : trivial distributivity (§3., part 1). To

understand 3 , let us put together two diagrams, (12) and (14), to get



This shows that "trivial distributivity" (§3., part 1) can be applied with the composites $\rho_j^{-1}\lambda_j$ as κ_j ; the result is 3 . For 4 : by 3.(19.2),

$$\overrightarrow{\alpha_j \square \lambda_j}\,\Phi \;=\; \vec{\alpha}_j \square_{\rho_j^{-1}\lambda_j}\,\Phi$$

which justifies 4 . 5 is just the definition of $\circ_k$ again.

This completes the proof of (13).

(**The domain/codomain laws**)

The domain/codomain laws: first, the case $k=n-1$.

Since $\mathrm{d}P\beta = \hat{\mathrm{c}}\beta = \hat{\mathrm{d}}\alpha = \mathrm{d}\alpha$, we have

$$\hat{\mathrm{d}}(\alpha \circ_{n-1} \beta) = \mathrm{d}(\alpha \circ_{n-1} \beta) = \mathrm{d}(\alpha \circ \vec{\beta}) = \mathrm{d}\alpha \square \mathrm{d}\vec{\beta}$$
$$= \mathrm{d}P\beta \square \mathrm{d}\vec{\beta} = \mathrm{d}(P\beta \circ \vec{\beta}) = \mathrm{d}\beta = \hat{\mathrm{d}}\beta \,,$$

and

$$\hat{c}(\alpha\circ_{n-1}\beta) = dP(\alpha\circ_{n-1}\beta) = dP(\alpha\circ\vec{\beta}) = dP\alpha = \hat{c}\alpha\ .$$

Next, the case $k\le n-2$:

$$\hat{d}(\alpha\circ_k\beta) = d(\alpha\circ_k\beta) = d((\hat{c}\alpha\circ_k\hat{c}\beta)^+\circ(\vec{\alpha}\wedge\vec{\beta})) = d((\hat{c}\alpha\circ_k\hat{c}\beta)^+)\square d(\vec{\alpha}\wedge\vec{\beta})$$
$$= (\hat{c}\alpha\circ_k\hat{c}\beta)\square d(\vec{\alpha}\wedge\vec{\beta}) = (\hat{c}\alpha\circ_k\hat{c}\beta)\square(d\vec{\alpha}\wedge d\vec{\beta}) \overset{*}{=} (\hat{c}\alpha\square d\vec{\alpha})\circ_k(\hat{c}\beta\square d\vec{\beta})$$
$$= ((dP\alpha)\square d\vec{\alpha})\circ_k((dP\beta)\square d\vec{\beta}) = d(P\alpha\circ\vec{\alpha})\circ_k d(P\beta\circ\vec{\beta}) = \hat{d}\alpha\circ_k\hat{d}\beta\ .$$

At $*$, we applied (13), with the following standing for (14):

$$\begin{array}{ccccc} |\langle\hat{c}\alpha\rangle^-| & \xrightarrow{\ \zeta\ } & |\langle\hat{c}\alpha\circ_k\hat{c}\beta\rangle^-| & \xleftarrow{\ \xi\ } & |\langle\hat{c}\beta\rangle^-| \\ \| & & \| & & \| \\ |d\vec{\alpha}| & \xrightarrow[\ \zeta\ ]{} & |d\vec{\alpha}\wedge d\vec{\beta}| & \xleftarrow[\ \xi\ ]{} & |d\vec{\beta}| \end{array}$$

On the other hand,

$$\hat{c}(\alpha\circ_k\beta) = dP((\hat{c}\alpha\circ_k\hat{c}\beta)^+\circ(\vec{\alpha}\wedge\vec{\beta})) \overset{*}{=} d(\hat{c}\alpha\circ_k\hat{c}\beta)^+ = \hat{c}\alpha\circ_k\hat{c}\beta\ ;$$

at $*$ we used 3.(17).

(**Associativity in Y**)

Suppose $\beta,\ \gamma\in Y_n$, $\beta\circ_{n-1}\gamma = \beta\circ\vec{\gamma}$ is well-defined. As before, $C=C_n(R^+)$, $s=s_C$, $t=t_C$. We have the inclusion

$$t\circ\vec{\gamma} \xrightarrow[\ \iota\ ]{} s(P\gamma)=\langle dP\gamma\rangle=\langle\hat{c}\gamma\rangle=\langle\hat{d}\beta\rangle=\langle d\beta\rangle\ ,$$

and the inclusion misses just one element of $|\langle d\beta\rangle|$, namely $p_{d\beta}$; therefore, we have $t\circ\vec{\gamma} = \langle d\beta\rangle^-$. The canonical isomorphism

$$\langle \mathrm{d}\vec{\beta}\rangle \xrightarrow[\cong]{\varphi} \langle \mathrm{d}\beta\rangle^{-} \qquad (17)$$

of 3.(16) gives

$$\mathrm{t}\circ\vec{\gamma} \xrightarrow[\cong]{\varphi^{-1}} \langle \mathrm{d}\vec{\beta}\rangle\ .$$

Let's write $\lambda=\varphi^{-1}$. This makes $\vec{\beta}\circ_{\lambda}\vec{\gamma}$ well-defined. Moreover, applying the GAL, we get

$$\mathrm{P}\beta\circ(\vec{\beta}\circ_{\lambda}\vec{\gamma}) \;=\; (\mathrm{P}\beta\circ\vec{\beta})\circ_{\bar{\lambda}}\vec{\gamma}$$

where $\bar{\lambda}=\varphi\circ\lambda$, with $\varphi:\mathrm{s}\vec{\beta}\longrightarrow \mathrm{s}(\mathrm{P}\beta\circ\vec{\beta})$ the canonical φ-map, which is

$$\varphi:\langle \mathrm{d}\vec{\beta}\rangle \longrightarrow \langle \mathrm{d}(\mathrm{P}\beta\circ\vec{\beta})\rangle = \langle \mathrm{d}\beta\rangle\ ,$$

the same map as in (1); thus, $\iota=\bar{\lambda}$ is an inclusion. We conclude that

$$\mathrm{P}\beta\circ(\vec{\beta}\circ_{\lambda}\vec{\gamma}) \;=\; \beta\circ\vec{\gamma}\ ,$$

which implies that

$$\mathrm{P}(\beta\circ_{n-1}\gamma) = \mathrm{P}\beta\ , \qquad \overrightarrow{\beta\circ_{n-1}\gamma} = \vec{\beta}\circ_{\lambda}\vec{\gamma} \qquad (18)$$

with $\lambda=\varphi^{-1}$ for φ as in (17).

Let us turn to the proof of associativity: for $\alpha,\ \beta,\ \gamma\in Y_n$,

$$(\alpha\circ_k\beta)\circ_k\gamma \overset{?}{=} \alpha\circ_k(\beta\circ_k\gamma)$$

$$(\ k<n\ ,\ \mathrm{c}^{(k)}\beta=\mathrm{d}^{(k)}\alpha\ ,\ \ \mathrm{c}^{(k)}\gamma=\mathrm{d}^{(k)}\beta\ );$$

Case 1: $k=n-1$. We have

$$(\alpha\circ_{n-1}\beta)\circ_{n-1}\gamma \;=\; (\alpha\circ\vec{\beta})\circ\vec{\gamma} \qquad (19)$$

$$\alpha\circ_{n-1}(\beta\circ_{n-1}\gamma) = \alpha\circ(\overrightarrow{\beta\circ\vec{\gamma}}) = \alpha\circ(\vec{\beta}\circ_{\lambda}\vec{\gamma}) = (\alpha\circ\vec{\beta})\circ_{\bar{\lambda}}\vec{\gamma}$$
$$\uparrow (21)$$

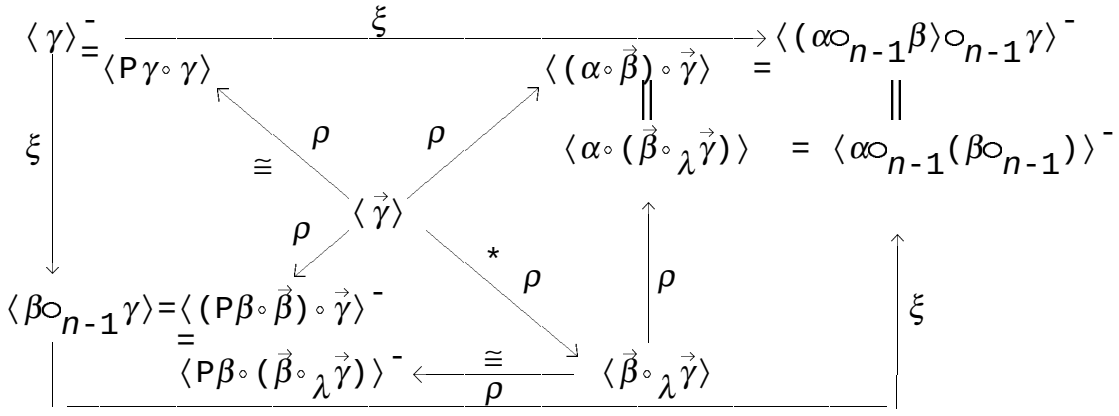
with $\bar{\lambda}=\varphi'\circ\lambda$, for $\varphi':\langle d\vec{\beta}\rangle \longrightarrow \langle d(\alpha\circ\vec{\beta})\rangle$, $\varphi'=\varphi[\alpha,\vec{\beta},\ \imath]_{\boldsymbol{C}_n^+}$. On the other hand, $\lambda=\varphi^{-1}$ with $\langle d\vec{\beta}\rangle \xrightarrow{\ \varphi\ } \langle d(P\beta\circ\vec{\beta})\rangle$, $\varphi=\varphi[P\beta,\vec{\beta},\ \imath]_{\boldsymbol{C}_n^+}$. By beginning part of Part 3 of §3 , $\varphi'=\varphi$, which makes $\bar{\lambda}$ an inclusion;

$$\alpha\circ_{n-1}(\beta\circ_{n-1}\gamma) = (\alpha\circ\vec{\beta})\circ\vec{\gamma}\ . \qquad (20)$$

(19) and (20) give the desired result.

Next, let's see the three coherence conditions for this case ($k=n-1$) of associativity. The conditions relate various ζ's and ξ's. Looking at the definitions of these items, we see that they depend on σ's and ρ's, that is, amalgamating maps for the higher arity structure of $\boldsymbol{C}_n(\boldsymbol{R}^+)$. The latter satisfy three associativity coherence conditions; as expected, the ones for (ζ,ξ) follow from those for (σ,ρ) . We show this for the third condition.

Consider the diagram



Using calculations and notation we just produced, this commutative diagram displays the definitions of the three ξ-arrows involved; and in addition, it has one more arrow, itself a ρ-arrow, the one marked $*$. The four triangles involving ρ's are commutative, by the coherence conditions for the ρ-structure (no σ is involved now). The commutativity of the outside triangle (square?) of ξ's, the desired fact, is a consequence. Thus, this coherence condition for ξ takes four instances of the corresponding condition for ρ.

Case 2: $k \leq n-2$. We have

$$(\alpha \circ_k \beta) \circ_k \gamma = (\hat{c}(\alpha \circ_k \beta) \circ_k \hat{c}\gamma) \circ ((\overrightarrow{\alpha \circ_k \beta}) \hat{} \vec{\gamma}) = ((\hat{c}\alpha \circ_k \hat{c}\beta) \circ_k \hat{c}\gamma) \circ ((\vec{\alpha} \hat{} \vec{\beta}) \hat{} \vec{\gamma})$$

and

$$\alpha \circ_k (\beta \circ_k \gamma) = (\hat{c}\alpha \circ_k \hat{c}(\beta \circ_k \gamma)) \circ (\vec{\alpha} \hat{} (\overrightarrow{\beta \circ_k \gamma})) = (\hat{c}\alpha \circ_k (\hat{c}\beta \circ_k \hat{c}\gamma)) \circ (\vec{\alpha} \hat{} (\vec{\beta} \hat{} \vec{\gamma}))$$

By induction hypothesis (for $n-1$), the first factors of the $\circ$-composites are equal. The second factors are equal too; this is because we have the coherence conditions for the instance of associativity at hand in dimension $n-1$. To see this, let us explicate those coherence conditions, for the general case, involving α, β and γ, in this manner.

Let us abbreviate $|\langle \alpha \rangle^-|$ by $[\alpha]$. $[(\alpha \circ_k \beta) \circ_k \gamma]$ is the coproduct of $[\alpha]$, $[\beta]$ and $[\gamma]$, with coprojections $\zeta \circ \zeta$, $\zeta \circ \xi$ and ξ, where, of course, appropriate instances of the ζ- and ξ-maps are meant. On the other hand, $[\alpha \circ_k (\beta \circ_k \gamma)]$ is also a coproduct of the same $[\alpha]$, $[\beta]$ and $[\gamma]$, with appropriate similar coprojections. The coherence conditions

assert that these two triple coproduct diagrams are the same: not only the coproduct objects are the same, but the respective coprojections are as well.

Now notice that $(\vec{\alpha}\hat{}\vec{\beta})\hat{}\vec{\gamma}$ is a function with domain $[(\hat{c}\alpha \circ_k \hat{c}\beta) \circ_k \hat{c}\gamma]$, which, by definition, when it is precomposed with the three coprojections, gives $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ with respective domains $[\hat{c}\alpha]$, $[\hat{c}\beta]$ and $[\hat{c}\gamma]$ in this order. The analogous statement can be made for $\vec{\alpha}\hat{}(\vec{\beta}\hat{}\vec{\gamma})$. But then, by the above explication for the coherence conditions for the composition of the $\hat{c}\alpha$, $\hat{c}\beta$ and $\hat{c}\gamma$, which are true by induction hypothesis, we conclude $(\vec{\alpha}\hat{}\vec{\beta})\hat{}\vec{\gamma} = \vec{\alpha}\hat{}(\vec{\beta}\hat{}\vec{\gamma})$.

We have proved $(\alpha \circ_k \beta) \circ_k \gamma = \alpha \circ_k (\beta \circ_k \gamma)$.

The associativity coherence conditions for the composition of α, β and γ are, in this case, are an immediate consequence of the corresponding conditions for $\hat{c}\alpha$, $\hat{c}\beta$ and $\hat{c}\gamma$. The reason is in the uniform nature of the definition of the (ζ, ξ)-structure for the composite $\alpha \circ_k \beta$ from that for $\hat{c}\alpha \circ_k \hat{c}\beta$; see (12). Notice that, in (12), $|\langle \vec{\alpha} \rangle| = |\langle \vec{c}\alpha \rangle^{-}| = [\hat{c}\alpha]$, $|\langle \vec{\beta} \rangle| = [\hat{c}\beta]$, $|\langle \vec{\alpha}\hat{}\vec{\beta} \rangle| = [\hat{c}\alpha \circ_k \hat{c}\beta]$; thus, (12) takes the form

$$\begin{array}{ccccc} [\alpha] & \xrightarrow{\ \zeta\ } & [\alpha \circ_k \beta] & \xleftarrow{\ \xi\ } & [\beta] \\ \rho \uparrow \cong & \circ & \rho \uparrow \cong & \circ & \rho \uparrow \cong \\ [\hat{c}\alpha] & \xrightarrow[\ \bar{\zeta}\ ]{} & [\hat{c}\alpha \circ_k \hat{c}\beta] & \xleftarrow[\ \bar{\xi}\ ]{} & [\hat{c}\beta] \end{array} \quad ;$$

moreover, the left-most ρ, say, is $\rho[P\alpha, \vec{\alpha}, \iota]$, something that depends on α only, not how α is put together; similarly for the others. Now it is clear that whatever diagrams for the $(\bar{\zeta}, \bar{\xi})$ commute, the corresponding ones for the (ζ, ξ) will commute too.

(**Middle-exchange in Y**)

This we will do in outline only (for now), and without the coherence conditions; the proof is

similar to the proof of associativity.

Let $\ell<k<n$; $\alpha, \beta, \gamma \in Y_n$; assume

$$\hat{c}(k)\beta=\hat{d}(k)\alpha\ ,\ \hat{c}(k)\delta=\hat{d}(k)\gamma\ ,\ (\hat{c}(\ell)\gamma=)\hat{c}(\ell)\delta=\hat{d}(\ell)\alpha(=\hat{d}(\ell)\beta)\ ,$$

to show $$(\alpha\circ_k\beta)\circ_\ell(\gamma\circ_k\delta) \overset{?}{=} (\alpha\circ_\ell\gamma)\circ_k(\beta\circ_\ell\delta)\ .$$

We have

$$\text{LHS} = (\alpha\circ_k\beta)\circ_\ell(\gamma\circ_k\delta) = (\hat{c}(\alpha\circ_k\beta))\circ_\ell(\hat{c}(\gamma\circ_k\delta))^+\circ((\overrightarrow{\alpha\circ_k\beta})\wedge(\overrightarrow{\gamma\circ_k\delta})) \quad (21)$$

by the definition of $\circ_\ell$. Let us distinguish two cases.

Case 1: $k=n-1$. Now, (21) equals

$$= (\hat{c}\alpha\circ_\ell\hat{c}\gamma)^+\circ(\vec{\alpha}\wedge\vec{\beta})\wedge(\vec{\gamma}\wedge\vec{\delta}) \quad (22;\ \text{Case 1})$$

with suitable, but here unspecified, placings marked ? (the various ?'s are not meant to mean equal placings), by the codomain law, (11) and (18).

Case 2 : $k\leq n-2$. Now, (21) equals

$$((\hat{c}\alpha\circ_k\hat{c}\beta)\circ_\ell(\hat{c}\gamma\circ_k\hat{c}\delta))^+\circ(\vec{\alpha}\wedge\vec{\beta})\wedge(\vec{\gamma}\wedge\vec{\delta})\ . \quad (23;\ \text{Case 2})$$

In Case 1,

$$\text{RHS} = (\alpha\circ_\ell\gamma)\circ_{n-1}(\beta\circ_\ell\delta) = (\alpha\circ_\ell\gamma)\circ(\overrightarrow{\beta\circ_\ell\delta}) =$$
$$((\hat{c}\alpha\circ_\ell\hat{c}\gamma)^+\circ(\vec{\alpha}\wedge\vec{\gamma}))\circ(\vec{\beta}\wedge\vec{\delta})$$
$$= (\hat{c}\alpha\circ_\ell\hat{c}\gamma)^+\circ((\vec{\alpha}\wedge\vec{\gamma})\wedge(\vec{\beta}\wedge\vec{\delta})) \quad (24;\ \text{Case 1})$$

by "trivial distributivity" read backwards.

In Case 2,

RHS $= (\alpha \circ_\ell \gamma) \circ_k (\beta \circ_\ell \delta) = (\hat{c}(\alpha \circ_\ell \gamma) \circ_k \hat{c}(\beta \circ_\ell \delta))^+ \circ (\overrightarrow{\alpha \circ_\ell \gamma}) \wedge (\overrightarrow{\beta \circ_\ell \delta})$

$= ((\hat{c}\alpha \circ_\ell \hat{c}\gamma) \circ_k (\hat{c}\beta \circ_\ell \hat{c}\delta))^+ \circ (\vec{\alpha} \wedge \vec{\gamma}) \wedge (\vec{\beta} \wedge \vec{\delta})$ (25; Case 2)

For the equalities (22)=(24) and (23)=(25), we need, for one thing, that the concatenations $(-)\wedge(-)$ appearing the same are in fact the same. The proof of this fact will use the four coherence conditions for the middle exchange law in dimension $n-1$. In Case 1, that is enough. In Case 2, for (23)=(25), we use the induction hypothesis that the middle exchange law holds in dimension $n-1$.

(**Identity cells in Y**)

We will write id_α for the identity $(n+1)$-cell on the n-cell α in the ω-category $\boldsymbol{Y}$; and 1_α is the identity multiarrow on the object α in the multicategory $\boldsymbol{C}_{n+1}(\boldsymbol{R})$.

For $\alpha \in Y_0$, $\mathrm{id}_\alpha = 1_\alpha$.

For $\alpha \in Y_n$, $n \geq 1$, $\mathrm{id}_\alpha = \alpha^+$. Note that since $\alpha^+ \in R_{n+1}$, also $\alpha^+ \in S_{n+1}$. It is clear, by the definition in the form (1), that $\alpha^+ \in Y_{n+1}$.

It is clear that $\hat{d}(\mathrm{id}_\alpha) = \hat{c}(\mathrm{id}_\alpha) = \alpha$.

Recursively, for $\gamma \in Y_k$, we write id^0_γ for γ, and id^m_γ for $\mathrm{id}_{\mathrm{id}^{m-1}_\gamma}$ when $m \geq 1$. We have $\mathrm{id}^m_\gamma \in Y_{k+m}$. We also write $\mathrm{id}^{(k+m)}_\gamma$ for id^m_γ if $\gamma \in Y_k$.

We leave the verification of the **identity laws** to the reader:

(26) $\mathrm{id}^{(n)}_\gamma \circ_k \alpha = \alpha = \alpha \circ_\ell \mathrm{id}^{(n)}_\delta$ ($k<n$, $\ell<n$, $\alpha \in Y_n$, $\gamma \in Y_k$, $\delta \in Y_\ell$, composites are well-defined)

(27) $\mathrm{id}_\gamma \circ_\ell \mathrm{id}_\delta = \mathrm{id}_{\gamma \circ_\ell \delta}$ ($\ell<n-1$, $\gamma, \delta \in Y_{n-1}$, $\gamma \circ_\ell \delta$ is well-defined)

We have verified that $\boldsymbol{Y}=\langle \boldsymbol{R} \rangle$ is an ω-category.

We have defined the object function of the functor $\langle \text{-} \rangle : \mathtt{multSet} \longrightarrow \omega\mathtt{Cat}$. Specifying the action of the functor on arrows is now easy.

Let $f : \boldsymbol{R}^1 \longrightarrow \boldsymbol{R}^2$ be an arrow, to define $\langle f \rangle : \boldsymbol{Y}^1 = \langle \boldsymbol{R}^1 \rangle \longrightarrow \boldsymbol{Y}^2 = \langle \boldsymbol{R}^2 \rangle$.

First of all, note the almost obvious fact that there is a unique map $f^+ : (\boldsymbol{R}^1)^+ \longrightarrow (\boldsymbol{R}^2)^+$ of ms's extending f such that $f^+(u)$ is an indeterminate in $(\boldsymbol{R}^2)^+$ for any indeterminate u in $(\boldsymbol{R}^1)^+$. This is the same thing as saying that the square

$$\begin{array}{ccc} \boldsymbol{R}^1 & \xrightarrow{\quad f \quad} & \boldsymbol{R}^2 \\ {\scriptstyle \mathtt{incl}}\downarrow & \square & \downarrow{\scriptstyle \mathtt{incl}} \\ (\boldsymbol{R}^1)^+ & \xrightarrow[\quad f^+ \quad]{} & (\boldsymbol{R}^2)^+ \end{array}$$

is a pullback diagram. The definition is, as always for a morphism of ms's, a dimensional recursion.

For $\alpha \in Y^1_n = \mathtt{C}_n(\boldsymbol{Y}^1)$, we put $\langle f \rangle(\alpha) = f^+(\alpha)$.

It is easy to check that $\langle f \rangle$ so defined is a morphism of ω-categories. To carry out the checking, one extends the definition of $\langle f \rangle$ to functions of the form $\langle f \rangle : \hat{Y}^1_n \longrightarrow \hat{Y}^2_n$; these extensions are defined as restrictions of the maps $f^+ : (\boldsymbol{N}^1_n)^+ \longrightarrow (\boldsymbol{N}^2_n)^+$.

The fact that $\langle \text{-} \rangle$:mltSet $\longrightarrow \omega$Cat so defined is a functor is easily seen.

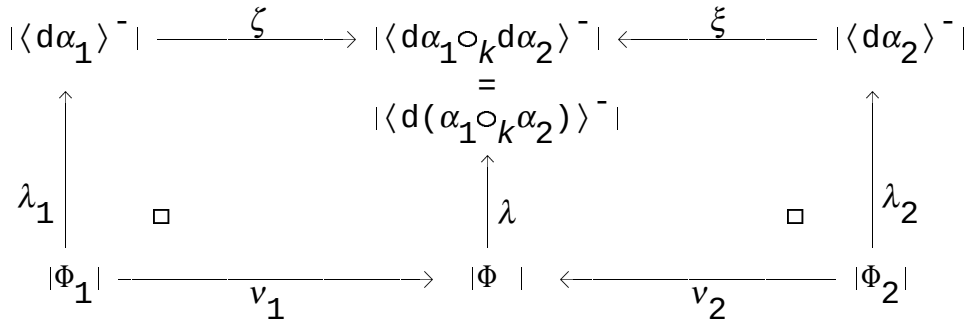
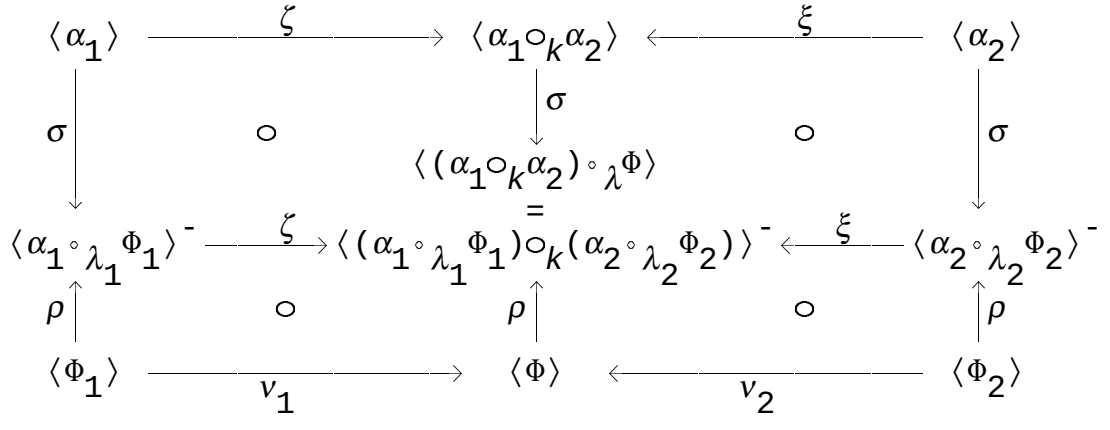
The following lemma, to be used in §5, is analogous to (13).

(28) Suppose $\alpha_1, \alpha_2 \in Y_n$, $k<n-1$, $\alpha_1 \circ_k \alpha_2 \in Y_n$ is well-defined. Suppose that

$\Phi \in \overrightarrow{S_n}$, and that $(\alpha_1 \circ_k \alpha_2) \circ_\lambda \Phi$ is well-defined. Then

$$(\alpha_1 \circ_k \alpha_2) \circ_\lambda \Phi = (\alpha_1 \circ_{\lambda_1} \Phi_1) \circ_k (\alpha_2 \circ_{\lambda_2} \Phi_2)$$

for $\Phi_1, \Phi_2, \lambda_1, \lambda_2$ determined by any diagram of the form



Moreover, the following commutes: